Axioms of Adaptivity (AoA) in Lecture 2 (sufficient for optimal convergence rates)

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Reduction Property

With Dörfler Marking (A1)+(A2)=(A12)

Abbreviate $\mathcal{T} := \mathcal{T}_\ell$ and $\widehat{\mathcal{T}} := \mathcal{T}_{\ell+1}$ in (A1)-(A2) with $\eta(K) := \eta_\ell(K)$ for $K \in \mathcal{T}$ resp. $\widehat{\eta}(T) := \eta_{\ell+1}(T)$ for $T \in \widehat{\mathcal{T}}$ and $\delta := \delta(\mathcal{T}, \widehat{\mathcal{T}})$. Then (A1)-(A2) read

$$\begin{aligned} |\widehat{\eta}(\widehat{\mathcal{T}} \cap \mathcal{T}) - \eta(\mathcal{T} \cap \widehat{\mathcal{T}})| &\leq \Lambda_1 \delta \\ \widehat{\eta}(\widehat{\mathcal{T}} \setminus \mathcal{T}) \leq \varrho_2 \eta(\mathcal{T} \setminus \widehat{\mathcal{T}}) &+ \Lambda_2 \delta \end{aligned}$$

Underlying sum convention $\qquad \eta^2(\mathcal{M}) := \sum_{M \in \mathcal{M}} \eta^2(M)$
Since marked T are refined, Dörfler marking leads to

$$\Theta \eta^2(\mathcal{T}) \leq \eta^2(\mathcal{T} \setminus \widehat{\mathcal{T}})$$

Weighted geometric-arithmetic mean inequality for $a, b \in \mathbb{R}$ reads

$$(a+b)^2 \leq (1+\lambda)\, a^2 + (1+1/\lambda)\, b^2 \quad \text{for all } \lambda > 0$$

(with equality if $a, b > 0$ for the minimizing choice of λ)
Application to (A1) shows for any $\lambda > 0$

$$\widehat{\eta}^2(\widehat{\mathcal{T}} \cap \mathcal{T}) \leq (1+\lambda)\eta^2(\mathcal{T} \cap \widehat{\mathcal{T}}) + (1+1/\lambda)\Lambda_1^2\delta^2$$

Application to (A2) shows for any $\mu > 0$

$$\widehat{\eta}^2(\widehat{\mathcal{T}} \setminus \mathcal{T}) \leq \varrho_2^2(1+\mu)\eta^2(\mathcal{T} \setminus \widehat{\mathcal{T}}) + (1+1/\mu)\Lambda_2^2\; \delta^2$$

The sum gives $\quad \widehat{\eta}^2(\widehat{\mathcal{T}}) = \widehat{\eta}^2(\widehat{\mathcal{T}} \cap \mathcal{T}) + \widehat{\eta}^2(\widehat{\mathcal{T}} \setminus \mathcal{T})$ on LHS

Hence

$$\widehat{\eta}^2(\widehat{\mathcal{T}}) \le (1+\lambda)\eta^2(\mathcal{T}\cap\widehat{\mathcal{T}}) + \varrho_2^2(1+\mu)\eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}}) + \underbrace{\Big((1+1/\lambda)\Lambda_1^2 + (1+1/\mu)\Lambda_2^2\Big)}_{\Lambda_{12}}\delta^2$$

Recall $0 < \Theta \le 1$ and $0 < \varrho_2 < 1$ so there exist

$$0 < \mu < \varrho_2^{-2} - 1 \quad \text{and} \quad 0 < \lambda < \Theta\frac{1-(1+\mu)\varrho_2^2}{1-\Theta}$$

Theorem (A12)

(A1)-(A2) and Dörfler marking imply

$$\widehat{\eta}^2(\widehat{\mathcal{T}}) \le \varrho_{12}\,\eta^2(\mathcal{T}) + \Lambda_{12}\delta^2$$

with $\varrho_{12} < 1$ and $\Lambda_{12} < \infty$ for any such choice of λ, μ

Proof of (A12)

Recall

$$\widehat{\eta}^2(\widehat{\mathcal{T}}) \le (1+\lambda)\eta^2(\mathcal{T}\cap\widehat{\mathcal{T}}) + \varrho_2^2(1+\mu)\eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}}) + \Lambda_{12}\delta^2$$

Recall $\eta^2(\mathcal{T}\cap\widehat{\mathcal{T}}) = \eta^2(\mathcal{T}) - \eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}})$ and rewrite

$$\begin{aligned}\widehat{\eta}^2(\widehat{\mathcal{T}}) - \Lambda_{12}\delta^2 &\le (1+\lambda)\eta^2(\mathcal{T}\cap\widehat{\mathcal{T}}) + \varrho_2^2(1+\mu)\eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}}) \\ &= (1+\lambda)\eta^2(\mathcal{T}) + \Big(\varrho_2^2(1+\mu) - (1+\lambda)\Big)\eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}})\end{aligned}$$

Since $\varrho_2^2(1+\mu) < 1 < 1+\lambda$, the factor in brackets is ≤ 0 and Dörfler marking with $\Theta\eta^2(\mathcal{T}) \le \eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}})$ leads to

$$
\begin{aligned}
&\widehat{\eta}^2(\widehat{\mathcal{T}}) - \Lambda_{12}\delta^2 \\
&\le (1+\lambda)\eta^2(\mathcal{T}) + \Big(\varrho_2^2(1+\mu) - (1+\lambda)\Big)\eta^2(\mathcal{T}\setminus\widehat{\mathcal{T}}) \\
&\le \underbrace{\Big((1+\lambda)(1-\Theta) + \Theta\varrho_2^2(1+\mu)\Big)}_{\varrho_{12}}\eta^2(\mathcal{T})
\end{aligned}
$$

The proof concludes with $\varrho_{12} < 1$ iff $\lambda < \Theta\frac{1-(1+\mu)\varrho_2^2}{1-\Theta}$ (by a minor calculation) $\square$

NB. λ, μ small reduce ϱ_{12} but increase Λ_{12}

Repeat (A12)

Abbreviate $\mathcal{T} := \mathcal{T}_\ell$ and $\widehat{\mathcal{T}} := \mathcal{T}_{\ell+1}$ in (A1)-(A2) with $\eta(K) := \eta_\ell(K)$ for $K \in \mathcal{T}$ resp. $\widehat{\eta}(T) := \eta_{\ell+1}(T)$ for $T \in \widehat{\mathcal{T}}$ and $\delta := \delta(\mathcal{T}, \widehat{\mathcal{T}}) =: \delta_{\ell,\ell+1}$.
Recall there exist

$$0 < \mu < \varrho_2^{-2} - 1 \quad \text{and} \quad 0 < \lambda < \Theta \frac{1 - (1+\mu)\varrho_2^2}{1 - \Theta}$$

Theorem (A12)

(A1)-(A2) and Dörfler marking imply

$$\eta_{\ell+1}^2 \le \varrho_{12}\, \eta_\ell^2 + \Lambda_{12} \delta_{\ell,\ell+1}^2$$

with $\varrho_{12} < 1$ and $\Lambda_{12} < \infty$ for any such choice of λ, μ

Plain Convergence

Convergence $\eta_k \to 0$ as $k \to \infty$

Theorem (plain convergence)

(A12), (A4), and $\Lambda := (1 + \Lambda_{12}\Lambda_4)/(1 - \varrho_{12}) < \infty$ *imply*

$$\sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda \eta_\ell^2 \quad \textit{for all } \ell \in \mathbb{N}_0.$$

Proof. Recall (A12) as $\eta_{k+1}^2 \le \varrho_{12}\eta_k^2 + \Lambda_{12}\delta_{k,k+1}^2$. Then

$$\sum_{k=\ell}^{\ell+m} \eta_k^2 \le \sum_{k=\ell}^{\ell+m+1} \eta_k^2 \le \eta_\ell^2 + \varrho_{12} \sum_{k=\ell}^{\ell+m} \eta_k^2 + \Lambda_{12} \sum_{k=\ell}^{\ell+m} \delta_{k,k+1}^2$$

Recall (A4) $\sum_{k=\ell}^{\infty} \delta_{k,k+1}^2 \le \Lambda_4 \eta_\ell^2$
for the last term and utilize $\varrho_{12} < 1$ to conclude the proof □

Linear Convergence

R-Linear Convergence Uniformly on Each Level

Theorem

(A12),(A4),Λ*, and* $q := 1 - 1/\Lambda < 1$ *imply*

$$\eta_{\ell+m}^2 \le q^m \Lambda \, \eta_\ell^2 \quad \textit{for all } \ell,\, m \in \mathbb{N}_0.$$

Proof. Plain convergence gives

$$\xi_\ell^2 := \sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda \, \eta_\ell^2 < \infty$$

and so $$\Lambda^{-1} \xi_\ell^2 \le \eta_\ell^2 = \xi_\ell^2 - \xi_{\ell+1}^2$$

This proves $$\xi_{\ell+1}^2 \le q \xi_\ell^2 \quad (\text{for each } \ell \in \mathbb{N}_0)$$

Recall $\qquad \xi_{\ell+1}^2 \le q\xi_\ell^2 \quad$ for all $\ell \in \mathbb{N}_0$

Mathematical induction leads to

$$\xi_{\ell+m}^2 \le q^m \xi_\ell^2 \qquad \text{for all } m \in \mathbb{N}_0$$

This and $\xi_\ell^2 := \sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda\, \eta_\ell^2$ show

$$\eta_{\ell+m}^2 \le \xi_{\ell+m}^2 \le q^m \xi_\ell^2 \le q^m \Lambda \eta_\ell^2 \quad \square$$

R-linear convergence uniformly on each level implies

$$\sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim \eta_\ell^{-1/s}$$

via a geometric series

Quasimonotonicity

(QM) Estimator Quasimonotonicity

Theorem (QM)

(A1)—(A3) and $\Lambda_{mon} := 1 + \sqrt{\Lambda_1^2 + \Lambda_2^2}\Lambda_3$ *imply*

$$\eta(\widehat{\mathcal{T}}) \leq \Lambda_{mon}\eta(\mathcal{T})$$

for any refinement $\widehat{\mathcal{T}}$ *of any* $\mathcal{T}$ *in* $\mathbb{T}$

Proof. For any $0 < \lambda < \infty$, utilize (A1)-(A2) in the decomposition

$$\widehat{\eta}^2(\widehat{\mathcal{T}}) = \widehat{\eta}^2(\widehat{\mathcal{T}} \cap \mathcal{T}) + \widehat{\eta}^2(\widehat{\mathcal{T}} \backslash \mathcal{T})$$

$$
\begin{aligned}
\widehat{\eta}^2(\widehat{\mathcal{T}}) &= \widehat{\eta}^2(\widehat{\mathcal{T}} \cap \mathcal{T}) + \widehat{\eta}^2(\widehat{\mathcal{T}} \backslash \mathcal{T}) \\
&\leq (1+\lambda)\Big(\underbrace{\eta^2(\mathcal{T} \cap \widehat{\mathcal{T}}) + \eta^2(\mathcal{T} \backslash \widehat{\mathcal{T}})}_{\eta^2(\mathcal{T})}\Big) \\
&\qquad + (1+1/\lambda)(\Lambda_1^2 + \Lambda_2^2)\delta^2
\end{aligned}
$$

(A3) reads $\delta^2 \leq \Lambda_3^2 \eta^2(\mathcal{T})$ and leads to

$$
\eta^2(\widehat{\mathcal{T}}) \leq (\underbrace{1 + \lambda + (1+1/\lambda)(\Lambda_1^2 + \Lambda_2^2)\Lambda_3^2}_{\Lambda^2_{\mathsf{mon}}})\eta^2(\mathcal{T}) \quad \square
$$

Comparison Lemma

Comparison Lemma

Given $0 < \varkappa < 1$ and $s > 0$ with

$$M := \sup_{N \in \mathbb{N}_0} (N+1)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T}) < \infty$$

$\exists\, 0 < \theta_0 < 1\ \forall \ell \in \mathbb{N}_0\ \exists\, \hat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$ s.t.

(a) $\eta(\hat{\mathcal{T}}_\ell) \leq \varkappa \eta(\mathcal{T}_\ell)$
(b) $\varkappa \eta_\ell |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|^s \leq \Lambda_{\mathsf{mon}} M$
(c) $\theta_0 \eta_\ell^2 \leq \eta_\ell^2(\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell)$

Proof. (1) W.l.o.g. $\eta_\ell \equiv \eta(\mathcal{T}_\ell) > 0$.

Then (QM) implies $0 < \eta_0 \leq M$

(2) Choose minimal $N_\ell \in \mathbb{N}_0$ s.t.

$$(N_\ell + 1)^{-s} \leq \frac{\varkappa \eta_\ell}{\Lambda_{\mathsf{mon}} M} < N_\ell^{-s} \leq 1$$

$(N_\ell \geq 1$ because $\eta_\ell \Lambda_{\mathsf{mon}}^{-1} / M \leq \eta_0 / M \leq 1)$

(3) Set $\widehat{\mathcal{T}_\ell} := \mathcal{T}_\ell \otimes \mathcal{T}'$ for $\mathcal{T}'$ with $\mathcal{T}' \in \mathbb{T}(N_\ell)$ s.t.

$$(N_\ell + 1)^s \eta(\mathcal{T}') \leq M$$

Quasimonotonicity and overlay control lead to (a),

$$\eta(\widehat{\mathcal{T}_\ell}) \leq \Lambda_{\mathsf{mon}} M (N_\ell + 1)^{-s} \leq \varkappa \eta_\ell \quad \text{and} \quad |\widehat{\mathcal{T}_\ell}| \leq |\mathcal{T}_\ell| + N_\ell$$

Proof of (b)-(c) in Comparison Lemma

(4) Proof of (b). Count triangles to verify

$$|\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \leq |\widehat{\mathcal{T}}_\ell| - |\mathcal{T}_\ell| \underset{\text{from } \otimes}{\leq} N_\ell \underset{\text{from (2)}}{<} \varkappa^{-1/s} \eta_\ell^{-1/s} \Lambda_{\text{mon}}^{1/s} M^{1/s} \quad \square$$

(5) Any $\widehat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$ with (a) allows for (c).

Given $0 < \mu < \varkappa^{-2} - 1$, (A1) plus (a) and (A3) imply

$$\begin{aligned} \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) &\leq (1+\mu)\eta^2(\widehat{\mathcal{T}}_\ell, \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + (1+1/\mu)\Lambda_1^2 \delta^2(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell) \\ &\leq (1+\mu)\varkappa^2 \eta_\ell^2 + (1+1/\mu)\Lambda_1^2 \Lambda_3^2 \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \end{aligned}$$

This and $\eta_\ell^2 = \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell)$ lead to

$$\left(1 - (1+\mu)\varkappa^2\right) \eta_\ell^2 \leq \left(1 + (1+1/\mu)\Lambda_1^2 \Lambda_3^2\right) \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \quad \square$$

NB. Θ_0 is a quotient that depends onκ and μ. Fix those paramaters.

Repetition on Optimal Rates

Optimality Analysis at a Glance II

$\mathcal{R}$ satisfies bulk criterion if $\theta_A \le \theta_0$ thus $|\mathcal{M}_\ell^\star| \le |\mathcal{R}|$ for optimal set $\mathcal{M}_\ell^\star$ of marked cells. AFEM utilizes almost minimal $\mathcal{M}_\ell$, whence

$$|\mathcal{M}_\ell| \lesssim |\mathcal{M}_\ell^\star| \le |\mathcal{R}|$$

Set $M := \sup_{N\in\mathbb{N}_0}(1+N)^s \min_{\mathcal{T}\in\mathbb{T}(N)} \eta(\mathcal{T})$ with (writing $\kappa \approx 1$)

$$|\mathcal{R}| \lesssim M^{1/s}\eta_\ell^{-1/s}$$

Recall closure overhead control and combine with aforementioned estimates for

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \lesssim M^{1/s} \sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim M^{1/s}\eta_\ell^{-1/s} \qquad \square$$

Separate Marking

Adaptive Algorithm: SAFEM

Input: $\mathcal{T}_0$, $0 < \kappa$, $\theta_A \ll 1$, $\rho_B < 1$

$\forall \ell = 0, 1, 2, 3, \dots$

Compute $\eta_\ell^2(K)$ and $\mu^2(K)$ for all $K \in \mathcal{T}_\ell$

if $\mu_\ell^2 := \mu^2(\mathcal{T}_\ell) \leq \kappa \eta_\ell^2$

$\mathcal{T}_{\ell+1} := \texttt{Dörfler\_marking}(\theta_A, \mathcal{T}_\ell, \eta_\ell^2)$

else

$\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \texttt{data\_approximation}(\rho_B \mu_\ell^2, \mathcal{T}_0, \mu_\ell^2)$

Output: Sequence $(\mathcal{T}_\ell)$, (η_ℓ), (μ_ℓ)

abreviate $\sigma_\ell^2 := \eta_\ell^2 + \mu_\ell^2$

Axioms (A1)–(A4)

Suppose $\rho_2 < 1, \Lambda_k < \infty$ s.t. $\forall \mathcal{T} \in \mathbb{T} \, \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \, \exists \mathcal{R} \subset \mathcal{T}$ s.t.
$\mathcal{T} \setminus \hat{\mathcal{T}} \subseteq \mathcal{R} \wedge |\mathcal{R}| \lesssim |\mathcal{T} \setminus \hat{\mathcal{T}}|$

$$|\eta(\hat{\mathcal{T}}, \mathcal{T} \cap \hat{\mathcal{T}}) - \eta(\mathcal{T}, \mathcal{T} \cap \hat{\mathcal{T}})| \leq \Lambda_1 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A1}$$

$$\eta(\hat{\mathcal{T}}, \hat{\mathcal{T}} \setminus \mathcal{T}) \leq \rho_2 \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_2 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A2}$$

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) \leq \Lambda_3 \big(\eta^2(\mathcal{T}, \mathcal{R}) + \mu^2(\mathcal{T})\big) + \widehat{\Lambda_3} \eta^2(\hat{\mathcal{T}}) \tag{A3}$$

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_4 \sigma_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0 \tag{A4}$$

Axioms (B1)–(B2)

$\forall \mathsf{Tol} > 0\ \mathcal{T}_{\mathsf{Tol}} = \mathtt{data\_approx}(\mathsf{Tol}, \mathcal{T}_0, \mu^2) \in \mathbb{T}$ satisfies $\mu^2(\mathcal{T}_{\mathsf{Tol}}) \leq \mathsf{Tol}$ and

$$|\mathcal{T}_{\mathsf{Tol}}| - |\mathcal{T}_0| \leq \Lambda_5 \mathsf{Tol}^{-1/(2s)} \tag{B1}$$

$$\forall \mathcal{T} \in \mathbb{T}, \hat{T} \in \mathbb{T}(\mathcal{T}) \quad \mu^2(\hat{\mathcal{T}}) \leq \Lambda_6 \mu^2(\mathcal{T}) \tag{B2}$$

(B1)–(B2) follow for Approx from subadditivity

$$\mu^2(\hat{\mathcal{T}}(\mathcal{M})) := \sum_{K \in \mathcal{M}} \sum_{T \in \hat{\mathcal{T}}(K)} \mu^2(T) \leq \Lambda_6 \mu^2(\mathcal{M}) \tag{SA}$$

Typical example TSA by Binev and DeVore but also Dörfler marking over sufficient levels could do [C-Rabus 2016]

Theorem (C-Rabus 2016)

SAFEM leads to optimal convergence rates in total estimators provided (A1)-(A4), (B1)-(B2),

$$0 < \theta_A < \theta_0 := \frac{1 - \kappa\Lambda_1^2\Lambda_3}{1 + \Lambda_1^2\Lambda_3} \quad \textit{and} \quad \kappa < \frac{1 - \rho_A}{\Lambda_6 - 1}$$

plus Quasimonotonicity (e.g. for $(\Lambda_1^2 + \Lambda_2^2)\widehat{\Lambda_3} < 1$*)* $\forall \mathcal{T} \in \mathbb{T}\, \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$

$$\sigma(\hat{\mathcal{T}}) \le \Lambda_7 \sigma(\mathcal{T}).$$

Applications to div-LSFEM and to MFEM with convergence rates in $H(\operatorname{div}, \Omega) \times L^2(\Omega)$ in [C-Rabus 2016]

Alternative Adaptive LS FEM 4 Laplace

Alternative A Posteriori Error Control

Estimator η and data approximation error μ in 2D

$$\eta^2(\mathcal{T},K) := \|(1-\Pi_0)p_{LS}\|^2_{L^2(K)}$$
$$+ |K|^{1/2}\sum_{E\in\mathcal{E}(K)}\Big(||[p_{LS}]_E\cdot\tau_E||^2_{L^2(E)}+||[\partial u_{LS}/\partial\nu_E]_E||^2_{L^2(E\setminus\partial\Omega)}\Big)$$

$$\mu^2(K) := \|f-\Pi_0 f\|^2_{L^2(K)} \quad \text{for } K\in\mathcal{T} \text{ [C-Park 2015]}$$

satisfy discrete reliability (A3) (for $k=0$) (Proof with explicit Crouzeix-Raviart and Raviart-Thomas functions, discrete Helmholtz decomposition, mixed intermediate solutions still leaves extra term)

References and Further Reading

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Prepare yourself for tomorrow

Conforming P_1 FEM for Poisson Model Problem (weak form, $H_0^1(\Omega)$, energy scalar product $a(\bullet,\bullet) := \int_\Omega \nabla \bullet \cdot \nabla \bullet \, dx$, energy norm)
Inverse estimates (for polynomials)
Trace inequality (for Sobolev functions)
Discrete trace inequality (for polynomials)
Shape regularity (for triangles, simplices)
Poincaré and Friedrichs inequality (for Sobolev functions)
Equivalence of norms in finite dimensional vector spaces
Scaling argument (for derivatives of Sobolev functions)
Triangle inequality (in normed linear spaces)
Cauchy inequality (in Hilbert spaces like L^2 or w.r.t. $a(\bullet,\bullet)$)

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