5. Applications

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Outline

1. Anderson localization
   - Electrons in disordered materials
     - Scattering and diffusion of electrons
     - Interference effects
     - Scaling theory of Anderson localization
   - Classical waves with disorder
     - Sound waves with disorder
     - Evaluation of the stochastic field theory: Nonlinear σ model
     - Saddle Point: Self-consistent Born approximation (SCBA)
     - Beyond the Saddle Point: wave diffusivity and localization
     - Experimental evidence for localized sound waves in 2d
   - Localization of light
     - Transverse localization of light
     - Two conflicting theories and *experimentum crucis*

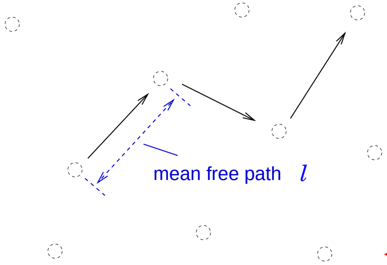
Anderson localization

Scattering and diffusion of electrons

- Electrons in a metal, which are occasionally scattered by an impurity, which results in a random walk
- Diffusivity

$$D = \frac{1}{3}\ell^2 \underbrace{\frac{1}{\tau}}_{\text{scattering rate}} \qquad \ell = v_F\tau \quad \text{mean-free path}$$

$$v_F = \frac{1}{m}\hbar k_F = \text{Fermi velocity}$$



- Einstein relation (degenerate statistics)

$$\sigma = N(E_F)e^2 D = \frac{3}{2}\frac{n}{E_F}e^2\frac{1}{3}\frac{\hbar^2 k_F^2}{m^2}\tau = \frac{1}{m}ne^2\tau \quad \text{Drude conductivity}$$

$$E_F = \frac{m}{2}v_F^2 = \text{Fermi energy}, \; k_F = \text{Fermi wavenumber}$$

Quantum diffusion of the electrons

- Electron-impurity Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} = E_{\text{kin}} + \sum_i v(\mathbf{r} - \mathbf{r}_i)$$

- Diffusivity and random walk

$$D = \lim_{t\to\infty} \frac{1}{2t}\langle r^2(t)\rangle = \lim_{t\to\infty} \int \mathrm{d}^3\mathbf{r}\frac{r^2}{2t}P(\mathbf{r},t)$$

- The "time" t runs along the diffusion path, i.e. $t = L_{\text{diff}}/v_F$. $P(\mathbf{r},t)$ is the diffusion probability density. Defining ϵ to be the Laplace variable (instead of s) we have

$$D = \lim_{\epsilon\to 0} \frac{\epsilon^2}{2}\langle r^2(\epsilon)\rangle = \lim_{\epsilon\to 0} \frac{\epsilon^2}{2} \int \mathrm{d}^3\mathbf{r}\, r^2 P(\mathbf{r},\epsilon) = \frac{\epsilon^2}{2} \int \mathrm{d}^3\mathbf{r}\, r^2 \langle |G(\underbrace{\mathbf{r}_1 - \mathbf{r}_0}_{\mathbf{r}})|^2\rangle$$

Quantum diffusion of the electrons

- Electron-impurity Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} = E_{\text{kin}} + \sum_i v(\mathbf{r} - \mathbf{r}_i)$$

- The "time" t runs along the diffusion path, i.e. $t = L_{\text{diff}}/v_F$. $P(\mathbf{r}, t)$ is the diffusion probability density. Defining ϵ to be the Laplace variable (instead of p) we have

$$D = \lim_{\epsilon \to 0} \frac{\epsilon^2}{2} \langle r^2(\epsilon) \rangle = \lim_{\epsilon \to 0} \frac{\epsilon^2}{2} \int \mathrm{d}^3\mathbf{r}\, r^2 P(\mathbf{r}, \epsilon) = \frac{\epsilon^2}{2} \int \mathrm{d}^3\mathbf{r}\, r^2 \langle | G(\underbrace{\mathbf{r}_1 - \mathbf{r}_0}_{\mathbf{r}}, \epsilon)|^2$$

- Green's function $G(\mathbf{r}_1, \mathbf{r}_0, \epsilon) = G(\mathbf{r}_1 - \mathbf{r}_0, \epsilon) =$

$$< \mathbf{r}_1 | \mathcal{G}(\epsilon) | \mathbf{r}_0 > = < \mathbf{r}_1 | \frac{1}{E_F - i\epsilon - \mathcal{H}} | \mathbf{r}_0 > = < \mathbf{r}_1 | \frac{1}{\underbrace{E_F - i\epsilon - \mathcal{H}_0}_{\mathcal{G}_0^{-1}} - \mathcal{V}} | \mathbf{r}_0 >$$

Expansion of the Green operator (Resolvent)

$$\begin{aligned}
\mathcal{G}(\epsilon) &= \mathcal{G}_0 + \mathcal{G}_0 \mathcal{V} \mathcal{G}_0 \cdots = \mathcal{G}_0 \sum_i v_i \mathcal{G}_0 \\
&= \mathcal{G}_0 + \sum_i \Big(\mathcal{G}_0 v_i \mathcal{G}_0 + \mathcal{G}_0 v_i \mathcal{G}_0 + \ldots \\
&\qquad + \sum_{j \neq i} \mathcal{G}_0 v_i \mathcal{G}_0 v_j \mathcal{G}_0 + \cdots + \sum_{k \neq j,i} \ldots \Big) \\
&= \mathcal{G}_0 + \sum_i \mathcal{G}_0 t_i \mathcal{G}_0 + \sum_i \sum_{j \neq i} \mathcal{G}_0 t_i \mathcal{G}_0 t_j \mathcal{G}_0 \\
&\qquad + \sum_i \sum_{j \neq i} \sum_{k \neq j,i} \mathcal{G}_0 t_i \mathcal{G}_0 t_j \mathcal{G}_0 t_k \mathcal{G}_0 \ldots
\end{aligned}$$

- Here we have introduced the so-called single-site t matrix t_i which collects all scattering contributions from impurity i.
- Every term defines a certain path along the impurities

Interference effects

- If, now, we approximate $|G|^2$ in such a way that we put

$$\left|\sum_{\text{paths}} G_0 t_1 G_0 t_2 \ldots t_{n-1} G_0\right|^2 \stackrel{\rightarrow}{=} \sum_{\text{paths}} |G_0|^2 |t_1|^2 |G_0|^2 |t_2|^2| \ldots |t_{n-1}|^2 |G_0|^2$$

  we obtain the diffusive description, which leads to the Drude formula ("incoherent approximation").

- By this procedure all interferences have been assumed to be cancelled out by the impurity averages, i.e. one has assumed that the $T_{ij} = A_{ij} e^{i\phi_{ij}}$ have all random phases ϕ_{ij}, which cancel out by the impurity average.

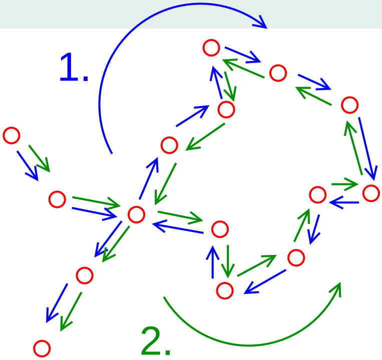
- As the free Green's functions have the form

$$G_0(r_{\nu,\nu+1}) \propto e^{ik_F r_{\nu,\nu+1}}$$

  the phases ϕ_{ij} are just given by k_F times the length s_{ij} of the path from i to j. This fact leads to an important exception from the rule that the interference terms cancel out.

Interference effects

- Assume that a multiple-scattering path contains a loop. As the positions of the impurities do not change in time, the phases along this path are fixed once forever.
- Let us now consider two paths one, which leads clockwise around the loop, the other anticlockwise. The phases along both paths are exactly the same so that we have for the intensities of the two processes



$$\begin{aligned} \big|T_1 + T_2\big|^2 &= \Big|A_1 e^{i\phi_1} + A_2 e^{i\phi_2}\Big|^2 \\ &= \big|A_1\big|^2 + \big|A_2\big|^2 + 2A_1A_2 \underbrace{\cos(\phi_1-\phi_2)}_{=1 \text{ for } \phi_1=\phi_2} \\ &= 4\big|A_1\big|^2 \qquad \text{for } A_1 = A_2 \end{aligned}$$

instead of $2|A_1|^2$ in the incoherent approximation

Interference effects

- Obviously there must be a correction to the scattering rate due to the closed-loops:

$$\tau^{-1} = \tau_0^{-1} + \Delta\tau^{-1}$$

which is of the form

$$\Delta\tau^{-1} \propto \overset{\curvearrowright}{\sum_{\text{loops}}} t_1 G_0 t_2 G_0 \cdots \overset{\curvearrowleft}{\sum_{\text{loops}}} t_1 G_0 t_2 G_0 \ldots$$

which can be approximated as

$$\Delta\tau^{-1} \propto P_{\text{diff}}(\mathbf{r} = 0, \epsilon \to 0) = \lim_{\epsilon \to 0} \int_0^\infty \mathrm{d}t \frac{1}{\left[4\pi D_0 t\right]^{d/2}} e^{-\epsilon t}$$

where we have used the expression for the d dimensional diffusion propagator

$$P_{\text{diff}}(\mathbf{r}, t) = \frac{1}{\left[4\pi D_0 t\right]^{d/2}} e^{-r^2/4D_0 t}$$

Interference effects

- For evaluating the interference contribution

$$\Delta\tau^{-1} \propto = \lim_{\epsilon\to 0}\int_0^\infty dt \frac{1}{\left[4\pi D_0 t\right]^{d/2}} e^{-\epsilon t}$$

  we first consider $d = 3$:

$$\int_0^\infty dt\, t^{-3/2} e^{-\epsilon t} = \epsilon^{1/2}\Gamma(-1/2) \stackrel{\epsilon\to 0}{\longrightarrow} 0$$

  So the contribution vanishes, unless there is a maximum loop length, given by the *inelastic mean-free path* $\ell_{\rm in}$ due to scattering from phonons. In this case ϵ is replaced by $\tau_{\rm in}^{-1} = v_F/\ell_{\rm in}$.

- Assuming $\tau_{\rm in}^{-1} \propto T^p$ we obtain a correction with a temperature dependence $\propto T^{p/2}$.
- Let us now consider the case $d = 2$. The integral

$$\int_0^\infty \frac{dt}{t}$$

  certainly diverges. This is a first indication for a disorder and interference induced localization transition in d = 2.

Interference effects

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  certainly diverges. This is a first indication for a disorder and interference induced localization transition in d = 2.

- If we insert a minimal and maximal scattering time (the minimal one corresponding to the elastic mean-free path $\ell_0 = v_F\tau_0$) we obtain

$$\Delta\tau^{-1} \propto \ln \tau_{\rm in}/\tau_0 \propto -\ln T$$

  Such a behavior is, indeed observed in two-dimensional devices.

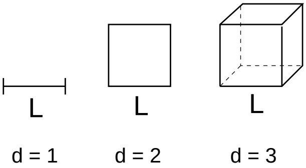
Anderson-localization

- In 1958 Anderson found by evaluating a tight-binding model with spatially fluctuating potentials that there can be absence of diffusion in quenched-disordered systems, i.e.
- $\langle r^2(t)\rangle$ can have an upper bound ξ^2, which is called localization length if the potential fluctuations $\langle(\Delta v)^2\rangle$ are large enough.
- In 1979 Abrahams, Anderson, Licciardello and Ramakrishnan showed that this localization effect is due to a quantum interference as discussed some minutes ago.
- In the case of localization the *conductance* G of the material will decay exponentially with the length L of a bar

$$G(L) \propto e^{-L/r_0}$$

Scaling theory of Anderson localization

- We consider the length dependence of the conductance a *metallic* piece of material of size L^d of resistivity $\rho = 1/\sigma$



$$G \propto \frac{1}{\rho L} \qquad G \propto \frac{1}{\rho} \qquad G \propto \frac{L}{\rho}$$

- We want to construct a dimensionless conductance and define

$$\frac{1}{G_0} = \frac{\hbar}{e^2} = \frac{1.05 \cdot 10^{-34}\mathrm{Ws}^2}{(1.6)^2 \cdot 10^{-38}(\mathrm{As})^2} = 410\Omega$$

so that we have in the metallic regime

$$\lim_{L\to\infty} g(L) = \lim_{L\to\infty} \frac{G(L)}{G_0} \propto L^{d-2}$$

Scaling theory of Anderson localization

- The *scaling hypothesis* now consists in postulating that in all dimension and whether there is localization or not $g(L)$ can be represented as

$$g(L) = g_0 L^{\beta} \qquad \text{or inversely} \qquad \beta(g) = \frac{\mathrm{d}\ln g}{\mathrm{d}\ln L}$$

  where $\beta(g)$ is supposed to be a *universal* function of g. If $\beta(g)$ is known, one can decide from the *sign* of β, whether

$$\lim_{L\to\infty} g(L) \quad \begin{cases} \longrightarrow \infty & \text{metallic for} \quad \beta > 0 \\ \longrightarrow 0 & \text{localized for} \quad \beta < 0 \end{cases}$$

- How can $\beta(g)$ look like? For large g we have

$$\lim_{g\to\infty} \beta = d - 2$$

Scaling theory of Anderson localization

- How can $\beta(g)$ look like? For large g we have

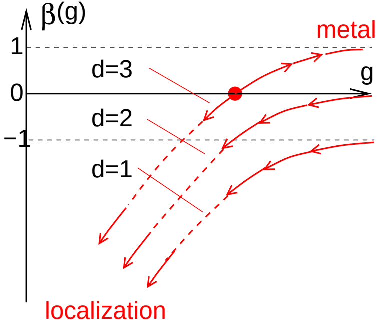
$$\lim_{g\to\infty} \beta = d - 2$$

- Abrahams *et al.* calculated the *corrections* to this asymptotic law:

$$\beta(g) = d - 2 - \frac{\mathrm{cons.}}{g}$$

- On the other hand, we have for small g $g(L) \propto e^{-L/r_0}$, from which follows

$$\lim_{g\to 0} \beta(g) = \ln g - \mathrm{const.}$$



- Result:
  - In $d = 1$ and $d = 2$ increasing L leads always towards $g = 0$.
  - Only in $d = 3$, depending on the initial conductance g_0 (the Drude conductance) g can scale towards the metallic value or towards zero.

Sound waves with disorder

- Model: Elasticity with fluctuating shear modulus $G(\mathbf{r})$

$$\rho_m \frac{\partial^2}{\partial t^2} u_i(\mathbf{r}, t) = \sum_j \partial_j \sigma_{ij}(\mathbf{r}, t)$$

- Stress tensor

$$\sigma_{ij} = \lambda \delta_{ij} \mathrm{Tr}\{\epsilon\} + 2G(\mathbf{r})\epsilon_{ij} \qquad \rho_m = \text{mass density}$$

Here $\lambda = K + \frac{2}{3}G$ is the longitudinal Lamé modulus and ϵ_{ij} is the strain tensor

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_j} u_i\right) .$$

- Separation into longitudinal and transverse displacements

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{u}_L(\mathbf{r}, t) + \mathbf{u}_T(\mathbf{r}, t) \qquad \nabla \times \mathbf{u}_L = 0; \qquad \nabla \cdot \mathbf{u}_T = 0$$

$\Rightarrow$ Two separate equations of motion $\tilde{G} = G/\rho_m = c_T^2;\ \tilde{\lambda} = \lambda/\rho_m = c_L^2 - 2\tilde{G}$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla\Big[\lambda + 2G(\mathbf{r})\Big]\nabla \cdot \right)\mathbf{u}_L(\mathbf{r}, t) = A_L[t, \mathbf{r}, G]\mathbf{u}_L(\mathbf{r}, t) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + \nabla \times G(\mathbf{r})\Big)\nabla \times \right)\mathbf{u}_T(\mathbf{r}, t) = A[t, \mathbf{r}, G]T\mathbf{u}_T(\mathbf{r}, t) = 0$$

Sound waves with disorder

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$$\left(\frac{\partial^2}{\partial t^2} - \nabla\left[\tilde{\lambda} + 2\tilde{G}(\mathbf{r})\right]\nabla \cdot\right)\mathbf{u}_L(\mathbf{r},t) = A_L[t,\mathbf{r},\tilde{G}]\mathbf{u}_L(\mathbf{r},t) = 0$$

$$\left(\frac{\partial^2}{\partial t^2} + \nabla \times \tilde{G}(\mathbf{r})\right)\nabla \times\Big)\mathbf{u}_T(\mathbf{r},t) = A_T[t,\mathbf{r},\tilde{G}]\mathbf{u}_T(\mathbf{r},t) = 0$$

Electromagnetic waves with disorder

- Maxwell's equations in a medium, which allows for spatial fluctuations of the dielectric constant

$$\begin{aligned} \nabla \times \vec{B}(\mathbf{r},t) &= \frac{\epsilon(\mathbf{r})}{c_0^2}\frac{\partial}{\partial t}\vec{E}(\mathbf{r},t) && (1) \\ \nabla \times \vec{E}(\mathbf{r},t) &= -\frac{\partial}{\partial t}\vec{B}(\mathbf{r},t) && (2) \end{aligned}$$

- If we divide (1) through $\epsilon(\mathbf{r})/c_0^2$ and then apply the curl we obtain from (2)

$$\left(\frac{\partial^2}{\partial t^2} + \nabla \times \frac{c_0^2}{\epsilon(\mathbf{r})}\nabla \times \right)\vec{B}(\mathbf{r},t) = 0$$

- We see that this is equivalent to the transverse part of the elastic equation.
- So we can treat both cases with the same theory.
- We shall discuss later, why this equation is not the basis of the current literature on light localization.

Evaluation of the stochastic field theory

- Statistics of the $\tilde{G}(\mathbf{r}) = \tilde{G}_0 + \Delta\tilde{G}(\mathbf{r})$

$$P[\Delta\tilde{G}(\mathbf{r})] = P_0 e^{-\frac{1}{2\gamma}\int d^d\mathbf{r}\Delta\tilde{G}(\mathbf{r})^2} \qquad \gamma \propto \langle(\Delta\tilde{G})^2\rangle/\tilde{G}_0^2$$

- Representation of the Green's function in terms of functional integrals with A_L and A_T as "Lagrangians"
- Replica trick in order to be able to perform the configuration average
  - Result: interacting field theory with γ as interaction parameter.
- Hubbard-Stratonovich transformation to "take the interaction apart".
  - Result: Field theory with matrix fields Q, which replace the fluctuations $\Delta\tilde{G}(\mathbf{r})$ with effective action

$$S_{\text{eff}}[Q] = -\frac{1}{2}\mathrm{Tr}\ln A(\omega,\mathbf{q},Q) - \frac{1}{2\gamma}\mathrm{Tr}Q^2$$

$$A_L(\omega,\mathbf{q},Q) = -\omega^2+q^2(\tilde{\lambda}+2[\tilde{G}_0+Q]) \qquad A_T(\omega,\mathbf{q},Q) = -\omega^2+q^2(\tilde{G}_0+Q)$$

Saddle Point: Self-consistent Born approximation (SCBA)

WS Europhys. Lett. 2006, WS et al. Phys. Rev. Lett 2007

$$S_{\text{eff}}[Q] = -\frac{1}{2}\text{Tr}\ln A(\omega,\mathbf{q},Q) - \frac{1}{2\gamma}\text{Tr}Q^2$$

$$A_L(\omega,\mathbf{q},Q) = -\omega^2 + q^2(\tilde{\lambda} + 2[\tilde{G}_0 + Q]) = G_L(\mathbf{q},\omega)^{-1}$$
$$A_T(\omega,\mathbf{q},Q) = -\omega^2 + q^2(\tilde{G}_0 + Q) = G_T(\mathbf{q},\omega)^{-1}$$

- Varying the Action with respect to Q

$$Q(\omega) = -\Sigma(\omega) = -\gamma\sum_{\mathbf{q}}\Big(q^2 G_L(\mathbf{q},\omega) + q^2 G_T(\mathbf{q},\omega)\Big)$$

- From this the density of states can be calculated (boson peak!)

$$\rho(\omega^2) = \frac{1}{2\omega}g(\omega) = \frac{1}{3\pi}\text{Im}\{\sum_{\mathbf{q}} G_L(\mathbf{q},\omega) + 2G_T(\mathbf{q},\omega)\}$$

- and the sound attenuation coefficient $\Gamma(\omega) = \frac{2\omega}{\underbrace{\tilde{\lambda} + 2\tilde{G}_0 - 2\Sigma'(\omega)}_{c_L(\omega)^2}}\Sigma''(\omega)$

Beyond the Saddle Point: wave diffusivity and localization

- For discussing the wave diffusivity (and whether or not it is finite) we have to go beyond the saddle point and discuss the fluctuations of the auxiliary fields $Q(\mathbf{r})$ beyond $Q_{\text{saddle}} = -\Sigma(\omega)$.
- $\hat{Q}(\mathbf{r}) = Q(\mathbf{r}) - Q_{\text{saddle}}$ exhibit a continuous symmetry in replica space, which is analogous to a continuous symmetry in Heisenberg-like spin models.
- These models are treated within the nonlinear-sigma-model field theory.

<u>Analogy</u>

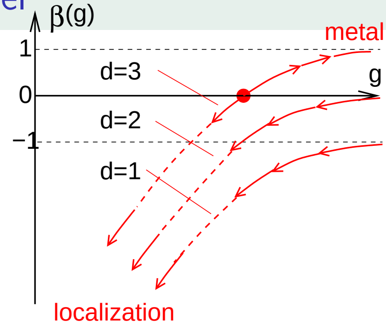
<u>spins</u>	$\longleftrightarrow$	<u>disordered waves</u>
ordered phase	$\longleftrightarrow$	delocalized phase
disordered phase	$\longleftrightarrow$	localized phase
<u>Mermin-Wagner theorem</u>		<u>scaling theory</u>
no ordered phase for $d \leq 2$	$\longleftrightarrow$	no delocalized phase for $d \leq 2$

Scaling of the nonlinear sigma model

- The scaling obtained from a renormalization analysis of the nonlinear sigma model is the same as that of the (ad-hoc formulated) scaling theory of Abrahams et al. :

$$\beta(g) = d - 2 - \frac{\mathrm{C}}{g}$$

- This is, however, a non-perturbative result!



- A detailed calculation gives for the reference "conductance"

$$g_0 = \rho(\omega^2) D_0(\omega) \propto \frac{c_T(\omega)^2}{\Sigma''(\omega)} \qquad c_T^2 = \tilde{\tilde{G}} - \Sigma'(\omega)$$

- critical g_0 in $d = 3$: $\qquad g_0^c = C$
- C must evaluated by numerics.
- The frequency at which localization occurs, lies above the boson-peak frequency near the upper band edge. WS et al. 1998

Situation in two dimensions

a

- Scaling equation

$$\begin{aligned} \beta(g) &= \underbrace{d-2}_{0} - \frac{\mathrm{C}}{g} = \frac{1}{g}\frac{dg}{\ln L} \quad \Rightarrow \quad \frac{dg}{d\ln L} = -C \\ g &= g_0 - C\ln(L/L_0) \\ \Rightarrow L(g) &= e^{[g_0-g]/C} = L_0\tilde{C}(g)e^{g_0/C} \end{aligned}$$

- The conductance goes always $\rightarrow 0$ for $L \rightarrow \infty$
- $g_0 = g(L_0)$ and L_0 are the reference conductances and length scales.
- The localization length is $L(g=1)$

$$\xi = L(1) = L_0\tilde{C}(1)e^{g_0(\omega)/C} \qquad g_0(\omega) = \frac{c_T^2}{\Sigma''(\omega)} \propto \omega^{-2}$$

- If ξ is bigger than the sample length L_{sample} then effectively the waves are delocalized
- Below a frequency given by $\xi = L_{\text{sample}}$ the states are predicted to be delocalized

Experimental evidence for localized sound waves in 2d

PAPER www.rsc.org/softmatter

Jamming for a 2D granular material†

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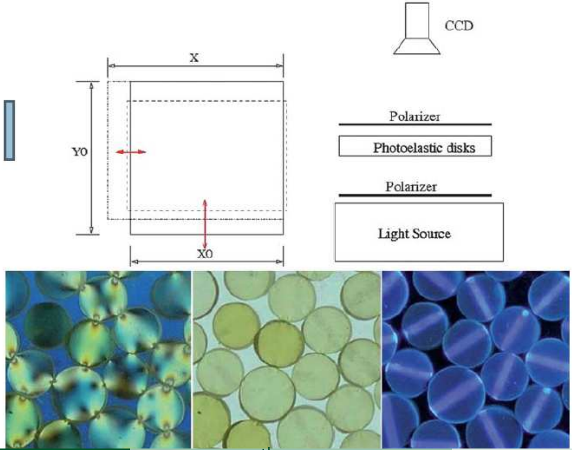
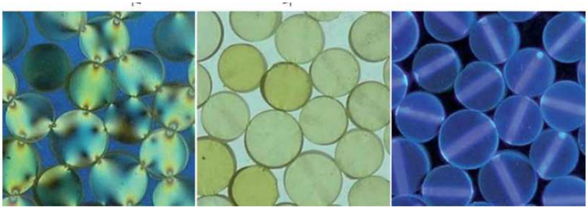


Experimental evidence for localized sound waves in 2d



- The plastic disks under pressure exert random forces among each other for a mixture of disks with different diameters
- The forces can be measured by image processing
- The dynamical matrix can be evaluated for the eigenvalues and eigenvectors
- The spatial extent of the eigenvectors determine whether or not the states are localized
- The following pictures have been made by Ling Zhang

Localization of light

- In the past 30 years, after the possibility of light localization was predicted by Saeev John in Phys. Rev. Letter 1987 many
  - experiments
  - simulations and
  - analytical theories have been put forward.
- Problem: The spectrum of light in disordered transparent materials has not an upper bound as the disordered solid, which is produced by the atoms, and where the localized sound waves are situated.
- So artificial atoms were introduced, which produced gaps in the spectrum, near which one hoped to find localized electromagnetic states, but until now with no success
- Finally the breakthrough came in dimensionally reduced systems.

Theory of Localization of light

- Maxwell's equations in a medium, which allows for spatial fluctuations of the dielectric constant

$$\nabla \times \vec{B}(\mathbf{r},t) = \frac{\epsilon(\mathbf{r})}{c_0^2}\frac{\partial}{\partial t}\vec{E}(\mathbf{r},t) \qquad (1)$$

$$\nabla \times \vec{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\vec{B}(\mathbf{r},t) \qquad (2)$$

- If we divide (1) through $\epsilon(\mathbf{r})/c_0^2$ and then apply the curl we obtain from (2)

$$\left(\frac{\partial^2}{\partial t^2} + \nabla \times \frac{c_0^2}{\epsilon(\mathbf{r})}\nabla \times\right)\vec{B}(\mathbf{r},t) = 0$$

- On the other hand, if we apply the curl to (2) we obtain from (1)

$$\frac{\epsilon(\mathbf{r})}{c_0^2}\left(\frac{\partial^2}{\partial t^2} + \nabla \times \nabla \times\right)\vec{E}(\mathbf{r},t) = 0$$

Theory of Localization of light

- On the other hand, if we apply the curl to (2) we obtain from (1)

$$\left(\frac{\epsilon(\mathbf{r})}{c_0^2}\frac{\partial^2}{\partial t^2} + \nabla \times \nabla \times \right)\vec{E}(\mathbf{r},t) = 0$$

- If we go to the frequency regime and apply $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$ we obtain

$$\left(-\omega^2 \frac{\epsilon(\mathbf{r})}{c_0^2} + \nabla\,\nabla \cdot - \nabla^2 \right)\vec{E}(\mathbf{r},t) = 0$$

- In the previous literature one has now taken $\nabla \cdot \vec{E} = 0$ with the result

$$\left(\underbrace{-\omega^2 \frac{\epsilon(\mathbf{r})}{c_0^2}}_{\text{``}\mathcal{V}(\mathbf{r})\text{''}} - \nabla^2 \right)\vec{E}(\mathbf{r},t) = 0$$

  This is effective a Anderson-type Schrödinger equation with a vector valued wave function. So, in the literature, the whole theory of electronic localization was taken over via this equation. BUT

- Is really $\nabla \cdot \vec{E} = 0$ in the disordered system??

Divergence of the electric field

- In the presence of a spatially fluctuating dielectric constant we get for the divergence of the electic field

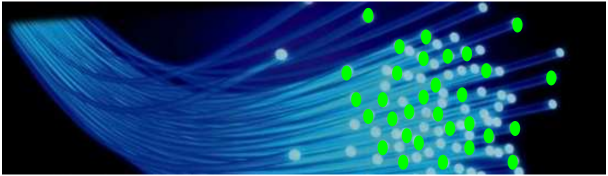
$$\nabla \cdot \vec{E} = -\frac{1}{\epsilon_0} \nabla \cdot \mathbf{P} = -\frac{1}{\epsilon_0} \nabla \cdot [\epsilon(\mathbf{r}) - \epsilon_0] \vec{E}$$

  from which follows

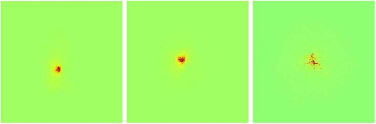
$$\nabla \cdot \vec{E} = -\frac{1}{\epsilon(\mathbf{r})} \vec{E} \cdot \nabla \epsilon(\mathbf{r}) \neq 0$$

- Setting $\nabla \cdot \vec{E} = 0$ is obviously not an approximation but an ad-hoc assumption, which makes this potential-type approach (PT) very different from our modulus-type approach (MT).

Transverse localization of light



- Karbasi and coworkers reported in Optics Letters 2012 on an experiment in which they
  - mixed microfibers of polysterene (PS, $n = 1.49$) and plexiglas, (polymethyl methacrylate, PMMA, $n =1.49$) and sintered them to a fiber, which is disordered in the transverse direction
  - transverse disorder
- They launched a beam from a single-mode fiber into the sample and observed the output at the end cross-section by near-field techniques



Transverse localization of light

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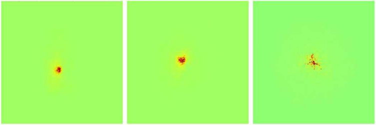
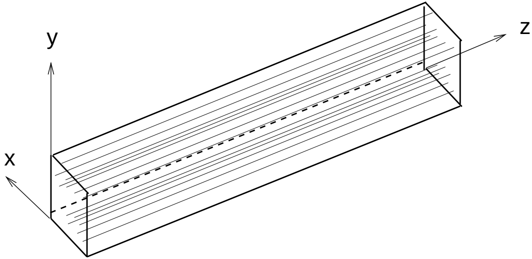


- This is evidence for transverse-localized light:
  - The light forms a channel like in a micro-waveguide.
  - Leonetti et al. 2016 found that the localized ray correspond to a single mode.
  - The localized modes “dont talk to each other” which is of importance for information processing, e.g. in endoscopy

Transverse localization of light



- Let's study a block of transparent material, in which the permittivity fluctuates in the x, y direction but not in the z direction. In the potential-type description of light with disorder the wave equation is

$$\Big[\underbrace{k_0^2 - k_z^2}_{E = k_0^2 \cos\theta^2} + k_0^2 \Delta\tilde{\epsilon}(\boldsymbol{\rho}) + \nabla^2_{\boldsymbol{\rho}}\Big] \vec{E}(k_z, \boldsymbol{\rho}, \omega) = 0 \qquad k_0^2 = \omega^2 \langle\epsilon\rangle / c_0^2$$

$$\tilde{\epsilon}(\mathbf{r}) = [\epsilon(\mathbf{r}) - \langle\epsilon\rangle](\langle\epsilon\rangle$$

where we have taken a Fourier transform with respect to z

- θ is the angle between the initial ray and the z axis (azimuthal angle)

Transverse localization of light

$$\Big[\underbrace{k_0^2 - k_z^2}_{E = k_0^2 \cos\theta^2} + k_0^2 \Delta\tilde{\epsilon}(\boldsymbol{\rho}) + \nabla^2_{\boldsymbol{\rho}}\Big]\vec{E}(k_z,\boldsymbol{\rho},\omega) = 0 \quad k_0^2 = \omega^2\langle\epsilon\rangle/c_0^2$$

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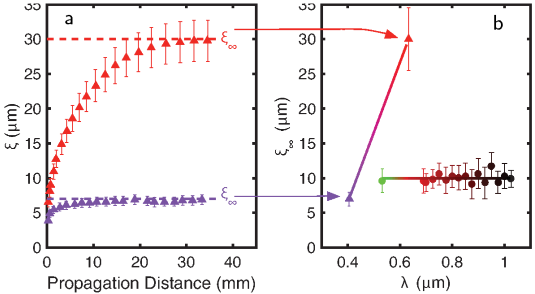
where we have taken a Fourier transform with respect to z

- θ is the angle between the initial ray and the z axis (azimuthal angle)
- In the paraxial limit $\theta \to 0$ E can approximated by $-2k_0\Delta k_z$, where $\Delta k_z = k_z - k_0$ is the Fourier wavenumber corresponding to the z dependence of the envelope of the electric field. Transformed back to the z dependence of the envelope one obtains

$$\Big[2ik_0\frac{\partial}{\partial z} + k_0^2\Delta\tilde{\epsilon}(\boldsymbol{\rho}) + \nabla^2_{\boldsymbol{\rho}}\Big]E_\alpha^{(0)}(z,\boldsymbol{\rho},\omega) = 0$$

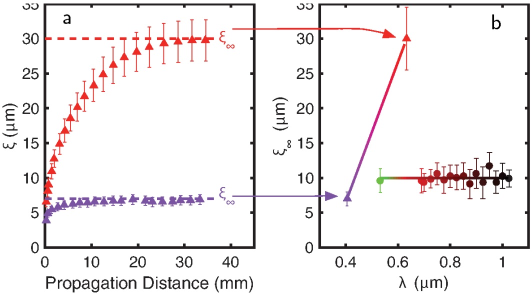
- This is the paraxial equation, which looks like a time-dependent Schrödinger equation. The “time” is just the z coordinate.
- This equation has been solved by deRaedt et al. (PRL 1989) and Karbasi in connection with his experiments

Transverse localization of light



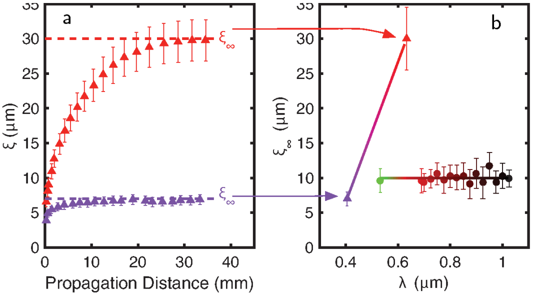
- Left picture: Karbasi's result of solving the paraxial equation in the presence of disorder for two different wavelengths of incident light $\lambda = 2\pi/k_0$. The plot shows the extent of the ray ξ as a function of z.
- The saturation value is the localization length ξ_∞ which is found to depend strongly on λ.
- Right picture: Results of measurements of Marco Leonetti of samples of the Mafi/Karbasi group, from which he extracted the average localization length for a large range of wavelengths λ
- Clearly no dependence on λ is found!

Transverse localization of light



- What can be the origin of the discrepance between theory and the experiment?
  - Marco Leonetti has repeated the experiment with the same result
  - Probably not the experiment but the theory is wrong
  - We conclude that neglecting the $\nabla \cdot \mathbf{P}$ term might be the reason
  - So let's take the modulus-type theory, taken from the acoustic localization theory and see, whether it is consistent with the experiment.

Comparison of the two wave equations for transverse disorder



$$\Big[\underbrace{k_0^2 - k_z^2}_{E = k_0^2 \cos\theta^2} + k_0^2 \Delta\tilde{\epsilon}(\boldsymbol{\rho}) + \nabla_{\boldsymbol{\rho}}^2\Big]\vec{E}(k_z, \boldsymbol{\rho}, \omega) = 0$$ Potential-type (PT)

$$\Big[E - \nabla_{\boldsymbol{\rho}} \times \frac{1}{\tilde{\epsilon}(\mathbf{r})} \nabla_{\boldsymbol{\rho}} \times\Big]\vec{B}(k_z, \boldsymbol{\rho}, \omega) = 0$$ Modulus-type (MT)

- In the PT equation $k_0 = 2\pi/\lambda$ appears as an external parameter in front of the fluctuating permittivity

Comparison of the two wave equations for transverse disorder

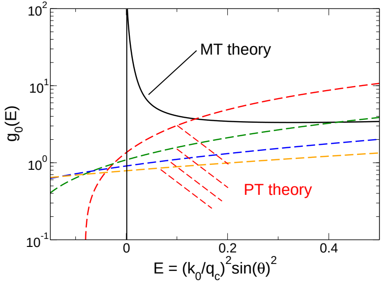
$$\big[\underbrace{k_0^2 - k_z^2}_{} + k_0^2 \Delta\tilde{\epsilon}(\boldsymbol{\rho}) + \nabla_{\boldsymbol{\rho}}^2\big]\vec{E}(k_z, \boldsymbol{\rho}, \omega) = 0 \qquad \text{Potential-type (PT)}$$

$$E = k_0^2 \cos\theta^2$$

$$\big[E - \nabla_{\boldsymbol{\rho}} \times \frac{1}{\tilde{\epsilon}(\mathbf{r})}\nabla_{\boldsymbol{\rho}} \times \big]\vec{B}(k_z, \boldsymbol{\rho}, \omega) = 0 \qquad \text{Modulus-type (MT)}$$

- In the PT equation $k_0 = 2\pi/\lambda$ appears as an external parameter in front of the fluctuating permittivity
- This is not so for the MT equation. k_0 only appears implicitly in the spectral parameter E.
- The scale change of E if k_0 is changed does not change the distribution of the localization lengths and their average.

Nonlinear-sigma-model calculation of the localization lengths for the two conficting theories



- Both calculations have been done for four different wavelengths.
- The PT curves (dashes) depend on k_0 (as could be guessed from the wave equation), the MT curves (line) is the same for all values of k_0.
- In PT there is no divergence of the localization length at $E = k_0^2 \sin^2\theta = 0$, meaning that in PT the sample is predicted not to be transparent in $z(\theta = 0)$ direction!
- This is strong evidence for the experiment and the MT theory to be right and the PT theory to be wrong!

Take-home messages

- Anderson-localization of waves is a disorder-induced interference phenomenon.
- Heterogeneous-elasticity theory provides via the lonlinear sigma model a theory of sound localization.
- The nonlinear sigma model provides an analogy between the Anderson transition and a continuous Heisenberg-like spin model.
- By this analogy there cannot be a delocalized phase in dimension $d \leq 2$.
- Sound localization appears only at the upper band edge.
- In two dimensions at $\omega \to 0$ there exist delocalized states for finite samples.
- Light localization exists only for transverse disorder.
- The correct theory for light localization follows the acoustic theory.