Mean-Field Limits for Large Particle Systems
Lecture 1: From Newton to Vlasov

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What is a mean-field dynamics?

System of N identical particles, with pairwise interactions; $N \gg 1$ (e.g. $N =$ Avogadro number $\simeq 6.02 \cdot 10^{23}$...)

Dynamics described
(a) either by the system of motion equations for each particle
(b) or by the motion equation for the "typical particle" driven by the collective interaction with all the other particles

Approach (b) is usually referred to as the "mean-field approximation" for the N-particle dynamics

Advantages/Drawbacks

(a) perfect in theory, unfeasible in practice (phase space of dimension $6N$, how to measure/observe initial data/trajectories?)
(b) only an approximation, but on a phase space of low (fixed) dimension 6

Problem
To justify approach (b) by a rigorous derivation from (a), possibly with a convergence rate as the particle number $N \to \infty$

Examples

Examples of mean-field equations in physics are
•the Vlasov-Poisson or Vlasov-Maxwell system used in the modeling of plasmas or ionized gases
•the Hartree or Hartree-Fock equations used in quantum chemistry ab initio computations

The Boltzmann equation of the kinetic theory of gases is not a mean-field equation: each gas molecule interacts only with another gas molecule at the same position at the same instant of time

The Boltzmann equation has been derived rigorously for short-range molecular interactions (Lanford 1975)
The Vlasov-Poisson system is used for the Coulomb interaction, or Newton's law of gravitation, which are long-range interactions

Syllabus of this course

•Mean-field limit of large particle systems in classical mechanics, with convergence rate

•Mean-field limit of large particle systems in quantum mechanics, with convergence rate

•Quantum dynamics of large particle systems in the semiclassical limit

Emphasis on the uniformity as $\hbar \to 0$ of the mean-field limit in quantum mechanics — the question of singular interactions in this context is still open and will be mostly left aside

Roughly speaking, the idea is to build an $\hbar$-deformation of the metric structure used in the derivation of the mean-field limit in classical mechanics

The diagram

$$\boxed{\textbf{Schrödinger}} \quad \xrightarrow{N\to\infty} \quad \boxed{\textbf{Hartree}}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\hbar \to 0 \qquad\qquad\qquad \hbar \to 0$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\boxed{\textbf{Liouville}} \quad \xrightarrow{N\to\infty} \quad \boxed{\textbf{Vlasov}}$$

Horizontal arrows correspond to the mean-field limit, whereas vertical arrows correspond to the semiclassical limit

Outline of this course

Lecture 1
From Newton's equation of classical mechanics to the Vlasov equation: Dobrushin's proof revisited

Lecture 2
From the N-particle Schrödinger equation to the Hartree equation: BBGKY hierarchy vs. Pickl's approach

Lecture 3
From the N-particle Schrödinger equation to the Vlasov equation — uniformity in $\hbar$ of the mean-field limit in quantum mechanics

DOBRUSHIN'S DERIVATION OF THE VLASOV EQUATION

R.L. Dobrushin: Funct. Anal. Appl. **13**, 115-123 (1979)

See also:
H. Neunzert, J. Wick: Springer LNM 395
W. Braun, K. Hepp: Commun. Math. Phys. 1977

The N-body problem in classical mechanics

System of N identical point particles of mass m, spatial domain $\mathbf{R}^d$
Pairwise interaction given by a potential $V(x_j - x_k)$, with $x_j, x_k \in \mathbf{R}^d$
Newton's second law for the motion of particle no. k:

$$m\dot{x}_j = \xi_j \,, \qquad \dot{\xi}_j = \sum_{\substack{k=1\\k\neq j}}^{N} -\nabla V(x_j - x_k)$$

Assumptions on V:

(H1) $\quad V(z) = V(-z) \quad$ for all $z \in \mathbf{R}^d$

(H2) $\quad V \in C^1(\mathbf{R}^d)$ with $\nabla V \in L^\infty(\mathbf{R}^d) \cap \mathrm{Lip}(\mathbf{R}^d)$

Mean-field scaling

Rescaled time, position and momentum:

$$\hat{t} = t/N\,, \quad \hat{x}_j(\hat{t}) = x_j(t)\,, \quad \hat{\xi}_j(\hat{t}) = \xi_j(t)$$

Motion equations

$$mN\frac{d\hat{x}_j}{d\hat{t}} = \hat{\xi}_j\,, \qquad N\frac{d\hat{\xi}_j}{d\hat{t}} = \sum_{\substack{k=1\\k\neq j}}^{N} -\nabla V(\hat{x}_j - \hat{x}_k)$$

Finite mass assumption

$$Nm = 1$$

N-particle flow

Henceforth drop hats on all variables; our starting point is

$$\dot{x}_j = \xi_j \,, \qquad \dot{\xi}_j = -\frac{1}{N}\sum_{k=1}^{N} \nabla V(x_j - x_k)$$

Notation

$$X_N = (x_1, \ldots, x_N)\,, \qquad \Xi_N := (\xi_1, \ldots, \xi_N)$$

By Cauchy-Lipschitz (see assumption (H2)), the differential system above generates a global flow on $\mathbf{R}^{2dN}$

$$t \mapsto (X_N(t, X_N^{in}, \Xi_N^{in}), \Xi_N(t, X_N^{in}, \Xi_N^{in}))$$

— solution of the differential system with initial data (X_N^{in}, Ξ_N^{in})

The Vlasov equation

Unknown $f(t, dxd\xi) =$ single-particle phase-space number density

$$(\partial_t + \xi \cdot \nabla_x) f - \nabla_x V_f \cdot \nabla_\xi f = 0\,, \qquad x, \xi \in \mathbf{R}^d$$

where $V_f \equiv V_f(t, x)$ is the mean-field potential

$$V_f(t, x) := \iint_{\mathbf{R}^d \times \mathbf{R}^d} V(x - y) f(t, dyd\eta)$$

Empirical measure

N-particle phase space empirical measure

$$\mu_{(X_N,\Xi_N)(t)} := \frac{1}{N}\sum_{k=1}^{N}\delta_{x_k(t),\xi_k(t)}$$

Key observation The two following conditions are equivalent:
(a) $t \mapsto (X_N, \Xi_N)(t)$ is a solution of Newton's differential system of motion equations, and
(b) $t \mapsto \mu_{(X_N,\Xi_N)(t)}$ is a weakly continuous in time, measure-valued solution of the Vlasov equation

Dobrushin's theorem (1979)

Assume that V satisfies (H1-2). Let f^{in} be a probability density on $\mathbf{R}^{2d}$ such that

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d}(|x|+|\xi|)f^{in}(x,\xi)dxd\xi<\infty$$

and let f be the solution of the Vlasov equation with initial data f^{in}. Let $t\mapsto(X_N,\Xi_N)(t)$ be the solution of Newton's differential system with initial data (X_N^{in},Ξ_N^{in}). Then

$$\mathrm{dist}_{\mathrm{MK},1}(\mu_{(X_N,\Xi_N)(t)},f(t,\cdot))\le\mathrm{dist}_{\mathrm{MK},1}(\mu_{(X_N^{in},\Xi_N^{in})},f^{in})e^{\max(1,2\,\mathrm{Lip}(\nabla V))t}$$

Remark Mean-field limit $\Leftrightarrow$ continuous dependence on the initial data in the weak topology of probability measures

Choice of the initial phase-space points

Consider a sequence of initial phase-space points $(x_j^{in}, \xi_j^{in})_{j\geq 1}$ s.t.

$$(x_j^{in}, \xi_j^{in}) \text{ are i.i.d., with distribution } f^{in}(dxd\xi)$$

By the strong LLN

$$\mu_{(X_N^{in}, \Xi_N^{in})} \rightharpoonup f^{in} \quad \text{a.s. in } (x_j^{in}, \xi_j^{in})_{j\geq 1} \text{ as } N \to \infty$$

Since $\operatorname{dist}_{\mathrm{MK},1}$ metrizes the weak topology of Borel probability measures on $\mathbf{R}^{2d}$, the strong LLN implies that, in the limit as $N \to \infty$

$$\operatorname{dist}_{\mathrm{MK},1}(\mu_{(X_N, \Xi_N)(t, X_N^{in}, \Xi_N^{in})}, f^{in}) \to 0 \quad \text{a.s. in } (x_j^{in}, \xi_j^{in})_{j\geq 1}$$

A CRASH COURSE ON MONGE-KANTOROVICH DISTANCES

C. Villani : "Topics in Optimal Transportation", AMS (2003)

Couplings of probability measures

Let $\mathcal{P}(\mathbf{R}^n)$ be the set of Borel probability measures on $\mathbf{R}^n$; for $p > 0$, denote

$$\mathcal{P}_p(\mathbf{R}^n) := \left\{ m \in \mathcal{P}(\mathbf{R}^n) \text{ s.t. } \int_{\mathbf{R}^n} |x|^p m(dx) < \infty \right\}$$

Given $\mu, \nu \in \mathcal{P}(\mathbf{R}^n)$, a coupling of μ and ν is an element $\pi \in \mathcal{P}(\mathbf{R}^n \times \mathbf{R}^n)$ such that

$$\iint_{\mathbf{R}^n \times \mathbf{R}^n} (\phi(x) + \psi(y))\pi(dxdy) = \int_{\mathbf{R}^n} \phi(x)\mu(dx) + \int_{\mathbf{R}^n} \psi(y)\nu(dy)$$

Set of couplings of μ, ν denoted $\Pi(\mu, \nu)$; obviously

$$\mu, \nu \in \mathcal{P}_p(\mathbf{R}^n) \Rightarrow \Pi(\mu, \nu) \subset \mathcal{P}_p(\mathbf{R}^n \times \mathbf{R}^n)$$

Monge-Kantorovich distances

Let $p \geq 1$; for each $\mu, \nu \in \mathcal{P}_p(\mathbf{R}^n)$, the Monge-Kantorovich distance between μ and ν is

$$\text{dist}_{\text{MK,p}}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^p \pi(dxdy) \right)^{1/p}$$

Monge-Kantorovich duality

$$\text{dist}_{\text{MK,p}}(\mu, \nu)^p = \sup_{\substack{\phi(x)+\psi(y) \leq |x-y|^p \\ \phi, \psi \in C_b(\mathbf{R}^n)}} \int_{\mathbf{R}^n} \phi(x)\mu(dx) + \int_{\mathbf{R}^n} \psi(x)\nu(dx)$$

In particular

$$\text{dist}_{\text{MK,1}}(\mu, \nu) = \sup_{\text{Lip}(\phi) \leq 1} \left| \int_{\mathbf{R}^n} \phi(z)\mu(dz) - \int_{\mathbf{R}^n} \phi(z)\nu(dz) \right|$$

Separation axiom $\mathrm{dist_{MK,p}}(\mu,\nu)=0 \Leftrightarrow \mu=\nu$
$\Leftarrow$ choose $\pi=\mu(dx)\delta(x-y)\in\Pi(\mu,\mu)$
$\Rightarrow$ if $\phi\in C^1_c(\mathbf{R}^n)$ and $\pi\in\Pi(\mu,\nu)$, then

$$\left|\int_{\mathbf{R}^n}\phi(z)\mu(dz)-\int_{\mathbf{R}^n}\phi(z)\nu(dz)\right|\le \mathrm{Lip}(\phi)\,\mathrm{dist_{MK,p}}(\mu,\nu)$$
$$\le \mathrm{Lip}(\phi)\left(\iint_{\mathbf{R}^n\times\mathbf{R}^n}|x-y|^p\pi(dxdy)\right)^{1/p}$$

Lemma Let $\lambda, \mu, \nu \in \mathcal{P}_p(\mathbf{R}^n)$, let $\pi \in \Pi(\lambda, \mu)$ and $\rho \in \Pi(\mu, \nu)$; there exists $\omega \in \mathcal{P}(\mathbf{R}^{3n})$ such that

$$\iiint_{\mathbf{R}^{3n}} (\phi(x,y) + \psi(y,z))\omega(dxdydz) = \iint_{\mathbf{R}^{2n}} \phi(x,y)\pi(dxdy) + \iint_{\mathbf{R}^{2n}} \psi(y,z)\rho(dydz)$$

Disintegrate π and ρ along μ:

$$\pi = \int_{\mathbf{R}^n} \pi_y \otimes \delta_y \mu(dy)\,, \quad \rho = \int_{\mathbf{R}^n} \delta_y \otimes \rho_y \mu(dy)$$

Set

$$\omega := \int_{\mathbf{R}^n} \pi_y \otimes \delta_y \otimes \rho_y \mu(dy)$$

Triangle inequality

Let ω be a glueing of $\pi \in \Pi(\lambda, \mu)$ and $\rho \in \Pi(\mu, \nu)$; then, one has

$$\begin{aligned}
\mathrm{dist_{MK,p}}(\mu, \nu) &\le \left(\iiint |x - z|^p \omega(dxdydz) \right)^{1/p} \\
&\le \left(\iiint (|x - y| + |y - z|)^p \omega(dxdydz) \right)^{1/p} \\
&\le \left(\iiint |x - y|^p \omega(dxdydz) \right)^{1/p} + \left(\iiint |y - z|^p \omega(dxdydz) \right)^{1/p} \\
&= \left(\iint |x - y|^p \pi(dxdy) \right)^{1/p} + \left(\iint |y - z|^p \rho(dydz) \right)^{1/p}
\end{aligned}$$

Minimizing over $\pi \in \Pi(\lambda, \mu)$ and $\rho \in \Pi(\mu, \nu)$ implies that

$$\mathrm{dist_{MK,p}}(\lambda, \nu) \le \mathrm{dist_{MK,p}}(\lambda, \mu) + \mathrm{dist_{MK,p}}(\mu, \nu)$$

APPLICATION OF MONGE-KANTOROVICH DISTANCES

CONVERGENCE RATE IN LLN

N. Fournier, A. Guillin, Prob. Theory Rel. Fields **162**, 707–738 (2015)

The Fournier-Guillin estimate

Let $P \in \mathcal{P}_q(\mathbf{R}^n)$ for some $q > 1$. For $Z_N := (z_1, \ldots, z_N) \in \mathbf{R}^{nN}$, we denote the empirical measure of the N-tuple Z_N by

$$\mu_{Z_N} := \frac{1}{N} \sum_{j=1}^{N} \delta_{z_j}$$

Then

$$\int_{\mathbf{R}^{nN}} \text{dist}_{\text{MK},1}(\mu_{Z_N}, P) P^{\otimes N}(dZ_N)$$
$$\leq C \left(\int_{\mathbf{R}^n} |z|^q P(dz) \right)^{\frac{1}{q}} \left(N^{-\frac{1}{q}} + N^{-(1-\frac{1}{q})} \right)$$

provided that $q \neq \frac{n}{n-1}$.

Consequence of Dobrushin's theorem

Assume that V satisfies (H1-2). Let f^{in} be a probability density on $\mathbf{R}^{2d}$ such that, for some $1 < q \neq \frac{2d}{2d-1}$

$$M_q := \iint_{\mathbf{R}^d \times \mathbf{R}^d} (|x| + |\xi|)^q f^{in}(x,\xi) dx d\xi < \infty$$

and let f be the solution of the Vlasov equation with initial data f^{in}. Let $t \mapsto (X_N, \Xi_N)(t)$ be the solution of Newton's differential system with initial data (X_N^{in}, Ξ_N^{in}). Then

$$\int_{\mathbf{R}^{2dN}} \mathrm{dist}_{\mathrm{MK},1}(\mu_{(X_N,\Xi_N)(t,X_N^{in},\Xi_N^{in})}, f(t,\cdot)) \prod_{j=1}^{N} f^{in}(dx_j d\xi_j)$$
$$\leq C M_q^{1/q} e^{\max(1,2\,\mathrm{Lip}(\nabla V))t} \left(N^{-\frac{1}{q}} + N^{-(1-\frac{1}{q})} \right)$$

Limitations of Dobrushin's approach

•Seems limited to Lipschitz continuous interaction forces (but can be modified to treat singular forces: see Hauray-Jabin, Lazarovici)
•Convergence rate estimate limited by quantization error for the initial distribution function f^{in}
•Extension to the quantum N-body probem very unclear — is there a natural notion of empirical measure in quantum mechanics? does it satisfy the Hartree equation? is there a notion of "particle trajectory" in quantum mechanics?

DOBRUSHIN'S ARGUMENT REVISITED

F. Golse, C. Mouhot, T. Paul:
Commun. Math. Phys. **343**, 165–205 (2016)

N-body Liouville equation

Hamiltonian formulation of Newton's equations

$$\dot{x}_j = \partial\mathcal{H}_N/\partial\xi_j\,, \quad \dot{\xi}_j = -\partial\mathcal{H}_N/\partial x_j$$

N-body Hamiltonian

$$\mathcal{H}_N(X_N,\Xi_N) := \sum_{j=1}^{N} \tfrac{1}{2}|\xi_j|^2 + \frac{1}{N}\sum_{1\le j<k\le N} V(x_j - x_k)$$

Liouville equation for the N-particle density $F_N \equiv F_N(t, dX_N d\Xi_N)$

$$\partial_t F_N + \{\mathcal{H}_N, F_N\}_N = 0$$

where

$$\{\Phi,\Psi\}_N := \sum_{j=1}^{N}\left(\nabla_{\xi_j}\Phi\cdot\nabla_{x_j}\Psi - \nabla_{x_j}\Phi\cdot\nabla_{\xi_j}\Psi\right)$$

The Cauchy problem for the Liouville equation

The system of Hamilton's equations is the system defining the characteristic curves of the Liouville equation
Weak solution of the Cauchy problem

$$\partial_t F_N + \{\mathcal{H}_N, F_N\}_N = 0\,, \qquad F_N\big|_{t=0} = F_N^{in} \in \mathcal{P}(\mathbf{R}^{2dN})$$

given by

$$F_N(t) = (X_N, \Xi_N)(t,\cdot)\# F_N^{in}\,, \quad F_N \in C(\mathbf{R}; w - \mathcal{P}(\mathbf{R}^{2dN}))$$

Notation if $T : X \to Y$ is $\mathcal{A} - \mathcal{B}$ measurable, and if μ is a measure on $(X, \mathcal{A})$, the image measure $\nu = T\#\mu$ on $(Y, \mathcal{B})$ is defined by

$$\nu(B) = \mu(T^{-1}(B))$$

Indistinguishable particles

For $\sigma \in \mathfrak{S}_N$, denote $\sigma \cdot X_N := (x_{\sigma(1)}, \ldots, x_{\sigma(N)})$, and set

$$T_\sigma(X_N, \Xi_N) := (\sigma \cdot X_N, \sigma \cdot \Xi_N)$$

Lemma Let $F_N^{in} \in \mathcal{P}(\mathbf{R}^{2dN})$, and let $F_N := (X_N, \Xi_N)(t, \cdot)\#F_N^{in}$ be the solution of the Cauchy problem for the Liouville equation with initial data F_N^{in}. Then, for each $\sigma \in \mathfrak{S}_N$ and all $t \in \mathbf{R}$

$$T_\sigma \# F_N^{in} = F_N^{in} \Rightarrow T_\sigma \# F_N(t) = F_N(t)$$

Idea of the proof: observe that $\mathcal{H}_N \circ T_\sigma = \mathcal{H}_N$, so that, by uniqueness of the solution of the Cauchy problem for Hamilton's equations

$$(X_N, \Xi_N)(t, \cdot) \circ T_\sigma = T_\sigma \circ (X_N, \Xi_N)(t, \cdot)$$

n-particle marginal of F_N

Let $F_N \in \mathcal{P}(\mathbf{R}^{2dN})$ satisfy $T_\sigma \# F_N = F_N$ for all $\sigma \in \mathfrak{S}_N$.

Definition For $n = 1, \ldots, N-1$, the n-particle marginal of F_N is

$$F_{N:n} := P_n^N \# F_N \in \mathcal{P}(\mathbf{R}^{2dn}) \quad \text{where } P_n^N(X_N, \Xi_N) := (X_n, \Xi_n)$$

If F_N has a density (w.r.t. the Lebesgue measure $dX_N d\Xi_N$, then $F_{N:n}$ is the probability density given by

$$F_{N:n}(X_n, \Xi_n) := \int_{\mathbf{R}^{2d(N-n)}} F_N(X_N, \Xi_N) dx_{n+1} d\xi_{n+1} \ldots dx_N d\xi_N$$

Hamiltonian formulation of Vlasov's equation

Mean-field Hamiltonian

$$\mathbf{H}_{f(t,\cdot)}(x,\xi) := \tfrac{1}{2}|\xi|^2 + V_{f(t,\cdot)}(x)$$

where

$$V_{f(t,\cdot)}(x) := \iint_{\mathbf{R}^{2d}} V(x-y) f(t, dy d\eta)$$

Vlasov's equation equivalent to

$$\partial_t f + \{\mathbf{H}_{f(t,\cdot)}, f\}_1 = 0\,, \quad f\big|_{t=0} = f^{in} \in \mathcal{P}(\mathbf{R}^{2d})$$

Lemma For each $f^{in} \in \mathcal{P}_1(\mathbf{R}^{2d})$, there exists a unique weak solution $f \in C(\mathbf{R}_+; w - \mathcal{P}(\mathbf{R}^{2d})$ of the Vlasov equation
Idea of the proof fixed point argument in the complete metric space $C([0,T], (\mathcal{P}_1(\mathbf{R}^{2d}), \mathrm{dist}_{\mathrm{MK},1}))$ (topology of t-uniform convergence)

Propagation of moment bounds

Lemma If $f^{in} \in \mathcal{P}_p(\mathbf{R}^{2d})$ for some $p \geq 1$, the solution of the Cauchy problem for the Vlasov equation with initial data f^{in} satisfies

$$\iint_{\mathbf{R}^{2d}} (|x|^p + |\xi|^p) f(t, dxd\xi) \leq e^{L_p t} \iint_{\mathbf{R}^{2d}} (|x|^p + |\xi|^p) f^{in}(dxd\xi)$$
$$+ 2^{p-1} K_p \|\nabla V\|^p_{L^\infty} \frac{e^{L_p t} - 1}{L_p}$$

with $K_p := \max(1, p-1)$ and $L_p := K_p(1+\max(1, 2^{p-1}\operatorname{Lip}(\nabla V)^p))$

Idea of the proof multiply both sides of the Vlasov equation by $|x|^p + |\xi|^p$, and use Young's inequality

$$pab^{p-1} \leq K_p(a^p + b^p), \qquad a, b > 0$$

to estimate

$$\{\mathbf{H}_{f(t,\cdot)}, |x|^p + |\xi|^p\}_1$$

Theorem

Assume that V satisfies (H1+2). Let f be the solution of the Vlasov equation with initial data $f^{in} \in \mathcal{P}_2(\mathbf{R}^{2d})$ and F_N be the solution of the Liouville equation with initial data F_N^{in}, satisfying $T_\sigma \# F_N^{in} = F_N^{in}$ for all $\sigma \in \mathfrak{S}_N$. Then, for all $t \geq 0$ and $n = 1, \ldots, N$

$$\frac{1}{n}\operatorname{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2 \leq \frac{1}{N}\operatorname{dist}_{\mathrm{MK},2}((f^{in})^{\otimes N}, F_N^{in})^2 e^{\Lambda t} + \frac{8\|\nabla V\|_{L^\infty}^2}{N}\frac{e^{\Lambda t}-1}{\Lambda}$$

where

$$\Lambda := (1 + \max(1, 8\,\mathrm{Lip}(\nabla V)^2))$$

Embedding Vlasov's dynamics in the N-body phase space

Let f be a solution of Vlasov's equation

$$\partial_t f + \{\mathbf{H}_{f(t,\cdot)}, f\}_1 = \partial_t f + \xi \cdot \nabla_x f - \nabla_x V_{f(t,\cdot)} \cdot \nabla_\xi f = 0$$

Equation satisfied by $f(t)^{\otimes N}$:

$$\partial_t f^{\otimes N} + \{\mathbf{H}^N_{f(t,\cdot)}, f^{\otimes N}\}_N = 0$$

where

$$\mathbf{H}^N_{f(t,\cdot)}(X_N, \Xi_N) := \sum_{j=1}^N \tfrac{1}{2}|\xi_j|^2 + \sum_{j=1}^N V_{f(t,\cdot)}(x_j)$$

Comparing $\mathbf{H}^N_{f(t,\cdot)}$ and $\mathcal{H}_N$

An elementary computation shows that

$$\mathbf{H}^N_{f(t,\cdot)}(X_N,\Xi_N) - \mathcal{H}_N(X_N,\Xi_N)$$
$$= \sum_{j=1}^N \left(V_{f(t,\cdot)}(x_j) - \frac{1}{N}\sum_{k=1}^N V(x_j - x_k) \right)$$

If $x_1,\ldots,x_N$ are i.i.d. random variables distributed under

$$\rho_{f(t,\cdot)}(x) = \int_{\mathbf{R}^d} f(t,x,\xi)d\xi$$

the following limit holds a.s. in $(x_j)_{j\geq 1}$ as $N\to\infty$

$$\frac{1}{N}\sum_{k=1}^N V(z-x_k) \to \int_{\mathbf{R}^d} V(z-x)\rho_{f(t,\cdot)}(x)dx$$
$$= \iint_{\mathbf{R}^{2d}} V(z-x)f(t,x,\xi)dxd\xi$$

Dynamics of couplings

Let $P^{in} \in \Pi((f^{in})^{\otimes N}, F_N^{in})$, and let $t \mapsto P(t, dX_N d\Xi_N dY_N dH_N)$ be the weak solution of

$$\partial_t P(t) + \{\mathbf{H}^N_{f(t,\cdot)}(X_N, \Xi_N) + \mathcal{H}(Y_N, H_N), P(t)\}_{2N} = 0$$
$$P\Big|_{t=0} = P^{in}$$

(a) $P(t) \in \Pi(f(t)^{\otimes N}, F_N(t))$ for each $t \in \mathbf{R}$
(b) for each $\sigma \in \mathfrak{S}_N$, set

$$\mathcal{T}_\sigma(X_N, \Xi_N, Y_N, H_N) = (\sigma \cdot X_N, \sigma \cdot \Xi_N, \sigma \cdot Y_N, \sigma \cdot H_N)$$

then

$$\mathcal{T}_\sigma \# P^{in} = P^{in} \Rightarrow \mathcal{T}_\sigma \# P(t) = P(t) \text{ for all } t \geq 0$$

For each $P^{in} \in \Pi((f^{in})^{\otimes N}, F_N^{in})$, set

$$D_N(t) := \int \frac{1}{N} \sum_{j=1}^{N} (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t, dX_N d\Xi_N dY_N dH_N)$$

Lemma Assume that $\mathcal{T}_\sigma \# P^{in} = P^{in}$ for all $\sigma \in \mathfrak{S}_N$. Then

$$D_N(t) \geq \frac{1}{n} \mathrm{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2 , \qquad \text{for } n = 1, \ldots, N$$

Proof Set $c_k := |x_k - y_k|^2 + |\xi_k - \eta_k|^2$; for each $k, n = 1, \ldots, N$

$$D_N(t) = \int c_k P(t) = \int \frac{1}{n} \sum_{j=1}^{n} c_j P(t) \geq \mathrm{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2$$

The dynamics of $D_N(t)$

Notation for $Y_N = (y_1, \ldots, y_N)$, set

$$\mu_{Y_N} := \frac{1}{N}\sum_{j=1}^{N} \delta_{y_j}$$

Multiplying each side of the equation for P by

$$\frac{1}{N}\sum_{j=1}^{N}(|x_j - y_j|^2 + |\xi_j - \eta_j|^2)$$

and integrating in all variables

$$\dot{D}_N(t) = \int \frac{1}{N}\sum_{j=1}^{N}(\xi_j \cdot \nabla_{x_j} + \eta_j \cdot \nabla_{y_j})|x_j - y_j|^2 P(t)$$

$$+\int \frac{1}{N}\sum_{j=1}^{N}\left(\nabla V \star_x \rho_f(x_j) \cdot \nabla_{\xi_j} + \nabla V \star \mu_{Y_N}(y_j) \cdot \nabla_{\eta_j}\right)|\xi_j - \eta_j|^2 P(t)$$

The dynamics of $D_N(t)$

Thus

$$\dot{D}_N(t) = \int \frac{2}{N} \sum_{j=1}^{N} (\xi_j - \eta_j) \cdot (x_j - y_j) P(t)$$

$$+ \int \frac{2}{N} \sum_{j=1}^{N} (\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j)) \cdot (\xi_j - \eta_j) P(t)$$

Using the elementary inequality $2ab \leq (a^2 + b^2)$ implies that

$$\dot{D}_N(t) \leq D_N(t) + \frac{1}{N} \sum_{j=1}^{N} \int |\xi_j - \eta_j|^2 P(t)$$

$$+ \frac{1}{N} \sum_{j=1}^{N} \int |\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j)|^2 P(t)$$

The dynamics of $D_N(t)$

$$\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j) = \nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{X_N}(x_j) + \nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)$$

so that, by convexity of $z \mapsto |z|^2$, one has

$$\dot{D}_N(t) \leq D_N(t) + \frac{1}{N} \sum_{j=1}^{N} \int |\xi_j - \eta_j|^2 P(t) + I_N(t) + J_N(t)$$

with

$$I_N(t) := \frac{2}{N} \sum_{j=1}^{N} \int |\nabla V \star_x \rho_f(t, x_j) - \nabla V \star \mu_{X_N}(x_j)|^2 P(t)$$

$$J_N(t) := \frac{2}{N} \sum_{j=1}^{N} \int |\nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)|^2 P(t)$$

Stability: controling J_N

By convexity of $z \mapsto |z|^2$, and since $\nabla V \in \mathrm{Lip}(\mathbf{R}^d)$, one has

$$
\begin{aligned}
&|\nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)|^2 \\
&= \left| \frac{1}{N} \sum_{k=1}^{N} (\nabla V(x_j - x_k) - \nabla V(y_j - y_k)) \right|^2 \\
&\leq \frac{1}{N} \sum_{k=1}^{N} |\nabla V(x_j - x_k) - \nabla V(y_j - y_k)|^2 \\
&\leq \frac{\mathrm{Lip}(\nabla V)^2}{N} \sum_{k=1}^{N} |(x_j - y_j) - (x_k - y_k)|^2 \\
&\leq 2\,\mathrm{Lip}(\nabla V)^2 \left(|x_j - y_j|^2 + \frac{1}{N} \sum_{k=1}^{N} |x_k - y_k|^2 \right)
\end{aligned}
$$

Since

$$\frac{1}{N}\sum_{j=1}^{N}\int\left(|x_j-y_j|^2+\frac{1}{N}\sum_{k=1}^{N}|x_k-y_k|^2\right)P_N(t)$$
$$=\frac{2}{N}\sum_{j=1}^{N}\int|x_j-y_j|^2P_N(t)$$

we conclude that

$$J_N(t)\le\frac{8\,\mathrm{Lip}(\nabla V)^2}{N}\sum_{j=1}^{N}\int|x_j-y_j|^2P_N(t)$$

Lemma Let ρ be a probability density on $\mathbf{R}^d$. For all $j = 1, \ldots, N$, one has

$$\int \left| \nabla V \star \rho(x_j) - \frac{1}{N} \sum_{k=1}^{N} \nabla V(x_j - x_k) \right|^2 \prod_{m=1}^{M} \rho(x_m) dx_m \leq \frac{4\|\nabla V\|_{L^\infty}^2}{N}$$

Proof Set

$$W(t, x_1, z) := (\nabla V \star_x \rho_f(t, x_1) - \nabla V(x_1 - z))$$

observe that

$$\int W(t, x_1, z) \rho(t, z) dz = 0$$

End of the proof of the Lemma

Therefore

$$\int |\nabla V \star_x \rho_f(t,x_1) - \nabla V \star \mu_{X_N}(x_j)|^2 \prod_{l=1}^N \rho_f(t,x_l)dx_l$$

$$= \frac{2}{N^2} \sum_{1\le j<k\le N} \underbrace{\int W(t,x_1,x_j)\cdot W(t,x_1,x_k) \prod_{l=1}^N \rho_f(t,x_l)dx_l}_{=0}$$

$$+ \frac{1}{N^2} \underbrace{\sum_{j=1}^N \int |W(t,x_1,x_j)|^2 \prod_{l=1}^N \rho_f(t,x_l)dx_l}_{\le N(2\|W\|_{L^\infty})^2}$$

Final estimate

$$\dot{D}_N(t) \le D_N(t) + \frac{1}{N}\sum_{j=1}^{N}\int |\xi_j - \eta_j|^2 P(t)$$
$$+ \frac{8\,\mathrm{Lip}(\nabla V)^2}{N}\sum_{j=1}^{N}\int |x_j - y_j|^2 P_N(t) + \frac{8\|\nabla V\|_{L^\infty}^2}{N}$$
$$\le \Lambda D_N(t) + \frac{8\|\nabla V\|_{L^\infty}^2}{N}$$

Gronwall $\Longrightarrow$

$$\frac{1}{n}\,\mathrm{dist}_{\mathrm{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2 \le D_N(t)$$
$$\le D_N(0)e^{\Lambda t} + \frac{8\|\nabla V\|_{L^\infty}^2}{N}\,\frac{e^{\Lambda t} - 1}{\Lambda}$$