Chapter 1

Metric spaces

1.1 Metric and convergence

We will begin with some basic concepts.

Definition 1.1. (Metric space) Metric space is a set X, with a metric

$$d : X \times X \to \mathbb{R}_+ = [0, +\infty)$$

satisfying:

1. $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

From now on, we will represent a metric space with (X, d). Here are some examples:

Example 1.2. $X = \mathbb{R}^n = (x_1, x_2, \cdots, x_n)$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$.

Example 1.3. $l_p = (x_1, x_2, \cdots, x_n, \cdots)$, $d(x, y) = (\sum_{i=1}^{\infty}(x_i - y_i)^p)^{\frac{1}{p}}, p \geq 1$.

Example 1.4. $X = C[a, b] = \{f(x)|f(x)$ is continuous in $[a, b]\}$, $d(f, g) = \max|f(x) - g(x)|, x \in [a, b]$.

Example 1.5. $X = C[a, b] = \{f(x)|f(x)$ is continuous in $[a, b]\}$, $d(f, g) = \int_{[a,b]} |f(x) - g(x)|dx$.

Definition 1.6. If there exists a sequence $\{x_n\} \subset (X, d)$ and $x \in (X, d)$, we will say $x_n \to x$, iff $d(x_n, x) \to 0$.

Definition 1.7. Suppose that (X, d) is a metric space, A is an open subset of X, if $\forall x \in A$, $\exists \delta > 0, B_\delta(x) = \{y|d(x, y) < \delta\} \subset A$. A subset $B \subset X$ is a closed subset of X, if $B^c = \{x \in X|x \notin B\}$ is open.

Example 1.8. $X = \mathbb{R}$, then the interval $(0, 1)$ is an open subset, the interval $[0, 1]$ is a closed subset. The interval $[0, 1)$ is neither an open subset nor a closed subset.

Proposition 1.9. *We have following basic properties for open and closed sets:*

1. *Unions of open sets is open.*

2. *Finite intersection of open sets is open.*

3. *Intersection of closed sets is closed.*

4. *Finite union of closed sets is closed.*

Proof. According to the De Morgan's law, we have

$$(\cap^\alpha A_\alpha)^c = \cup_\alpha A_\alpha^c \quad \text{and} \quad (\cup_\alpha A_\alpha)^c = \cap^\alpha A_\alpha^c.$$

Hence we only need to prove the first two properties.

Suppose that $A = \bigcup_{\alpha \in I} A_\alpha$, A_α is open. $\forall x \in A$, $\exists \alpha_0 \in I$, such that $x \in A_{\alpha_0}$. Since A_{α_0} is open, $\exists \delta > 0$ such that $B_\delta(x) \subset A_{\alpha_0}$. Then $B_\delta(x) \subset A$, i.e. $A = \bigcup_{\alpha \in I} A_\alpha$ is an open set.

Suppose that $B = \bigcap_{i=1}^{n} A_i$ where A_i $(i = 1, \cdots, n)$ is an open set. $\forall x \in B$, so for every $i = 1, 2, \cdots, n$, there exists δ_i, such that $B_{\delta_i}(x) \subset A_i$, then take $\delta = \min\{\delta_1, \delta_2, \cdots, \delta_n\}$, we can derive that $B_\delta(x) \subset B$, i.e. $B = \bigcap_{i=1}^{n} A_i$ is an open subset. $\square$

Definition 1.10. The interior $\mathring{A}$ of a set A is the union of all open sets which are contained in A. The closure $\bar{B}$ of a set B is the intersection of all closed sets which contain B.

Definition 1.11. If a set A satisfies $\bar{A} = X$, then A is said to be dense in X.

Definition 1.12. Metric space (X, d) is said to be separable, if there exists a countable dense subset $A \subset X$.

1.2 Completeness

Definition 1.13. (X, d) is a metric space, if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, we say that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. The metric space (X, d) is called a complete metric space if every Cauchy sequence converges in X, i.e. $\exists x \in X$, s.t. $d(x_n, x) \to 0, n \to \infty$.

Example 1.14. $\mathbb{R}$ is a complete metric space and $[0, 1]$ is also a complete metric space.

Proof. First, let $\{x_n\}$ be a Cauchy sequence in $[0, 1]$. Then $\{x_n\}$ is a Cauchy sequence in R. Hence $x_n \to x \in R$.

Second, $[0, 1]$ is closed $\Rightarrow x \in [0, 1] \Rightarrow x_n$ converges in $[0, 1]$. $\square$

Example 1.15. $(0,1)$ is not a complete metric space.

Proof. Take $x_n = \frac{1}{n}$, $x_n \to 0 \notin (0,1)$, but x_n is a Cauchy sequence in $(0,1)$, then we will say that $(0,1)$ is not a complete metric space. □

We have the following theorem, whose proof is one of our exercises.

Theorem 1.16. *Given a subset Y of a complete metric space (X,d), the metric space (Y,d) is complete iff Y is a closed subset.*

Definition 1.17. Let X be a metric space. A set $E \subset X$ is said to be *nowhere dense* if its closure $\overline{E}$ has an empty interior. The sets of the *first category* in X are that are *countable* unions of nowhere dense sets. Any subset of X that is not of the first category is said to be of the *second category* in X.

Proposition 1.18. *We have the following properties on the category.*

(a) If $A \subset B$ and B is of the first category in X, so is A.

(b) Any countable union of sets of the first category is of the first category.

(c) Any closed set $E \subset X$ whose interior is empty is of the first category in X.

Theorem 1.19. *(Baire Category Theorem) If (X,d) is a complete metric space, then the intersection of every countable collection of dense open subsets of X is dense in X.*

Corollary 1.20. *A complete metric space is of the second category.*

Proof of the corollary. Suppose not. Let $\{E_i\}$ be a countable collection of nowhere dense subset of X and $X = \cup_i E_i$. Denote $V_i = X \setminus \overline{E_i}$. $\forall x \in X$, if $x \notin V_i$, then $x \in \overline{E_i}$. Since E_i is nowhere dense, $\overline{E_i}$ is nowhere dense. Hence $\forall n \in \mathbb{N}$, $\exists x_n \in \overline{E_i}^c = V_i$ such that $x_n \in B_{1/n}(x)$. Hence $x \in \overline{V_i}$. Thus each V_i is dense. By Baire category Theorem, $\cap_{i=1}^{\infty} V_i \neq \emptyset$. Therefore,

$$\cup_i \overline{E_i} = (\cap_i V_i)^c \subset X.$$

Hence X cannot be the union of countable nowhere dense sets. Therefore, X is of the second category. □

Proof of Baire Category Theorem. Suppose that $V_1, V_2, \ldots$ are dense open subsets of X. Let U_0 be an arbitrary nonempty open set in X, If $n \geq 1$ and an open set $U_{n-1} \neq \emptyset$ has been chosen, then there exists an open set $\overline{U}_n \subset V_n \cap U_{n-1}$ where we use the fact that V_n is dense and U_n may be taken to be a ball of radius less than $\frac{1}{n}$. Put

$$K = \bigcap_{n=1}^{\infty} \overline{U_n}$$

Note that the centers of the balls U_n form a Cauchy sequence which converges to some point of K, and so $K \neq \emptyset$. Our construction shows that $K \subset U_0$ and $K \subset V_n$ for each n. Hence U_0 intersect $\cap V_n$. □

Definition 1.21. (X, d_X), (Y, d_Y) are two metric spaces. The mapping $T : X \to Y$ is called an *isometry* if

$$d_Y(Tx_1, Tx_2) = d_X(x_1, x_2)$$

If it is also onto, then it is called *isometric-isomorphism*, we represent this by $X \cong T(X)$.

As we mentioned in the above, not all the metric spaces are complete metric spaces(see the example 1.15), however, by the definition of *isometry*, we can define the completion of a metric space.

Definition 1.22. A metric space $(\widetilde{X}, \widetilde{d})$ is a completion of (X, d) if the following conditions are satisfied:

(a) there is an *isometry* $\sigma : X \to \widetilde{X}$;

(b) $\sigma(X)$ is dense in $\widetilde{X}$, i.e. $\overline{\sigma(X)} = \widetilde{X}$;

(c) $(\widetilde{X}, \widetilde{d})$ is complete.

Now we introduce the main theorem of the completion of metric spaces.

Theorem 1.23. *Every metric space has a completion.*

Example 1.24. $\mathbb{Q}$ represents the set of rational numbers, so $\mathbb{R}$ is the completion of $\mathbb{Q}$.

Example 1.25. Let

$$\mathcal{C}[0,1] = \{f(x)|f(x) \text{ is continuous in } [0,1]\} \text{ with metric } d_1(f,g) = \int_0^1 |f(x) - g(x)|dx.$$

Then

$$L^1[0,1] = \{f(x)\Big| \int_{[0,1]} |f(x)|dx < \infty\}$$

is the completion of the metric space $(\mathcal{C}[0,1], d_1)$.

1.3 Compactness

Compactness is one of the most important concepts in analysis. We will define compact sets in a metric space by means of sequence.

Definition 1.26. $K \subset X$ is sequentially compact if every sequence of K has a convergent subsequence whose limit is also in K.

Example 1.27. $X = \mathbf{R}$, $K = [0,1]$ is a sequentially compact subset, but $K = (0,1)$ is not a sequentially compact subset.

By the well known *Bolzano-Weistrass* theorem which we have already learnt in the mathematical analysis, the following theorem holds

Theorem 1.28. *$K \subset R^n$ is sequentially compact iff it is bounded and closed.*

In the following, we will find an equivalent criterion for the sequentially compactness of a metric space that is easier to verify. First, we introduce the definition of ε-net of a subset Ω.

Definition 1.29. Let $\Omega \subset X$, we say that $F \subset X$ is an ε-net of Ω if $\Omega \subset \bigcup_{x \in F} B_\varepsilon(x)$ (*F is not required to be a subset of* Ω).

Definition 1.30. A subset of a metric space is totally bounded if it has a finite ε-net for any $\varepsilon > 0$.

Theorem 1.31. *A subset $K \subset X$ as a metric space is sequentially compact iff it is complete and totally bounded.*

Proof. Step 1. K is sequentially compact $\Rightarrow$ K is complete and totally bounded.

If $\{x_n\}$ is a Cauchy sequence in K $\Rightarrow$ $\{x_n\}$ has convergent subsequence $x_{n_k} \to x \in K$ $\Rightarrow x_n \to x \in K \Rightarrow K$ is complete.

$\forall \varepsilon > 0$, choose any $x_1 \in K$, if $K \subset B_\varepsilon(x_1)$, then we can end the proof. If it is not true, then $\exists x_2 \in K \backslash B_\varepsilon(x_1)$, we continue this step, if it can be ended in finite steps, i.e. $K \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$.

If it can't be ended in finite steps, we can find a sequence $\{x_i\}_{i=1}^{+\infty}$ with $d(x_i, x_j) > \varepsilon$, $\forall i, j$, if $j > i$, then $x_j \notin \bigcup_{k=1}^{i} B_\varepsilon(x_k)$, since K is sequentially compact$\Rightarrow \exists x_{n_k} \to x$, this is a contradiction with $d(x_{n_k}, x_{n_l}) > \varepsilon$, $n_k \neq n_l$, then we can find finite many balls such that $K \subset \bigcup_{i=1}^{n} B_\varepsilon(x_i)$, i.e. K is totally bounded.

Step 2. K is totally bounded and complete$\Rightarrow$ K is sequentially compact.

Let $\{x_n\}$ be a sequence in K, for $\varepsilon_n = \frac{1}{2^n}$, $\exists$ a finite ε_n-net, $\{x_1^n, \cdots, x_{l_n}^n\}$, such that $K \subset \bigcup_{i=1}^{l_n} B_{\frac{1}{2^n}}(x_i^n)$, when $n = 1$, $K \subset \bigcup_{i=1}^{l_1} B_{\frac{1}{2}}(x_i^1)$, because there are only finite many balls that contain $\{x_n\}_{n=1}^{+\infty} \Rightarrow \exists x_{i_1}^1$, s.t. $B_{\frac{1}{2}}(x_{i_1}^1)$ contains infinite many $\{x_n\}$, we denote it by $\{x_{n_k^1}\}$, $k = 1, 2, \cdots$

When $n = 2$, $\varepsilon = \frac{1}{4}$, $K \subset \bigcup_{i=1}^{l_2} B_{\frac{1}{4}}(x_i^2)$; $\exists x_{i_2}^2 \in \{x_{n_k^1}\}$, s.t. $B_{\frac{1}{4}}(x_{i_2}^2)$ contains infinite many of $\{x_{n_k^1}\}$, we denote it by $\{x_{n_k^2}\}$, which is a subsequence of $\{x_{n_k^1}\}$.

Repeat the step again and again, we can see that for every $q \geq 1$, $d(x_{n_k^q}, x_{n_m^q}) < \frac{1}{2^q} + \frac{1}{2^q} = \frac{1}{2^{q-1}}$, pick one element from each $\{x_{n_k^q}\}$ diagonally, we will get a new subsequence $\{x_{n_q^q}\}$, then $\{x_{n_q^q}\}$ is a subsequence of $\{x_n\}$ with the property that $d(x_{n_q^q}, x_{n_p^p}) < \frac{1}{2^{q-1}}$, for any $p \geq q$, i.e. $\{x_{n_q^q}\}$ is a Cauchy sequence, by the condition that K is complete, then $\{x_{n_q^q}\} \to x \in K$, i.e. K is sequentially compact. □

The proof of the following lemma is an exercise.

Lemma 1.32. *A sequentially compact metric space is separable.*

Then we will introduce the definition of *compactness* and claim the equivalence between *compactness* and *sequentially compactness.*

Definition 1.33. Let $\{A_\alpha\}$ be a collection of open sets, if $U \subset \cup_\alpha A_\alpha$, then A_α is called an open cover of U.

Definition 1.34. A subset $K \subset X$ is compact if every open cover has a finite sub-cover.

Theorem 1.35. *A subset K of a metric space is compact iff it is sequentially compact.*

Proof. We first show that K is sequentially compact as long as K is compact.

Step 1. Every sequence has a convergent subsequence. If $\{x_n\}$ does not exist a convergent subsequence, denote

$$A_n = \{x_1, x_2, \cdots, x_{n-1}, x_{n+1}, \cdots\}$$

Then A_n is closed. On the other hand,

$$\cup_{n=1}^{\infty}(X \setminus A_n) = X \setminus \cap_{n=1}^{\infty} A_n = X \setminus \emptyset = X \supset K.$$

Since K is compact, one has $\cup_{n=1}^{N}(X \setminus A_n) \supset K$. Therefore, we have $X \setminus \{x_n\}_{n=N+1}^{\infty} \supset K$. But $\{x_n\}_{n=N+1}^{\infty} \subset K$. There is a contradiction.

Step 2. K is closed. $\forall x_0 \in X \setminus K$, we have

$$K \subset \cup_{x \in K} B_{\frac{1}{2}d(x,x_0)}(x)$$

Since K is compact, there is a finite open cover for K. Assume that

$$K \subset \cup_{i=1}^{n} B_{\frac{1}{2}d(x_i,x_0)}(x_i)$$

Let $\delta = \frac{1}{4}\min_{1 \le i \le n} d(x_i, x_0)$. Then for any $x \in B_\delta(x_0)$, we have

$$d(x_k, x) \ge d(x_k, x_0) - d(x_0, x) \ge d(x_k, x_0) - \frac{1}{4}d(x_k, x_0) = \frac{3}{4}d(x_k, x_0)$$

This implies that $x \notin B_{\frac{1}{2}d(x_i,x_0)}(x_i)$ for all $i = 1, \cdots, n$. Hence $B_\delta(x_0) \cap K = \emptyset$. Therefore, K is a closed set.

Step 3. Sequentially compact must be compact. Suppose not, there exists open cover $\{A_\alpha\}$ which does not have a finite cover. Since K is sequentially compact, $\forall n \in \mathbb{N}$, $\exists$ an $\frac{1}{2n}$-net

$$\tilde{\mathcal{N}}_n = \{y_1^{(n)}, y_2^{(n)}, \cdots, y_{k(n)}^{(n)}\}$$

Therefore, we have

$$K \subset \cup_{i=1}^{k(n)} B_{\frac{1}{2n}}(y_i^n).$$

Hence, for $i = 1, \cdots, k(n)$, one can choose $x_i^n \in B_{\frac{1}{2n}}(y_i^n) \cap K$ such that

$$K \subset \cup_{i=1}^{k(n)} B_{\frac{1}{n}}(x_i^n).$$

Denote

$$\mathcal{N}_n = \{x_1^{(n)}, x_2^{(n)}, \cdots, x_{k(n)}^n\}$$

Therefore, $\forall n \in \mathbb{N}$, $\exists x_n \in \mathcal{N}_n$ such that $B_{1/n}(x_n)$ cannot be covered by finite sets in A_α. Since K is sequentially compact, there exists a subsequence $\{x_{n_k}\}$ such that

$$x_{n_k} \to x_0 \in A_{\alpha_0}$$

Since A_{α_0} is open, there exists a $\delta > 0$ such that $B_\delta(x_0) \subset A_{\alpha_0}$. If n_k is sufficiently large, we have $d(x_{n_k}, x_0) < \delta/2$. Therefore,

$$\forall x \in B_{\frac{1}{n_k}}(x_{n_k}) \subset B_\delta(x_0)$$

This yields a contradiction. □

Definition 1.36. A subset K of a metric space is *pre-compact* if its closure $\overline{K}$ is compact.

1.4 Continuous functions

Definition 1.37. (Continuous functions) Given two metric spaces (X, d_X) and (Y, d_Y), we say that $f : X \to Y$ is continuous, if $\forall x \in X$ $\lim_{n\to\infty} f(x_n) = f(x)$, whenever $x_n \to x$.

Proposition 1.38. *$f : X \to Y$ is continuous at x_0 if and only if $\forall \epsilon > 0$, $\exists \delta > 0$, one has $d_Y(f(x), f(x_0)) < \epsilon$ provided that $d_X(x, x_0) < \delta$.*

Proof. The sufficient part is trivial.

Now let's prove the necessary part. Suppose not, then $\exists \epsilon_0$, $\forall n$, $\exists x_n$ such that $d_X(x_n, x_0) < 1/n$, $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$. □

In fact, we have the following characterization for the continuous functions.

Proposition 1.39. *f is continuous function from (X, d_X) to (Y, d_Y) iff $f^{-1}(U)$ is an open set for any open set $U \subset X$.*

We can also define so called lower and upper semicontinuous functions from a metric space (X, d) to $\mathbb{R}$.

Definition 1.40. A function $f : X \to \mathbb{R}$ is called lower (upper) semicontinuous if

$$\liminf_{n\to\infty} f(x_n) \geq f(x) \quad \text{and} \quad \limsup_{n\to\infty} f(x_n) \leq f(x),$$

respectively, whenever $x_n \to x$ as $n \to \infty$.

We will denote the class of all continuous functions from X to Y by $\mathcal{C}(X, Y)$, i.e.

$$\mathcal{C}(X, Y) = \{f : X \to Y | f \quad \text{is continuous}\}$$

Example 1.41. We define *discrete metric space*(X, d_D) as:

$$d_D(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

Then, we can say that $C((X, d_D), Y) =$ *all maps from* X *to* Y.

We can also have the notation as uniform continuity.

Definition 1.42. Given $f \in \mathcal{C}(X, Y)$, we say that f is uniformly continuous, if $\forall \varepsilon > 0$, $\exists \delta > 0$, for any x_1, $x_2 \in X$ satisfying $d_X(x_1, x_2) < \delta$, one has

$$d_Y(f(x_1), f(x_2)) < \varepsilon.$$

The proof of the following two theorems is similar to the ones of mathematical analysis. We leave them as exercises.

Theorem 1.43. *$f \in C(K, \mathbb{R})$, K is compact, then f is uniformly continuous.*

Theorem 1.44. *$f \in C(K, \mathbb{R})$, K is compact, then f is bounded and attains its maximum and minimum.*

We will discuss the compactness of a special metric space that is composed of the *continuous* functions.

$$\mathcal{C}_{\mathcal{B}}(X) = \{f : X \to \mathbb{R} | f \text{ is continuous and bounded}\} \text{ with metric } d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

Then we have the following theorem.

Theorem 1.45. *The metric space $(\mathcal{C}_{\mathcal{B}}(X), d)$ is complete.*

When X is compact, we just write the above metric space $\mathcal{C}(X)$ for simplicity.

Definition 1.46. $\mathcal{F} \subset C(X, Y)$ is *equi-continuous* if for any $x \in X$, $\forall \varepsilon > 0$, $\exists \delta(\varepsilon, x)$ such that $\forall y \in X$ with $d_X(x, y) < \delta$, one has $d_Y(f(x), f(y)) \leq \varepsilon$ for any $f \in \mathcal{F}$. If δ does not depend on x, then we say that $\mathcal{F}$ is uniformly equicontinuous.

Theorem 1.47. *An equi-continuous family $\mathcal{F} \subset \mathcal{C}(K, Y)$ of functions from a compact metric space K to a complete metric space Y is uniformly-equi-continuous.*

Proof. We prove the theorem by contradiction argument.

Step 1. Suppose the theorem is not true, $\exists \varepsilon_0 > 0$, for $\delta_n = \frac{1}{n} > 0$, $\exists x_n, y_n \in K$, $f_n \in \mathcal{F}$, such that

$$d(f_n(x_n), f_n(y_n)) \geq \varepsilon_0 \quad \text{with} \quad d(x_n, y_n) \leq \frac{1}{n}.$$

Since K is compact, $\{x_n\}$ has a convergent subsequence $x_{n_k} \to \overline{x} \in K$.

Step 2. By the condition $\mathcal{F}$ is equi-continuous at $\overline{x}$, for $\overline{\varepsilon} = \frac{\varepsilon_0}{10} > 0$, $\exists N$, such that if $d(x, \overline{x}) < \frac{1}{N}$, then $d(f(\overline{x}), f(x)) < \frac{\varepsilon_0}{10}$. For k sufficiently large, we have

$$\begin{aligned}\varepsilon_0 \leq & d(f_{n_k}(x_{n_k}), f_{n_k}(y_{n_k})) \\ \leq & d(f_{n_k}(x_{n_k}), f_{n_k}(\overline{x})) + d(f_{n_k}(y_{n_k}), f_{n_k}(\overline{x})) \\ \leq & \frac{\varepsilon_0}{10} + \frac{\varepsilon_0}{10} \\ < & \varepsilon_0.\end{aligned}$$

This is a contradiction. Therefore, $\mathcal{F}$ is *uniformly-equi-continuous.* □

Here, we will state an important theorem which describes the compact criteria for $\mathcal{C}(K)$.

Theorem 1.48. *(Arzela-Ascoli theorem) Let K be a compact metric space. A subset $\mathcal{F}$ of $\mathcal{C}(K)$ is compact iff it is closed, bounded, and equi-continuous.*

Proof. Step 1. compact set is closed, bounded, and equi-continuous. Note that a compact set is sequentially compact, hence it is complete and totally bounded. Since $\mathcal{F}$ is compact, then $\mathcal{F}$ is complete, i.e. $\mathcal{F}$ is a closed subset of a complete metric space $\mathcal{C}(K)$ and it is totally bounded. What left is to show that $\mathcal{F}$ is *equi-continuous.* In fact, because $\mathcal{F}$ is totally bounded. $\forall \varepsilon > 0$, $\exists$ a finite $\frac{\varepsilon}{100}$-net, i.e. $\exists f_1, f_2, \cdots, f_N$, such that $\mathcal{F} \subset \bigcup_{i=1}^{N} B_{\frac{\varepsilon}{100}}(f_i)$, i.e. for any $f \in \mathcal{F}$, $\exists f_{i_0}$, $i_0 \in \{1, 2 \cdots, N\}$, s.t.

$$|f(x) - f_{i_0}(x)| < \frac{\varepsilon}{100}, \quad \forall x \in K.$$

On the other hand, $\forall x \in K$, $\exists \delta > 0$, if $d(x, y) < \delta$, one has

$$|f_i(x) - f_i(y)| < \frac{\varepsilon}{100} (i = 1, 2, \cdots, N).$$

Hence for any $f \in \mathcal{F}$, one has

$$\begin{aligned}|f(x) - f(y)| \leq & |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \\ \leq & \frac{\varepsilon}{100} + \frac{\varepsilon}{100} + \frac{\varepsilon}{100} \\ < & \varepsilon,\end{aligned} \tag{1.1}$$

i.e. $\mathcal{F}$ is *equi-continuous.*

Step 2. closed, bounded and equi-continuous set is compact. Suppose that $\mathcal{F}$ is a bounded, equicontinuous subset of $\mathcal{C}(K)$. We will show that every sequence $\{f_n\}$ in $\mathcal{F}$ has a convergent subsequence. There is a countable dense set $\{x_1, x_2, x_3 \ldots\}$ in the compact domain K. We choose a subsequence $\{f_{1,n}\}$ of $\{f_n\}$ such that the sequence of values $\{f_{1,n}(x_1)\}$ converges in $\mathbb{R}$, Such a subsequence exists because $\{f_n(x_1)\}$ is bounded in $\mathbb{R}$, since $\mathcal{F}$ is bounded in $C(K)$. We choose a subsequence $\{f_{2,n}\}$ of $\{f_{1,n}\}$ such that $\{f_{2,n}(x_2)\}$ converges, which exists for the same reason. Repeating this procedure, we obtain sequences $\{f_{k,n}\}_{n=1}^{\infty}$ for $k = 1, 2, \ldots$ such that $\{f_{k,n}\}$ is a subsequence of $\{f_{k-1,n}\}$, and $\{f_{k,n}(x_k)\}$ converges as $n \to \infty$. Finally, we define a "diagonal" subsequence $\{g_k\}$ by $g_k = f_{k,k}$. By construction, the sequence $\{g_k\}$ is a subsequence of $\{f_n\}$ with the property that $\{g_k(x_i)\}$ converges in $\mathbb{R}$ as $k \to \infty$ for all x_i in a dense subset of K.

So far, we have only used the boundedness of $\mathcal{F}$. The equicontinuity of $\mathcal{F}$ is needed to ensure the uniform convergence of $\{g_k\}$. Let $\epsilon > 0$. Since $\mathcal{F}$ is equicontinuous and K is compact, therefore, $\mathcal{F}$ is uniformly equicontinuous. Consequently, $\forall \epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies

$$|g_k(x) - g_k(y)| < \frac{\epsilon}{3}.$$

Since $\{x_i\}$ is dense in K, we have

$$K \subset \bigcup_{i=1}^{\infty} B_\delta(x_i).$$

Since K is compact, there is a finite subset of $\{x_i\}$, which we denote by $\{x_1, \ldots, x_n\}$, such that

$$K \subset \bigcup_{i=1}^{n} B_\delta(x_i).$$

The sequence $\{g_k(x_i)\}_{k=1}^{\infty}$ is convergent for each $i = 1, \cdots, n$, and hence is a Cauchy sequence for each $i = 1, \cdots, n$, so there is an N such that

$$|g_j(x_i) - g_k(x_i)| < \frac{\epsilon}{3}.$$

for all $j, k \geq N$ and $i = 1, \ldots, n$. For any $x \in K$, there is an i such that $x \in B_\delta(x_i)$. Then, for $j, k \geq N$, we have

$$|g_j(x) - g_k(x)| \leq |g_j(x) - g_j(x_i)| + |g_j(x_i) - g_k(x_i)| + |g_k(x_i) - g_k(x)| < \epsilon.$$

It follow that $\{g_k\}$ is a Cauchy sequence in $C(K)$. Since $\mathcal{F}$ is closed set in the complete space $C(K)$, it converges to a limit in $\mathcal{F}$. Hence $\mathcal{F}$ is compact. □

Now we take an example to see the applications of Arzela-Ascoli theorem.

Example 1.49. Prove that

$$\mathcal{F} = \{f | f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x) \text{ with } \sum_{n=1}^{\infty} n|a_n| \leq 1 \text{ for } x \in [0, 1]\}$$

is a compact set in $C[0, 1]$.

Proof. Step 1. It is easy to see that $\mathcal{F}$ is bounded in $C[0, 1]$. Let $\{f_k\} \subset \mathcal{F}$ converge to $f \in C[0, 1]$. Suppose that f_k is represented as

$$f_k = \sum_{n=1}^{\infty} a_{k,n} \sin(n\pi x).$$

Then we have

$$a_{k,n} = \int_0^1 f_k(x)\sin(n\pi x)dx.$$

Since $\{f_k\}$ converges to f uniformly on $[0,1]$, one has

$$\lim_{k\to\infty} a_{k,n} = \int_0^1 f(x)\sin(n\pi x)dx := a_n.$$

Therefore, for any $N \in \mathbb{N}$, we have

$$\sum_{n=1}^{N} n|a_n| = \lim_{k\to\infty}\sum_{n=1}^{N} n|a_{k,n}| \le 1.$$

This implies that

$$\sum_{n=1}^{\infty} n|a_n| \le 1.$$

Hence $f \in \mathcal{F}$. This yields that $\mathcal{F}$ is closed.

Step 2. By the mean value theorem, for any $x < y \in \mathbb{R}$ there is a $x < \xi < y$ with

$$\sin x - \sin y = (\cos\xi)(x-y).$$

Hence, for all $x, y \in \mathbb{R}$ we have

$$|\sin x - \sin y| \le |x-y|.$$

Thus, every $f \in \mathcal{F}$ satisfies

$$|f(x)-f(y)| \le \sum_{n=1}^{\infty}|a_n||\sin(n\pi x)-\sin(n\pi y)| \le \sum_{n=1}^{\infty}\pi n|a_n||x-y| \le \pi|x-y|.$$

Therefore, given $\epsilon > 0$, we can pick $\delta = \epsilon/\pi$, and then $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon$ for all $f \in \mathcal{F}$. From the Arzleà-Ascoli theorem, $\mathcal{F}$ is a compact subset of $C([0,1])$. □

In the above example, we can see that a special class of functions f, Lipschitz functions, which satisfy

$$d(f(x), f(y)) \le Ld(x,y)$$

must be uniformly equi-continuous. We will study a special class of Lipschitz mapping in the next section.

1.5 Contraction mapping theorem

Definition 1.50. (Contraction map) Let(X,d) be a metric space. A map $T : X \to X$ is a contraction map if there exists a fixed $\theta \in [0,1)$, such that

$$d(Tx,Ty) \le \theta d(x,y), \quad \forall x,y \in X. \tag{1.2}$$

Now let us state a *fixed point theorem* of the contraction map which can be a useful tool to prove the existence of solutions of the equations.

Theorem 1.51. *(Contraction map theorem) Let* (X, d) *be a complete metric space,* $T : X \to X$ *is a contraction map satisfying* (1.2)*, then* T *has a* ***unique*** *fixed point* $x \in X$ *such that* $Tx = x$.

Proof. Step 1. (Existence). Pick any $x_0 \in X$, and define

$$x_n = Tx_{n-1} \quad \text{for } n \geq 1.$$

It follows from (1.2) that one has

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq \theta d(x_n, x_{n-1}) \leq \theta^2 d(x_{n-1}, x_{n-2}) \cdots \leq \theta^n d(x_1, x_0).$$

Then for $m = n + k > n$

$$\begin{aligned} d(x_m, x_n) \leq & d(x_{n+k}, x_{n+k-1}) + d(x_{n+k-1}, x_{n+k-2}) + \cdots + d(x_{n+1}, x_n) \\ \leq & \theta^n d(x_1, x_0)(\theta^{k-1} + \cdots + 1) \\ \leq & \frac{\theta^n}{1-\theta} d(x_1, x_0). \end{aligned} \tag{1.3}$$

Thus $\{x_n\}$ is a Cauchy sequence. Then we take limit on $x_{n+1} = T(x_n) \Rightarrow \overline{x} = T(\overline{x})$, where $\overline{x} = \lim\limits_{n\to+\infty} x_n$.

Step 2. (Uniqueness). If $\exists z$ such that $T(z) = z$, then

$$d(\overline{x}, z) = d(T(\overline{x}), T(z)) \leq \theta d(\overline{x}, z).$$

Therefore, we have $d(\overline{x}, z) = 0$. Hence $\overline{x} = z$. □

Then let us see two examples of the applications of the above *fixed point theorem* of contraction map.

Example 1.52. Consider the initial value problem for the ordinary differential equation,

$$\begin{cases} u'(t) = f(t, u), \\ u(t_0) = u_0, \end{cases} \tag{1.4}$$

where $f(t, u)$ is continuous on $[t_0 - h, t_0 + h] \times [u_0 - b, u_0 + b]$(then $\exists M > 0$, s.t. $|f(t, u)| \leq M$), what's more, $f(t, u)$ is *Lipchitz* continuous with respect to u, i.e. $\exists L > 0$, s.t. $|f(t, u_1) - f(t, u_2)| \leq L|u_1 - u_2|$. Then we can use the contraction mapping theorem to prove the existence and uniqueness of a solution for the problem (1.4) on $[t_0 - \epsilon, t_0 + \epsilon]$ for some $\epsilon > 0$.

Proof. Step 1. Define

$$X = \{u \in \mathcal{C}[t_0 - \epsilon, t_0 + \epsilon] \Big| u(t_0) = u_0, |u(t) - u_0| \leq b \text{ for } t \in [t_0 - \epsilon, t_0 + \epsilon]\},$$

where $\epsilon \in (0, h]$ is to be determined. Define $T : X \to X$ as

$$(Tu)(t) = u_0 + \int_{t_0}^{t} f(\tau, u(\tau))d\tau.$$

We claim that T maps X into itself. It is easy to see that for any $u \in X$, we have $Tu \in C[t_0-\epsilon, t_0+\epsilon]$ and $(Tu)(t_0) = u_0$. Furthermore, we have

$$|Tu(t) - u_0| = \left|\int_{t_0}^{t} f(\tau, u(\tau))d\tau\right| \leq M|t-t_0| \leq M\epsilon.$$

Therefore, if $\epsilon \leq M/b$, then T maps X to itself.

Step 2. If ϵ is sufficiently small, then T is a contraction map. Indeed,

$$d(Tu, Tv) = \left|\int_{t_0}^{t} (f(\tau, u(\tau)) - f(\tau, v(\tau)))d\tau\right| \leq \left|\int_{t_0}^{t} L|u(\tau) - v(\tau)|d\tau\right| \leq L|t-t_0|d(u,v).$$

Hence if

$$\epsilon \leq \min\left\{h, \frac{M}{b}, \frac{1}{2L}\right\},$$

then we have

$$d(Tu, Tv) \leq \frac{1}{2}d(u,v).$$

So T is a contraction map.

Step 3. By *Contraction map theorem*, $\exists \overline{u}$, such that $T(\overline{u}) = \overline{u}$, i.e.,

$$\bar{u} = u_0 + \int_{t_0}^{t} f(\tau, \bar{u}(\tau))d\tau.$$

This is exact the solution of the problem (1.4). Furthermore, the contraction map theorem also gives the *uniqueness* of the solution. □

Example 1.53. Let K be a convex and compact subset of $\mathbb{R}^n$, $T : K \to K$ satisfies the condition $|T(s) - T(t)| \leq |s-t|$, show that T has at least one fixed point(Be careful: the map T here may not be a contraction map, so we can not use the *contraction map theorem* directly.)

Proof. Pick up any $\bar{x} \in K$. Define $T_n : K \to K$ as follows

$$T_n x = (1 - \frac{1}{n})Tx + \frac{1}{n}T\bar{x}.$$

We claim that T_n is a contraction map. In fact

$$|T_n(x) - T_n(y)| \leq (1-\frac{1}{n})|Tx - Ty| \leq (1-\frac{1}{n})|x-y|.$$

This means that $\forall n$, $\exists x_n$, such that

$$(1-\frac{1}{n})Tx_n + \frac{1}{n}T\bar{x} = T_n(x_n) = x_n. \tag{1.5}$$

Since K is compact, then $\exists \{x_{n_j}\}$ and $x \in K$, s.t. $\{x_{n_j}\} \to x$, then take limit on both sides of (1.5), we can derive that $Tx = x$. So T has at least one fixed point. □