THE OBSTACLE PROBLEM

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Introduction

The material presented here, corresponds to the Fermi lectures, that I was invited to deliver at the Scuola Normale de Pisa in the spring of 1998.

It was for me a great honor, for the prestige of the Scuola and the lectures, and a touching experience, because of the enormous influence that Italian mathematics, particularly the schools of Stampacchia and De Giorgi, had in my work.

The obstable problem, in fact, was one of the main motivations for the development of the theory of variational inequalities (of which Stampacchia was one of the main architects) and the problematic of free boundary problems, in the late '60's early '70's.

On the other hand many of the themes of De Giorgi's work appear in these lectures; Boundary Harnack inequalities, classification of global solutions, flatness and regularity, etc.

The lectures are almost self-contained, except for the work of De Giorgi and Weinberger-Littman-Stampacchia quoted in the Appendix. Finally, I would like to thank my colleagues at Pisa for their usual, warm hospitality.

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The obstacle problem consists in studying the properties of minimizers of the Dirichlet integral

$$D(u) = \int_D (\nabla u)^2 \, dX$$

in a domain, D, of R^n, among all those configurations $u(X)$, with prescribed boundary values: $u|_{\partial D} = f(X)$, and constrained to remain, in D, above a prescribed obstacle $\varphi(X)$.

More precisely, we are given:

a) A (smooth) domain, D, of R^n.

b) A (smooth) function $f(X)$ on ∂D.

c) A (smooth) function $\varphi(X)$ on D, with $\varphi|_{\partial D} < f(X)$.

In the Hilbert space, $H^1(D)$, of all those functions, u, with square integrable gradient, we define K to be the closed convex set

$$K = \{u \in H^1, \ u|_{\partial D} = f(X), \ u \geq \varphi\}$$

On K, there is a unique point u_0 that minimizes the Dirichlet integral

$$D(u) = \int (\nabla u)^2 \, dX \ .$$

Such a point u_0 is called the "solution to the obstacle problem." Such a problem is motivated by the description of the equilibrium position of a membrane (the graph of u) that is "attached" at level $f(X)$ along the boundary of D, and is restricted to remain above φ, (the obstacle).

Such a membrane will minimize area integral

$$A(u) = \int \sqrt{1 + (\nabla u)^2} \, dX$$

that is linearized to Dirichlet integral for small deflections. In any case, the theory developed here applies to the "minimal surface", i.e., the non linearized case, but for simplicity we will restrict here to the "linear" case.

The same mathematical problem appears in many other contexts: fluid filtration in porous media, elasto-plasticity, optimal control and financial math. See for instance the book of Friedman ([Fr]) where many of these applications are described, as well as the classical literature on this problem.

We start with some classical statements about u_0:

Lemma 1.

a) u_0 *stays between* $\lambda_1 = \min f(X)$, *and* $\lambda_2 = \max(f(X), \varphi(X))$.

b) u_0 *is superharmonic, and support* $\Delta u_0 \subset \{u_0 = \varphi\}$.

Point a) is a standard application of the weak maximum principle for H^1 functions.

The minimum of two such functions is again in H^1, so, for instance, $\bar{u} = \min(u_0, \lambda_2)$, is an admissible function and

$$D(u_0) = D(\bar{u}) + \int_{\{u_0 > \lambda_2\}} (\nabla u_0)^2 \, dX \geq D(\bar{u}) \ .$$

Therefore $u_0 = \bar{u}$, a.e.

About point b) more than a proof, it is a matter of definition: We would like to say that a function v is super harmonic if $\Delta v \leq 0$. Unfortunately this requires to take two derivatives of v.

Instead we use the weak definition provided by integrating by parts: If v were still smooth, and ψ a C_0 function, we have the (Green formula)

$$\int (\Delta v)\psi = \int v \Delta \psi \ .$$

This allows us to say, for a function v, in L^1_{loc}, that v is super harmonic without taking any derivative.

Definition. v in L^1 loc is super harmonic in D, if, for any $\psi \in C_0^{1,1}(D)$, φ *non negative*, we have

$$\int v \Delta \psi \leq 0 \ .$$

(That is, we have said, heuristically, for any positive φ, $\int(\Delta v)\psi \leq 0$, and if φ "approaches" Dirac's δ, this seems to imply $Deltav \leq 0$. Indeed, from the theory of distributions, Δv is (as a distribution) a negative measure.)

An Appart on Super Harmonic Functions

Let us show that this coincides with the classical definition of superharmonicity.

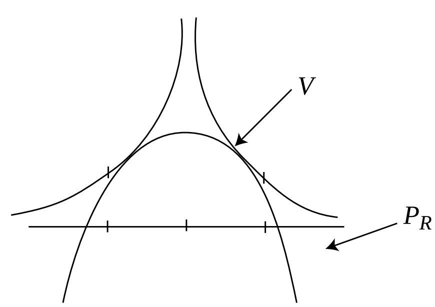
Lemma 2. *If v is weakly superharmonic then its average is a decreasing function of R. More precisely, if $0 < R < S$*

$$\fint_{B_R(X_0)} = \frac{1}{|B_R|}\int_{B_R(X_0)} v(y)\,dy \geq \fint_{B_S(X_0)} v\ .$$

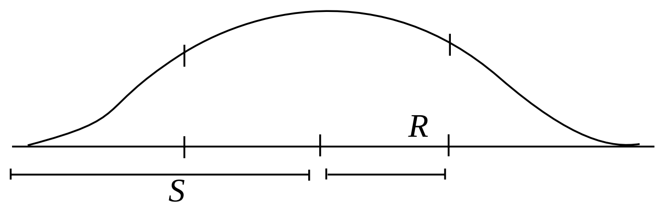
Proof. To that effect we construct a test function ψ that relates both averages. We start with the fundamental solution $V = \frac{1}{|X|^{n-2}}$, and "fit" under its graph a paraboloid, P_0, tangent to V at $|X| = R$ that can be easily computed. We define

$$V_R(X) = \begin{cases} V - P_R \text{ for } X < R \\ \equiv 0 \text{ otherwise} \end{cases}$$

We note that, except at the origin, V_R is $C^{1,1}$ and $\Delta V_R = -C(R)\chi_{B_R}$. We also note that for $R < S$, $V_R \leq V_S$.



Let us take as a test function $\psi = V_S - V_R$



graph of ψ

Notice that ψ vanishes outside B_S, so ψ is now $C_0^{1,1}(R^n)$ and

$$\Delta\psi = C(S)\chi_{B_S} - C(R)\chi_{B_R} \ .$$

Finally, applying the formula, notice that the constants 1 and -1 are both superharmonic, so we get

$$C(S)|B_S| - C(R)|B_R| \left.\begin{matrix}\leq\\=\\\geq\end{matrix}\right\} = 0$$

That is

$$\Delta\psi = C\left[\frac{1}{|B_S|}\chi_{B_S} - \frac{1}{|B_R|}\chi_{B_R}\right] \ .$$

We now use the definition of superharmonicity

$$0 \geq \int v\Delta\psi = C\left[\frac{1}{|B_S|}\int_{B_S} v - \frac{1}{|B_R|}\int_{B_R} v\right]$$

Remark. The function V_R seems to be particular of the Laplacian. It is not so, one can construct such a V_R for any divergence operator, $D_i a_{ij} D_j$, (uniformly elliptic, bounded measurable coefficients) by solving the variational inequality:

$$D_i a_{ij} D_j(V_R) = \frac{1}{R^n}\chi_{\{V_R>0\}} - \delta_{X_0}$$

(δ_{X_0}: Diracs delta at X_0). From the Littman, Stampacchia Weinberger theory, that says the fundamental solution V behaves like $\frac{1}{|X|^{n-2}}$, it can be seen that

$$\{V_R > 0\} \sim \chi_{B_R} \ .$$

In fact, by scaling, it is enough to prove it for $R = 1$.

This provides a "mean value theorem" for general divergence equations, i.e., given X_0, there exists an increasing family of sets $D_R(X_0)$ each one comparable to $B_R(X_0)$ such that if v is a supersolution of $D_i a_{ij} D_j v = 0$, the average, $⨍_{D_R(X)} v \, dX$, is decreasing with R.

Corollary 1. *Any super harmonic function v, has a unique pointwise defined representative as*

$$v(X_0) = \lim_{R\to 0} \fint_{B_R(X_0)} v(X)\, dX \ .$$

Also, v is semicontinuous, that is

$$v(X_0) \leq \varliminf_{X\to X_0} v(X) \ .$$

We now go back to our solution u_0 of the obstacle problem:

About b) u_0 is subharmonic. Indeed any positive $\varphi \in C_0^1$ is an admissible perturbation, i.e.,

$$\begin{aligned} \int (\nabla u_0)^2\, dX &\leq \int (\nabla u_0 + \varepsilon\varphi)^2 \\ &= \int (\nabla u_0)^2 + 2\varepsilon \int \nabla u_0 \nabla\varphi + \varepsilon^2 \int (\nabla\varphi)^2 \ . \end{aligned}$$

Therefore

$$0 \leq \int \nabla u_0 \nabla \varphi \ .$$

Corollary 2. *u_0 is pointwise defined, semicontinuous, and the set $\{u_0 > \varphi\}$ is open. (More precisely if $u_0(X_0) \geq \varphi(X_0) + \delta$, there exists a neighborhood of X_0, where $u_0(X) \geq \varphi(X) + \delta/2$.)*

Corollary 3. *Δu_0 is supported in the closed set $\{u_0 = \varphi\}$.*

Corollary 4. *u_0 is continuous.*

The proof of Corollary 4 is a consequence of Evans theorem, that says:

Theorem 1. *Let v be a superharmonic function, and suppose that $v/\text{support}(\Delta v)$ is continuous. Then v is continuous.*

Let me sketch the *proof of the Theorem 1*. Suppose that there exists a sequence $X_k \to X_0$, such that $\lim v(X_k) \neq v(X_0)$.

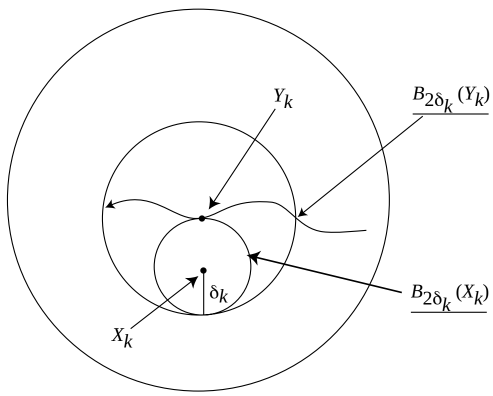
Then a) $X_0 \in \text{support}(\Delta v)$, if not v would be harmonic in a neighborhood of X_0 and thus continuous. (Remember that by definition, support(Δv) is closed.)

b) $X_k \notin \text{support}(\Delta v)$, since v is continuous there.

c) $\lim v(X_k) = a > v(X_0)$ (no loss of generality) $= 0$ by semincontinuity.

d) Further, by semicontinuity, given $\varepsilon > 0$, we may assume $v(X) \geq -\varepsilon$, for $|X - X_0| \leq \delta$.

Let Y_k be the closest point to X_k in support(Δv) (in particular $\delta_k = |Y_k - Y_k| \leq |X_0 - X_k| \to 0$.)



Then $v(Y_k) \to v(X_0) = 0$. Let us find a contradiction: By superharmonicity

$$v(Y_k) \geq \frac{1}{|B_{2\delta_k}(Y_k)|} \int_{B_{2\delta_k}(Y_k)} v(Y)\, dY \ .$$

But we evaluate

$$\int_{B_{2\delta_k}(Y_k)} v(y)\, dY = \int - B_{2\delta_k}(Y_k) \setminus B_{\delta_k}(X_k) + \int_{B_{\delta_k}(X_k)} = I_1 + I_2 \ .$$

For I_1, we use that $v \geq -\varepsilon$, once our configuration is close to X_0. Therefore $I_1 \geq -\varepsilon |B_{2\delta_k}(Y_k)|$.

For I_2, we use that v is harmonic in $B_{\delta_k}(X_k)$. Thus the mean value theorem holds and

$$I_2 = |B_{\delta_k}| v(X_k)$$

Therefore

$$v(Y_k) \geq -\varepsilon + \frac{1}{2^n} v(X_k) \ .$$

that converges to $\frac{a}{2^n}$, a contradiction.

Before we go on, let me list a few *properties of harmonic and superharmonic functions* that are an easy consequence of the mean value theorem:

a) Harnack inequality.

If v is harmonic and non-negative in B_1, then for $R < 1$

$$\sup_{B_R} v \leq C(R) \inf_{B_R} v \ .$$

($C(R)$ goes to infinite when R goes to one as a (negative) power of $(1 - R)$).

b) Derivative estimates.

If v is harmonic in $B_1(0)$

$$|\nabla v(0)| \leq \operatorname*{osc}_{B_1(0)} v \ .$$

Note that harmonicity is a linear, translation invariant property, thus derivatives of harmonic functi ons are again harmonic, and by iterating b), and scaling, we get

b′) $|D^{(k)} v(0)| \leq C(k) \dfrac{1}{r^k} \operatorname*{osc}_{B_r(0)} v$. (Check it!).

c) Strict maximum principle.

Given a superharmonic function v in D, v cannot have a local minimum, $v(X_0)$ in D, unless v is identically a constant.

Finally, we will also need the

d) Solvability of Laplace's equation in a ball.

Given f continuous in ∂B_1, then there exists a unique harmonic function v, such that

$$v|_{\partial B_1} = f \ .$$

We are now ready to launch into the study of the regularity properties of u and the set $u = \varphi$. We start with the regularity of u. The $C^{1,1}$ regularity of u, when φ is $C^{1,1}$ is due to Frehse [F], The proof below is in [C-K]. We will always stay away from ∂D.

Theorem 2. *"Up to $C^{1,1}$, u is as regular as φ". More precisely*

a) *Assume that φ has a modulus of continuity $\sigma(r)$, then u has mod ulus of continuity $C\sigma(2r)$*
b) *Assume now that $\nabla\varphi$ has modulus of continuity $\sigma(r)$, then ∇v has modulus $C\sigma(2r)$.*

To prove Theorem 2, we start with

Lemma 3. *Let $u(X_0) = \varphi(X_0)$. Then "u separates from φ with the speed dictated by $\sigma(r)$", more precisely in case a)*

$$\sup_{B_r(X_0)} u - \varphi \leq C\sigma(2r) \ .$$

In case b)

$$\sup_{B_r(X_0)} u - \varphi \leq Cr\sigma(2r) \ .$$

Proof. We prove case b). Let $L = \varphi(X_0) + \langle \nabla\varphi(X_0), X - X_0 \rangle$ be the linear part of φ at X_0. Then, by definition of $\sigma(r)$, on $B_r(X_0)$

$$L - r\sigma(r) \leq \varphi(X) \leq u(X) \ .$$

Let us show that on $B_{r/2}$,

$$u(X) \leq L + Cr\sigma(r) \ .$$

Consider

$$w = u - (L - r\sigma(r)) \ .$$

This is a non-negative superharmonic function in B_r.

Let us split it in $w = w_1 + w_2$, with w_1 harmonic and equal to w in

Thus, since w is superharmonic and non negative

$$0 \leq w_1 \leq w \ ,$$

and hence also

$$0 \le w_2 \le w \ .$$

We have that

$$w_1(X_0) \le u(X_0) - (L - r\sigma(r)) = \varphi(X_0) - (L - r\sigma(r)) = r\sigma(r) \ .$$

By Harnack inequality

$$w_1|_{B_{r/2}} \le Cr\sigma(r) \ .$$

About w_2, it is superharmonic and vanishes on ∂B_r. Thus, it attains its maximum in the support of its Laplacian. But

$$\Delta w_2 = \Delta u \ .$$

So it attains its maximum at a point X_1, where $u = \varphi$.

But remember that $w_2 \le w = u - (L - r\sigma(r))$. Thus

$$w(X_1) \le \varphi(X_1) - (L - r(\sigma(r)) \le C\sigma(r) \ .$$

The proof of the lemma is complete.

In particular if for instance φ is $C^{1,1}$, u "lifts away" from φ in a quadratic fashion, that is $(u - \varphi)(X) \le C|X - X_0|^2$.

From part b) of the Theorem, let me just show that if φ is $C^{1,1}$, then u is $C^{1,1}$ (away from ∂D). This just follow s by scaling: Let $X_1 \in \Omega = \{u > \varphi\}$, $d(X_1, \Lambda) = d(X, X_0) = \rho$ $(\Lambda = \{u = \varphi\}, X_0 \in \Lambda)$. Then on $B_\rho(X_1)$, u is harmonic. But on $B_{4\rho}(X_1)$, u has quadratic bounds away from L_{X_0}. (Now, φ being $C^{1,1}$, we can take $r\sigma(r) = Cr^2$.) Thus,

$$\|D^2 u(X_1)\| \le \frac{1}{\rho^2} \operatorname*{osc}_{B_\rho(X_1)} (u - L_{X_0}) \le C \frac{1}{r^2} r^2 \le C \ .$$

We have now completed the local regularity theory of u, i.e., $u \in C^{1,1}$ is as good an estimate as we may hope for, since Δu jumps from zero to $\Delta\varphi$ across $\partial\Omega$.

We now begin to study the regularity of $\partial\Omega$, that is

Free Boundary Regularity: Part I, generalities.

The material in this part can be found in [C-1] and [C-3].

To study the free boundary regularity, we reduce it to a local problem, and consider the new variable $w = u - \varphi$. Then, we have the following local problem.

Definition (Normalized solutions). In the unit ball of R^n we are given a function w with the following properties:

a) $w \geq 0$, w is $C^{1,1}$.

b) On the set $\Omega = \{w > 0\}$, $\Delta w \equiv 1$

c) The point 0 belongs to $\partial\Omega$ (i.e., is a free boundary point).

Question. What can we say about the geometry of $\partial\Omega$?

Some Remarks. a). On $\Omega = \{w > 0\}$ we really have

$$\Delta w = \Delta(u - \varphi) = -\Delta\varphi = g(X) \ .$$

Since u is superharmonic, it cannot touch φ at a point where $\Delta\varphi \geq 0$, so near the free boundary we should expect $g(X) \geq 0$. In fact, if $\Delta\varphi$ and $\nabla\Delta\varphi$ do not vanish simultaneously (a necessary non-degeneracy condition), a variation of Hopf's principle shows that $g(X) > 0$ near the free boundary.

Remark b). We have made $g(X) \equiv 1$. All it is necessary is $g(X) =$ of class C^α for the general theory and $g(X)$ of class $C^{1,\alpha}$, to show that singul ar points lay in smooth manifolds, but these assumptions would fill the proofs of little technicalities.

Remark c): Important Rescaling Observations. The function $w_\lambda = \frac{1}{\lambda^2} w(\lambda X)$ satisfies the same conditions, in B_λ instead of B_1.

We start with the following

Lemma 4: Optimal regularity. *w restricted to $B_{1/2}$ is bounded by a universal constant and its $C^{1,1}$ norm is also bounded by a universal constant.*

Proof. Apply Theorem 2 to $u = w - \frac{1}{2n}|X|^2$, $\varphi = -\frac{1}{2n}|X|^2$.

Lemma 5: Optimal gradient bound.

$$|\nabla w(X_0)| \leq C(w(X_0))^{1/2} \ .$$

Proof. Let $w(X_0) = h > 0$. Then, from the second derivative bounds, $B_{(Ch)^{1/2}}(X_0) \subset \Omega$. (If $\exists\ X_1 \in B_{Ch^{1/2}}(X_0) \cap \Lambda$, find a contradiction!). In $B_{(Ch)^{1/2}}(X_0)$, $\Delta w = 1$ and

$$v = w + \frac{(Ch - |X - X_0|^2)}{2n}$$

is har monic and non-negative.

From Harnack and Interior estimates

$$\begin{aligned} |\nabla w(X_0)| = |\nabla v(X_0)| &\leq \frac{C}{(Ch)^{1/2}} \operatorname*{osc}_{B_{(Ch)^{1/2}}} v \leq \frac{C}{(Ch)^{1/2}} v(X_0) \\ &\leq \frac{C}{h^{1/2}} h = Ch^{1/2} \ . \end{aligned}$$

Lemma 6: (Maximum growth). *Let* $X_0 \in \overline{\Omega}$, *then* $\sup_{B_r(X_0)} w \geq Cr^2$.

Proof. It is enough, by continuity, to prove it for $X_0 \in \Omega$. Let $v = w - \frac{1}{2n}|X - X_0|^2$. Then v is harmonic in $\Omega \cap B_r(X_0)$, and positive at X_0. It should take a positive maximum on $\partial(\Omega \cap B_r(X_0))$. But on $(\partial\Omega) \cap B_r(X_0)$, $w \equiv 0$, so v is negative.

Thus the maximum takes place at a point $X_1 \in \partial B_r$. There

$$0 < v = w(X_1) - \frac{1}{2n} r^2 \ .$$

Lemma 7. *The "level strip",* $S_h = \{0 < w < h^2\} \subset \Omega$, *satisfies*

$$\text{meas}(S_h \cap B_r) = |S_h \cap B_r| \leq Ch\, r^{n-1} \ .$$

We split t he proof into two steps:

Lemma 7a. *Let $w_e = D_e w$ be the directional derivative of w in the direction e. Then*

$$\int_{\{0\le w_e\le h\}\cap B_r} |\nabla w_e|^2 \le C h\, r^{n-1} \ .$$

Proof. By the rescaling properties of w (i.e., by looking at $w_r = \frac{1}{r^2} w(rX)$) it is enough to look at $r = 1$ (Check this fact!!). We truncate w_e at levels ε and h: $\overline{w}_e = \min[(w_e - \varepsilon)^+, h]$, and write the usual formula

$$\int_{B_1} \nabla \overline{w}_e \nabla w_e + \overline{w}_e \Delta w_e = \int_{\partial B_1} \overline{w}_\varepsilon D_\nu w_e \ .$$

Since $\Delta w_e \equiv 0$ in Ω (in particular for $w_e > \varepsilon$) and $|D_\nu w_e| \le |D^2 w| \le C$, we immediately get

$$\int_{\{\varepsilon < w_e < h\}} |\nabla w_e|^2 \le Ch \ .$$

Proof of Lemma 6. Again we prove it for $r = 1$

$$S_h \subset \{|\nabla w| < h\} \text{ (from Lemma 4) } \subset \cap\{w_{\pm e_n} < h\} \ .$$

Thus

$$|S_h \cap B_1| = \int_{S_h\cap B_1} \Delta w \le C \int_{S_h \cap B_1} |D^2 w|^2 \le \Sigma \int_{\{w_{\pm e_n} < h\}} |\nabla w_e|^2 \, dX \ .$$

Corollary 5. *The neighborhood N_δ of the free boundary*

$$(N_\delta(S)) = \{X : d(X,S) \le \delta\}$$

has measure

$$|N_\delta \cap B_r| \le \delta\, r^{n-1} \ .$$

In particular, the free boundary has locally finite $n-1$ directional Hausdorff measure, and

$$H^{n-1}(\partial\Omega \cap B_r) \le C\, r^{n-1} \ .$$

Proof. $(N_\delta \cap \Omega) \subset S_\delta$, and thus

$$|N_\delta \cap \Omega \cap B_r| \leq \delta\, m r^{n-1} \ .$$

But Ω has uniform positive density along the free boundary, i.e., for $X_0 \in \partial\Omega$, we have

$$\frac{|B_r(X_0) \cap \Omega|}{|B_r|} \geq \mu > 0$$

for some universal constant μ.

Indeed, from Lemma 6 (maximum growth), $w(X_1) = \sup_{B_r(X_0)} w \geq Cr^2$, and from $C^{1,1}$ estimates, $|\nabla w|_{B_{2r(X_0)}} \leq Cr$. Thus w must remain positive in $B_{Cr}(X_1)$ for C small enough.

This completes the "generalities" part of our discussion, that is those properties and techniques common to many free boundary problems: optimal regularity, to allow for rescalings, maximum possible growth to provide stability of the free boundaries and coincidence sets under rescalings and measure theoretical properties of the free boundaries.

Free Boundary Regularity: Part II.

We now go to a special issue in each free boundary problem, that is the classification of global solutions to our problem. I.e., if we plan, as in the theory of minimal surfaces to prove local regularity by a blow up argument we want to see what is special of the "blow up limits" of our problem.

The main theorem in this regard is that global solutions are convex. More precisely

Theorem 3. ([C1]) *Let, as before, w be a normalized solution of our problem in $B_1(0)$. Then, there exists a universal modulus of continuity $\sigma(r)$ ($\sigma(0^+) = 0$), such that second pure derivatives of w*

$$D_{ee} w$$

satisfy

$$D_{ee} w(X) \geq -\sigma(|X|) \ .$$

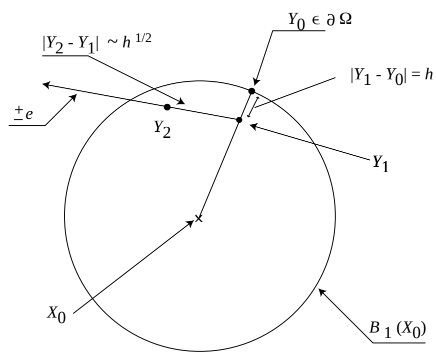
We prove the theorem t hrough the following lemma applied inductively.

Lemma 8. *Assume that $B_r(X_0) \subset \Omega(w)$, and tangent to the free boundary, $\partial\Omega$, at a point Y_0.*

Let $-\alpha = \inf_{B_r(X_0)} D_{ee} w$.

Then $D_{ee}w(X_0) \geq -\alpha + C\alpha^M$ *for some C, M depending only on dimension.*

Proof of Lemma 8. By rescaling w to $\overline{w} = \frac{1}{r^2} w(rx)$, we may assume $r = 1$. (Note that this rescaling does not change bounds on second derivatives.



We do the following construction: In the ray (X_0, Y_0) choose a point, $Y_1 = X_0 + (1 - h)(Y_0 - X_0)$, at distance h from Y_0. Since Y_0 belongs to $\partial\Omega$

$$w(Y_1) \leq Ch^2 \ \text{ and } \ \nabla w(Y_1) \leq Ch \ .$$

Starting at Y_1, the direction $+e$ or $-e$ points "inwards" to the ball (i.e., $\langle + \text{ or } -e, Y_0 - X_0\rangle \leq 0$). Say $+e$. (Note that $D_{ee} = D_{-e,-e}$.) Therefore if $Y_2 = Y_1 + \frac{1}{4}h^{1/2}e$, the segment from Y_1 to Y_2, remains at distance at least $h/4$ from ∂B_1 (for h small).

At Y_2, w is still positive. Let us show that this means that $D_{ee}w$ must be "almost positive" at some point in the segment $I = (Y_1, Y_2)$.

Indeed we represent $w(Y_2) - w(Y_1)$ as a double integral of D_{ee} along I

$$0 \leq w(Y_2) = w(Y_1) + \langle \nabla w(Y_1) \cdot Y_2 - Y_1 \rangle$$
$$+ \iint_I D_{ee} w \leq h^2 + h \cdot \overbrace{|Y_2 - Y_1|}^{h^{1/2}} + \iint_I D_{ee}$$

That is

$$\iint_I D_{ee}w \geq -Ch^2 - Ch^{3/2} \ .$$

But I has length $h^{1/2}$. Therefore

$$(h^{1/2})^2 \sup_I D_{ee}w \geq -Ch^2 - Ch^{3/2} \ ,$$

or

$$\sup_I D_{ee}w \geq -Ch^{1/2} \ .$$

We have, therefore at least, a point $Y_3 \in I$, with $D_{ee}w(Y_3) \geq -Ch^{1/2}$. We want now to choose h so that we have an actual gain over the previous bound:

$$D_{ee}w + \alpha \geq 0 \ .$$

Thus if we choose $-Ch^{1/2} = -\alpha/2$, i.e., $h = \left(\frac{\alpha}{2C}\right)^2$. We have that

$$D_{ee}w(Y_3) + \alpha \geq \alpha/2 \ .$$

We now apply Harnack inequality to the non-negative harmonic function

$$v(X) = D_{ee}w + \alpha \ .$$

It says that

$$D_{ee}w(X_0) + \alpha = v(X_0) \geq Cv(Y_3) \cdot (1 - |Y_3|)^{\bar{M}} = C\frac{\alpha}{2}(h)^{\bar{M}} = C\alpha^{1+2\bar{M}} \ .$$

The lemma is complete.

Corollary 6. *Let w be a normalized solution.*

If $D_{ee}w|_\Omega \geq -\alpha$, then $D_{ee}w|_{\Omega\cap B_{1/2}}$.

Proof. Let $X_0 \in \Omega \cap B_{1/2}$, and $B_r(X_0)$ be the largest ball in Ω containing X_0. Since $0 \in \partial\Omega$, $B_r(X_0) \subset B_1$ and the lemma applies.

Proof of Theorem. By induction we have that, for w a normalized solution, since $w|_{B_{1/2}}$ is $C^{1,1}$

$$D_{ee}w|_{B_{1/2}} \geq -C \equiv -\alpha_0 \ .$$

We apply the lemma inductively and we get

$$D_{ee}w|_{B_{2^{-k}}} \geq -\alpha_k$$

with

$$-\alpha_{k+1} \geq -\alpha_k + C\alpha_k^M \ .$$

This implies $\alpha_k \geq -k^{-\varepsilon}$ for some sma ll ε, or $\sigma(r) = -|\log r|^{-\varepsilon}$.

Corollary 7. *Let w be a solution in B_M, i.e., $\bar{w} = \frac{1}{M^2}w(MX)$ is a normalized solution. Then*

$$D_{\alpha\alpha}w|_{B_1} \geq \sigma\left(\frac{1}{M}\right) \ .$$

Corollary 8. *Let w be a solution in R^n, then w is convex, and $\Lambda(w) = \{w = 0\}$ is convex.*

At this point let us pause for a moment and study what are the possible candidates for global solutions w:

a) $w = \frac{1}{2}(x^+)^2$ (in some systems of coordinates).

These are the "blow up" limits that we expect to get if we start from a point where $\partial\Omega$ is a smooth surface separating Ω from Λ.

b) $w = \sum_1^n \lambda_i \frac{x_i^2}{2}$ with $\lambda_i \geq 0$, $\sum \lambda_i = 1$, always in some system of coordinates.

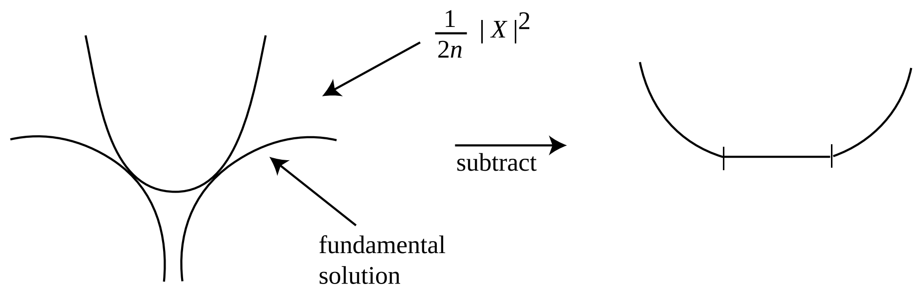
These are the solutions we expect if we started from a point where Λ had very little density or further an isolated point of Λ.

In fact:

Remark. If w is global and Λ a half space, w is as in a) from Cauchy-Kovalevski theorem.

If w is global and Λ has empty interior then w is as in b) from Liouville theorem.

c) We can construct a radial solution with Λ a ball:



So life is not so simple. But at least we can say that if w is a global solution, $0 \in \Lambda$ and the trace of Λ in say ∂B_1 is not "so thin" that is "almost" lower dimensional, then near the origin all level surfaces of w are smooth Lipschitz surfaces.

Let us write, from now on $X = (x', x_n)$ with $x' = (x_1, \ldots, x_{n-1})$.

Lemma 9. *Let w be a global solution (with $0 \in \partial\Omega$) and assume that $B_\rho(-te_n) \subset \Lambda$, for some $0 \leq t \leq 1/2$.*

Then a) for $|x'| \leq \frac{\rho}{8}$, $-t < x_n < 1$, for any unit vec tor σ, with $\sigma_n > 0$, $|\sigma'| \leq \rho/8$, we have

a$_1$) $D_\sigma w \geq 0$

a$_2$) All level surfaces, $w = \lambda$ are Lipschitz graphs.

$$x_n = f(x', \lambda) \quad \text{with} \quad \|f\|_{\text{Lip}} \leq \frac{C}{\rho} \ .$$

a$_3$) $D_{e_n} w(X) \geq C(\rho) d(X, \partial\Lambda)$.

a$_4$) For $|\sigma'| \leq \rho/16$

Note. a$_2$), a$_3$) and a$_4$) follow from a$_1$).

Proof. Since w is convex, the directional derivative, $D_\sigma w$ is monotone along any line in the direction σ, and thus, it becomes positive once such a line intersects Λ.

This proves a$_1$) and thus a$_2$). This also proves that w grows quadratically in the vertical direction that is, if $X = (x', x_n)$, then $w(X) \geq C\rho^2 |x_n - f(x', 0)|^2$. Indeed, consider the ball

$$B = B_{\frac{1}{16}\rho|x_n - f|}(x', f(x', 0)) \ .$$

From maximal growth

$$w(Y) = \sup_B w \geq C\rho^2 |x_n - f|^2$$

but w is monotone increasing along the s egment that joins Y to X, from a_1).

Finally, this implies a_3: Indeed

$$\int_{(x', f(x'))}^{(x', x_n)} D_{e_n} \, dy_n \geq C\rho^2 |x_n - f|^2 \ ,$$

therefore, $\sup_I D_{e_n} \geq C\rho^2 |x_n - f|$. But this can only happen at a point along I, at distance $C\rho^2 |x_n - f|$ from the free boundary.

From Harnack inequality, this also holds for all of the segment between say $C_1 \rho^2 |x_n - f|$ and $|x_n - f|$ with some large constant $C(\rho)$. (Notice that the factor $|x_n - f|$ scales out of the computation.)

Finally, a_4) follows by expressing any such direction σ, as

$$\sigma = a\tilde{\sigma} + b e_n$$

with $\tilde{\sigma}$ a direction for which a_1) applies.

Then a, b can be chosen positive and $b \geq b_0 > 0$, b_0 a universal constant (1/8?). (Check this out)

We will now invoke the theory of harmonic functions in Lipschitz domains to deduce that all these level surfaces are uniformly $C^{1,\alpha}$ all the way to the free boundary. We will develop this theory as soon as we complete the free boundary regularity theory.

Let me state the main theorems (see ¸iteCFMS, [K-J] and [A-C]).

Theorem 4. *Consider the domain $B_1^+ = B_1 \cap \{x_n > 0\}$. Let v_1, v_2 be two non-negative solutions of a divergence operator (a_{ij} bounded measurable)*

$$D_i a_{ij} D_j v = 0$$

and

$$v_i|_{x_n=0} \equiv 0 \ .$$

Let us normalize them so $v_1(\frac{1}{2}e_n) = v_2(\frac{1}{2}e_n) = 1$. *Then, the quotient*

$$u(X) = \frac{v_1(X)}{v_2(X)}$$

is bounded and Hölder continuous up to $x_n = 0$ *in* $B^+_{1/2}$, *with* $\|u\|_{L^\infty}, \|u\|_{C^\alpha} \leq C$ *depending only on the ellipticity of* a_{ij}.

Remark. The hypothesis of v_i are invariant under bilipschitz transformations of B_1^+. Indeed, we transfer the weak formulation of $D_i a_{ij} D_j v = 0$. That is

$$\int D_i \psi a_{ij} D_j v = 0$$

for any H_0^1 test function ψ or

$$\int (\nabla\psi)^T RA\nabla v \, dX = 0 \ .$$

Change variables $Y = Y(X)$. Then

$$\nabla_X \psi = \mathcal{T}_Y^X \nabla\psi(Y)$$

and

$$dX = \det \mathcal{T}_X^Y \, dY \ .$$

So

$$\int (\nabla_Y \psi)^T A^* \nabla_Y v \, dY = 0$$

for a bounded measurable A^*.

Corollary 10. *Same result holds if instead of* B_1^+, *we consider the domain*

$$D = \{|x'| \leq 1, f(x') \leq x \leq M\}$$

for some $f(x')$ *Lipschitz,* $|f(x')| \leq M/2$.

Application to our free boundary problem (still global solutions)**.**

Theorem 5. *Under the hypothesis of Theorem 4, the level surfaces* $\{x_n = f(x', \lambda)\}$ *are uniformly* $C^{1,\alpha}$ *up to* $\lambda = 0$.

Proof. We show that $\frac{w_\sigma}{w_n}$ are uniformly Hölder continuous in $\{|x'| \le \frac{\rho}{8}, |x_n| \le 1\}$ for any "horizontal" σ.

To apply the theorem we miss the positivity of w_σ. Write $\overline{\sigma} = \frac{\rho}{16}\sigma + e_n$. Then a$_4$) applies to $\overline{\sigma}$ (although it is not unitary) and according to Corollary 9, the quotient $\frac{w_{\overline{\sigma}}}{w_{e_n}}$ is C^α up to $\partial\Omega$.

That is $\frac{\rho}{16}\left(\frac{w_\sigma}{w_{e_n}}\right) + 1$ is C^α up to the free boundary and hence

$$u_\sigma = \frac{w_\sigma}{w_{e_n}}$$

is C^α up to the free boundary.

But u_σ is simply, $D_\sigma f(x', \lambda)$. Therefore the graphs $x_n = f(x', \lambda)$ are uniformly $C^{1,\alpha}$ all the way up to $\lambda = 0$, i.e., up to the free boundary.

We next show, in order to complete the theory, that it is not necessary to pass to the limit, i.e., require w to be a global solution to reproduce the geometry above. This is a rather unusual fact, i.e., that in a finite approximation we can directly get regularity, characteristic of this particular problem. First an approximation lemma that says that we can almost reproduce the geometry of the global solutions if our w is a solution on a large enough ball.

Lemma 10. *Fix* $\rho > 0$, $\varepsilon > 0$. *assume that we are given a solution* w *in a ball* $B_M(0)$, *with* $M = M(\rho, \varepsilon)$ *"very" large.*

Assume also that $w|_{B_1(0)}$ *has the following property:*

> *"The set* $\Lambda_w \cap B_1$ *cannot be enclosed in any strip* $\{\alpha < x_n < \beta\}$ *of width,* $(\beta - \alpha) \le 4n\rho$*"*

Then, for $M \ge M(\rho, \varepsilon)$ *large there exist a global solution* $w_\infty(X)$*, that satisfies the hypothesis of Theorem 4, and such that*

$$\|w - w_\infty\|_{L^\infty(B_1)} \quad , \quad \|\nabla w - \nabla w_\infty\|_{L^\infty(B_1)} \le \varepsilon \ .$$

Proof. The proof is by compactness: Assume that such an $M(\rho, \varepsilon)$ does not exist. That means that if we fix (ρ, ε), we can find a sequence of solutions w_k defined in balls $B_k(0)$ that satisfy the hypothesis of the theorem and not the conclusion.

The w_k are a compact family in $C^{1,\alpha}$ in compact sets (they all vanish with its gradient at the origin and are universally $C^{1,1}$ in $B_{k/2}$. Thus a subsequence converges uniformly in compact sets to some function w_∞. To get a contradiction, it suffices to show that w is a global solution satisfy ing the hypothesis of Theorem 4. Indeed

a) $w_\infty \geq 0$, $w_\infty \in C^{1,1}$, and $\Delta w_\infty(x) = 1$, whenever $w(X) > 0$
b) $w_\infty(0) = 0$ and $0 \in \partial\Omega$ because $\sup_{B_r(0)} w_k \geq Cr^2$ independently of k and therefore $\sup_{B_r(0)} w_\infty \geq Cr^2$.

Also, w_∞ and $\Lambda(w)$ are convex and further:

"$\Lambda(w_\infty) \cap B_1$ cannot be enclosed in any strip of width $\frac{7}{2} n\rho$".

Indeed of if $\Lambda(w_\infty)$ is enclosed in such a strip, say

$$\alpha < x_n < \beta \ , \qquad \beta - \alpha = \frac{7}{2} n\rho$$

then $w_\infty \geq \delta > 0$ outside $\{\alpha - \varepsilon \leq x_n \leq \beta + \varepsilon\} \cap B_1$. But w_k is converging uniformly to w in compact sets, thus $w_k \geq \frac{\delta}{2} > 0$ outside

$$\{\alpha - \varepsilon \leq x_n \leq \beta + \varepsilon\} \cap B_1$$

contradicting one of the hypothesis. To complete the proof of the theorem, we need to show that the "strip property" implies that $\Lambda(w_\infty) \cap B_1$ contains a ball of radius ρ.

For that we invoke a lemma of F. John that says that if E is the ellipsoid of largest volume contained in a convex set, in our case, $(\Lambda(w_\infty) \cap B_1)$ then $nE \supset (\Lambda(w_\infty) \cap B_1)$.

With this lemma at hand, if E has one of its diameters smaller than 2ρ, nE has one of its diameters smaller than $2n\rho$ and we can trap $\Lambda(w_\infty) \cap B_1$ in a $2n\rho$ strip, a contradiction.

In order to be able to jump now from the limiting configuration to an approximating one, we need the following curious property of the set $\Omega(w)$ for a normalized solution w.

Lemma 11. *Let h be a harmonic function in $\Omega(w)$. Assume*

a) $h \geq 0$ *on* $\partial\Omega$ *(for instance* $\underline{\lim}\, h \geq 0$*)*

b) *If* N_σ *denotes the* σ *neighborhood of* $\partial\Omega$*,*

  b_1*)* $h|_{N_\sigma} \geq -\sigma$

  b_2*)* $h|_{\Omega \setminus N_\sigma} \geq 1$

Then there exists a universal σ_0 *such that for* $\sigma < \sigma_0$*, the hypothesis above imply that* $h \geq 0$ *in* $\Omega \cap B_{1/2}$*.*

Proof. Suppose not, then there exists an X_0, in $N_\sigma \cap B_{1/2}$, where $h \leq 0$ (since outside of N_σ, $h \geq 1$). Consider, in $B_{1/4}(X_0)$, the function

$$v = h(X) - \delta \left[w(X) - \frac{1}{2n}|X - X_0|^2 \right] .$$

Then (this must sound familiar!)

a) v is harmonic in $B_{1/4}(X_0) \cap \Omega$

b) $v(X_0) \leq 0$

Thus

c) v must have a negative minimum in $\partial(B_{1/4} \cap \Omega)$.

But along $\partial\Omega$, $v \geq 0$, thus the minimum occurs along $\partial B_{1/4}$.

Let us see that this is not possible. On $\partial B_{1/4} \cap N_\sigma$,

$$v \geq -\sigma - C\delta\sigma^2 + \frac{\delta}{2n}\left(\frac{1}{4}\right)^2 > 0$$

if $\delta \geq 200n\sigma$, and σ small. On $\partial B_{1/4} \cap (\Omega \setminus N_\sigma)$

$$v \geq 1 - C\delta^2 \geq 0$$

if δ (universally) small. $\square$

As a corollary we get the following.

Theorem 6. *There exists an $\varepsilon = \varepsilon(\rho)$ such that*

If *w satisfies the conditions of Lemma 9, that is: w is a solution in a ball B_M, $M(\rho, \varepsilon(\rho)) = M(\rho)$ and $\Lambda(w) \cap B_1$ cannot be trapped in a strip of width $2n\rho$.*

Then *in an appropriate system of coordinates.*

a) *Conclusions of Lemma 8 hold for w (with $|x'| < \frac{\rho}{8}$ substituted by $|x'| < \frac{\rho}{16}$ and $|\sigma'| < \frac{\varphi}{16}$ substituted by $|\sigma'| < \frac{\rho}{32}$ in a_4.*

b) *Conclusions of Theorem 5 hold, that is all level surfaces $f(x', \lambda)$ are uniformly $C^{1,\alpha}$ up to $\lambda = 0$.*

Proof. From Lemma 10, we have that there exists a w_∞, satisfying the hypothesis of Lemma 8, such that $|w - w_\infty|_{L^\infty}, |\nabla w - \nabla w_\infty|_{L^\infty} \leq \varepsilon$. Therefore, the crucial pro perties a_3), a_4) that make $f(x', \lambda)$ uniformly Lipschitz graphs, are "almost" satisfied. We will now use Lemma 11, to show that these properties are "fully" satisfied. We start by organizing the information we have by putting Lemma 9 and Lemma 10 together.

Lemma 12. *Let w satisfy the hypothesis of Lemma 9, and let w_∞ be its ε global approximation. Then, in the domain $|x'| \leq \frac{\rho}{8}$, $-t \leq x_n \leq 1$, we have ($N_\varepsilon(S)$ the ε-neighborhood of S)*

a) $\partial\Omega(w) \subset N_{\bar{C}(\rho)\sqrt{\varepsilon}} \quad (\partial\Omega(w_\infty))$

b) $D_{e_n} w(X) \geq C(\rho)[d(X, \partial\Lambda) - C\sqrt{\varepsilon}]$

c) *For* $|\sigma'| \leq \frac{\rho}{16}$

$$D_\sigma w(X) \geq \bar{C}(\rho)[d(X, \partial\Lambda) - C\sqrt{\varepsilon}]$$

d) $w(X) \geq C(\rho)(d(X, \partial\Lambda) - C\sqrt{\varepsilon})^2$

Proof. All we have to prove is a) and then we just put the estimates i n Lemmas 8 and 9 together.

To prove a) we note that if X_0 is in $\Omega(w_\infty)$ and $d(X_0, \partial\Omega(w_\infty)) \geq \bar{C}(\rho)\sqrt{\varepsilon}$ then $w_\infty(X_0) \geq C(\rho)(\bar{C}(\rho)\sqrt{\varepsilon}\,)^2 \geq 2\varepsilon$. Thus, $w(X_0) > 0$ and X_0 cannot belong to $\partial\Omega$.

On the other hand if $X_0 \in \bar{\Omega}(w) \cap \Lambda(w_\infty)$, from non-degeneracy,

$$\sup_{B_{\bar{C}(\rho)\sqrt{\varepsilon}}} w \geq C(\bar{C}(\rho)\sqrt{\varepsilon}\,)^2 \geq 2\varepsilon \ ,$$

thus $B_{\bar{C}(\rho)\sqrt{\varepsilon}}(X_0)$ cannot be contained in $\Lambda(w_\infty)$. $\square$

Now, to complete the proof of Theorem 6, all we have to prove is that w satisfies a_1) of Lemma 8. (Now with $|\sigma'| \leq \frac{\rho}{16}$.) Since a_2), a_3) and a_4) follow from it.

From c) of Lemma 12,

$$D_\sigma w(X) \geq [\bar{C}(\rho) d(X, \partial\Lambda) - C\sqrt{\varepsilon}\,] \ .$$

We apply Lemma 11 to $D_\sigma w$ in $B_{\rho/8}$. Let $h = \frac{D_\sigma w}{\varepsilon^{1/4}}$.

Then if $d(X, \partial\Lambda) \geq \frac{2\varepsilon^{1/4}}{\bar{C}(\rho)}$

$$h(X) \geq 2 - C\varepsilon^{1/4} \geq 1 \ .$$

If $d(X, \partial\Lambda) \leq \frac{2\varepsilon^{1/4}}{\bar{C}(\rho)}$,

$$h(X) \geq -C\varepsilon^{1/4} \ .$$

If we choose ε small enough so that $C(\rho)\varepsilon^{1/4} < (\rho/8)\sigma_0$, we have $h(X) \geq 0$ in $B_{\rho/16}(0)$. Since outside of $B_{\rho/16}, D_\sigma$ is already non-negative, since we are away from Λ, the proof is complete.

By inverting the relation $M = \bar{M}(\rho)$ into $\rho = \rho(M)$, we have proven the following theorem.

Theorem 7. *Let w be a normalized solution.*

Then there is a universal modulus of continuity $\sigma(r)$. (more precisely $\sigma(r) = \rho(\frac{1}{r})$) such that if for one value of r, say r_0, $\Lambda(w) \cap B_{r_0}$ cannot be enclosed in a strip of width $r_0\sigma(r_0)$, then, in an r_0^2 neighborhood of the origin, the free boundary is a $C^{1,\alpha}$ surface $x_n = f(X')$ with

$$\|f\|_{C^{1,\alpha}} \leq \frac{C}{r_0} \ .$$

P roof. Let us renormalize w by r_0, i.e., consider

$$\bar{w} = \frac{1}{r_0^2} w(r_0, X) \ .$$

Then $\bar{w}$ is defined in a ball B_M of radius $M = \frac{1}{r_0}$ and $\Lambda(\bar{w}) \cap B_1$ cannot be enclosed in any strip of width $\sigma(r_0) = \rho(M)$. Thus, Lemma 10 applies.

The Structure of the set of singular points.

Now, it only remains to be studied the structure of the set of singular points of N, $\mathrm{Sing}(w)$ that is those X_0, in $\partial\Omega$, for which $|\Lambda \cap B_r(X_0)| \subset S_{r\sigma(r)}$ (a strip of width $r\sigma(r)$) for every positive r.

Our main objective is to prove the following

Theorem 8. *Given a singular point, say* $X_0 = 0 \in \partial\Omega$

a) *There exists a unique* non-negative *quadratic polynomial*

$$Q_{X_0} = \frac{1}{2}(X^T M X)$$

with $\Delta Q_{X_0} = \mathrm{trace}\ M = 1$, *such that*

$$|(w - Q_{X_0})(X)| \leq |X|^2 \sigma(|X|)$$

for some (universal) modulus of continuity $\sigma(|X|)$.

b) $M(X_0)$ *is continuous on* X_0 *(for* X_0 *in Sing*(w)*) ıtem"c)" If* $\dim \ker M = k$, *the singular set* $\mathrm{Sing}(w)$, *lays in a neighborhood of* X_0, *in a* k*-dimensional* C^1 *manifold.*

The size of the neighborhood depends on the smallest non zero eigenvalue of M.

Before entering into the proof of this theorem, let us point out that, because of compactness, if X_0 is a singular point, then $w|_{B_r(X_0)}$ looks "more and more" as a quadratic polynomial:

Lemma 11. *Given ε, there exists an $M(\varepsilon)$, such that, if w is a solution in B_M and if* 0 *is a singular point of $\partial\Omega$, then, for some non-negative quadratic polynomial $Q = X^T AX$, with $\Delta Q = 1$, we have that*

$$|w - Q|_{B_1} \leq \varepsilon \ .$$

Proof. Suppose not, then there is a sequence of solutions $w_k(X)$ defined in $B_k(0)$, that have zero as a singular point, and for which there is no such polynomial.

Let us take a subsequence, w_k, that converges uniformly in compact sets t o a global solution, w_∞. We prove: w_∞ is a quadratic polynomial Q as above.

Remark 1. Λ_∞ has empty interior. If not, Λ_∞ being convex, $\Lambda_\infty \cap B_1$ will also have nonempty interior, i.e., will contain a ball $B_{r_0}(X_0)$, where $w_\infty \equiv 0$. From nondegeneracy $w_k \equiv 0$ on $B_{r_0/2}(X_0)$ for k large. If not

$$\sup_{B_{\frac{1}{4}r_0}(Y_k)} w_k \geq Cr_0^2 \ \text{ for any } \ Y_k \in B_{r_0/2}(X_0) \cap \Omega(w_k) \ ,$$

contradicting the uniform convergence of w_k to w_0.

But, since 0 is a singular point of w_k, $\Lambda(w_k) \cap B_1$ must be contained in a strip of width $\sigma(\frac{1}{k})$ according to Theorem 7, a contradiction as soon as $\sigma(\frac{1}{k})$ becomes smaller than r_0.

Remark 2. w_∞ is a quadratic polynomial. Indeed $\Delta w_\infty \equiv 1$ and has quadratic growth, so

$$h = w_\infty - \frac{1}{2n}|X|^2$$

is globally ha rmonic with quadratic growth, thus h and hence w_∞ is a quadratic polynomial.

Corollary 11. *Let w be a normalized solution, and* 0 *a singular point. Then given ε, there exists a $r_0(\varepsilon)$ so that for any $r \leq r_0$, we have a quadratic polynomial $Q^r = X^T AX$, with*

$$|w - Q^r|_{B_r} \leq \varepsilon r^2 \ .$$

As before, we invert the relation $\varepsilon(r) = \sigma(r)$ and say: If w is a normalized solution and zero is a singular point, there exists a Q_r such that

$$|w - Q^r|_{B_r} \leq r^2\sigma(r)\ .$$

The problem with Lemma 11, is the standard problem in singularity theory, for instance in minimal surface theory: You take a sequence of blow ups of a minimal surfaces, and you get a minimal cone.

The problem is that different sequences may give different cones, that is the cone (or in our case the quadratic polynomial) may slowly rotate. The question is thus how to "glue" the polynomials Q_r that approximate a normal ized solution at all different levels B_r.

This is solved by the use of a *monotonicity formula*:

Theorem 9 ([ACF]). *In $B_1(0)$, let u_1, u_2 be two continuous functions such that*

a) *Have disjoint supports:* $u_1 \cdot u_2 = 0$

b) $u_1(0) = u_2(0) = 0$

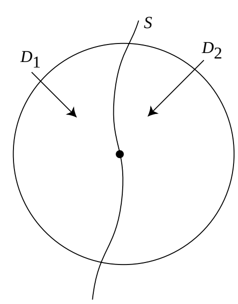
c) $u_i \Delta u_i \geq 0$

Then

$$\mathcal{T}(R) = \left(\frac{1}{R^2}\int_{B_R}\frac{(\nabla u_1)^2}{r^{n-2}}\,dX\right)\left(\frac{1}{R^2}\int_{B_R}\frac{(\nabla u_2)^2}{r^{n-2}}\,X\right) = \mathcal{T}_1 \cdot I_2$$

is monotone increasing in R.

Let me make several remarks about this theorem whose proof we postpone to the end of the discussion. The standard picture to understand this theorem, is to have in $B_1(0)$ a (relatively nice) surface S, through the origin, separating B_1 in two domains D_1 and D_2.



In each of them we have a harmonic function v_i that vanishes along S. Then

a) Each of the terms $\mathcal{T}_i$ can be understood as an average of $(\nabla v_i)^2$, i.e., we are dividing the volume integral in a domain of size $\sim R^n$ by a factor $R^2 r^{n-2} \sim R^n$.

In fact if v_1 is the positive part of a linear function, $v_1(X) = \alpha x_1^+$

$$\mathcal{T}_1(R) \equiv C(n)\alpha^2 \ .$$

with $C(n)$ a precise constant related to the volume of the unit ball of R^n.

b) $\mathcal{T}_i$ has linear scaling, i.e., if $\bar{u}(X) = \frac{1}{\lambda}u(\lambda X)$,

$$\mathcal{T}_i\left(\frac{R}{\lambda}, \bar{u}\right) = \mathcal{T}_i(R, u) \ .$$

c) If S is smooth at zero, i.e., $D_\nu v_i$ exists, then

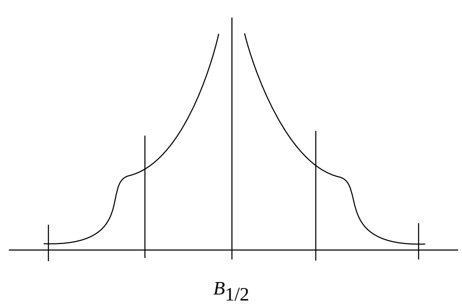
$$\lim_{R\to 0} \mathcal{T}_i = C(n)(D_\nu v_i)^2 \ .$$

In particular

$$C^2(n)(D_\nu v_1)^2(D_\nu v_2)^2 \leq \mathcal{T}(1/2) \ .$$

d) On the other hand one may suspect that the quantity $\mathcal{T}_i(R)$ could be uncontrolled for some "not too bad" v_i. That is not so: Consider the function $\frac{1}{r^{n-2}}$ in $B_{1/2}$, and extended to a function V on B_1 in a smooth, non-negative way, so that $V \equiv 0$ near B_1.

Then



"graph of V"

$$
\begin{aligned}
\mathcal{T}_i(1/2) &= 2^2 \int_{B_{1/2}} \left(\frac{(\nabla v_i)^2}{r^{n-2}} \right) dX \\
&\le 2^2 \int_{B_1} \Delta \left(\frac{v_i^2}{2} \right) \cdot V \, dX \\
&= 2^2 \int_{B_1 \setminus B_{1/2}} (\Delta V) \frac{v_i^2}{2} \, dX
\end{aligned}
$$

e) Finally, a remark on homogeneous solutions to harmonic functions in cones. Let Γ be a cone with vertex at the origin, i.e., given a subset $\Sigma_0 \subset S_1$,

$$
\Gamma = \left\{ X : \frac{X}{|X|} \in \Sigma_0 \subset S_1 \right\} .
$$

Then we can look for homogeneous harmonic functions on Γ, $h(X) = r^\alpha f(\sigma)$, vanishing on $\partial\Gamma$ by finding eigenfunctions on Σ_0:

$$
\begin{aligned}
\Delta h &= D_{rr} h + \frac{(n-1)}{r} D_r h + \frac{1}{r^2} \Delta_\sigma h \\
&= r^{\alpha-2} \left[(\alpha(\alpha-1) + (n-1)\alpha) f(\sigma) + \Delta_\sigma f(\sigma) \right]
\end{aligned}
$$

So if f is an eigenfunction of $\lambda f(\sigma) + \Delta_\sigma f(\sigma) = 0$, in Σ_0, we can associate to f the homogeneous harmonic function $h(X) = r^\alpha f(\sigma)$, where α is related to λ by

$$
\alpha(\alpha + n - 2) = \lambda .
$$

For any positive λ, we have two roots, one bigger than zero and one less than $-(n-2)$ (the extremal case being a constant and the fundamental solution).

We sketch now the proof of the theorem.

Proof. It is enough to prove it for $r = 1$, by scaling

$$
\mathcal{T}'(1) = \mathcal{T}_1' \mathcal{T}_2 + \mathcal{T}_1 \mathcal{T}_2' - 4 \mathcal{T}_1 \mathcal{T}_2 ,
$$

so we want to prove

$$
\frac{\mathcal{T}_1'}{\mathcal{T}_1} + \frac{\mathcal{T}_2'}{\mathcal{T}_2} - 4 \ge 0 .
$$

We now reduce the formula to integrals on $\Sigma_i = D_i \cap S_1$

$$\mathcal{T}_i = \int \frac{(\nabla u_i)^2}{r^{n-2}}\, dX \geq \int \frac{\Delta \frac{(u_i)^2}{2}}{r^{n-2}}\, dX = \int_{\Sigma_i} \left(u_i D_r u_i + \frac{n-2}{2} u_i^2 \right) d\sigma$$

(We use $\int u\Delta v - v\Delta u = \int u v_\nu - v u_\nu$) while

$$\mathcal{T}_i' = \int_\Sigma (\nabla u_i)^2\, dX \ .$$

So

$$\frac{\mathcal{T}_i'}{\mathcal{T}_i} = \frac{\int (u_r)^2 + (u_\sigma)^2\, d\sigma}{\int u u_r + \frac{n-2}{2} u^2\, d\sigma} \ .$$

Note at this point that $\frac{\int (u_\sigma)^2}{\int u^2}$ is minimized by λ_1, the first eigenfunction of the Spherical Laplacian Δ_σ, in Σ_1, so we want to split $u u_r$ in an optimal fashion to spread its control between $\int (u_r)^2$ and $\int u_\sigma^2$, i.e.,

$$\int u u_r = \frac{1}{2}\left[A \int u^2 + \frac{1}{A} \int u_1^2 \ . \right.$$

That will leave us with

$$2\left[\frac{\int u_r^2 + \int (u_\sigma)^2}{\frac{1}{A}\int u_r^2 + [A + (n-2)] \int (u)^2} \ . \right.$$

To perfectly balance both terms, we want

$$\frac{1}{A} = \frac{[A + (n-2)]}{\lambda_1} \ , \quad \text{or } A[A + n - 2] = \lambda_1 \ .$$

If we choose A in such a way we can say that

$$\frac{\mathcal{T}_i'}{\mathcal{T}_i} \geq 2A_i \ .$$

Thus

$$\frac{\mathcal{T}_1'}{\mathcal{T}_1} + \frac{\mathcal{T}_2'}{\mathcal{T}_2} - 4 \geq 2(A_1 + A_2 - 2) \ .$$

But note that A_i, from the formula above, is precisely the homogeneity in r of the function h_i, harmonic in the cone Γ_i, generated by Σ_i, vanishing on $\partial\Gamma_i$.

That is, the monotonicity formula has been reduced to the following question:

> Given two disjoint cones Γ_i in R^n, let h_i be the corresponding homog eneous, non-negative harmonic functions, is it true that the sum of their homogeneities, A_i, is always bigger or equal than two?

Note that for a linear function we have equality (i.e., we have homogeneity one on both sides) and it is natural to guess that this is the extremal configuration. We sketch the main ideas of the proof.

Step a). (Sperner [S]) By symmetrization, among all domains Σ in S_1 of prescribed measure ($|\Sigma|$), the one that minimizes λ, and thus A, is the spherical cup

$$\Sigma_1 = S_1 \cap \{x_1 > \alpha\} \quad , \quad \Sigma_2 = S_2 \cap \{x_1 < \alpha\} \ .$$

There

$$\lambda_1 = \inf \frac{\int_{-1}^{\alpha} (u_{x_1})^2 (1-(x_1)^2)^{(n-2)/2}\, dx_1}{\int_{-1}^{\alpha} u^2 (1-(x_1)^2)^{n/2}\, dx_1}$$

(by changing variables $x_1 = \cos\theta$ in polar coordinates).

Step b). (Friedland and Hayman [F-H]) The minimum decreases with dimension, since an n_1 dimensio nal configuration, can be extended (without changing the homogeneities) to a higher, $n_2 = n_1 + k$ dimension, so we may try to scale properly and study the problem for large dimension.

Note that since $A(A+(n-2)) = \lambda$,

$$A = \frac{1}{2}\left[\sqrt{(n-2)^2 - 4\lambda} - (n-2)\right] = \frac{\lambda}{(n-2)} + \left(\frac{1}{(n-2)^2}\right)$$

for n large, so it is enough to study $\tilde{A}_1 + \tilde{A}_2$ for n going to infinity, with

$$\tilde{A}_1 = \frac{1}{n}\left[\frac{\int_{-1}^{\alpha} (u_{x_1})^2 (1-(x_1)^2)^{(n-2)/2}\, dx_1}{\int_{-1}^{\alpha} u^2 (1-(x_1)^2)^{n/2}\, dx_1}\right.$$

This suggests the change of variable

$$y = n^{1/2} x_1 \ .$$

Then $u_y = \frac{1}{n^{1/2}} u_{x_1}$, and

$$\tilde{A}_1 = \frac{\int_{-n^{1/2}}^{S} (u_y)^2 (1 - y^2/n)^{(n-2)/2}\, dy}{\int_{-n^{1/2}}^{S} u^2 (1 - y^2/n)^{n/2}\, dy} \ .$$

But then $(1 - y^2/n)^{(n-2)/2}$ converges in compact sets to $e^{-y^2/2}$ and $\tilde{A}_1$ converges to the eigenvalue of the Gaussian

$$\frac{\int_{-\infty}^{\infty} (u_y)^2 e^{-y^2/2}\, dy}{\int_{-\infty}^{s} u^2 e^{-y^2/2}\, dy} \ .$$

Our problem thus becomes: Given $s \in R$, let λ_1 be the Gaussian eigenvalue for $(-\infty, s)$, λ_2 for (s, ∞), is it true that

$$\lambda_1 + \lambda_2 \geq 2\lambda(-\infty, 0) \ ?$$

(Remember that $\lambda(-\infty, 0)$ corresponds to the plane solution, for which the desired inequality holds.)

Step c). ([B-K-P]) For the Gaussian, λ is a convex function of s.

Idea of the Proof. The eigenfunction u_λ satisfies an equation

$$D_X e^{-X^2} D_X u = e^{-X^2} u \ .$$

Such an equation

$$D_X f D_X u = -\lambda f u \ .$$

Can always be reduced, by taking a new variable $v = f^{1/2} u$, to

$$D_{XX} v - \frac{(f^{1/2})''}{f^{1/2}} v = \lambda v \ .$$

In our case

$$D_{XX} v - \left(\frac{|x|^2}{4} - \frac{1}{4} \right) v = \lambda v \ .$$

A theorem of Brascamp and Lieb, then says that, if U_1, U_2 are convex sets of R^n, V is a convex potential and *lambda* the first Dirichlet eigenvalue in U of

$$\Delta v - V v = \lambda v \ .$$

Then

$$\lambda(tU_1 + (1-t)U_2) \le t\lambda(U_1) + (1-t)\lambda(U_2)\ .$$

We apply this theorem for $\lambda(a)$ above.

We now go back to the classification theorem for singular points.

Proof of the main theorem.

Some previous remarks. Let 0 be a singular point of $\partial\Omega$. Recall that

a) $(\Lambda \cap B_r) \subset S_{r\sigma(r)}$ (a strip of width $r\sigma(r)$, and that

b) $\exists\ Q^r = \frac{1}{2}(X^T M^r X)$, with M non-negative, trace of $M = 1$, such that

$$|w - Q^r| \le r^2\sigma(r)$$
$$|\nabla w - Q^r| \le r\sigma^{1/2}(r)\ .$$

In rescaled terms, if $w^r = \frac{1}{r^2} w(rX)$ and $\Lambda^r = \Lambda(w^r)$,

$$|w^r - Q^r|_{B_1} \le \sigma(r)\ , \quad |\nabla(w^r - Q^r)| \le \sigma^{1/2}(r)$$
$$(\Lambda^r \cap B_1) \subset S_{\sigma(r)}\ .$$

In particular, outside of $S_{\sigma(r)^{1/3}}$, from a priori estimates:

$$|D_{ij}(w^r - Q^r)| \le \frac{1}{\sigma(r)^{2/3}} \operatorname{osc}(w^r - Q^r) \le \sigma(r)^{1/3}\ .$$

We are now ready for Step 1: Let $u_e = D_e w$, be a directional derivative of w. Then $\Delta u_e = 0$ in Ω, and $u_e \equiv 0$ on Λ, so we can apply the monotonicity formula to u_e^+, u_e^-: The function

$$\mathcal{T}(R) = \left(\frac{1}{R^2}\int \frac{(\nabla u_e^+)^2}{r^{n-2}}\,dX\right)\left(\frac{1}{R^2}\int \frac{(\nabla u_e^-)^2}{r^{n-2}}\,dx\right) = \mathcal{T}^+(R,e)\mathcal{T}^-(R,e)$$

is monotone in R.

In Step 1, we will evaluate $\mathcal{T}^{\pm}(R,e)$ in terms of M^R (i.e., D^2Q^R)

$$\begin{aligned}\mathcal{T}^+(R,e) &= \frac{1}{R^2}\int_{\{(D_e w)>0\}\cap B_R} \frac{|D_{je}w|^2}{r^{n-2}}\,dX \\ &= \int_{\{D_e w^R>0\}\cap B_1} \frac{|D_{je}w^R|^2}{r^{n-2}}\,dX \ .\end{aligned}$$

We want to substitute $D_{je}w^R$ by $D_{je}Q^R = M^R_{je} = e_j^T M^R e$. From the general remarks above

$$\mathcal{T}^+(R,e) = \int_{D_e w^R>0} \frac{\|M^R e\|^2}{r^{n-2}}\,dX + \sigma(r)^{1/3}$$

(we estimate the integral splittings in $S_{\sigma(R)^{1/3}}$ and $\mathcal{C}S_{\sigma(R)^{1/3}}$).

Next we want to substitute the domain

$$D = \{D_e w^R > 0\} \ , \quad \text{by } D^0 = \{D_e Q^R = X^T M^R e > 0\} \ .$$

If we denote by $D^{\pm} = \{D_e Q^R = X^T M^R e > \underset{(-)}{+}\sigma^{1/2}(R)\}$ we have, from the estimates above

$$D^+ \subset D \subset D^- \ .$$

Thus

$$\begin{aligned}&\|M^R e\|^2\left(\int_{D^+}\frac{1}{r^{n-2}}\,dX\right) - \sigma(r)^{1/3} \\ &\qquad \le \mathcal{T}^+(R,e) \le \|M^R e\|^2\left(\int_{D^-}\frac{1}{r^{n-2}}\,dX\right) + \sigma(r)^{1/3} \ .\end{aligned}$$

But $D^- \setminus D^+$ consists of a strip of width $\frac{\sigma(r)^{1/2}}{\|Me\|}$, hence

$$\Big|\int_{D^{\pm}} - \int_{D^0}\Big| \le C\min\left(\frac{\sigma(r)^{1/2}}{\|Me\|}, 1\right) \ .$$

Therefore, with

$$C^*(n) = \int_{B_1^+}\frac{dX}{r^{n-2}}$$

we have the final estimate

$$\begin{aligned}\mathcal{T}^+(R,e) &= \|M^Re\|^2\left(C^*+0\Big(\frac{\sigma(r)^{1/2}}{\|Me\|}\Big)\right)+0(\sigma(r)^{1/3})) \\ &= C^*\|M^Re\|+0(\sigma(r)^{1/3}) \ \text{ and} \\ \mathcal{T}(R,e) &= (C^*)^2\|M^Re\|^2+0(\sigma(r)^{1/3})\ .\end{aligned}$$

(Remember that $\|M^R\|$ is universally bounded since $0\le M^R$, and trace $M^R=1$.

This completes Step 1.

In Step 2, we "glue" all of the M^R by the monotonicity formula.

Corollary of Step 1. *If $R_1\le R_2$, for any unit vector e,*

$$\|M^Re\|^2\le\|M^{R_2}e\|^2+0(\sigma^{1/3}(R_2))\ .$$

Proof. It follows from the fact that $\mathcal{T}(R,e)$ is monotone in R.

Lemma 12. $\|M^{R_1}-M^{R_2}\|\le 0(\sigma^{1/6}(R_2))$.

Proof. Let $N=M^{R_2}-M^{R_1}$. Then N is symmetric, and trace $N=0$. We have

$$\|(M^{R_2}-N)e\|^2\le\|M^{R_2}e\|^2+\sigma^{1/3}(R_2)$$

or

$$-2\langle M^{R_2}e,Ne\rangle+\|Ne\|^2\le\sigma^{1/3}(R_2)\ .$$

Choose e the eigenvector, e_λ, corresponding to $\lambda\le 0$, the smallest eigenvalue of N. Then

$$-2\lambda e^TM^{R_2}e+\lambda^2\le 0^{1/3}(R_2)\ .$$

Since M^{R_2} is non-negative, we have $\lambda^2\le\sigma^{1/3}(R_2)$ or $|\lambda|\le\sigma^{1/6}(R_2)$. Since $\operatorname{tr}N=0$, $\|N\|\le\sigma^{1/6}(R_2)$.

Corollary 12.

a) *As R goes to zero M^R has a unique limit M^0, and*

b) $\|M^R - M_0\| \leq \sigma^{1/6}(R)$

c) $\|w - X^T M_0 X\|\big|_{B_R} \leq \|w - Q^R\| + \|Q^R - Q^0\| \leq R^2\sigma^{1/6}(R)$.

d) *Let $Q^R_{X_0}$ denote the polynomial corresponding to center X_0, then for*

$$\begin{aligned}\|Q^R_{X_0} - Q^R_{X_1}\|_{B_R(X_0)\cap B_R(X_1)} &\leq \|Q^R_{X_0} - w\| + \|Q^R_{X_1} - w\| \\ &\leq R^2\sigma(R)\ .\end{aligned}$$

In particular if $\|X_1 - X_0\| \leq \delta R$ for δ small

$$\|M^0_{X_0} - M^0_{X_1}\| \leq \sigma(r)\ .$$

The rest of the theorem now follows.

Appendix

Boundary values of harmonic functions in Lipschitz Domains

In this section we discuss the boundary regularity properties of harmonic functions in Lipschitz domains, or more generally of solutions of elliptic equations with bounded measurable coefficients.

Since the family of solutions, u, to an equation

$$Lu = D_i a_{ij} D_j u = 0$$

is invariant under bilipschitz transformations (just write the weak formulation)

$$\int_\Omega \nabla \varphi^T \, A \nabla u \, dX = 0$$

for any φ in $H^1(\Omega)$).

We may choose as our basic domain

$$\Omega = B_1^+ = B_1(0) \cap \{x_n > 0\} \ .$$

The main theorem I want to prove is that:

Theorem A. *Let u, v, be two positive solutions to*

a) $Lu = Lv = 0$ *in* Ω

b) u, v *take continuously the value zero on* $\{x_n = 0\}$

Normalize them so

c) $u(\frac{1}{2}e_n) = v(\frac{1}{2}e_n) = 1$

Then

$$\frac{u(X)}{v(X)} \text{ is of class } C^\alpha \text{ in } (B_{1/2}^+)$$

(all the way to $x_n = 0$) and

$$\left\| \frac{u(X)}{v(X)} \right\|_{L^\infty}, \left\| \frac{u}{v} \right\|_{C^\alpha} \le C$$

where C is a universal constant depending only on the ellipticity of $A = [a_{ij}]$.

In order to prove this theorem we w ill need several classical results on the theory of solutions to $Lu = 0$.

Theorem A-1 (DeGiorgi oscillation lemma). *Let v be a subsolution of $Lv = 0$ in B_1, satisfying*

a) $v \leq 1$

b) $|\{v \leq 0\}| = \mu > 0$

Then $\sup_{B_{1/2}} v \leq \lambda(\mu) < 1$.

Theorem A-2 (DeGiorgi-Nash-Moser Interior Harnack inequality). *Let v be a non-negative solution in $B_1(0)$, then for $r < 1$*

$$\sup_{B_r(0)} u \leq (1-r)^{-P} \inf_{B_r(0)} u$$

(for r close to one, we may choose constant one by making p large).

Theorem A-3 (Littman Stampacchia Weinberger — Behavior of the fundamental solution). *The fundamental solution of L behaves like that of the Laplacian, more precisely: Let $B_1 \subset \Omega$, and V satisfy*

a) $L(V) = -\delta_0$ *(Dirac's)*

b) $V\big|_{\partial\Omega} = 0$

Then on $B_{1/2}$

$$\frac{C_1}{r^{n-2}} \leq V \leq \frac{C_2}{r^{n-2}} \ .$$

With these tools at hand, we can prove the theorem. The proof is divided into two main steps:

Step 1: Boundary Harnack principle. If u is as in Theorem A, above. Then

$$u\big|_{B^+_{1/2}} \leq M \ , \text{ a universal constant.}$$

Step 2. Show that $\frac{v}{u}$ remains bounded in $B^+_{1/2}$ all the way to $X_n = 0$.

Step 3. Iterate steps 1 and 2.

Proof of Step 1. We start by noticing

a) If $Y_0 \in \{x_n = 0\}$, then $\sup_{B_r(Y_0)} u$ decreases polynomially, i.e., for $r < R$

$$\sup_{B_r(Y_0)} u \leq C \left(\frac{r}{R}\right)^\alpha \sup_{B_R} u$$

indeed, when extended by $u \equiv 0$ for $x_n < 0$, u is a subsolution of $Lu = 0$ and $|\{u < 0\} \cap B_r| = \frac{1}{2}|B_r|$. From DeGiorgi oscillation Lemma, with $\lambda = \lambda(1/2)$, we get:

$$\sup u\Big|_{B_{r/2}} \leq \lambda \sup u\Big|_{B_r} \ .$$

b) From the interior Harnack inequality

$$\sup_{B_{3/4} \cap \{x_n > S\}} u \leq s^{-p} u \left(\frac{1}{2} e_n\right) = s^{-p} \ .$$

c) Since u takes continuously the value zero at $\{x_n\} = 0$, the sup of u in $B^+_{1/2}$ is attained, i.e.,

$$\sup_{B^+_{1/2}} u = u(X_0) = M \ .$$

We will now show that if $M \geq M_0$ large, we can construct a sequence of points, X_k all contained in $B^+_{3/4}$, $X_k \to \{x_n = 0\}$, and such that $u(X_k)$ goes to $+\infty$.

Construction. We will denote by Y_k the projection of X_k in the $\{x_n = 0\}$ axis.

From interior Harnack,

$$M = u(X_0) \leq |X_0 - Y_0|^{-p} \ .$$

Thus, with $\varepsilon = 1/p$

$$d_0 = |X_0 - Y_0| \leq M^{-\varepsilon}$$

that is X_0 is very close to the $\{x_n = 0\}$ plane.

Now we use the oscillation lemma backwards: Since,

$$\sup_{B_{d_0}(Y_0)} u \geq u(X_0) \geq M \ .$$

This implies that

$$\sup_{B_{d_0}(Y_0)} u = u(X_1) \geq TM$$

(for $T = \frac{1}{\lambda(1/2)}$, a universal constant bigger than one).

Again, by Harnack, as with d_0, we obtain that

$$d_1 = |X_1 - Y_1| \leq (TM)^{-\varepsilon}$$

and, by oscillation backwards, as with $u(x_1)$, we obtain that

$$u(X_2) = \sup_{B_{4d_1}} u \geq Tu(X_1) \geq T^2 M \ .$$

Once more, by Harnack,

$$d_2 = |X_2 - Y_2| \leq (T^2 M)^{-\varepsilon} \ .$$

We repeat inductively the process, and we get a sequence of points X_k, satisfying

a) $u(X_k) \geq T^k M$

b) $|X_k - Y_k| \leq (T^k M)^{-\varepsilon}$

c) $|X_k - X_{k-1}| \leq 4(T^{k-1}M)^{-\varepsilon}$

All we have to make sure is that in this construction we always stayed inside, say $B_{9/16}$. But T is universal, ε is universal and M we can choose as large as we please, so we can make $\sum |X_k - X_{k-1}| \leq 1/16$, and get, for $M \geq M_0$, a contradiction. This proves Step 1.

Proof of Step 2. We want to show now that $\frac{v}{u}$ remains bounded. Since $v(\frac{1}{2}e_n) = u(\frac{1}{2}e_n) = 1$, we have, from Step 1, that $v\big|_{B^+_{1/2}} \leq M$, and from interior Harnack, that $u\big|_{B^+_{1/2} \cap \{x_n \geq 1/8\}} \geq \frac{1}{M}$. So our Step 2 reduces to the following question: Take in R^n, the cube $Q_2(e_n) = \{0 <$

$x_n < 2, |x_j| < 1$ for $j < n\}$ and $Q_{1/2}(\frac{1}{4}e_n) = \{0 < x_n < \frac{1}{2}, |x_j| < \frac{1}{4}$ for $j < n\}$. Let F_1, F_2 be two faces of $Q_2(e_n)$, different from $\{x_n = 0\}$. Let v_i be the function satisfying

a) $Lv_i = 0$ in Q_2

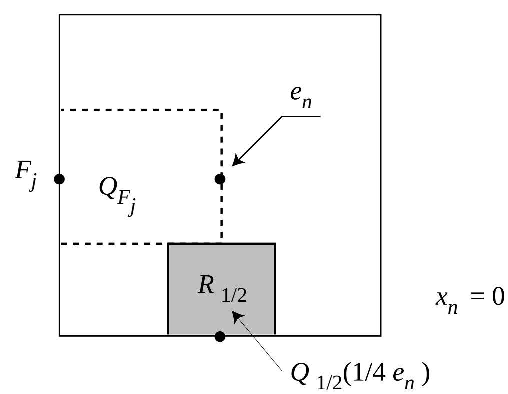
b) $v_i\big|_{\partial Q_2} = \chi_{F_i}$

Show that in $Q_{1/2}$,

$$v_1 \le Cv_2 \ .$$

(This is called the doubling property of harmonic measure. It states that two adjacent "balls" along the boundary of a Lipschitz domain have comparable L-(harmonic) measure. L-harmonic measure may be absolutely singular with respect to Lebesgue measure so this is a nontrivial result.)

Proof. From the oscillation lemma, by extending $(1 - v_i)$ identically zero across F_i we get that $(1 - v_i) \le \lambda < 1$, near F_i, say on the cube Q_{F_i} of sides one with one face lying on F_i (see figure).



Thus $v_i(e_n) \ge (1 - \lambda) > 0$ and thus v_i is strictly positive inside Q_2, say in $Q_1(e_n)$, the cube of sides one centered on e_n.

Let $G(X, Y)$ denote the (L) Green function in the unit cube. From [L-S-W], $G(X, e_1)$ is bounded for X on the boundary of $Q_1(e_n)$, vanishes on ∂Q_2 and hence

$$G(X, e_1) \le (v_1(X))$$

in $Q_2(e_n) - Q_1(e_n)$.

We now show that for X in $R_{1/2} = Q_{1/2}(\frac{1}{4}e_n)$, also $v_1(X) \leq CG(X, e_1)$. For that, we "freeze" X in $Q_{1/2}(\frac{1}{4}e_n)$ and recall that $L_Y G(X,Y) = 0$, i.e., G is a solution in Y for X fixed, as long as $X \neq Y$, in particular for $Y \notin Q_{1/2}(\frac{1}{4}e_n)$.

Therefore, Step 1 applies and with X always frozen in $R_{1/2}$

$$G(X,Y) \leq CG(X, e_n)$$

for say, Y in $Q_{3/4}(\frac{3}{8}e_n)$.

Since G vanishes in $\partial Q_2(e_n)$, the standard energy estimate says that

$$\int_{Q_2(e_n)\smallsetminus Q_1(\frac{1}{2}e_n)} [\nabla_Y G(X,Y)]^2 \, dX \leq C \int_{Q_2(e_n)\smallsetminus Q_{3/4}(\frac{3}{8}e_n)} G^2 \leq CG^2(X, e_n) \ .$$

We now take a $C^{\in} fty$ function η, vanishing in a $\frac{1}{4}$ neighborhood of F_1, and $\eta \equiv 1$ on $Q_1(\frac{1}{2}e_n)$ and represent $v_1(X)$ for X in $Q_{1/4}(\frac{1}{8}e_n)$, by the formulas (no boundary terms left)

$$\int (\eta v_1)(Y) L_Y G(X,Y) + \int \nabla^T (\eta v_1) A \nabla_Y G(X,Y) = 0$$

and

$$\int \eta(Y) G(X,Y) L v_1(Y) + \int \nabla^T (\eta G) A \nabla v_1 = 0$$

That give us, after subtracting

$$v_1(X) = \int \nabla^T \eta A [v \nabla_Y G - G \nabla v] \ .$$

But on the support of $\nabla \eta$,

$$\|v\|_{H^1} \leq C \ , \quad \text{and} \quad \|G\|_{H^1} \leq CG(X, e_n) \ .$$

So $v_1(X) \leq CG(X, e_n)$. (Note that $\int (\nabla \eta v_1)^2$ is bounded from the standard energy inequality since η vanishes near F_1.

Step 2 is complete, since, going back to our u and v, we can say that

$$v \le M\left(\sum_{F_i} v_i\right)$$

and

$$u \ge \frac{1}{M} v_1 \ ,$$

where F_1 is the face opposite to $x_n = 0$.

Step 3. Consists in showing a C^α estimate by iteration, the following way.

Lemm a. *There are constants a_k, b_k, $\frac{1}{M} \le a_k \le b_k \le M$, and a constant $\lambda < 1$, such that:*
On B^+_{2-k},

$$a_k u \le v \le b_k u \ , \quad \textit{and} \ \ (b_k - a_k) \le \lambda(b_{k-1} - a_{k-1}) \ .$$

Proof. By induction: Renormalize $B^+_{2^{-k}}$ to B^+_1 by the transformation $\bar u = u(2^{-k}x)$, define the positive functions

$$w_1(X) = \frac{(\bar v - a_k \bar u)(X)}{b_k - a_k} \ .$$

$$w_2(X) = \frac{(b_k \bar u - \bar v)(X)}{b_k - a_k}$$

and look at the positive numbers $w_1(\frac{1}{2}e_n)$, $w_2(\frac{1}{2}e_n)$. One of them is bigger than $\frac{1}{2}\bar u(\frac{1}{2}e_n)$ since $w_1 + w_2 = \bar u(\frac{1}{2}e_n)$.

Say $w_1(\frac{1}{2}e_n)$. Then by inductive hypothesis $2w_1(X)$ is a non-negative solution of $L = 0$, vanishes on $\{x_n = 0\}$ an $2w_1(\frac{1}{2}e_n) \ge \bar u(\frac{1}{2}e_n)$. Hence

$$\frac{2w_1}{\bar u}\bigg|_{B_{1/2}} \ge \frac{1}{M}$$

or, renormalizing back, in $B^+_{2^{-(k+1)}}$,

$$\frac{v - a_k u}{(b_k - a_k)u} \ge \frac{1}{2M}$$

that is in $B_{2^{-(k+1)}}$

$$\left[a_k + \frac{1}{2M}(b_k - a_k)\right] u \leq v \leq b_k \ .$$

So $b_{k+1} = b_k$ and $a_{k+1} = a_k + \frac{1}{2M}(b_k - a_k)$.

References

[ACF] Alt, H.W., Caffarelli, L.A., Friedman, A., *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461.

[A-C] Athanasopoulos, I., Caffarelli, L.A., *A theorem of real analysis and its application to free boundary problems*, Comm. Pure Appl. Math. **38** (1985), no. 5, 499–502.

[B-K-P] W. Beckner, C.E. Kenig and J. Piper, *A convexity property of eigenvalues, with applications* (to appear).

[C1] Caffarelli, L.A., *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), no. 3–4, 155–184.

[C2] Caffarelli, L.A., *Compactn ess methods in free boundary problems*, Comm. Partial Differential Equations **5** (1980), no. 4, 427–448.

[C3] Caffarelli, L.A., *A remark on the Hausdorff measure of a free boundary, and the convergence of coincidence sets*, Boll. Un. Mat. Ital. A (5) **18** (1981), no. 1, 109–113.

[CK] Caffarelli, L.A., Kinderlehrer, D., *Potential methods in variational inequalities*, J. Analyse Math. **37** (1980), 285–295.

[CFMS] Caffarelli, L.A., Fabes, E., Mortola, S., Salsa, S., *Boundary behavior of nonnegative solutions of elliptic operators in divergence form*, Indiana Univ. Math. J. **30** (1981), no. 4, 621–640.

[F] Frehse, J., *On the regularity of solutions of a second order variational inequality*, Boll. Un. Mat. Ital (4) **6** (1972).

[Fr] Friedman, A., *Variational Principles and free boundary problems*, Wiley, 1982.

[F-H] S. Friedland and W.K. Hayman, *Eigenvalue Inequalities for the Dirichlet problem on spheres and the growth of subharmonic functions*, Comment. Math. Helv. **51** (1979), 133–161.

[J-K] Jerison, D., Kenig, C., *Boundary behaviour of harmonic functions in nontangentially accessible domains*, Adv. in Math. **46** (1982), no. 1, 80–147.

[S] E. Sperner, *Fur symmetrisierung von Functionen auf Sphären*, Math. Z. (1973), 317–327.