# Mean-field theories for the description of waves in disordered materials

*Walter Schirmacher, Universität Mainz*

2018 International Summer School on "Frontiers of Applied and Computational Mathematics"

Shanghai Jiao Tong University July 9 – 21 2nd Lecture

# Contents

# 3 Functional integral treatment: Self-consistent Born approximation

## 3.1 Gaussian integrals

- We want to generalize the following well-known Gaussian integrals:

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2} = \sqrt{2\pi} \qquad \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} e^{bx} = \sqrt{\frac{2\pi}{a}} e^{\frac{1}{2}b^2/a} \tag{3.1}$$

Theorem: Let $A$ be an $N \times N$ matrix, and $\underline{x}$ und $\underline{b}$ vectors with $N$ components $x_i$, $b_i$, then

$$\boxed{\int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}\sum\limits_{ij} x_i A_{ij} x_j} e^{\sum\limits_i b_i x_i} = C e^{\frac{1}{2}\sum\limits_{ij} b_i [A^{-1}]_{ij} b_j}} \tag{3.2}$$

with

$$C = (2\pi)^{N/2} \prod_{i=1}^{N} \frac{1}{\sqrt{detA}} = (2\pi)^{N/2} \prod_{i=1}^{N} e^{-\frac{1}{2}\mathrm{tr}\{\ln A\}} \tag{3.3}$$

Proof: Let $U$ be the matrix which diagonalizes $A$:

$$U^T A U = A_d = \begin{pmatrix} e_1 & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & e_N \end{pmatrix} \qquad \text{and} \qquad \underline{c} = U\underline{b}. \tag{3.4}$$

where $e_1 \dots e_N$ are the eigenvalues of $A$. We make a variable transformation $\underline{y} = U\underline{x}$. The Jacobian determinant is $= 1$ because $U$ is orthogonal. Then we have

$$\int_{-\infty}^{\infty} \prod_{i=1}^{N} dx_i e^{-\frac{1}{2}\sum\limits_{ij} x_i A_{ij} x_j} e^{\sum\limits_i b_i x_i} = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dy_i e^{-\frac{1}{2}\sum\limits_i (y_i)^2 e_i} e^{\sum\limits_i c_i y_i} = (2\pi)^{N/2} \prod_{i=1}^{N} \sqrt{e_i^{-1}} \tag{3.5}$$

$\prod_i e_i$ is the determinant of $A_d$. Because this is invariant under change of basis we have

$$\prod_{i=1}^{N} \sqrt{e_i^{-1}} = \frac{1}{\sqrt{detA}} \tag{3.6}$$

On the other hand we have, because of the invariance of the trace with respect of basis change

$$\prod_{i=1}^{N} \sqrt{e_i^{-1}} = e^{-\frac{1}{2}\sum_{i=1}^{N} \ln e_i} = e^{-\frac{1}{2}\mathrm{tr}\{\ln A\}} \tag{3.7}$$

## 3.2 Discrete version: Anderson model on a lattice

### A Hamiltonian and representation of the Green's function as a Gaussian integral

If we discretize the Schrödinger equation with a random potential we arrive at the original tight-binding Hamiltonian of Anderson's publication 1958 for which he obtained the Nobel prize for predicting the possibility of localization:

$$\mathcal{H}_{ii} = 6t - v_i \qquad \mathcal{H}_{ij} = -t \quad i \neq j, \quad (i,j) \text{ nearest neighbors} \tag{3.8}$$

where $t$ is the tight-binding transfer matrix element between nearest neighbors and $v_i$ is the random potential with

$$\langle v_i \rangle = 0 \qquad \langle v_i v_j \rangle = \sigma^2 \delta_{ij} \tag{3.9}$$

corresponding to a distribution density

$$P(v_1 \dots v_i \dots v_N) = P_0 e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} v_i^2} \tag{3.10}$$

Here $\sigma^2$ is the mean-square fluctuation of the random potentials.
The discretized version of the Green's function is the Green matrix, which obeys the equation

$$-zG_{ij}(z) - \sum_{\ell} \mathcal{H}_{i\ell} G_{\ell j}(z) \equiv \sum_{\ell} A_{i\ell}(z) G_{\ell j}(z) = \delta_{ij} \tag{3.11}$$

$$z = -\omega^2 - i\epsilon$$

from which follows

$$A_{ij}(z) = -(z + v_i)\delta_{ij} - t(1 - \delta_{ij}) \tag{3.12}$$

and

$$G_{ij}(z) = \left[A^{-1}(z)\right]_{ij} \tag{3.13}$$

The corresponding ordered system has the Hamiltonian

$$\mathcal{H}_{ij}^{(0)} = t \quad (i,j) \text{ nearest neighbors} \qquad H_{ii}^{(0)} = 0 \tag{3.14}$$

On the simple-cubic lattice it has the band structure

$$E_{\mathbf{k}}^{(0)} = 6t - 2t\Big[cos(ak_x) + cos(ak_y) + cos(ak_z)\Big] \tag{3.15}$$

where $a$ is the lattice constant and we have

$$G_0(\mathbf{k}, z) = \sum_{ij} e^{i\mathbf{k}[\mathbf{r}_i - \mathbf{r}_j]} G_{ij}(z) = \frac{1}{z + E_{\mathbf{k}}^{(0)}} \tag{3.16}$$

For small $|\mathbf{k}| = k$ the unperturbed band structure becomes

$$E_{\mathbf{k}}^{(0)} \approx ta^2 k^2 \tag{3.17}$$

so that we can make correspondence to the continuum Schrödinger equation, making the identifications

$$\boxed{\frac{\hbar^2}{2m} \quad \leftrightarrow \quad ta^2}$$

We now use the formula proven the last subsection and change notation

$$x_i \longrightarrow u_i \qquad b_i \longrightarrow J_i$$

$$\int_{-\infty}^{\infty}\prod_{i=1}^{N} du_i e^{-\frac{1}{2}\sum_{ij} u_i A_{ij} u_j} e^{\sum_i J_i u_i} = \underbrace{\left(\tfrac{1}{2\pi}\right)^{N/2} e^{-\mathrm{tr}\frac{1}{2}\{\ln A\}}}_{\tilde{C}} e^{\frac{1}{2}\sum_{ij} J_i [A^{-1}]_{ij} J_j} \tag{3.18}$$

Defining "field" vectors

$$\{\ldots u_i \ldots\} \equiv |u) \qquad \{\ldots u_i \ldots\}^T \equiv (u| \tag{3.19}$$

and "source" vectors

$$\{\ldots J_i \ldots\} \equiv |J) \qquad \{\ldots J_i \ldots\}^T \equiv (J| \tag{3.20}$$

(3.18) now takes the compact form

$$\int_{-\infty}^{\infty}\prod_{i=1}^{N} du_i e^{-\frac{1}{2}(u|A|u)} e^{(J|u)i} = \underbrace{\left(\tfrac{1}{2\pi}\right)^{N/2} e^{-\mathrm{tr}\frac{1}{2}\{\ln A\}}}_{\tilde{C}} e^{\frac{1}{2}(J|[A^{-1}]|J)} \tag{3.21}$$

We now define the following <u>moment generating function</u>

$$Z[J] = \int_{-\infty}^{\infty}\prod_{i=1}^{N} du_i e^{-\frac{1}{2}(u|A(z)|u)} e^{(J|u)} = \tilde{C} e^{\frac{1}{2}(J|G(s)|J)} \tag{3.22}$$

where we observe $\tilde{C} = Z[J=0]$. We now have

<u>Theorem:</u>

$$\begin{aligned} G_{ij}(s) &= \frac{1}{Z[0]}\frac{\partial^2}{\partial J_i \partial J_j} Z[J]\Big|_{J=0} \\ &= \frac{1}{Z[0]}\int_{-\infty}^{\infty}\prod_{\ell=1}^{N} du_\ell\, u_i\, u_j\, e^{-\frac{1}{2}(u|A(z)|u)} = \frac{\partial^2}{\partial J_i \partial J_j}\ln Z[J]\Big|_{J=0}, \end{aligned} \tag{3.23}$$

which states that the Green matrix can be represented as the second moment of the function $Z[0]$. Obviously $Z[0] = \tilde{C}$ acts as a sort of partition function for the Green matrix.

<u>Proof:</u>

$$\begin{aligned} \frac{\partial}{\partial J_j}\ln Z[J] &= \frac{1}{Z[J]}\frac{\partial}{\partial J_j} Z[J] \\ &= \frac{1}{Z[J]}\frac{\partial}{\partial J_j}\tilde{C} e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m} \\ &= \frac{1}{Z[J]}\tilde{C}\sum_m G_{jm} J_m e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m} \\ &\longrightarrow 0 \quad \text{for} \quad J_i \to 0 \\ \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}\ln Z[J] &= G_{ji}\frac{1}{Z[J]}\tilde{C} e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m} + \tilde{C}\sum_m G_{jm} J_m \frac{\partial}{\partial J_j}\frac{1}{Z[J]} e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m} \end{aligned}$$

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_i}\ln Z[J]\Big|_{J=0} = G_{ji}}$$

$$
\begin{aligned}
\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z[J] &= \frac{\partial}{\partial J_i}\tilde{C}\sum_m G_{jm}J_m e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m} \\
&= \tilde{C}G_{ji}e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m} + \tilde{C}\sum_m G_{jm}J_m \frac{\partial}{\partial J_j} e^{\frac{1}{2}\sum_{\ell m} J_\ell G_{\ell m} J_m}
\end{aligned}
$$

$$
\Rightarrow \qquad \boxed{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_i}Z[J] = \tilde{C}G_{ji} = Z(0)G_{ji}}
$$

$$
\begin{aligned}
\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_i}Z[J] &= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_i}\int_{-\infty}^{\infty}\prod_{\ell=1}^{N} du_\ell e^{-\frac{1}{2}(u|A(z)|u)}e^{(J|u)} \\
&= \int_{-\infty}^{\infty}\prod_{\ell=1}^{N} du_\ell u_i u_j e^{-\frac{1}{2}(u|A(z)|u)}e^{(J|u)}
\end{aligned}
$$

$$
\Rightarrow \qquad \boxed{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j}Z[J] = Z[0]G_{ij} = Z[0]G_{ji}}
$$

# B Removal of the denominator $1/Z(0)$ by the replica trick

Let us look again at our integral representation of the Green matrix:

$$
\begin{aligned}
G_{ij}(s) &= \frac{1}{Z[0]}\frac{\partial^2}{\partial J_i \partial J_j}Z[J]\bigg|_{J=0} \qquad (4.23)\\
&= \frac{1}{Z[0]}\int_{-\infty}^{\infty}\prod_{\ell=1}^{N} du_\ell\, u_i\, u_j\, e^{-\frac{1}{2}(u|A(z,v_i)|u)} = \frac{\partial^2}{\partial J_i \partial J_j}\ln Z[J]\bigg|_{J=0},
\end{aligned}
$$

with $\qquad Z(0) = \int_{-\infty}^{\infty}\prod_{\ell=1}^{N} du_\ell\, e^{-\frac{1}{2}(u|A(z,v_i)|u)}$

and $\qquad A_{ij}(z,v_i) = -(z+v_i)\delta_{ij} - t(1-\delta_{ij})$

If it were not for the denominator $1/Z(0)$ in Formula (3.23), we could perform the average of the Green's function over the distribution density $P[v_i] = P_0 e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} v_i^2}$ straight away.
As we rather would like to be able to do so, we use the so-called replica trick:

$$
\boxed{\ln Z = \int_1^Z dx\,\frac{1}{x} = \lim_{n\to 0}\int_1^Z dx\, x^{n-1} = \lim_{n\to 0}\frac{1}{n}(Z^n - 1)} \qquad (3.24)
$$

from which follows
Theorem:

$$G_{ij}(s) = \lim_{n\to 0} Z[0]^{n-1} \frac{\partial^2}{\partial J_i \partial J_j} Z[J]\Big|_{J=0} = \lim_{n\to 0} \int_{-\infty}^{\infty} \prod_{\alpha=1}^{n} \prod_{\ell=1}^{N} du_\ell^\alpha \, u_i^1 \, u_j^1 \, e^{-\frac{1}{2}\sum_\alpha (u^\alpha|A(z)|u^\alpha)}, \tag{3.25}$$

where $\alpha$ labels the different replicas.

<u>Proof:</u>

$$\begin{aligned} G_{ij}(z) &= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_i} \ln Z[J]\Big|_{J=0} \\ &= \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} \lim_{n\to 0} \frac{1}{n}\Big(Z^n - 1\Big)\Big|_{J=0} \\ &= \lim_{n\to 0} \frac{\partial}{\partial J_i} Z^{n-1} \frac{\partial}{\partial J_j} Z\Big|_{J=0} \\ &= \lim_{n\to 0} \Big[(n-1)Z^{n-2} \frac{\partial}{\partial J_i} Z\Big)\frac{\partial}{\partial J_j} Z\Big) + Z^{n-1} \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} Z\Big]\Big|_{J=0} \end{aligned} \tag{3.26}$$

$$\Rightarrow \qquad \boxed{G_{ij}(z) = Z^{n-1} \frac{\partial}{\partial J_i}\frac{\partial}{\partial J_j} Z} \tag{3.27}$$

On the other hand, we have shown already in the previous proof that

$$\boxed{\frac{\partial}{\partial J_i}\frac{\partial}{\partial J_i} Z[J] = \int_{-\infty}^{\infty} \prod_{\ell=1}^{N} du_\ell u_i u_j e^{-\frac{1}{2}(u|A(z)|u)} e^{(J|u)}} \tag{3.28}$$

The replica trick has been invented to be able to calculate the quenched-disorder average $\langle \ldots \rangle$ of thermodynamic quantities, which usually appear in the form

$$\overline{\mathcal{O}} = \frac{1}{Z(0)} \mathrm{Tr}\Big\{\mathcal{O} e^{-\mathcal{H}/k_B T}\Big\} = k_B T \frac{\partial}{\partial h} \ln Z(h)\Big|_{h=0} \tag{3.29}$$

with

$$Z(h) = \mathrm{Tr}\Big\{e^{-[\mathcal{H}-\mathcal{O}h]/k_B T}\Big\} \tag{3.30}$$

Here $\overline{[\ldots]}$ denotes a thermal average. In order to be able to perform a Gaussian integral over some quenched disorder appearing in the Hamiltonian one has to resort to the replica trick. The replica trick formally amounts to calculating the partition function or generating function for $n$ replicas (replications) of the system and in the end letting $n \to 0$.
One must make sure that the final results of any calculation

- allow for such an analytical continuation,
- give results, which are nontrivial in this limit and that
- the replica limit commutes with the thermodynamic limit (if required).

Remark

There exist two other methods to remove the unwanted denominator $\frac{1}{Z[0]}$ from the Green matrix:

— the supersymmetry method

— the Keldysh method.

Returning now to our Green matrix we are able to perform the average over the quenched disorder (3.10) to obtain

$$\begin{aligned} \langle G_{ij}(z)\rangle &= P_0 \lim_{n\to 0} \int \prod_{\ell=1}^{N} dv_\ell e^{-\frac{1}{2\gamma}\sum_\ell v_\ell^2} \int_{-\infty}^{\infty} \prod_{\alpha=1}^{n} \prod_{\ell=1}^{N} du_\ell^\alpha \\ &\qquad \times u_i^1\, u_j^1\, e^{-\frac{1}{2}\sum_\alpha (u^\alpha|[s-H_0]|u^\alpha)} e^{-\frac{1}{2}\sum_\ell v_\ell \sum_\alpha (u_\ell^\alpha)^2} \\ &= P_0 \lim_{n\to 0} \int_{-\infty}^{\infty} \prod_{\alpha=1}^{n} \prod_{\ell=1}^{N} du_\ell^\alpha\, u_i^1\, u_j^1\, e^{-\frac{1}{2}\sum_\alpha (u^\alpha|[s-H_0]|u^\alpha)} e^{\frac{\sigma^2}{8}\sum_\ell \sum_{\alpha\alpha'} (u_\ell^\alpha)^2 (u_\ell^{\alpha'})^2} \end{aligned} \tag{3.31}$$

We see that

- the integral representation of $\langle G_{ij}(z)\rangle$ is not a Gaussian integral but involves an exponential function with an argument, which is quartic in the "fields" $u_\ell^\alpha$ and has a positive sign;
- so the integral does not look convergent.

We postpone this problem to the next but one subsection and perform the continuum limit of the Anderson problem first.

## 3.3 Continuum version: One electron in a random potential

### A Continuum limit and functional integral

We now perform the continuum limit of the formalism developed in the last subsection, i.e. We consider the following Schrödinger equation, introduced in the beginning:

$$\mathcal{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}) = -\frac{\hbar^2\nabla^2}{2m} - v(\mathbf{r}) \tag{3.32}$$

where $v(\mathbf{r})$ is assumed to be a *random potential*, i.e. a potential varying randomly in space.
For a moment we still keep the cubic lattice of the last subsection and again draw a cubic $V_i$ of size $a^3$ around each size $i$ with position vector $\mathbf{r}_i$ lattice, centered at $\mathbf{r}_i$. We assume that the random potential is constant inside $V_i$ and has the value $v_i = v(\mathbf{r}_i)$, which are subject to Gaussian disorder

$$\langle v_i v_j\rangle = \sigma^2 \delta_{ij} \tag{3.33}$$

$$P(\{v_i\}) = \prod_i p(v_i) = P_0 e^{-\frac{1}{2\sigma^2}(\sum_i v_i^2)} \tag{3.34}$$

Going over to the continuum limit $V_i \to 0$ we have now for the spatially varying fields $v(\mathbf{r})$

$$P[v(\mathbf{r})] = P_0 e^{-\frac{1}{2\gamma}\int d^3\mathbf{r} v(\mathbf{r})^2} \qquad \gamma = \sigma V_\xi \tag{3.35}$$

and we have

$$\langle v(\mathbf{r})v(\mathbf{r}') = \gamma\delta(\mathbf{r}-\mathbf{r}')\,. \tag{3.36}$$

(see section 3.1B).
A physical variable $X[v(\mathbf{r})]$, which depends on the random potential, is now averaged over according to

$$\begin{aligned}\langle X[v(\mathbf{r})]\rangle &= \lim_{V_i\to 0}\int\cdots\int\prod_i dv_i X(v_i)P(\{v_i\})e^{-\frac{1}{2\sigma}\sum_i \mathbf{r}v(\mathbf{r}_i)^2} \\ &= \int\mathcal{D}[v(\mathbf{r})]X[v(\mathbf{r})]P_0 e^{-\frac{1}{2\gamma}\int d^3\mathbf{r}v(\mathbf{r})^2}\end{aligned} \tag{3.37}$$

Such an integral over a field $v(\mathbf{r})$ is called a functional integral.

# B Gaussian functional integral and Green's function

Going to the continuum limit $V_i \to 0$ we generalize formula (3.2) as follows

$$\begin{aligned}&\lim_{V_i\to 0}\int\cdots\int\prod_i du_i\, e^{-\frac{1}{2\sigma}\sum_{ij}u_iA_{ij}u_j}e^{-\sum_i u_iJ_i} \\ =\;&\lim_{V_i\to 0}\left(\frac{1}{2\pi}\right)^{\frac{N}{2}}\prod_i\sqrt{\frac{1}{e_i}}\,e^{\frac{1}{2}\sum_{ij}J_iA^{-1}_{ij}J_j} \\ \equiv\;&\int\mathcal{D}[u(\mathbf{r})]e^{-\frac{1}{2}<u|A|u>}e^{<u|J>} \\ =\;& C_0\,e^{-\frac{1}{2}\mathrm{tr}\{\ln A\}}\,e^{\frac{1}{2}<J|A^{-1}|J>}\end{aligned} \tag{3.38}$$

where, again $e_i$ are the Eigenvalues of the matrix $A$.
In the continuum limit $A$ can be a *linear operator* with discrete or continuous spectrum and $C_0\,e^{-\frac{1}{2}\mathrm{tr}\{\ln A\}}$ is the appropriate generalization of the prefactor in the right-hand side of (3.2). $u(\mathbf{r})$ is the field and $J(\mathbf{r})$ is the source field.
We now deal with the usual *bra-ket* algebra of quantum mechanics, i.e. we have

$$\begin{aligned}<u|J> &= \int d^3\mathbf{r}u(\mathbf{r})J(\mathbf{r}) \qquad <u|A|u>=\int d^3\mathbf{r}u(\mathbf{r})A(\mathbf{r})u(\mathbf{r}) \\ &\int d^3\tilde{\mathbf{r}} <\mathbf{r}|A|\tilde{\mathbf{r}}><\tilde{\mathbf{r}}|A^{-1}|\mathbf{r}'>=\delta(\mathbf{r}-\mathbf{r}')\end{aligned} \tag{3.39}$$

The *Green's function* is given as a matrix element of the resolvent ($z_\pm = E\pm i\epsilon$):

$$\mathcal{G}(z_\pm) = \frac{1}{z_\pm + \mathcal{H}} \tag{3.40}$$

$$G(\mathbf{r},\mathbf{r}',z_\pm) = <\mathbf{r}|\mathcal{G}(z_\pm)|\mathbf{r}'> \equiv <\mathbf{r}|A(z_\pm)^{-1}|\mathbf{r}'> \tag{3.41}$$

where we have introduced an operator $A(z) = z - \mathcal{H}$.
Like $\mathcal{H}$ the operator $A(z)$ is obviously local in the $|\mathbf{r}>$ basis:

$$<\mathbf{r}|A(z)|\mathbf{r}'> = -\big[z + \frac{\hbar^2}{2m}\nabla^2 - v(\mathbf{r})\big]\delta(\mathbf{r}-\mathbf{r}') \tag{3.42}$$

Therefore we can define a *generating functional* $Z[J(\mathbf{r})]$ as follows

$$Z[J] \equiv C_0\,e^{-\frac{1}{2}\mathrm{tr}\{\ln A(z)\}}\,e^{\frac{1}{2}<J|\mathcal{G}(z)|J>} = \int\mathcal{D}[u(\mathbf{r})]e^{-\frac{1}{2}<u|A(z)|u>}e^{<u|J>} \tag{3.43}$$

where

$$<u|A(z)|u> \equiv \int d^3\mathbf{r}u(\mathbf{r})A(z)u(\mathbf{r})$$

$$\begin{aligned}<J|\mathcal{G}(z)|J> &\equiv \iint d^3\mathbf{r}d^3\mathbf{r}'J(\mathbf{r})<\mathbf{r}|A^{-1}(z)|\mathbf{r}'>J(\mathbf{r}') \\ &= \iint d^3\mathbf{r}d^3\mathbf{r}'J(\mathbf{r})G(\mathbf{r},\mathbf{r}',z)J(\mathbf{r}')\end{aligned}$$

## C Mathemathical digression: Variational derivative

Let $F\big[f(\mathbf{r})\big]$ be a functional of the function $f(\mathbf{r})$ defined in $\mathrm{R}^3$. The *variational derivative* is then defined by

$$\frac{\delta F\big[f(\mathbf{r})\big]}{\delta f(\mathbf{r}_0)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(F\big[f(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_0)\big] - F\big[f(\mathbf{r})\big]\Big)\,. \tag{3.44}$$

<u>*Examples:*</u>

$$\begin{aligned}
1)\qquad F\big[f(\mathbf{r})\big] &= \int d^3\mathbf{r}\, f(\mathbf{r})\\
\frac{\delta F\big[f(\mathbf{r})\big]}{\delta f(\mathbf{r}_0)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int d^3\mathbf{r}\big[f(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_0)\big] - \int d^3\mathbf{r}\, f(\mathbf{r})\Big) = 1\\
2)\qquad F\big[f(\mathbf{r})\big] &= \int d^3\mathbf{r}\, g(\mathbf{r}) f(\mathbf{r})\\
\frac{\delta F\big[f(\mathbf{r})\big]}{\delta f(\mathbf{r}_0)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int d^3\mathbf{r}\big[g(\mathbf{r})f(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_0)\big] - \int d^3\mathbf{r}\, g(\mathbf{r})f(\mathbf{r})\Big) = g(\mathbf{r}_0)\\
3)\qquad F\big[f(\mathbf{r})\big] &= \int d^3\mathbf{r}\, f(\mathbf{r})^2\\
\frac{\delta F\big[f(\mathbf{r})\big]}{\delta f(\mathbf{r}_0)} &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int d^3\mathbf{r}\big[f(\mathbf{r})+\epsilon\delta(\mathbf{r}-\mathbf{r}_0)\big]^2 - \int d^3\mathbf{r}\, f(\mathbf{r})^2\Big)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big(\int d^3\mathbf{r}\big[f(\mathbf{r})^2+2\epsilon f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0)\big] - \int d^3\mathbf{r}\, f(\mathbf{r})^2\Big) = 2f(\mathbf{r}_0)
\end{aligned}$$

I.e. for performing the functional derivative the integral is removed and the integrand is differentiated with respect to $f(\mathbf{r})$.

## D Green's function as variational derivative with respect to source fields

We now take the derivative of $< J|\mathcal{G}|J >$ with respect to the source field $J(\mathbf{r})$:

$$\frac{\delta}{\delta J(\mathbf{r})} < J|\mathcal{G}|J > = \frac{\delta}{\delta J(\mathbf{r})}\iint d^3\mathbf{r} d^3\mathbf{r}'\, J(\mathbf{r}) G(\mathbf{r},\mathbf{r}',z) J(\mathbf{r}') = 2\int d^3\mathbf{r}'\, G(\mathbf{r},\mathbf{r}',z)$$

and the double derivative

$$\frac{\delta}{\delta J(\tilde{\mathbf{r}})}\left(\frac{\delta}{\delta J(\mathbf{r})} < J|\mathcal{G}|J >\right) = 2G(\mathbf{r},\tilde{\mathbf{r}},z)$$

We recall

$$Z(J) = Z(J=0)e^{\frac{1}{2} < J|\mathcal{G}(z)|J >}$$

We therefore obtain

$$\begin{aligned}
G(\mathbf{r},\mathbf{r}',z) &= \frac{\delta^2}{\delta J(\mathbf{r})\delta J(\mathbf{r}')}\ln\frac{Z[J]}{Z[0]}\Big|_{J(0)} = \frac{\delta^2}{\delta J(\mathbf{r})\delta J(\mathbf{r}')}\ln Z[J]\Big|_{J(0)}\\
&\overset{!}{=} \frac{1}{Z[0]}\frac{\delta^2}{\delta J(\mathbf{r})\delta J(\mathbf{r}')} Z[J]\Big|_{J(0)}
\end{aligned} \tag{3.46}$$

where the last equality is subject to an exercise.
Taking now (4.5) into account we have

$$G(\mathbf{r},\mathbf{r}',z) = \frac{1}{Z(0)}\int \mathcal{D}\big[u(\mathbf{r})\big]\, u(\mathbf{r})\, u(\mathbf{r}')\, e^{-\frac{1}{2} < u|A(z)|u >} \tag{3.47}$$

We now understand the meaning of "generating functional": The functional derivative with respect to the source field $J(\mathbf{r})$ generates moments of

$$Z[0] = \int \mathcal{D}\big[u(\mathbf{r})\big]\, e^{-\frac{1}{2} < u|A(z)|u >}\,. \tag{3.48}$$

## E Replica trick

Applying now the replica trick (3.25) we obtain for the Green's function

$$
\begin{aligned}
G(\mathbf{r},\mathbf{r}',z) &= \lim_{n\to 0} Z[0]^{n-1}\frac{\delta^2}{\delta J(\mathbf{r})\delta J(\mathbf{r}')}\ln Z[J]\Big|_{J(0)} \\
&= \lim_{n\to 0}\int\prod_{\alpha=1}^{n}\mathcal{D}\big[u^\alpha(\mathbf{r})\big]\,u^1(\mathbf{r})u^1(\mathbf{r}')e^{-\sum_{\alpha\alpha'}\frac{1}{2}< u^\alpha|A(z)|u^{\alpha'} >} \\
&\equiv \lim_{n\to 0}\int\prod_{\alpha=1}^{n}\mathcal{D}\big[u^\alpha(\mathbf{r})\big]\,u^1(\mathbf{r})u^1(\mathbf{r}')e^{-\frac{1}{2}(u|A(z)|u)} \qquad (3.50)
\end{aligned}
$$

where we have defined new *bra's* and *ket's* in combined Hilbert and replica space as

$$
(a|B|c) = \sum_{\alpha\alpha'}\iint d^3\mathbf{r}d^3\mathbf{r}'a^\alpha(\mathbf{r})B_{\alpha\alpha'}(\mathbf{r},\mathbf{r}')c^{\alpha'}(\mathbf{r}') \qquad (3.50)
$$

and define $A$ to be diagonal in replica space

We are now able to perform the disorder average over the Green's function as

$$
\begin{aligned}
\mathsf{G}(\mathbf{r},\mathbf{r}',z) &= \langle G(\mathbf{r},\mathbf{r}',z)\rangle = \int\mathcal{D}[v(\mathbf{r})]P[v(\mathbf{r})] \\
&\times\int\prod_{\alpha}\mathcal{D}[u^\alpha(\mathbf{r})]u^1(\mathbf{r})u^1(\mathbf{r}')e^{-(u|A(z)|u)} \qquad (3.51)\\
&= \int\mathcal{D}[v(\mathbf{r})]u^1(\mathbf{r})u^1(\mathbf{r}')e^{-\frac{1}{2}(u|A_0(z)|u)}e^{S_{\text{int}}[u]}
\end{aligned}
$$

where

$$
A_0(z)_{\alpha\alpha'} = \big[z - \frac{\hbar^2}{2m}\nabla^2\big]\delta_{\alpha\alpha'} \qquad (3.52)
$$

and the disorder-induced "interaction" is given by

$$
S_{\text{int}}[u] = \frac{\gamma}{8}\sum_{\alpha\alpha'}\int d^3\mathbf{r}u^\alpha(\mathbf{r})^2u^{\alpha'}(\mathbf{r}')^2 \qquad (3.53)
$$

We see that

- the integral representation of $\langle G_{ij}(z)\rangle$ is not a Gaussian integral but involves an exponential function with an argument, which is quartic in the "fields" $u_\ell^\alpha$ and has a positive sign;
- so the integral does not look convergent.

We deal with this problem by performing the

## F Hubbard-Stratonovich transformation

As our integral is obviously ill-defined, because its interaction term has the wrong sign, we use a trick common in field theory, which is the application of another Gaussian functional integral, called "Hubbard-Stratonovich transformation:"

$$
\begin{aligned}
e^{S_{\text{int}}[u]} &= e^{\frac{\gamma}{8}\sum_{\alpha\alpha'}\int d^3\mathbf{r}u^\alpha(\mathbf{r})^2 u^{\alpha'}(\mathbf{r}')^2} \\
&= \prod_{\alpha\alpha'}\int \mathcal{D}[Q_{\alpha\alpha'}]e^{-\frac{1}{2\gamma}\sum_{\alpha\alpha'}\int d^3\mathbf{r}Q_{\alpha\alpha'}(\mathbf{r})^2} \\
&\qquad\qquad \times e^{\frac{1}{2}\sum_{\alpha\alpha'}\int d^3\mathbf{r}u^\alpha(\mathbf{r})Q_{\alpha\alpha'}(\mathbf{r})u^{\alpha'}(\mathbf{r})} \\
&\equiv \prod_{\alpha\alpha'}\int \mathcal{D}[Q_{\alpha\alpha'}]e^{-\frac{1}{2\gamma}\text{Tr}\{Q^2\}}e^{\frac{1}{2}(u|Q|u)}
\end{aligned}
\tag{3.54}
$$

Here

- $Q_{\alpha\alpha'}(\mathbf{r})$ is an auxiliary matrix field in replica space.
- Tr means a trace in replica and $\mathbf{r}$ space.
- We note that $Q$ is in general *non-diagonal* in replica space but *diagonal* in $\mathbf{r}$ space.

If we now define new source fields $\tilde{J}_{\alpha\alpha}(\mathbf{r},\mathbf{r}')$ (diagonal in replica and $\mathbf{r}$ space) and a new generating functional

$$
\tilde{Z}[\tilde{J}] = \int \mathcal{D}[u(\mathbf{r})]\int \mathcal{D}[Q(\mathbf{r})]e^{-\frac{1}{2}(u|A_0 - Q|u)}e^{(u|\tilde{J}|u)}e^{-\frac{1}{2\gamma}\text{Tr}\{Q^2\}} \tag{3.55}
$$

we are able to represent the averaged Green's function as

$$
\mathsf{G}(\mathbf{r},\mathbf{r}',z) = \frac{\delta}{\delta\tilde{J}_{11}(\mathbf{r},\mathbf{r}')}Z[\tilde{J}]\Big|_{J_{11}=0} \tag{3.56}
$$

The exponents in this multiple functional integral are *bilinear* in the fields $u^\alpha(\mathbf{r})$, so they can be integrated out to obtain

$$
\tilde{Z}[\tilde{J}] = \int \mathcal{D}[Q]e^{-S_{\text{eff}}[Q,\tilde{J}]} \tag{3.57}
$$

where the effective action is given by

$$
S_{\text{eff}}[Q,\tilde{J}] = \frac{1}{2}\text{Tr}\left\{\ln\left(A_0 - Q + 2\tilde{J}\right) + \frac{1}{\gamma}Q^2\right\} \tag{3.58}
$$

Remark

- This effective field theory is the starting point for the *nonlinear $\sigma$ model approach* to the Anderson-localization problem.

# G Mathematical digression: Saddle-point approximation

We want to calculate an improper integral

$$
I = \int_0^\infty e^{-pS(x)} \tag{3.59}
$$

in the limit of large $p > 0$. We take the example of the factorial

$$
n! = \int_0^\infty x^n e^{-x} = \int_0^\infty e^{-nS(x,n)} \tag{3.60}
$$

with

$$S(x,n) = \frac{x}{n} - \ln x \tag{3.61}$$

For large $n$ this function has an extremely narrow peak at a position $x_0$ given by

$$\left.\frac{dS(x,n)}{dx}\right|_{x=x_0} = 0 = \frac{1}{n} - \frac{1}{x_0} \qquad \Rightarrow \qquad x_0 = n \tag{3.62}$$

We now make a Taylor expansion around $x_0$

$$\begin{aligned} S(x,n) &= S(x_0,n) + \frac{1}{2}\frac{d^2}{dx^2}S(x,n)\bigg|_{x=x_0}(x-x_0)^2 \\ &= S(n,n) + \frac{1}{2}\frac{1}{n^2}(x-n)^2 \end{aligned} \tag{3.63}$$

inserting this in (3.60) we obtain

$$\begin{aligned} n! &\approx e^{-nS(x_0,n)} \int_0^\infty e^{-\frac{1}{2n}(x-n)^2} n! \approx e^{-n\overbrace{S(n,n)}^{1-\ln n}} \int_{-\infty}^{\infty} e^{-\frac{1}{2n}(x-n)^2} \\ &= n^n e^{-n}\sqrt{2\pi n} \end{aligned} \tag{3.64}$$

This is the celebrated Stirling formula for the factorial.
For very large $n$ the formula without the Gaussian correction

$$n! \approx e^{-nS(x_0),n} = e^{n\ln n - n} = n^n e^{-n} \tag{3.65}$$

is sufficient.

Remarks

- In our derivation of the mean-field theories we shall restrict ourselves to the method without the Gaussian corrections.
- If $x$ is allowed to have a complex value, a minimum along the real direction is a maximum along the imaginary direction, i.e we have a saddle in the complex plane. Therefore the name "saddle point".
- If we do quantum theory à la Feynman, the phase of the wave function may be expressed in terms of the action of classical mechanics

$$\Psi \propto e^{\frac{i}{\hbar} S\{p_i, q_i\}} \tag{3.66}$$

where $\{p_i, q_i\}$ are the generalized coordinates. In the classical limit $\frac{1}{\hbar} \to \infty$

we get the equations of motion by performing a generalized saddle-point approximation, which is called stationary-phase approximation.

from making the phase i.e. the action stationary

$$\delta S = 0 \tag{3.67}$$

we obtain the equations of motion of classical mechanics, i.e. the classical mechanics results in a saddle-point approximation of quantum mechanics without the Gaussian corrections. Including these corrections we arrive at the semiclassical quantum theory.

- In the same way one obtains the geometical optics from the wave optics: The action to be minimized is here the time along a light ray (Fermat's principle).

## H Saddle-point : Self-consistent Born approximation

We now perform a saddle-point approximation with respect to the effective field theory

$$S_{\text{eff}}[Q, \tilde{J}] = \frac{1}{2} \text{Tr} \left\{ \ln \left( A_0 - Q + 2\tilde{J} \right) + \frac{1}{\gamma} Q^2 \right\} \tag{3.68}$$

$$\begin{aligned} \delta S_{\text{eff}} &= 0 \\ \Leftrightarrow \qquad 0 &= \frac{\delta}{\delta Q_{\alpha\alpha'}(\mathbf{r}, z)} S_{\text{eff}} = \frac{1}{\gamma} Q_{\alpha\alpha'}(\mathbf{r}, z) - \frac{1}{2} (\alpha, \mathbf{r} | \left[ A - Q(z) - 2\tilde{J} \right]^{-1} | \alpha', \mathbf{r}') \Big|_{\mathbf{r}' \to \mathbf{r}} \end{aligned}$$

- Note that $\frac{1}{\gamma} = \frac{1}{\langle v^2 \rangle V_\xi}$ serves as the large parameter, which validates the saddle-point approximation.
- Therefore this approach is limited to

$$\frac{\langle v^2 \rangle}{\langle v \rangle^2} \ll 1 \tag{3.69}$$

We now take the simplest possible solution of the saddle-point approximation, namely a *spatially uniform* and *replica-diagonal* field

$$Q(\mathbf{r}, z) \equiv \Sigma(z) \delta_{\alpha\alpha'} \tag{3.70}$$

The field $Q(z) = \Sigma(z)$ constitutes an effective homogeneous medium in which the fluctuating potential has been replaced by the self-energy $\Sigma(z)$.

As now the effective Hamiltonian $\mathcal{H}[\Sigma] = \mathcal{H}_0 - \Sigma(z)$ is translational invariant we get the following self-consistent equation for $\Sigma(z)$

$$\Sigma(z) = \frac{\gamma}{2} < \mathbf{r}|\frac{1}{z - \Sigma(z) + \mathcal{H}_0}|\mathbf{r} >= \frac{\gamma}{2}\frac{1}{V}\sum_{|\mathbf{k}|\leq k_\xi} \frac{1}{z - \Sigma(z) + \frac{\hbar^2 k^2}{2m}} \tag{3.71}$$

which is identical to the uncorrelated version of the self-consistent Born approximation, (SCBA).
As before the upper cutoff in the $\mathbf{k}$ sum $k_\xi \sim [V_\xi]^{-1/3}$ is roughly the inverse correlation length of the potential fluctuations.

- All these results survive the Replica limit $n \to 0$ as they do not depend on $n$.

# 3.4 Diffusion and scalar waves

## A Model

We now consider simultaneously the following stochastic Helmholtz equations

$$-i\omega\rho(\mathbf{r},\omega) = \nabla D(\mathbf{r})\nabla\rho(\mathbf{r},\omega) \qquad \text{diffusion} \tag{3.72}$$

$$-\omega^2 u(\mathbf{r},\omega) = \nabla M(\mathbf{r})\nabla u(\mathbf{r},\omega) \qquad \text{scalar waves} \tag{3.73}$$

with

$$M(\mathbf{r}) = M_0 + \Delta(\mathbf{r}) \qquad\qquad D(\mathbf{r}) = D_0 + \Delta(\mathbf{r}) \tag{3.74}$$

- In the following we shall identify $M_0 = D_0$.

We again assume that the fluctuations $\Delta(\mathbf{r})$ obey

$$\langle\Delta(\mathbf{r})\Delta(\mathbf{r}')\rangle = \gamma\delta(\mathbf{r} - \mathbf{r}') \qquad\qquad \gamma = \langle\Delta^2\rangle V_\xi \tag{3.75}$$

which leads to the distribution density

$$P[\Delta(\mathbf{r})] = P_0 e^{-\frac{1}{2\gamma}\int d^3\mathbf{r}\Delta(\mathbf{r})^2} \tag{3.76}$$

## B Gaussian functional for the Green's function

The Green's function of (3.72) and (3.73) obeys the equation of motion

$$\begin{aligned} zG(\mathbf{r},\mathbf{r}',z) - \nabla M(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}',z) &= \Big(z - \nabla^2 M_0 - \nabla\Delta(\mathbf{r})\nabla\Big)G(\mathbf{r},\mathbf{r}',z) \\ &= \Big(A_0(\mathbf{r},z) - \nabla\Delta(\mathbf{r})\nabla\Big)G(\mathbf{r},\mathbf{r}',z) \\ &= A_\Delta(\mathbf{r},z)G(\mathbf{r},\mathbf{r}',z) = \delta(\mathbf{r} - \mathbf{r}') \end{aligned} \tag{3.77}$$

with

$$z = -i\omega + \epsilon \quad \text{diffusion} \qquad z = -\omega^2 - i\epsilon \quad \text{scalar waves} \tag{3.78}$$

Using again the replica trick we can represent the Green's function as

$$\begin{aligned} G(\mathbf{r},\mathbf{r}',z) &= \lim_{n\to 0}\int\prod_{\alpha=1}^{n}\mathcal{D}\big[u^\alpha(\mathbf{r})\big]\, u^1(\mathbf{r})u^1(\mathbf{r}')e^{-\sum_\alpha \frac{1}{2} < u^\alpha|A_\Delta(z)|u^\alpha >} \\ &\equiv \lim_{n\to 0}\int\prod_{\alpha=1}^{n}\mathcal{D}\big[u^\alpha(\mathbf{r})\big]\, u^1(\mathbf{r})u^1(\mathbf{r}')e^{-\frac{1}{2}(u|A_\Delta(z)|u)} \qquad (3.80) \end{aligned}$$

We can perform an integration by part for the matrix element of the disorder term in the operator $A$:

$$< u^\alpha|(A_\Delta - A_0)|u^\alpha >= -\int d^3\mathbf{r}u^\alpha(\mathbf{r})\nabla\Delta(\mathbf{r})\nabla u^\alpha(\mathbf{r}) = \int d^3\mathbf{r}\Delta(\mathbf{r})\left[\nabla u^\alpha(\mathbf{r})\right]^2 \tag{3.80}$$

Now the disorder average can be performed as follows

$$\begin{aligned} \mathsf{G}(\mathbf{r},\mathbf{r}',z) &= \langle G(\mathbf{r},\mathbf{r}',z)\rangle = \int \mathcal{D}[\Delta(\mathbf{r})]P[\Delta(\mathbf{r})]\int\prod_\alpha \mathcal{D}[u^\alpha(\mathbf{r})]u^1(\mathbf{r})u^1(\mathbf{r}) \\ &\times e^{-(u|A_0(z)|u)}e^{\int d^3\mathbf{r}\Delta(\mathbf{r})[\nabla u^\alpha(\mathbf{r})]^2} \\ &= \int \mathcal{D}[\Delta(\mathbf{r})]u^1(\mathbf{r})u^1(\mathbf{r}')e^{-\frac{1}{2}(u|A_0(z)|u)}e^{S_{\text{int}}[u]} \end{aligned} \tag{3.81}$$

where the disorder-induced interaction is given by

$$S_{\text{int}}[u] = \frac{\gamma}{8}\sum_{\alpha\alpha'}\int d^3\mathbf{r}[\nabla u^\alpha(\mathbf{r})]^2[\nabla u^{\alpha'}(\mathbf{r}')]^2 \tag{3.82}$$

The Hubbard-Stratonovich transformation now takes the form

$$\begin{aligned} e^{S_{\text{int}}[u]} &= e^{\frac{\gamma}{8}\sum_{\alpha\alpha'}\int d^3\mathbf{r}[\nabla u^\alpha(\mathbf{r})]^2[\nabla u^{\alpha'}(\mathbf{r}')]^2} \\ &= \prod_{\alpha\alpha'}\int \mathcal{D}[Q_{\alpha\alpha'}(z)]e^{-\frac{1}{2\gamma}\text{Tr}\{Q^2(z)\}}e^{-\frac{1}{2}(u|A_Q(z)|u)} \end{aligned} \tag{3.83}$$

with

$$A_Q(z)^{\alpha\alpha'} = A_0(z) - \nabla Q(\mathbf{r},z)^{\alpha\alpha'}\nabla \tag{3.84}$$

where we again performed an integration by part for the matrix elements of $A_Q(z)$.

Let us remark that the auxiliary fields $Q(\mathbf{r},z)^{\alpha\alpha'}$ appear exactly at the same place at which originally the stochastic field $\Delta(\mathbf{r})$ had been.

Defining again new source fields $\tilde{J}_{\alpha\alpha'}(\mathbf{r},\mathbf{r}')$ and a new generating functional

$$\tilde{Z}[\tilde{J}] = \int \mathcal{D}[u(\mathbf{r})]\int \mathcal{D}[Q(\mathbf{r},z)]e^{-\frac{1}{2}(u|A_Q(z)|u)}e^{(u|\tilde{J}|u)}e^{-\frac{1}{2\gamma}\text{Tr}\{Q^2(z)\}} \tag{3.85}$$

we are able to represent the averaged Green's function as

$$\mathsf{G}(\mathbf{r},\mathbf{r}',z) = \frac{\delta}{\delta\tilde{J}_{11}(\mathbf{r},\mathbf{r}')}Z[\tilde{J}]\Big|_{J_{11}=0} \tag{3.86}$$

The Gaussian integral over the fields $u^\alpha(\mathbf{r})$ in (3.85) can be performed with the result

$$\tilde{Z}[\tilde{J}] = \int \mathcal{D}[Q(z)]e^{-S_{\text{eff}}[Q(z),\tilde{J}]} \tag{3.87}$$

where the effective action is given by

$$S_{\text{eff}}[Q(z),\tilde{J}] = \frac{1}{2}\text{Tr}\Big\{\ln\left(A_Q(z)+2\tilde{J}\right) + \frac{1}{\gamma}Q^2(z)\Big\} \tag{3.88}$$

## C Saddle-point and Self-consistent Born approximation

We now perform again a saddle-point approximation with respect to the effective field theory (3.88):

$$\begin{aligned} \delta S_{\text{eff}} &= 0 \\ \Leftrightarrow \quad 0 &= \frac{\delta}{\delta Q(z)_{\alpha\alpha'}(\mathbf{r})} S_{\text{eff}}[Q(z), \tilde{J}] \\ &= \frac{1}{\gamma} Q(z)_{\alpha\alpha'}(\mathbf{r}) - \frac{1}{2}(\alpha, \mathbf{r}|\nabla[A_Q(z) - 2\tilde{J}]^{-1}\nabla|\alpha', \mathbf{r}')\Big|_{\mathbf{r}'\to\mathbf{r}} \end{aligned} \tag{3.89}$$

We now choose again a *spatially uniform* and *replica-diagonal* field as saddle-point solution

$$Q(z)(\mathbf{r}) \equiv -\Sigma(z)\delta_{\alpha\alpha'} \tag{3.90}$$

The field $Q(z) = -\Sigma(z)$ constitutes an effective homogeneous medium in which the fluctuation of the modulus $\Delta(\mathbf{r})$ has been replaced by the self-energy $\Sigma(z)$.

As now the operator $A_Q(z)$ is translational invariant, we obtain the self-consistent set of equations (*self-consistent Born approximation, SCBA*)

$$\Sigma(z) = -\frac{\gamma}{2} < \mathbf{r}|\nabla A_Q(\mathbf{r}, z)^{-1}\nabla|\mathbf{r} >= \frac{\gamma}{2}\frac{1}{V}\sum_{|\mathbf{k}|\leq k_\xi} k^2 \frac{1}{z + k^2\left[(\tilde{M}_0 - \Sigma(z)\right]} \tag{3.91}$$

Applying now (3.86) we obtain

$$\mathsf{G}(\mathbf{r}, \mathbf{r}', z) = (\mathbf{r}, \alpha|A_Q(z)^{-1}||\mathbf{r}', \alpha)\delta_{\alpha\alpha'} \tag{3.92}$$

As this matrix element is of a translationally invariant operator it depends only on $\mathbf{r} - \mathbf{r}' \equiv \tilde{\mathbf{r}}$ and has a Fourier transform

$$\mathsf{G}(\mathbf{k}) = \frac{1}{z + k^2\left[(\tilde{M}_0 - \Sigma(z)\right]} = \mathsf{G}(k) \tag{3.93}$$

## D Simplified variational scheme for obtaining the SCBA

We start again with our stochastic wave equation

$$0 = A_\Delta(z)u(\mathbf{r}, z) \qquad A_\Delta(z) = z - \nabla^2\tilde{M}_0 - \nabla\Delta(\mathbf{r})\nabla \tag{3.94}$$

Because, for obtaining our mean-field SCBA theory we are only interested in homogeneous and replica-diagonal auxiliary fields

$$Q_{\alpha\alpha'}(\mathbf{r}, z) = Q(z)\delta_{\alpha\alpha'} = -\Sigma(z)\delta_{\alpha\alpha'} \tag{3.95}$$

we can write down an <u>effective mean-field action</u>

$$\begin{aligned} \widetilde{S}_{\text{eff}}\left[Q(z), J\right] &= \frac{1}{2}\text{Tr}\left\{\frac{1}{\gamma}Q^2(z) + \ln\left(A_Q(z) + 2\tilde{J}\right)\right\} \\ &\to \frac{1}{2}\left(\frac{1}{\gamma}Q^2(z) + \sum_{|\mathbf{k}|\leq k_\xi} \ln[A_Q(k, z) + 2J(\mathbf{k})]\right) \end{aligned} \tag{3.96}$$

where $J(\mathbf{k})$ is a $\mathbf{k}$ dependent source field and

$$A_Q(k,z) = <\mathbf{k}|A_{\Delta(\mathbf{r})\to Q(z)}(z)|\mathbf{k}> = z + k^2\left[\tilde{M}_0 + Q(z)\right] \tag{3.97}$$

Varying now $\widetilde{S}_{\text{eff}}\left[Q(z), J=0\right]$ with respect to $Q(z)$ (which involves instead of a functional derivative an ordinary derivative with respect to the number $Q(z)$) we get the self-consistent SCBA equation

$$Q(z) = -\Sigma(z) = -\frac{\gamma}{2}\frac{1}{V}\sum_{|\mathbf{k}|\le k_\xi} k^2 \frac{1}{z + k^2\left[(\tilde{M}_0 - \Sigma(z)\right]} \tag{3.98}$$

The $\mathbf{k}$ dependent Green's function is obtained as

$$\mathsf{G}(\mathbf{k},z) = \frac{\delta}{\delta J(\mathbf{k})}\widetilde{S}_{\text{eff}}(Q,J)\bigg|_{J=0} = \frac{1}{A_Q(k,z)} \tag{3.99}$$

This scheme will be very helpful for deriving the appropriate versions of the SCBA for disordered vector Helmholtz equations.

## 3.5 Vector waves with disorder

### A Heterogeneous Elasticity

The equations of linear elasticity with spatially fluctuating elastic moduli can be written as

$$-\omega^2\mathbf{u}_{L,T}(\mathbf{r},\omega) = \nabla v_{L,T}^2(\mathbf{r})\nabla^2\mathbf{u}_{L,T}(\mathbf{r},\omega) \tag{3.100}$$

with

$$v_L^2(\mathbf{r}) = \frac{1}{\rho}\big(\lambda(\mathbf{r}) + 2G(\mathbf{r})\big) \equiv \frac{1}{\rho}M(\mathbf{r}) = \frac{1}{\rho}(K(\mathbf{r}) + \frac{4}{3}G(\mathbf{r})) \qquad v_T^2 = \frac{1}{\rho}G(\mathbf{r})$$

In realistic systems (glasses) the dominant heterogeneous elasticity is given by the fluctuations of the shear modulus. So we write

$$v_T(\mathbf{r})^2 = \frac{1}{\rho}G(\mathbf{r}) = \tilde{G}(\mathbf{r}) = \tilde{G}_0 + \Delta(\mathbf{r}) \tag{3.101}$$

If we assume that the bulk modulus $K = \rho\tilde{K}$ does not fluctuate we have

$$v_L(\mathbf{r})^2 = \tilde{M}(\mathbf{r}) = \tilde{K} + \frac{4}{3}\tilde{G}(\mathbf{r}) \tag{3.102}$$

and we put the Helmholtz equations into the form

$$0 = A_\Delta^{L,T}(\mathbf{r},z)\mathbf{u}_{L,T} \quad \text{with} \quad A_\Delta^{(L,T)}(\mathbf{r},z) = z - \nabla v_{L,T}^2(\mathbf{r})\nabla \tag{3.103}$$

we have now one longitudinal field and two transverse fields, i.e. the field-theoretical trace runs over all these fields. so we write our mean-field action as

$$
\begin{aligned}
\widetilde{S}_{\rm eff}\left[Q(z),J\right] &= \frac{1}{2}\Big(\frac{1}{\gamma}Q^2(z) + \sum_{|\mathbf{k}|\le k_\xi} \ln[A_Q(k,z)^{(L)}) + 2J(\mathbf{k})^{(L)}] \\
&+ 2\sum_{|\mathbf{k}|\le k_\xi} \ln[A_Q(k,z)^{(T)} + 2J(\mathbf{k})^{(T)}]\Big)
\end{aligned} \tag{3.104}
$$

with

$$
A_Q^{(L)}(k,z) = z + k^2\big[\tilde{K} + \frac{4}{3}\big(\tilde{G}_0 + Q(z)\big) \qquad A_Q^{(T)}(k,z) = z + k^2\big(\tilde{G}_0 + Q(z)\big) \tag{3.105}
$$

From this we get the SCBA for heterogeneous elasticity theory by varying $S_{\rm eff}$ with respect to $Q(z) = -\Sigma(z)$:

$$
\Sigma(z) = \gamma \sum_{|\mathbf{k}|\le k_\xi} k^2 \left(\frac{2/3}{z + k^2\big[\tilde{K} + \frac{4}{3}\big(\tilde{G}_0 - \Sigma(z)\big)} + \frac{1}{z + k^2\big(\tilde{G}_0 - \Sigma(z)\big)}\right) \tag{3.106}
$$

By performing the functional derivatives with respect to the longitudinal and transverse source fields $J^{(L,T)}(\mathbf{k})$ we get the longitudinal and transverse Green's function

$$
G_L(k,z) = \frac{1}{z + k^2\big[\tilde{K} + \frac{4}{3}\big(\tilde{G}_0 - \Sigma(z)\big)} \tag{3.107}
$$

and

$$
G_T(k,z) = \frac{1}{z + k^2\big(\tilde{G}_0 - \Sigma(z)\big)} \tag{3.108}
$$

The vibrational density of states is given by

$$
g(\omega) = 2\omega g(\omega^2) = \frac{2\omega}{\pi}\frac{1}{3}\mathcal{I}\mathrm{m}\Big\{\sum_{|\mathbf{k}|\le k_D}\big(G_L(k,z) + 2G_T(k,z)\big)\Big\} \tag{3.109}
$$

It is important to mention that the upper cutoff for the density of states is <u>not</u> the inverse correlation length but the <u>Debye wavenumber</u>

$$
k_D = \sqrt[3]{6\pi^2 N/V} \tag{3.110}
$$

where $N$ is the number of particles in the volume $V$.

# 4 Coherent-Potential Approximation (CPA)

## 4.1 Scalar fields and diffusion

We again consider our scalar Helmholtz equation for scalar fields and diffusion as before

$$0 = A_D(\mathbf{r}, z) G(\mathbf{r}, \mathbf{r}', z) = \delta(\mathbf{r} - \mathbf{r}') \tag{4.1}$$

with

$$A_D(\mathbf{r}, z) = z - \nabla D(\mathbf{r}) \nabla \tag{4.2}$$

and

$$z = -i\omega + \epsilon \text{ diffusion} \qquad - \omega^2 - i\epsilon \text{ scalar waves} \tag{4.3}$$

In the case of scalar waves $D(\mathbf{r})$ is the fluctuating elastic modulus, e.g. $D(\mathbf{r}) \equiv M(\mathbf{r})$.
We write again the replica integral representation of the Green's function

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}', z) &= \lim_{n\to 0} \int \prod_{\alpha=1}^{n} \mathcal{D}\big[u^\alpha(\mathbf{r})\big]\, u^1(\mathbf{r}) u^1(\mathbf{r}') e^{-\sum_\alpha \frac{1}{2} < u^\alpha | A_D(z) | u^\alpha >} \\ &\equiv \lim_{n\to 0} \int \prod_{\alpha=1}^{n} \mathcal{D}\big[u^\alpha(\mathbf{r})\big]\, u^1(\mathbf{r}) u^1(\mathbf{r}') e^{-\frac{1}{2}(u|A_D(z)|u)} \\ &= \lim_{n\to 0} \frac{\delta}{\delta J^1(\mathbf{r}, \mathbf{r}')} Z[J] \end{aligned} \tag{4.4}$$

with the moment generating functional

$$Z[J] = \int \prod_{\alpha=1}^{n} \mathcal{D}\big[u^\alpha(\mathbf{r})\big] e^{-\frac{1}{2}(u|A_D(z)|u)} e^{(u|J|u)} \tag{4.5}$$

Here we have introduced the source term

$$(u|J|u) = \sum_\alpha \int d^3\mathbf{r}\, u^\alpha(\mathbf{r}) J^\alpha(\mathbf{r}, \mathbf{r}') u^\alpha(\mathbf{r}) \tag{4.6}$$

We now use a method, originally introduced by Fadeev and Popov for high-energy field theory, i.e we we introduce a functional $\delta$ function in order to replace the fluctuating quantity $D(\mathbf{r})$ by auxiliary fields $Q(\mathbf{r}, z)^\alpha$ and represent this $\delta$ function, in turn by a further integral representation using a second set of auxiliary fields $\Lambda(\mathbf{r},z)^\alpha$:

$$\begin{aligned} Z[J] &= \int \mathcal{D}[u^\alpha(\mathbf{r})] \int \mathcal{D}[Q] e^{-\frac{1}{2}(u|A_Q(z)|u)} e^{(u|J|u)} \delta(D-Q) \qquad (4.7) \\ &= \int \mathcal{D}[u^\alpha(\mathbf{r})] \int \mathcal{D}[Q] \int \mathcal{D}[\Lambda] e^{-\frac{1}{2}(u|A_Q(z)|u)} e^{(u|J|u)}\, e^{(\Lambda|D-Q)} \\ &= \int \mathcal{D}[Q] \int \mathcal{D}[\Lambda] e^{\frac{1}{2}\sum_\alpha \mathrm{tr}\{\ln[A_Q(z)-2J]\}} e^{-(u|J)}\, e^{(\Lambda|D-Q)} \end{aligned}$$

where the last line results in integrating out the original fields $u^\alpha(\mathbf{r})$. The matrix element involving the auxiliary fields is defined by

$$(\Lambda|D-Q) = \frac{1}{V}\sum_{\alpha=1}^{n} \int d^3\mathbf{r} \Lambda^\alpha(\mathbf{r}) \Big( D(\mathbf{r}) - Q^\alpha(\mathbf{r}) \Big) \qquad (4.8)$$

We now devise the following coarse-graining procedure:

- We tile the total space into $N_c$ cells of (approximate) volume $V_c = V/N_c$, where $V = L^3$ is the total volume.
- Within a cell with label $i$ we replace the diffusivity by their average in each cell and assume that a regular diffusion equation (or scalar wave equation) holds within a cell $i$:

$$\big(z - D_i \nabla^2\big) u^\alpha(\mathbf{r},z) = 0 \qquad (4.9)$$

- The fluctuations of $D_i$ are assumed to be uncorrelated, i.e.

$$P(D_1 \ldots D_{N_c}) = \prod_i p(D_i) \qquad (4.10)$$

Within our model $D(\mathbf{r})$ is now a piecewise constant function in real space and the same should hold for the auxiliary fields $Q$ and $\Lambda$, which are now labeled as $Q_i^{(\alpha)}$, $\Lambda_i^{(\alpha)}$. Taking tis into account, the scalar product, which appears in the exponential in Eq. (4.7) can be written as:

$$(\Lambda|D-Q) \to \frac{V_c}{V} \sum_\alpha \sum_i \Lambda_i^{(\alpha)}(\mathbf{r}) \left( D_i^{(\alpha)} - Q_i^{(\alpha)} \right) \qquad (4.11)$$

We now start to evaluate the configurational average. Due to the Fadeev-Popov transformation the only term to be averaged over is the term

$$e^{(\Lambda|D-Q)}\,.$$

Assuming that all the $N_c$ coarse-graining cubes behave the same on average and using that the individual cubes are not correlated, we can write

$$\left\langle e^{(\Lambda|D-Q)}\right\rangle = \prod_\alpha \prod_i \left\langle e^{\frac{V_c}{V}\Lambda_i^{(\alpha)}(D_i^{(\alpha)}-Q_i^{(\alpha)})}\right\rangle_i \tag{4.12}$$
$$= e^{\sum_\alpha \frac{V}{V_c}\ln\left(\left\langle \exp\left[-\frac{V_c}{V}\Lambda_i^{(\alpha)}(D_i^{(\alpha)}-Q_i^{(\alpha)})\right]\right\rangle_i\right)}$$

where $\langle\dots\rangle_i$ denotes an average with $p(D_i)$. Note that the two occurring volume ratios do not cancel each other due to the average inside the logarithm. Using (4.12) the generating functional (4.5) can be written as

$$Z[\tilde{J}] = \int \mathcal{D}[Q,\Lambda]\, e^{-\sum_\alpha S^\alpha_{\text{eff}}[Q,\Lambda,\tilde{J}]} \tag{4.13}$$

with the effective action

$$S^\alpha_{\text{eff}}[Q,\Lambda,\tilde{J}] = \frac{1}{2}\text{tr}\{\ln\left(A_Q-2J\right)\} - \sum_{\alpha=1}^{n}\frac{V}{V_c}\ln\left(\left\langle e^{-\frac{V_c}{V}\Lambda_i^{(\alpha)}(D_i^{(\alpha)}-Q_i^{(\alpha)})}\right\rangle_i\right) \tag{4.14}$$

We now can use the prefactor $\frac{V}{V_c}$ of the action as large parameter for the saddle-point approximation.
Before writing down the saddle-point equations we go – as in the previous section – onto the effective medium, i.e. replace the auxiliary fields $\Lambda_i^\alpha(\mathbf{r},z)$ and $Q_i^\alpha(\mathbf{r},z)$ by homogeneous ones $\Lambda_i^\alpha(z)$ and $Q_i^\alpha(z)$ giving

$$\widetilde{S}^\alpha_{\text{eff}}[Q,\Lambda,\tilde{J}] = \frac{1}{2}\sum_{\mathbf{k}}\{\ln\left(A_Q(k,z)-2J\right)\} - \sum_{\alpha=1}^{n}\frac{V}{V_c}\ln\left(\left\langle e^{-\frac{V_c}{V}\Lambda_i^{(\alpha)}(D_i^{(\alpha)}-Q_i^{(\alpha)})}\right\rangle_i\right) \tag{4.15}$$
$$\equiv \widetilde{S}^{(1)}_{\text{eff}} - \widetilde{S}^{(2)}_{\text{eff}} \tag{4.16}$$

with

$$A_Q(k,z) = z + Q(z)k^2 \tag{4.17}$$

Varying $\widetilde{S}_{\text{eff}}$ with respect to $Q$ we get ( we suppress now the upper indices $\alpha$):

$$\frac{\partial \widetilde{S}_{\text{eff}}}{\partial Q_i} = 0 = \frac{\partial \widetilde{S}^{(1)}_{\text{eff}}}{\partial Q_i} - \frac{\partial \widetilde{S}^{(2)}_{\text{eff}}}{\partial Q_i} \tag{4.18}$$

We start with $\widetilde{S}^{(1)}_{\text{eff}}$:

$$\frac{\partial \widetilde{S}^{(1)}_{\text{eff}}}{\partial Q_i} = \frac{1}{2}\sum_{\mathbf{k}} \frac{k^2}{z + k^2 Q(z)} \tag{4.19}$$

Now keep going with $\widetilde{S}^{(2)}_{\text{eff}}$:

$$\begin{aligned} \frac{\partial \widetilde{S}^{(1)}_{\text{eff}}}{\partial Q_i} &= \frac{\frac{V_c}{V}\Lambda_{i,s}\left\langle e^{-\frac{V_c}{V}\Lambda_i(D_i - Q_{i,s}}\right\rangle_i}{\left\langle e^{-\frac{V_c}{V}\Lambda_{i,s}(D_i - Q_{i,s}}\right\rangle_i} \\ &= \frac{V_c}{V}\Lambda_i \end{aligned} \tag{4.20}$$

from which follows with (4.19)

$$\Lambda_(z) = \frac{V}{2V_c}\sum_{\mathbf{k}} \frac{k^2}{z + k^2 Q(z)} \tag{4.21}$$

Varying the action with respect to $\Lambda_i(z)$ we obtain

$$\begin{aligned} \frac{\partial \widetilde{S}_{\text{eff}}}{\partial \Lambda_i} = 0 \;&=\; \frac{\left\langle -\frac{V_c}{V}\Lambda_i(z)(D_i - Q_i(z))e^{-\frac{V_c}{V}\Lambda_i(z)[D_i - Q_i(z)]}\right\rangle_i}{\left\langle e^{-\frac{V_c}{V}\Lambda_i(z)[D_i - Q_i(z)]}\right\rangle_i} \\ \Rightarrow \qquad 0 \;&=\; \left\langle \frac{D_i - Q_i(z)}{\exp[\frac{V_c}{V}\Lambda_i(s)(D_i - Q_i(z))]}\right\rangle_i \end{aligned} \tag{4.22}$$

Since $\frac{V_c}{V} \ll 1$ the exponential in the denominator can be expanded to first order [1]:

$$0 = \left\langle \frac{D_i - Q_i(z)}{1 + \frac{V_c}{V}(D_i - Q_i(z))\Lambda_i(z)}\right\rangle_i \tag{4.23}$$

For the mean-field averaged Green's function we get

$$G(\mathbf{k}) = \frac{\delta}{\delta}J(k)\widetilde{Z}[J] = \frac{1}{z + k^2 Q(z)} \tag{4.24}$$

[1] If the exponential in the denominator of Eq. (4.22) would remain in the numerator and then expanded, we would obtain the self-consistent Born approximation (see below)

We can now write down the result for the self-consistent equations of the coherent potential approximation (CPA) (we drop now the indices $i$ at $Q$ and $\Lambda$ but keep it for $D_i$ to indicate that this quantity is fluctuating):

$$0 = \left\langle \frac{D - Q(z)}{1 + \nu(D_i - Q(z))\widetilde{\Lambda}(z)} \right\rangle_i \tag{4.25a}$$

$$\widetilde{\Lambda}_i(z) = \frac{1}{\nu}\Lambda(z) = \frac{1}{3\xi^3}\int_0^{k_\xi} dk k^2 \frac{k^2}{z + k^2 Q(z)} \tag{4.25b}$$

where we have identified $V_c$ with the correlation volume:

$$V_c = \frac{\nu}{3\pi^2 k_\xi^3} \tag{4.26}$$

with $\nu$ being an adjustable parameter of order unity and we have used $\sum_{\mathbf{k}} = \frac{V}{2\pi^2}\int_0^{k_\xi} dk k^2$
The CPA equation (4.25a) can be cast into the following equivalent forms: (we drop now the indices $i$ at $Q$ and $\Lambda$ but keep it for $D_i$ to indicate that this quantity is fluctuating):

$$Q(z) = \left\langle \frac{D_i}{1 + \nu\left[D_i - Q(z)\right]\widetilde{\Lambda}(z)} \right\rangle_i \tag{4.27a}$$

$$1 = \left\langle \frac{1}{1 + \nu\left[D_i - Q(z)\right]\widetilde{\Lambda}(z)} \right\rangle_i \tag{4.27b}$$

Proof: CPA equation:

$$Q(z)\left\langle \frac{1}{1+\nu(D_i-Q(z))\widetilde{\Lambda}_i(z)} \right\rangle_i = \left\langle \frac{D_i}{1+\nu(D_i-Q(z))\widetilde{\Lambda}_i(z)} \right\rangle_i \tag{4.28a}$$

$$= Q(z)\left\langle \frac{1+\nu(D_i-Q(z))\widetilde{\Lambda}_i(z) - \overbrace{\nu(D_i-Q(z))}^{=0}\widetilde{\Lambda}(z)}{1+\nu(D_i-Q(z))\widetilde{\Lambda}(z)} \right\rangle_i \tag{4.28b}$$

$$= Q(z) \tag{4.28c}$$

Comparing the left-hand side of (4.28a) with (4.28c) we get

$$1 = \left\langle \frac{1}{1+\nu(D_i-Q(z))\widetilde{\Lambda}_i(z)} \right\rangle_i \tag{4.28d}$$

## 4.2 Heterogeneous elasticity

We recall the equations for heterogeneous elasticity

$$\boxed{-\omega^2 \mathbf{u}_{L,T}(\mathbf{r},\omega) = \nabla v_{L,T}^2(\mathbf{r})\nabla^2 \mathbf{u}_{L,T}(\mathbf{r},\omega)} \tag{4.29}$$

with

$$v_L^2(\mathbf{r}) = \frac{1}{\rho}\big(\lambda(\mathbf{r})+2G(\mathbf{r})\big) \equiv \frac{1}{\rho}M(\mathbf{r}) = \frac{1}{\rho}(K(\mathbf{r})+\frac{4}{3}G(\mathbf{r})) \qquad v_T^2 = \frac{1}{\rho}G(\mathbf{r})$$

We again assume that only the shear modulus exhibits spatial fluctuations

$$v_T(\mathbf{r})^2 = \frac{1}{\rho}G(\mathbf{r}) = \tilde{G}(\mathbf{r}) = \tilde{G}_0 + \Delta(\mathbf{r}) \tag{4.30}$$

$$v_L(\mathbf{r})^2 = \tilde{M}(\mathbf{r}) = \tilde{K} + \frac{4}{3}\tilde{G}(\mathbf{r}) \tag{4.31}$$

and we put the Helmholtz equations into the form

$$0 = A_\Delta^{L,T}(\mathbf{r},z)\mathbf{u}_{L,T} \quad \text{with} \quad A_\Delta^{(L,T)}(\mathbf{r},z) = z - \nabla v_{L,T}^2(\mathbf{r})\nabla \tag{4.32}$$

with

$$A^L_\Delta(z) \;=\; z - \nabla[\tilde{K} + \frac{4}{3}\tilde{G}(\mathbf{r})]\nabla A^T = z - \nabla\tilde{G}(\mathbf{r})\nabla \qquad (4.33a)$$

Again the trace runs over one longitudinal and two transverse fields. Performing exactly the same steps done for the scalar theory we obtain the effective mean-field action as

$$\begin{aligned}\widetilde{S}_{\text{eff}}\left[Q(z), J\right] \;=\; & \frac{1}{2} + \sum_{|\mathbf{k}|\le k_\xi} \ln[A_Q(k,z)^{(L)}) + 2J(\mathbf{k})^{(L)}] \\ & +2\sum_{|\mathbf{k}|\le k_\xi} \ln[A_Q(k,z)^{(T)} + 2J(\mathbf{k})^{(T)}] \\ & -\sum_{\alpha=1}^{n}\frac{V}{V_c}\ln\left(\left\langle e^{-\frac{V_c}{V}\Lambda_i^{(\alpha)}(D_i^{(\alpha)}-Q_i^{(\alpha)})}\right\rangle_i\right) \qquad (4.34)\end{aligned}$$

with

$$A^{(L)}_Q(k,z) = z + k^2\big[\tilde{K} + \frac{4}{2}\big(\tilde{G}_0 + Q(z)\big) \qquad A^{(T)}_Q(k,z) = z + k^2\big(\tilde{G}_0 + Q(z)\big) \qquad (4.35)$$

from which we get now, varying with $Q$ and $\Lambda$, the following CPA equations

$$0 = \left\langle \frac{D - Q(z)}{1 + \nu(D_i - Q(z))\widetilde{\Lambda}(z)} \right\rangle_i \qquad (4.36a)$$

$$\begin{aligned}\widetilde{\Lambda}_i(z) \;=\; & \frac{1}{\nu}\Lambda(z) = \frac{1}{3\xi^3}\int_0^{k_\xi} dk k^2 \left(\frac{4}{3}\frac{k^2}{z + k^2[K + \frac{4}{3}Q(z)]}\right. \\ & \left. +2\frac{k^2}{z + k^2 Q(z)}\right) \qquad (4.36b)\end{aligned}$$

# 5 Applications

## 5.1 Vibrational anomalies of glasses and the boson peak

## 5.2 Diffusion in quenched-disordered systems

## 5.3 Heterogeneous viscoelasticity and the Glass transition

## 5.4 Localization of light: Potential-type vs. modulus-type disorder