交作业时间: 2018年12月23日，星期一

1. Compute the Fourier transforms for the following functions defined on $\mathbb{R}$.
   (a) $f(x) = e^{-a|x|}$, $a > 0$;
   (b) $f(x) = e^{-ax^2}$, $a > 0$;
   (c) $f(x) = \begin{cases} A, & 0 < x < 1, \\ 0, & x \notin (0,1). \end{cases}$

2. Compute the following limit
$$\lim_{\alpha \to 0} \int_0^{1+\alpha} \frac{dx}{1 + x^2 + \alpha^2}.$$

3. Compute the following integrals
   (a) $\int_0^1 \sin\left(\ln \frac{1}{x}\right) \frac{x^b - x^a}{\ln x} dx$ where $b > a > 0$;
   (b) $\int_0^{\frac{\pi}{2}} \ln \frac{1+a\sin x}{1-a\sin x} \frac{dx}{\sin x}$ where $1 > a > 0$;
   (c) $\int_0^{\frac{\pi}{2}} \ln(a^2 - \sin^2 x) dx$ where $a > 1$;
   (d) $\int_0^{\pi} \ln(1 - 2a\cos x + a^2) dx$ where $|a| < 1$.

4. Suppose that $f(x)$ is twice continuous differentiable and $F(x)$ is differentiable. Prove that
$$u(x,t) = \frac{1}{2}(f(x-at) + f(x+at)) + \frac{1}{2a}\int_{x-at}^{x+at} F(y)dy$$
satisfies the wave equation
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$
supplemented with the initial conditions
$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = F(x).$$

5. Prove that the elliptic integral of the second kind
$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t}\, dt \quad (0 < k < 1)$$
satisfies the differential equation
$$E''(k) + \frac{1}{k}E'(k) + \frac{E(k)}{1-k^2} = 0.$$

6. Suppose that the function $f(u, v)$ is twice continuous differentiable. Prove that the function

$$w(x, y, z) = \int_0^{2\pi} f(x + z\cos\varphi, y + z\sin\varphi)d\varphi$$

satisfies the partial differential equation

$$z\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w}{\partial z}.$$

7. Prove that the following improper integral depending on a parameter is uniformly convergent in the given interval,

$$\int_0^{+\infty} x\sin x^4 \cos\alpha x dx, \quad a \leq \alpha \leq b.$$

8. Discuss whether the following improper integrals are uniform convergent:
   (a) $\int_0^{+\infty} \frac{x\sin\alpha x}{\alpha(1+x^2)}dx$, $0 < \alpha < +\infty$;
   (b) $\int_0^{+\infty} \frac{\cos xy}{\sqrt{x}}dx$, $y \geq y_0 > 0$;
   (c) $\int_0^1 x^{p-1}\ln^2 x dx$, $p \geq p_0 > 0$;
   (d) $\int_0^1 x^{p-1}\ln^2 x dx$, $p > 0$;
9. Prove that the function $F(\alpha) = \int_0^{\pi} \frac{\cos x}{x^\alpha}dx$ is continuous on $(0, +\infty)$.
10. Suppose that $f(x)$ is continuous on $(0, +\infty)$ and that the improper integral $\int_0^{+\infty} f(x)dx$ is convergent. Prove that the Laplace transform of f defined by

$$F(s) = \int_0^{+\infty} e^{-sx}f(x)dx$$

is continuous on $[0, +\infty)$.