

Mean-Field Limits for Large Particle Systems

Lecture 1: From Newton to Vlasov

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What is a mean-field dynamics?

System of N identical particles, with pairwise interactions; $N \gg 1$
(e.g. $N = \text{Avogadro number} \simeq 6.02 \cdot 10^{23} \dots$)

Dynamics described

(a) either by the system of **motion equations for each particle**

(b) or by the **motion equation for the “typical particle”** driven by the collective interaction with all the other particles

Approach (b) is usually referred to as the **“mean-field approximation”** for the N -particle dynamics

- (a) perfect in theory, unfeasible in practice (phase space of dimension $6N$, how to measure/observe initial data/trajectories?)
- (b) only an approximation, but on a phase space of low (fixed) dimension 6

Problem

To justify approach (b) by a rigorous derivation from (a), possibly with a convergence rate as the particle number $N \rightarrow \infty$

Examples of mean-field equations in physics are

- the Vlasov-Poisson or Vlasov-Maxwell system used in the modeling of plasmas or ionized gases
- the Hartree or Hartree-Fock equations used in quantum chemistry ab initio computations

The Boltzmann equation of the kinetic theory of gases is not a mean-field equation: each gas molecule interacts only with another gas molecule at the same position at the same instant of time

The Boltzmann equation has been derived rigorously for short-range molecular interactions (Lanford 1975)

The Vlasov-Poisson system is used for the Coulomb interaction, or Newton's law of gravitation, which are long-range interactions

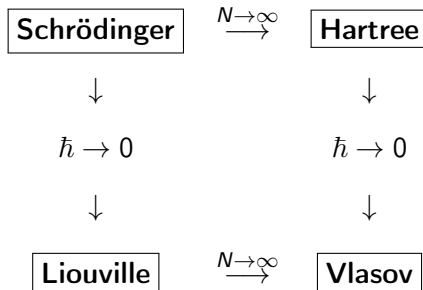
Syllabus of this course

- Mean-field limit of large particle systems in **classical mechanics**, with convergence rate
- Mean-field limit of large particle systems in **quantum mechanics**, with convergence rate
- Quantum dynamics of large particle systems in the **semiclassical limit**

Emphasis on the **uniformity as $\hbar \rightarrow 0$ of the mean-field limit** in quantum mechanics — the question of singular interactions in this context is still open and will be mostly left aside

Roughly speaking, the idea is to build an **\hbar -deformation of the metric structure** used in the derivation of the mean-field limit in classical mechanics

The diagram



Horizontal arrows correspond to the mean-field limit, whereas vertical arrows correspond to the semiclassical limit

Lecture 1

From Newton's equation of classical mechanics to the Vlasov equation: Dobrushin's proof revisited

Lecture 2

From the N -particle Schrödinger equation to the Hartree equation: BBGKY hierarchy vs. Pickl's approach

Lecture 3

From the N -particle Schrödinger equation to the Vlasov equation — uniformity in \hbar of the mean-field limit in quantum mechanics

DOBRUSHIN'S DERIVATION OF THE VLASOV EQUATION

R.L. Dobrushin: *Funct. Anal. Appl.* **13**, 115-123 (1979)

See also:

H. Neunzert, J. Wick: Springer LNM 395

W. Braun, K. Hepp: *Commun. Math. Phys.* 1977

The N -body problem in classical mechanics

System of N identical point particles of mass m , spatial domain \mathbf{R}^d
Pairwise interaction given by a potential $V(x_j - x_k)$, with $x_j, x_k \in \mathbf{R}^d$
Newton's second law for the motion of particle no. k :

$$m\dot{x}_j = \xi_j, \quad \dot{\xi}_j = \sum_{\substack{k=1 \\ k \neq j}}^N -\nabla V(x_j - x_k)$$

Assumptions on V :

$$(H1) \quad V(z) = V(-z) \quad \text{for all } z \in \mathbf{R}^d$$

$$(H2) \quad V \in C^1(\mathbf{R}^d) \text{ with } \nabla V \in L^\infty(\mathbf{R}^d) \cap \text{Lip}(\mathbf{R}^d)$$

Rescaled time, position and momentum:

$$\hat{t} = t/N, \quad \hat{x}_j(\hat{t}) = x_j(t), \quad \hat{\xi}_j(\hat{t}) = \xi_j(t)$$

Motion equations

$$mN \frac{d\hat{x}_j}{d\hat{t}} = \hat{\xi}_j, \quad N \frac{d\hat{\xi}_j}{d\hat{t}} = \sum_{\substack{k=1 \\ k \neq j}}^N -\nabla V(\hat{x}_j - \hat{x}_k)$$

Finite mass assumption

$$Nm = 1$$

Henceforth drop hats on all variables; our starting point is

$$\dot{x}_j = \xi_j, \quad \dot{\xi}_j = -\frac{1}{N} \sum_{k=1}^N \nabla V(x_j - x_k)$$

Notation

$$X_N = (x_1, \dots, x_N), \quad \Xi_N := (\xi_1, \dots, \xi_N)$$

By Cauchy-Lipschitz (see assumption (H2)), the differential system above generates a global flow on \mathbf{R}^{2dN}

$$t \mapsto (X_N(t, X_N^{in}, \Xi_N^{in}), \Xi_N(t, X_N^{in}, \Xi_N^{in}))$$

— solution of the differential system with initial data (X_N^{in}, Ξ_N^{in})

The Vlasov equation

Unknown $f(t, dx d\xi) =$ single-particle phase-space number density

$$(\partial_t + \xi \cdot \nabla_x) f - \nabla_x V_f \cdot \nabla_\xi f = 0, \quad x, \xi \in \mathbb{R}^d$$

where $V_f \equiv V_f(t, x)$ is the mean-field potential

$$V_f(t, x) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} V(x - y) f(t, dy d\eta)$$

N -particle phase space empirical measure

$$\mu_{(X_N, \Xi_N)}(t) := \frac{1}{N} \sum_{k=1}^N \delta_{x_k(t), \xi_k(t)}$$

Key observation The two following conditions are equivalent:

- (a) $t \mapsto (X_N, \Xi_N)(t)$ is a solution of Newton's differential system of motion equations, and
- (b) $t \mapsto \mu_{(X_N, \Xi_N)}(t)$ is a weakly continuous in time, measure-valued solution of the Vlasov equation

Dobrushin's theorem (1979)

Assume that V satisfies (H1-2). Let f^{in} be a probability density on \mathbf{R}^{2d} such that

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} (|x| + |\xi|) f^{in}(x, \xi) dx d\xi < \infty$$

and let f be the solution of the Vlasov equation with initial data f^{in} . Let $t \mapsto (X_N, \Xi_N)(t)$ be the solution of Newton's differential system with initial data (X_N^{in}, Ξ_N^{in}) . Then

$$\text{dist}_{\text{MK},1}(\mu_{(X_N, \Xi_N)(t)}, f(t, \cdot)) \leq \text{dist}_{\text{MK},1}(\mu_{(X_N^{in}, \Xi_N^{in})}, f^{in}) e^{\max(1, 2 \text{Lip}(\nabla V))t}$$

Remark Mean-field limit \Leftrightarrow continuous dependence on the initial data in the weak topology of probability measures

Choice of the initial phase-space points

Consider a sequence of initial phase-space points $(x_j^{in}, \xi_j^{in})_{j \geq 1}$ s.t.

(x_j^{in}, ξ_j^{in}) are i.i.d., with distribution $f^{in}(dx d\xi)$

By the strong LLN

$$\mu_{(X_N^{in}, \Xi_N^{in})} \rightarrow f^{in} \quad \text{a.s. in } (x_j^{in}, \xi_j^{in})_{j \geq 1} \text{ as } N \rightarrow \infty$$

Since $\text{dist}_{\text{MK},1}$ metrizes the weak topology of Borel probability measures on \mathbf{R}^{2d} , the strong LLN implies that, in the limit as $N \rightarrow \infty$

$$\text{dist}_{\text{MK},1}(\mu_{(X_N, \Xi_N)}(t, X_N^{in}, \Xi_N^{in}), f^{in}) \rightarrow 0 \quad \text{a.s. in } (x_j^{in}, \xi_j^{in})_{j \geq 1}$$

A CRASH COURSE ON MONGE-KANTOROVICH DISTANCES

C. Villani : "Topics in Optimal Transportation", AMS (2003)

Couplings of probability measures

Let $\mathcal{P}(\mathbf{R}^n)$ be the set of Borel probability measures on \mathbf{R}^n ; for $p > 0$, denote

$$\mathcal{P}_p(\mathbf{R}^n) := \left\{ m \in \mathcal{P}(\mathbf{R}^n) \text{ s.t. } \int_{\mathbf{R}^n} |x|^p m(dx) < \infty \right\}$$

Given $\mu, \nu \in \mathcal{P}(\mathbf{R}^n)$, a coupling of μ and ν is an element $\pi \in \mathcal{P}(\mathbf{R}^n \times \mathbf{R}^n)$ such that

$$\iint_{\mathbf{R}^n \times \mathbf{R}^n} (\phi(x) + \psi(y)) \pi(dx dy) = \int_{\mathbf{R}^n} \phi(x) \mu(dx) + \int_{\mathbf{R}^n} \psi(y) \nu(dy)$$

Set of couplings of μ, ν denoted $\Pi(\mu, \nu)$; obviously

$$\mu, \nu \in \mathcal{P}_p(\mathbf{R}^n) \Rightarrow \Pi(\mu, \nu) \subset \mathcal{P}_p(\mathbf{R}^n \times \mathbf{R}^n)$$

Monge-Kantorovich distances

Let $p \geq 1$; for each $\mu, \nu \in \mathcal{P}_p(\mathbf{R}^n)$, the Monge-Kantorovich distance between μ and ν is

$$\text{dist}_{\text{MK},p}(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \left(\iint_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^p \pi(dx dy) \right)^{1/p}$$

Monge-Kantorovich duality

$$\text{dist}_{\text{MK},p}(\mu, \nu)^p = \sup_{\substack{\phi(x) + \psi(y) \leq |x - y|^p \\ \phi, \psi \in C_b(\mathbf{R}^n)}} \int_{\mathbf{R}^n} \phi(x) \mu(dx) + \int_{\mathbf{R}^n} \psi(x) \nu(dx)$$

In particular

$$\text{dist}_{\text{MK},1}(\mu, \nu) = \sup_{\text{Lip}(\phi) \leq 1} \left| \int_{\mathbf{R}^n} \phi(z) \mu(dz) - \int_{\mathbf{R}^n} \phi(z) \nu(dz) \right|$$

Separation axiom $\text{dist}_{\text{MK},p}(\mu, \nu) = 0 \Leftrightarrow \mu = \nu$

\Leftarrow choose $\pi = \mu(dx)\delta(x - y) \in \Pi(\mu, \mu)$

\Rightarrow if $\phi \in C_c^1(\mathbf{R}^n)$ and $\pi \in \Pi(\mu, \nu)$, then

$$\begin{aligned} \left| \int_{\mathbf{R}^n} \phi(z) \mu(dz) - \int_{\mathbf{R}^n} \phi(z) \nu(dz) \right| &\leq \text{Lip}(\phi) \text{dist}_{\text{MK},p}(\mu, \nu) \\ &\leq \text{Lip}(\phi) \left(\iint_{\mathbf{R}^n \times \mathbf{R}^n} |x - y|^p \pi(dx dy) \right)^{1/p} \end{aligned}$$

Glueing couplings

Lemma Let $\lambda, \mu, \nu \in \mathcal{P}_p(\mathbf{R}^n)$, let $\pi \in \Pi(\lambda, \mu)$ and $\rho \in \Pi(\mu, \nu)$; there exists $\omega \in \mathcal{P}(\mathbf{R}^{3n})$ such that

$$\begin{aligned} \iiint_{\mathbf{R}^{3n}} (\phi(x, y) + \psi(y, z)) \omega(dx dy dz) &= \iint_{\mathbf{R}^{2n}} \phi(x, y) \pi(dx dy) \\ &\quad + \iint_{\mathbf{R}^{2n}} \psi(y, z) \rho(dy dz) \end{aligned}$$

Disintegrate π and ρ along μ :

$$\pi = \int_{\mathbf{R}^n} \pi_y \otimes \delta_y \mu(dy), \quad \rho = \int_{\mathbf{R}^n} \delta_y \otimes \rho_y \mu(dy)$$

Set

$$\omega := \int_{\mathbf{R}^n} \pi_y \otimes \delta_y \otimes \rho_y \mu(dy)$$

Triangle inequality

Let ω be a glueing of $\pi \in \Pi(\lambda, \mu)$ and $\rho \in \Pi(\mu, \nu)$; then, one has

$$\begin{aligned} \text{dist}_{\text{MK},p}(\mu, \nu) &\leq \left(\iiint |x - z|^p \omega(dx dy dz) \right)^{1/p} \\ &\leq \left(\iiint (|x - y| + |y - z|)^p \omega(dx dy dz) \right)^{1/p} \\ &\leq \left(\iiint |x - y|^p \omega(dx dy dz) \right)^{1/p} + \left(\iiint |y - z|^p \omega(dx dy dz) \right)^{1/p} \\ &= \left(\iint |x - y|^p \pi(dx dy) \right)^{1/p} + \left(\iint |y - z|^p \rho(dy dz) \right)^{1/p} \end{aligned}$$

Minimizing over $\pi \in \Pi(\lambda, \mu)$ and $\rho \in \Pi(\mu, \nu)$ implies that

$$\text{dist}_{\text{MK},p}(\lambda, \nu) \leq \text{dist}_{\text{MK},p}(\lambda, \mu) + \text{dist}_{\text{MK},p}(\mu, \nu)$$

APPLICATION OF MONGE-KANTOROVICH DISTANCES

CONVERGENCE RATE IN LLN

N. Fournier, A. Guillin, *Prob. Theory Rel. Fields* **162**, 707–738 (2015)

The Fournier-Guillin estimate

Let $P \in \mathcal{P}_q(\mathbf{R}^n)$ for some $q > 1$. For $Z_N := (z_1, \dots, z_N) \in \mathbf{R}^{nN}$, we denote the empirical measure of the N -tuple Z_N by

$$\mu_{Z_N} := \frac{1}{N} \sum_{j=1}^N \delta_{z_j}$$

Then

$$\begin{aligned} & \int_{\mathbf{R}^{nN}} \text{dist}_{\text{MK},1}(\mu_{Z_N}, P) P^{\otimes N}(dZ_N) \\ & \leq C \left(\int_{\mathbf{R}^n} |z|^q P(dz) \right)^{\frac{1}{q}} \left(N^{-\frac{1}{q}} + N^{-(1-\frac{1}{q})} \right) \end{aligned}$$

provided that $q \neq \frac{n}{n-1}$.

Consequence of Dobrushin's theorem

Assume that V satisfies (H1-2). Let f^{in} be a probability density on \mathbb{R}^{2d} such that, for some $1 < q \neq \frac{2d}{2d-1}$

$$M_q := \iint_{\mathbb{R}^d \times \mathbb{R}^d} (|x| + |\xi|)^q f^{in}(x, \xi) dx d\xi < \infty$$

and let f be the solution of the Vlasov equation with initial data f^{in} . Let $t \mapsto (X_N, \Xi_N)(t)$ be the solution of Newton's differential system with initial data (X_N^{in}, Ξ_N^{in}) . Then

$$\begin{aligned} & \int_{\mathbb{R}^{2dN}} \text{dist}_{\text{MK},1}(\mu_{(X_N, \Xi_N)(t, X_N^{in}, \Xi_N^{in})}, f(t, \cdot)) \prod_{j=1}^N f^{in}(dx_j d\xi_j) \\ & \leq CM_q^{1/q} e^{\max(1, 2 \text{Lip}(\nabla V))t} \left(N^{-\frac{1}{q}} + N^{-(1-\frac{1}{q})} \right) \end{aligned}$$

Limitations of Dobrushin's approach

- Seems limited to Lipschitz continuous interaction forces (but can be modified to treat singular forces: see Hauray-Jabin, Lazarovici)
- Convergence rate estimate limited by quantization error for the initial distribution function f^{in}
- Extension to the quantum N -body problem very unclear — is there a natural notion of empirical measure in quantum mechanics? does it satisfy the Hartree equation? is there a notion of “particle trajectory” in quantum mechanics?

DOBRUSHIN'S ARGUMENT REVISITED

F. Golse, C. Mouhot, T. Paul:
Commun. Math. Phys. **343**, 165–205 (2016)

N -body Liouville equation

Hamiltonian formulation of Newton's equations

$$\dot{x}_j = \partial \mathcal{H}_N / \partial \xi_j, \quad \dot{\xi}_j = -\partial \mathcal{H}_N / \partial x_j$$

N -body Hamiltonian

$$\mathcal{H}_N(X_N, \Xi_N) := \sum_{j=1}^N \frac{1}{2} |\xi_j|^2 + \frac{1}{N} \sum_{1 \leq j < k \leq N} V(x_j - x_k)$$

Liouville equation for the N -particle density $F_N \equiv F_N(t, dX_N d\Xi_N)$

$$\partial_t F_N + \{\mathcal{H}_N, F_N\}_N = 0$$

where

$$\{\Phi, \Psi\}_N := \sum_{j=1}^N (\nabla_{\xi_j} \Phi \cdot \nabla_{x_j} \Psi - \nabla_{x_j} \Phi \cdot \nabla_{\xi_j} \Psi)$$

The Cauchy problem for the Liouville equation

The system of Hamilton's equations is the system defining the characteristic curves of the Liouville equation

Weak solution of the Cauchy problem

$$\partial_t F_N + \{\mathcal{H}_N, F_N\}_N = 0, \quad F_N|_{t=0} = F_N^{in} \in \mathcal{P}(\mathbf{R}^{2dN})$$

given by

$$F_N(t) = (X_N, \Xi_N)(t, \cdot) \# F_N^{in}, \quad F_N \in C(\mathbf{R}; w - \mathcal{P}(\mathbf{R}^{2dN}))$$

Notation if $T : X \rightarrow Y$ is $\mathcal{A} - \mathcal{B}$ measurable, and if μ is a measure on (X, \mathcal{A}) , the image measure $\nu = T \# \mu$ on (Y, \mathcal{B}) is defined by

$$\nu(B) = \mu(T^{-1}(B))$$

Indistinguishable particles

For $\sigma \in \mathfrak{S}_N$, denote $\sigma \cdot X_N := (x_{\sigma(1)}, \dots, x_{\sigma(N)})$, and set

$$T_\sigma(X_N, \Xi_N) := (\sigma \cdot X_N, \sigma \cdot \Xi_N)$$

Lemma Let $F_N^{in} \in \mathcal{P}(\mathbf{R}^{2dN})$, and let $F_N := (X_N, \Xi_N)(t, \cdot) \# F_N^{in}$ be the solution of the Cauchy problem for the Liouville equation with initial data F_N^{in} . Then, for each $\sigma \in \mathfrak{S}_N$ and all $t \in \mathbf{R}$

$$T_\sigma \# F_N^{in} = F_N^{in} \Rightarrow T_\sigma \# F_N(t) = F_N(t)$$

Idea of the proof: observe that $\mathcal{H}_N \circ T_\sigma = \mathcal{H}_N$, so that, by uniqueness of the solution of the Cauchy problem for Hamilton's equations

$$(X_N, \Xi_N)(t, \cdot) \circ T_\sigma = T_\sigma \circ (X_N, \Xi_N)(t, \cdot)$$

n -particle marginal of F_N

Let $F_N \in \mathcal{P}(\mathbf{R}^{2dN})$ satisfy $T_\sigma \# F_N = F_N$ for all $\sigma \in \mathfrak{S}_N$.

Definition For $n = 1, \dots, N - 1$, the n -particle marginal of F_N is

$$F_{N:n} := P_n^N \# F_N \in \mathcal{P}(\mathbf{R}^{2dn}) \quad \text{where } P_n^N(X_N, \Xi_N) := (X_n, \Xi_n)$$

If F_N has a density (w.r.t. the Lebesgue measure $dX_N d\Xi_N$), then $F_{N:n}$ is the probability density given by

$$F_{N:n}(X_n, \Xi_n) := \int_{\mathbf{R}^{2d(N-n)}} F_N(X_N, \Xi_N) dx_{n+1} d\xi_{n+1} \dots dx_N d\xi_N$$

Hamiltonian formulation of Vlasov's equation

Mean-field Hamiltonian

$$\mathbf{H}_{f(t,\cdot)}(x, \xi) := \frac{1}{2}|\xi|^2 + V_{f(t,\cdot)}(x)$$

where

$$V_{f(t,\cdot)}(x) := \iint_{\mathbf{R}^{2d}} V(x-y)f(t, dyd\eta)$$

Vlasov's equation equivalent to

$$\partial_t f + \{\mathbf{H}_{f(t,\cdot)}, f\}_1 = 0, \quad f|_{t=0} = f^{in} \in \mathcal{P}(\mathbf{R}^{2d})$$

Lemma For each $f^{in} \in \mathcal{P}_1(\mathbf{R}^{2d})$, there exists a unique weak solution $f \in C(\mathbf{R}_+; w - \mathcal{P}(\mathbf{R}^{2d}))$ of the Vlasov equation

Idea of the proof fixed point argument in the complete metric space $C([0, T], (\mathcal{P}_1(\mathbf{R}^{2d}), \text{dist}_{\text{MK},1}))$ (topology of t -uniform convergence)

Propagation of moment bounds

Lemma If $f^{in} \in \mathcal{P}_p(\mathbf{R}^{2d})$ for some $p \geq 1$, the solution of the Cauchy problem for the Vlasov equation with initial data f^{in} satisfies

$$\iint_{\mathbf{R}^{2d}} (|x|^p + |\xi|^p) f(t, dx d\xi) \leq e^{L_p t} \iint_{\mathbf{R}^{2d}} (|x|^p + |\xi|^p) f^{in}(dx d\xi) + 2^{p-1} K_p \|\nabla V\|_{L^\infty}^p \frac{e^{L_p t} - 1}{L_p}$$

with $K_p := \max(1, p-1)$ and $L_p := K_p(1 + \max(1, 2^{p-1} \text{Lip}(\nabla V)^p))$

Idea of the proof multiply both sides of the Vlasov equation by $|x|^p + |\xi|^p$, and use Young's inequality

$$pab^{p-1} \leq K_p(a^p + b^p), \quad a, b > 0$$

to estimate

$$\{\mathbf{H}_{f(t, \cdot)}, |x|^p + |\xi|^p\}_1$$

Theorem

Assume that V satisfies (H1+2). Let f be the solution of the Vlasov equation with initial data $f^{in} \in \mathcal{P}_2(\mathbb{R}^{2d})$ and F_N be the solution of the Liouville equation with initial data F_N^{in} , satisfying $T_\sigma \# F_N^{in} = F_N^{in}$ for all $\sigma \in \mathfrak{S}_N$. Then, for all $t \geq 0$ and $n = 1, \dots, N$

$$\frac{1}{n} \text{dist}_{\text{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2 \leq \frac{1}{N} \text{dist}_{\text{MK},2}((f^{in})^{\otimes N}, F_N^{in})^2 e^{\Lambda t} + \frac{8 \|\nabla V\|_{L^\infty}^2}{N} \frac{e^{\Lambda t} - 1}{\Lambda}$$

where

$$\Lambda := (1 + \max(1, 8 \text{Lip}(\nabla V)^2))$$

Let f be a solution of Vlasov's equation

$$\partial_t f + \{\mathbf{H}_{f(t,\cdot)}, f\}_1 = \partial_t f + \xi \cdot \nabla_x f - \nabla_x V_{f(t,\cdot)} \cdot \nabla_\xi f = 0$$

Equation satisfied by $f(t)^{\otimes N}$:

$$\partial_t f^{\otimes N} + \{\mathbf{H}_{f(t,\cdot)}^N, f^{\otimes N}\}_N = 0$$

where

$$\mathbf{H}_{f(t,\cdot)}^N(X_N, \Xi_N) := \sum_{j=1}^N \frac{1}{2} |\xi_j|^2 + \sum_{j=1}^N V_{f(t,\cdot)}(x_j)$$

Comparing $\mathbf{H}_{f(t,\cdot)}^N$ and \mathcal{H}_N

An elementary computation shows that

$$\begin{aligned} & \mathbf{H}_{f(t,\cdot)}^N(X_N, \Xi_N) - \mathcal{H}_N(X_N, \Xi_N) \\ &= \sum_{j=1}^N \left(V_{f(t,\cdot)}(x_j) - \frac{1}{N} \sum_{k=1}^N V(x_j - x_k) \right) \end{aligned}$$

If x_1, \dots, x_N are i.i.d. random variables distributed under

$$\rho_{f(t,\cdot)}(x) = \int_{\mathbf{R}^d} f(t, x, \xi) d\xi$$

the following limit holds a.s. in $(x_j)_{j \geq 1}$ as $N \rightarrow \infty$

$$\begin{aligned} \frac{1}{N} \sum_{k=1}^N V(z - x_k) &\rightarrow \int_{\mathbf{R}^d} V(z - x) \rho_{f(t,\cdot)}(x) dx \\ &= \iint_{\mathbf{R}^{2d}} V(z - x) f(t, x, \xi) dx d\xi \end{aligned}$$

Dynamics of couplings

Let $P^{in} \in \Pi((f^{in})^{\otimes N}, F_N^{in})$, and let $t \mapsto P(t, dX_N d\Xi_N dY_N dH_N)$ be the weak solution of

$$\begin{aligned} \partial_t P(t) + \{ \mathbf{H}_{f(t, \cdot)}^N(X_N, \Xi_N) + \mathcal{H}(Y_N, H_N), P(t) \}_{2N} &= 0 \\ P|_{t=0} &= P^{in} \end{aligned}$$

- (a) $P(t) \in \Pi(f(t)^{\otimes N}, F_N(t))$ for each $t \in \mathbf{R}$
- (b) for each $\sigma \in \mathfrak{S}_N$, set

$$\mathcal{T}_\sigma(X_N, \Xi_N, Y_N, H_N) = (\sigma \cdot X_N, \sigma \cdot \Xi_N, \sigma \cdot Y_N, \sigma \cdot H_N)$$

then

$$\mathcal{T}_\sigma \# P^{in} = P^{in} \Rightarrow \mathcal{T}_\sigma \# P(t) = P(t) \text{ for all } t \geq 0$$

The functional $D_N(t)$

For each $P^{in} \in \Pi((f^{in})^{\otimes N}, F_N^{in})$, set

$$D_N(t) := \int \frac{1}{N} \sum_{j=1}^N (|x_j - y_j|^2 + |\xi_j - \eta_j|^2) P(t, dX_N d\Xi_N dY_N dH_N)$$

Lemma Assume that $\mathcal{T}_\sigma \# P^{in} = P^{in}$ for all $\sigma \in \mathfrak{S}_N$. Then

$$D_N(t) \geq \frac{1}{n} \text{dist}_{\text{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2, \quad \text{for } n = 1, \dots, N$$

Proof Set $c_k := |x_k - y_k|^2 + |\xi_k - \eta_k|^2$; for each $k, n = 1, \dots, N$

$$D_N(t) = \int c_k P(t) = \int \frac{1}{n} \sum_{j=1}^n c_j P(t) \geq \text{dist}_{\text{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2$$

The dynamics of $D_N(t)$

Notation for $Y_N = (y_1, \dots, y_N)$, set

$$\mu_{Y_N} := \frac{1}{N} \sum_{j=1}^N \delta_{y_j}$$

Multiplying each side of the equation for P by

$$\frac{1}{N} \sum_{j=1}^N (|x_j - y_j|^2 + |\xi_j - \eta_j|^2)$$

and integrating in all variables

$$\begin{aligned} \dot{D}_N(t) &= \int \frac{1}{N} \sum_{j=1}^N (\xi_j \cdot \nabla_{x_j} + \eta_j \cdot \nabla_{y_j}) |x_j - y_j|^2 P(t) \\ &+ \int \frac{1}{N} \sum_{j=1}^N (\nabla V \star_x \rho_f(x_j) \cdot \nabla_{\xi_j} + \nabla V \star \mu_{Y_N}(y_j) \cdot \nabla_{\eta_j}) |\xi_j - \eta_j|^2 P(t) \end{aligned}$$

The dynamics of $D_N(t)$

Thus

$$\begin{aligned}\dot{D}_N(t) &= \int \frac{2}{N} \sum_{j=1}^N (\xi_j - \eta_j) \cdot (x_j - y_j) P(t) \\ &+ \int \frac{2}{N} \sum_{j=1}^N (\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j)) \cdot (\xi_j - \eta_j) P(t)\end{aligned}$$

Using the elementary inequality $2ab \leq (a^2 + b^2)$ implies that

$$\begin{aligned}\dot{D}_N(t) &\leq D_N(t) + \frac{1}{N} \sum_{j=1}^N \int |\xi_j - \eta_j|^2 P(t) \\ &+ \frac{1}{N} \sum_{j=1}^N \int |\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j)|^2 P(t)\end{aligned}$$

The dynamics of $D_N(t)$

$$\begin{aligned}\nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{Y_N}(y_j) &= \nabla V \star_x \rho_f(x_j) - \nabla V \star \mu_{X_N}(x_j) \\ &\quad + \nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)\end{aligned}$$

so that, by convexity of $z \mapsto |z|^2$, one has

$$\dot{D}_N(t) \leq D_N(t) + \frac{1}{N} \sum_{j=1}^N \int |\xi_j - \eta_j|^2 P(t) + I_N(t) + J_N(t)$$

with

$$I_N(t) := \frac{2}{N} \sum_{j=1}^N \int |\nabla V \star_x \rho_f(t, x_j) - \nabla V \star \mu_{X_N}(x_j)|^2 P(t)$$

$$J_N(t) := \frac{2}{N} \sum_{j=1}^N \int |\nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)|^2 P(t)$$

By convexity of $z \mapsto |z|^2$, and since $\nabla V \in \text{Lip}(\mathbf{R}^d)$, one has

$$\begin{aligned} & |\nabla V \star \mu_{X_N}(x_j) - \nabla V \star \mu_{Y_N}(y_j)|^2 \\ &= \left| \frac{1}{N} \sum_{k=1}^N (\nabla V(x_j - x_k) - \nabla V(y_j - y_k)) \right|^2 \\ &\leq \frac{1}{N} \sum_{k=1}^N |\nabla V(x_j - x_k) - \nabla V(y_j - y_k)|^2 \\ &\leq \frac{\text{Lip}(\nabla V)^2}{N} \sum_{k=1}^N |(x_j - y_j) - (x_k - y_k)|^2 \\ &\leq 2 \text{Lip}(\nabla V)^2 \left(|x_j - y_j|^2 + \frac{1}{N} \sum_{k=1}^N |x_k - y_k|^2 \right) \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N \int \left(|x_j - y_j|^2 + \frac{1}{N} \sum_{k=1}^N |x_k - y_k|^2 \right) P_N(t) \\ = \frac{2}{N} \sum_{j=1}^N \int |x_j - y_j|^2 P_N(t) \end{aligned}$$

we conclude that

$$J_N(t) \leq \frac{8 \text{Lip}(\nabla V)^2}{N} \sum_{j=1}^N \int |x_j - y_j|^2 P_N(t)$$

Lemma Let ρ be a probability density on \mathbf{R}^d . For all $j = 1, \dots, N$, one has

$$\int \left| \nabla V \star \rho(x_j) - \frac{1}{N} \sum_{k=1}^N \nabla V(x_j - x_k) \right|^2 \prod_{m=1}^M \rho(x_m) dx_m \leq \frac{4 \|\nabla V\|_{L^\infty}^2}{N}$$

Proof Set

$$W(t, x_1, z) := (\nabla V \star_x \rho_f(t, x_1) - \nabla V(x_1 - z))$$

observe that

$$\int W(t, x_1, z) \rho(t, z) dz = 0$$

Therefore

$$\begin{aligned}
 & \int |\nabla V \star_x \rho_f(t, x_1) - \nabla V \star \mu_{X_N}(x_j)|^2 \prod_{l=1}^N \rho_f(t, x_l) dx_l \\
 = & \frac{2}{N^2} \sum_{1 \leq j < k \leq N} \underbrace{\int W(t, x_1, x_j) \cdot W(t, x_1, x_k) \prod_{l=1}^N \rho_f(t, x_l) dx_l}_{=0} \\
 & + \frac{1}{N^2} \sum_{j=1}^N \underbrace{\int |W(t, x_1, x_j)|^2 \prod_{l=1}^N \rho_f(t, x_l) dx_l}_{\leq N(2\|W\|_{L^\infty})^2}
 \end{aligned}$$

$$\begin{aligned}
 \dot{D}_N(t) &\leq D_N(t) + \frac{1}{N} \sum_{j=1}^N \int |\xi_j - \eta_j|^2 P(t) \\
 &\quad + \frac{8 \operatorname{Lip}(\nabla V)^2}{N} \sum_{j=1}^N \int |x_j - y_j|^2 P_N(t) + \frac{8 \|\nabla V\|_{L^\infty}^2}{N} \\
 &\leq \Lambda D_N(t) + \frac{8 \|\nabla V\|_{L^\infty}^2}{N}
 \end{aligned}$$

Gronwall \implies

$$\begin{aligned}
 \frac{1}{n} \operatorname{dist}_{\text{MK},2}(f(t)^{\otimes n}, F_{N:n}(t))^2 &\leq D_N(t) \\
 &\leq D_N(0) e^{\Lambda t} + \frac{8 \|\nabla V\|_{L^\infty}^2}{N} \frac{e^{\Lambda t} - 1}{\Lambda}
 \end{aligned}$$