

# Axioms of Adaptivity (AoA) in Lecture 2 (sufficient for optimal convergence rates)

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2018 International Graduate Summer School on  
Frontiers of Applied and Computational Mathematics



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Open-Access Reference: C-Feischl-Page-Praetorius: AoA.

Comp Math Appl 67 (2014) 1195—1253

# Reduction Property

# With Dörfler Marking (A1)+(A2)=(A12)

Abbreviate  $\mathcal{T} := \mathcal{T}_\ell$  and  $\widehat{\mathcal{T}} := \mathcal{T}_{\ell+1}$  in (A1)-(A2) with  $\eta(K) := \eta_\ell(K)$  for  $K \in \mathcal{T}$  resp.  $\widehat{\eta}(T) := \eta_{\ell+1}(T)$  for  $T \in \widehat{\mathcal{T}}$  and  $\delta := \delta(\mathcal{T}, \widehat{\mathcal{T}})$ . Then (A1)-(A2) read

$$\begin{aligned} |\widehat{\eta}(\widehat{\mathcal{T}} \cap \mathcal{T}) - \eta(\mathcal{T} \cap \widehat{\mathcal{T}})| &\leq \Lambda_1 \delta \\ \widehat{\eta}(\widehat{\mathcal{T}} \setminus \mathcal{T}) &\leq \varrho_2 \eta(\mathcal{T} \setminus \widehat{\mathcal{T}}) + \Lambda_2 \delta \end{aligned}$$

Underlying sum convention  $\eta^2(\mathcal{M}) := \sum_{M \in \mathcal{M}} \eta^2(M)$   
Since marked  $T$  are refined, Dörfler marking leads to

$$\Theta \eta^2(\mathcal{T}) \leq \eta^2(\mathcal{T} \setminus \widehat{\mathcal{T}})$$

Weighted geometric-arithmetic mean inequality for  $a, b \in \mathbb{R}$  reads

$$(a + b)^2 \leq (1 + \lambda) a^2 + (1 + 1/\lambda) b^2 \quad \text{for all } \lambda > 0$$

(with equality if  $a, b > 0$  for the minimizing choice of  $\lambda$ )

Application to (A1) shows for any  $\lambda > 0$

$$\hat{\eta}^2(\hat{\mathcal{T}} \cap \mathcal{T}) \leq (1 + \lambda)\eta^2(\mathcal{T} \cap \hat{\mathcal{T}}) + (1 + 1/\lambda)\Lambda_1^2 \delta^2$$

Application to (A2) shows for any  $\mu > 0$

$$\hat{\eta}^2(\hat{\mathcal{T}} \setminus \mathcal{T}) \leq \varrho_2^2(1 + \mu)\eta^2(\mathcal{T} \setminus \hat{\mathcal{T}}) + (1 + 1/\mu)\Lambda_2^2 \delta^2$$

The sum gives  $\hat{\eta}^2(\hat{\mathcal{T}}) = \hat{\eta}^2(\hat{\mathcal{T}} \cap \mathcal{T}) + \hat{\eta}^2(\hat{\mathcal{T}} \setminus \mathcal{T})$  on LHS

Hence

$$\hat{\eta}^2(\hat{\mathcal{T}}) \leq (1 + \lambda)\eta^2(\mathcal{T} \cap \hat{\mathcal{T}}) + \varrho_2^2(1 + \mu)\eta^2(\mathcal{T} \setminus \hat{\mathcal{T}}) + \underbrace{\left( (1 + 1/\lambda)\Lambda_1^2 + (1 + 1/\mu)\Lambda_2^2 \right)}_{\Lambda_{12}} \delta^2$$

Recall  $0 < \Theta \leq 1$  and  $0 < \varrho_2 < 1$  so there exist

$$0 < \mu < \varrho_2^{-2} - 1 \quad \text{and} \quad 0 < \lambda < \Theta \frac{1 - (1 + \mu)\varrho_2^2}{1 - \Theta}$$

### Theorem (A12)

(A1)-(A2) and Dörfler marking imply

$$\hat{\eta}^2(\hat{\mathcal{T}}) \leq \varrho_{12} \eta^2(\mathcal{T}) + \Lambda_{12} \delta^2$$

with  $\varrho_{12} < 1$  and  $\Lambda_{12} < \infty$  for any such choice of  $\lambda, \mu$

# Proof of (A12)

Recall

$$\hat{\eta}^2(\hat{\mathcal{T}}) \leq (1 + \lambda)\eta^2(\mathcal{T} \cap \hat{\mathcal{T}}) + \varrho_2^2(1 + \mu)\eta^2(\mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_{12}\delta^2$$

Recall  $\eta^2(\mathcal{T} \cap \hat{\mathcal{T}}) = \eta^2(\mathcal{T}) - \eta^2(\mathcal{T} \setminus \hat{\mathcal{T}})$  and rewrite

$$\begin{aligned}\hat{\eta}^2(\hat{\mathcal{T}}) - \Lambda_{12}\delta^2 &\leq (1 + \lambda)\eta^2(\mathcal{T} \cap \hat{\mathcal{T}}) + \varrho_2^2(1 + \mu)\eta^2(\mathcal{T} \setminus \hat{\mathcal{T}}) \\ &= (1 + \lambda)\eta^2(\mathcal{T}) + \left(\varrho_2^2(1 + \mu) - (1 + \lambda)\right)\eta^2(\mathcal{T} \setminus \hat{\mathcal{T}})\end{aligned}$$

Since  $\varrho_2^2(1 + \mu) < 1 < 1 + \lambda$ , the factor in brackets is  $\leq 0$  and Dörfler marking with  $\Theta\eta^2(\mathcal{T}) \leq \eta^2(\mathcal{T} \setminus \hat{\mathcal{T}})$  leads to

$$\begin{aligned}
& \widehat{\eta}^2(\widehat{\mathcal{T}}) - \Lambda_{12}\delta^2 \\
& \leq (1 + \lambda)\eta^2(\mathcal{T}) + \left(\varrho_2^2(1 + \mu) - (1 + \lambda)\right)\eta^2(\mathcal{T} \setminus \widehat{\mathcal{T}}) \\
& \leq \underbrace{\left((1 + \lambda)(1 - \Theta) + \Theta\varrho_2^2(1 + \mu)\right)}_{\varrho_{12}}\eta^2(\mathcal{T})
\end{aligned}$$

The proof concludes with  $\varrho_{12} < 1$  iff  $\lambda < \Theta \frac{1 - (1 + \mu)\varrho_2^2}{1 - \Theta}$  (by a minor calculation) □

NB.  $\lambda, \mu$  small reduce  $\varrho_{12}$  but increase  $\Lambda_{12}$



## Repeat (A12)

Abbreviate  $\mathcal{T} := \mathcal{T}_\ell$  and  $\widehat{\mathcal{T}} := \mathcal{T}_{\ell+1}$  in (A1)-(A2) with  $\eta(K) := \eta_\ell(K)$  for  $K \in \mathcal{T}$  resp.  $\widehat{\eta}(T) := \eta_{\ell+1}(T)$  for  $T \in \widehat{\mathcal{T}}$  and  $\delta := \delta(\mathcal{T}, \widehat{\mathcal{T}}) :=: \delta_{\ell, \ell+1}$ .

Recall there exist

$$0 < \mu < \varrho_2^{-2} - 1 \quad \text{and} \quad 0 < \lambda < \Theta \frac{1 - (1 + \mu)\varrho_2^2}{1 - \Theta}$$

### Theorem (A12)

(A1)-(A2) and Dörfler marking imply

$$\eta_{\ell+1}^2 \leq \varrho_{12} \eta_\ell^2 + \Lambda_{12} \delta_{\ell, \ell+1}^2$$

with  $\varrho_{12} < 1$  and  $\Lambda_{12} < \infty$  for any such choice of  $\lambda, \mu$

# Plain Convergence

# Convergence $\eta_k \rightarrow 0$ as $k \rightarrow \infty$

## Theorem (plain convergence)

(A12), (A4), and  $\Lambda := (1 + \Lambda_{12}\Lambda_4)/(1 - \varrho_{12}) < \infty$  imply

$$\sum_{k=\ell}^{\infty} \eta_k^2 \leq \Lambda \eta_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0.$$

Proof. Recall (A12) as  $\eta_{k+1}^2 \leq \varrho_{12}\eta_k^2 + \Lambda_{12}\delta_{k,k+1}^2$ . Then

$$\sum_{k=\ell}^{\ell+m} \eta_k^2 \leq \sum_{k=\ell}^{\ell+m+1} \eta_k^2 \leq \eta_\ell^2 + \varrho_{12} \sum_{k=\ell}^{\ell+m} \eta_k^2 + \Lambda_{12} \sum_{k=\ell}^{\ell+m} \delta_{k,k+1}^2$$

Recall (A4)

$$\sum_{k=\ell}^{\infty} \delta_{k,k+1}^2 \leq \Lambda_4 \eta_\ell^2$$

for the last term and utilize  $\varrho_{12} < 1$  to conclude the proof □

# Linear Convergence

# R-Linear Convergence Uniformly on Each Level

## Theorem

(A12), (A4),  $\Lambda$ , and  $q := 1 - 1/\Lambda < 1$  imply

$$\eta_{\ell+m}^2 \leq q^m \Lambda \eta_{\ell}^2 \quad \text{for all } \ell, m \in \mathbb{N}_0.$$

Proof. Plain convergence gives

$$\xi_{\ell}^2 := \sum_{k=\ell}^{\infty} \eta_k^2 \leq \Lambda \eta_{\ell}^2 < \infty$$

and so

$$\Lambda^{-1} \xi_{\ell}^2 \leq \eta_{\ell}^2 = \xi_{\ell}^2 - \xi_{\ell+1}^2$$

This proves

$$\xi_{\ell+1}^2 \leq q \xi_{\ell}^2 \quad (\text{for each } \ell \in \mathbb{N}_0)$$

Recall  $\xi_{\ell+1}^2 \leq q\xi_\ell^2$  for all  $\ell \in \mathbb{N}_0$

Mathematical induction leads to

$$\xi_{\ell+m}^2 \leq q^m \xi_\ell^2 \quad \text{for all } m \in \mathbb{N}_0$$

This and  $\xi_\ell^2 := \sum_{k=\ell}^{\infty} \eta_k^2 \leq \Lambda \eta_\ell^2$  show

$$\eta_{\ell+m}^2 \leq \xi_{\ell+m}^2 \leq q^m \xi_\ell^2 \leq q^m \Lambda \eta_\ell^2 \quad \square$$

R-linear convergence uniformly on each level implies

$$\sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim \eta_\ell^{-1/s}$$

via a geometric series

# Quasimonotonicity

# (QM) Estimator Quasimonotonicity

## Theorem (QM)

(A1)—(A3) and  $\Lambda_{mon} := 1 + \sqrt{\Lambda_1^2 + \Lambda_2^2} \Lambda_3$  imply

$$\eta(\widehat{\mathcal{T}}) \leq \Lambda_{mon} \eta(\mathcal{T})$$

for any refinement  $\widehat{\mathcal{T}}$  of any  $\mathcal{T}$  in  $\mathbb{T}$

Proof. For any  $0 < \lambda < \infty$ , utilize (A1)-(A2) in the decomposition

$$\widehat{\eta}^2(\widehat{\mathcal{T}}) = \widehat{\eta}^2(\widehat{\mathcal{T}} \cap \mathcal{T}) + \widehat{\eta}^2(\widehat{\mathcal{T}} \setminus \mathcal{T})$$



$$\begin{aligned}
\hat{\eta}^2(\hat{\mathcal{T}}) &= \hat{\eta}^2(\hat{\mathcal{T}} \cap \mathcal{T}) + \hat{\eta}^2(\hat{\mathcal{T}} \setminus \mathcal{T}) \\
&\leq (1 + \lambda) \underbrace{\left( \eta^2(\mathcal{T} \cap \hat{\mathcal{T}}) + \eta^2(\mathcal{T} \setminus \hat{\mathcal{T}}) \right)}_{\eta^2(\mathcal{T})} \\
&\quad + (1 + 1/\lambda)(\Lambda_1^2 + \Lambda_2^2)\delta^2
\end{aligned}$$

(A3) reads  $\delta^2 \leq \Lambda_3^2 \eta^2(\mathcal{T})$  and leads to

$$\eta^2(\hat{\mathcal{T}}) \leq \underbrace{(1 + \lambda + (1 + 1/\lambda)(\Lambda_1^2 + \Lambda_2^2)\Lambda_3^2)}_{\Lambda_{\text{mon}}^2} \eta^2(\mathcal{T}) \quad \square$$

# Comparison Lemma

# Comparison Lemma

Given  $0 < \varkappa < 1$  and  $s > 0$  with

$$M := \sup_{N \in \mathbb{N}_0} (N + 1)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T}) < \infty$$

$\exists 0 < \theta_0 < 1 \forall \ell \in \mathbb{N}_0 \exists \hat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$  s.t.

- (a)  $\eta(\hat{\mathcal{T}}_\ell) \leq \varkappa \eta(\mathcal{T}_\ell)$
- (b)  $\varkappa \eta_\ell |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|^s \leq \Lambda_{\text{mon}} M$
- (c)  $\theta_0 \eta_\ell^2 \leq \eta_\ell^2 (\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell)$

Proof. (1) W.l.o.g.  $\eta_\ell \equiv \eta(\mathcal{T}_\ell) > 0$ .

Then (QM) implies  $0 < \eta_0 \leq M$

(2) Choose minimal  $N_\ell \in \mathbb{N}_0$  s.t.

$$(N_\ell + 1)^{-s} \leq \frac{\varkappa \eta_\ell}{\Lambda_{\text{mon}} M} < N_\ell^{-s} \leq 1$$

$$(N_\ell \geq 1 \text{ because } \eta_\ell \Lambda_{\text{mon}}^{-1} / M \leq \eta_0 / M \leq 1)$$

(3) Set  $\widehat{\mathcal{T}}_\ell := \mathcal{T}_\ell \otimes \mathcal{T}'$  for  $\mathcal{T}'$  with  $\mathcal{T}' \in \mathbb{T}(N_\ell)$  s.t.  
 $(N_\ell + 1)^s \eta(\mathcal{T}') \leq M$

Quasimonotonicity and overlay control lead to (a),

$$\eta(\widehat{\mathcal{T}}_\ell) \leq \Lambda_{\text{mon}} M (N_\ell + 1)^{-s} \leq \varkappa \eta_\ell \quad \text{and} \quad |\widehat{\mathcal{T}}_\ell| \leq |\mathcal{T}_\ell| + N_\ell$$

# Proof of (b)-(c) in Comparison Lemma

(4) Proof of (b). **Count triangles** to verify

$$|\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell| \leq |\widehat{\mathcal{T}}_\ell| - |\mathcal{T}_\ell| \underset{\text{from } \otimes}{\leq} N_\ell \underset{\text{from (2)}}{<} \varkappa^{-1/s} \eta_\ell^{-1/s} \Lambda_{\text{mon}}^{1/s} M^{1/s} \quad \square$$

(5) Any  $\widehat{\mathcal{T}}_\ell \in \mathbb{T}(\mathcal{T}_\ell)$  with (a) allows for (c).

Given  $0 < \mu < \varkappa^{-2} - 1$ , (A1) plus (a) and (A3) imply

$$\begin{aligned} \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) &\leq (1+\mu)\eta^2(\widehat{\mathcal{T}}_\ell, \mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + (1+1/\mu)\Lambda_1^2\delta^2(\mathcal{T}_\ell, \widehat{\mathcal{T}}_\ell) \\ &\leq (1+\mu)\varkappa^2\eta_\ell^2 + (1+1/\mu)\Lambda_1^2\Lambda_3^2\eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \end{aligned}$$

This and  $\eta_\ell^2 = \eta_\ell^2(\mathcal{T}_\ell \cap \widehat{\mathcal{T}}_\ell) + \eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell)$  lead to

$$(1 - (1+\mu)\varkappa^2)\eta_\ell^2 \leq (1 + (1+1/\mu)\Lambda_1^2\Lambda_3^2)\eta_\ell^2(\mathcal{T}_\ell \setminus \widehat{\mathcal{T}}_\ell) \quad \square$$

NB.  $\Theta_0$  is a quotient that depends on  $\kappa$  and  $\mu$ . Fix those parameters.

# Repetition on Optimal Rates

## Optimality Analysis at a Glance II

$\mathcal{R}$  satisfies bulk criterion if  $\theta_A \leq \theta_0$  thus  $|\mathcal{M}_\ell^*| \leq |\mathcal{R}|$  for optimal set  $\mathcal{M}_\ell^*$  of marked cells. AFEM utilizes almost minimal  $\mathcal{M}_\ell$ , whence

$$|\mathcal{M}_\ell| \lesssim |\mathcal{M}_\ell^*| \leq |\mathcal{R}|$$

Set  $M := \sup_{N \in \mathbb{N}_0} (1 + N)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T})$  with (writing  $\kappa \approx 1$ )

$$|\mathcal{R}| \lesssim M^{1/s} \eta_\ell^{-1/s}$$

Recall closure overhead control and combine with aforementioned estimates for

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \lesssim \sum_{k=0}^{\ell-1} |\mathcal{M}_k| \lesssim M^{1/s} \sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim M^{1/s} \eta_\ell^{-1/s} \quad \square$$

# Separate Marking



# Adaptive Algorithm: SAFEM

**Input:**  $\mathcal{T}_0$ ,  $0 < \kappa$ ,  $\theta_A \ll 1$ ,  $\rho_B < 1$

$\forall \ell = 0, 1, 2, 3, \dots$

Compute  $\eta_\ell^2(K)$  and  $\mu^2(K)$  for all  $K \in \mathcal{T}_\ell$

**if**  $\mu_\ell^2 := \mu^2(\mathcal{T}_\ell) \leq \kappa \eta_\ell^2$

$\mathcal{T}_{\ell+1} := \text{Dörfler\_marking}(\theta_A, \mathcal{T}_\ell, \eta_\ell^2)$

**else**

$\mathcal{T}_{\ell+1} := \mathcal{T}_\ell \oplus \text{data\_approximation}(\rho_B \mu_\ell^2, \mathcal{T}_0, \mu_\ell^2)$

**Output:** Sequence  $(\mathcal{T}_\ell)$ ,  $(\eta_\ell)$ ,  $(\mu_\ell)$

abbreviate  $\sigma_\ell^2 := \eta_\ell^2 + \mu_\ell^2$

# Axioms (A1)–(A4)

Suppose  $\rho_2 < 1, \Lambda_k < \infty$  s.t.  $\forall \mathcal{T} \in \mathbb{T} \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \exists \mathcal{R} \subset \mathcal{T}$  s.t.  
 $\mathcal{T} \setminus \hat{\mathcal{T}} \subseteq \mathcal{R} \wedge |\mathcal{R}| \lesssim |\mathcal{T} \setminus \hat{\mathcal{T}}|$

$$|\eta(\hat{\mathcal{T}}, \mathcal{T} \cap \hat{\mathcal{T}}) - \eta(\mathcal{T}, \mathcal{T} \cap \hat{\mathcal{T}})| \leq \Lambda_1 \delta(\mathcal{T}, \hat{\mathcal{T}}) \quad (\text{A1})$$

$$\eta(\hat{\mathcal{T}}, \hat{\mathcal{T}} \setminus \mathcal{T}) \leq \rho_2 \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_2 \delta(\mathcal{T}, \hat{\mathcal{T}}) \quad (\text{A2})$$

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) \leq \Lambda_3 (\eta^2(\mathcal{T}, \mathcal{R}) + \mu^2(\mathcal{T})) + \widehat{\Lambda}_3 \eta^2(\hat{\mathcal{T}}) \quad (\text{A3})$$

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_4 \sigma_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0 \quad (\text{A4})$$

## Axioms (B1)–(B2)

$\forall \text{Tol} > 0 \mathcal{T}_{\text{Tol}} = \text{data\_approx}(\text{Tol}, \mathcal{T}_0, \mu^2) \in \mathbb{T}$  satisfies  $\mu^2(\mathcal{T}_{\text{Tol}}) \leq \text{Tol}$  and

$$|\mathcal{T}_{\text{Tol}}| - |\mathcal{T}_0| \leq \Lambda_5 \text{Tol}^{-1/(2s)} \quad (\text{B1})$$

$$\forall \mathcal{T} \in \mathbb{T}, \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T}) \quad \mu^2(\hat{\mathcal{T}}) \leq \Lambda_6 \mu^2(\mathcal{T}) \quad (\text{B2})$$

(B1)–(B2) follow for APPROX from subadditivity

$$\mu^2(\hat{\mathcal{T}}(\mathcal{M})) := \sum_{K \in \mathcal{M}} \sum_{T \in \hat{\mathcal{T}}(K)} \mu^2(T) \leq \Lambda_6 \mu^2(\mathcal{M}) \quad (\text{SA})$$

Typical example TSA by Binev and DeVore but also Dörfler marking over sufficient levels could do [C-Rabus 2016]

## Theorem (C-Rabus 2016)

*SAFEM leads to optimal convergence rates in total estimators provided (A1)-(A4), (B1)-(B2),*

$$0 < \theta_A < \theta_0 := \frac{1 - \kappa \Lambda_1^2 \Lambda_3}{1 + \Lambda_1^2 \Lambda_3} \quad \text{and} \quad \kappa < \frac{1 - \rho_A}{\Lambda_6 - 1}$$

*plus Quasimonotonicity (e.g. for  $(\Lambda_1^2 + \Lambda_2^2) \widehat{\Lambda}_3 < 1$ )  $\forall \mathcal{T} \in \mathbb{T} \forall \widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$*

$$\sigma(\widehat{\mathcal{T}}) \leq \Lambda_7 \sigma(\mathcal{T}).$$

Applications to div-LSFEM and to MFEM with convergence rates in  $H(\text{div}, \Omega) \times L^2(\Omega)$  in [C-Rabus 2016]

# Alternative Adaptive LS FEM 4 Laplace

# Alternative A Posteriori Error Control

Estimator  $\eta$  and data approximation error  $\mu$  in 2D

$$\begin{aligned}\eta^2(\mathcal{T}, K) &:= \|(1 - \Pi_0)p_{LS}\|_{L^2(K)}^2 \\ &+ |K|^{1/2} \sum_{E \in \mathcal{E}(K)} \left( \| [p_{LS}]_E \cdot \tau_E \|_{L^2(E)}^2 + \| [\partial u_{LS} / \partial \nu_E]_E \|_{L^2(E \setminus \partial\Omega)}^2 \right) \\ \mu^2(K) &:= \|f - \Pi_0 f\|_{L^2(K)}^2 \quad \text{for } K \in \mathcal{T} \text{ [C-Park 2015]}\end{aligned}$$

satisfy discrete reliability (A3) (for  $k = 0$ ) (Proof with explicit Crouzeix-Raviart and Raviart-Thomas functions, discrete Helmholtz decomposition, mixed intermediate solutions still leaves extra term)

## References and Further Reading

- C, M. Feischl, M. Page, D. Praetorius, Axioms of adaptivity, *Comput. Math. Appl.*, 67 (2014), pp. 1195-1253.
- C, H. Rabus. Axioms of adaptivity with separate marking for data resolution. *SIAM J. Numer. Anal.* 55(6) (2018) 2644-2665
- P. Bringmann , C, G. Starke: An adaptive least-squares FEM for linear elasticity with optimal conv rates, *SIAM J. Numer. Anal.* 56 (2018)
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- C, E. J. Park, P. Bringmann: Convergence of natural adaptive least squares FEMs, *Numer. Math.* 136 (2017)
- C, E.-J. Park: Convergence and optimality of adaptive least squares FEMs, *SIAM J. Numer. Anal.* 53 (2015)

# Prepare yourself for tomorrow

- Conforming  $P_1$  FEM for Poisson Model Problem (weak form,  $H_0^1(\Omega)$ , energy scalar product  $a(\bullet, \bullet) := \int_{\Omega} \nabla \bullet \cdot \nabla \bullet \, dx$ , energy norm)
- Inverse estimates (for polynomials)
- Trace inequality (for Sobolev functions)
- Discrete trace inequality (for polynomials)
- Shape regularity (for triangles, simplices)
- Poincaré and Friedrichs inequality (for Sobolev functions)
- Equivalence of norms in finite dimensional vector spaces
- Scaling argument (for derivatives of Sobolev functions)
- Triangle inequality (in normed linear spaces)
- Cauchy inequality (in Hilbert spaces like  $L^2$  or w.r.t.  $a(\bullet, \bullet)$ )

C, F. Hellwig: Constants in Discrete Poincaré and Friedrichs Inequalities and Discrete Quasi-interpolation, CMAM (arXiv:1709.00577), 2017.