Functional Analysis **Homework 3** **Spring 2019**

Due: Monday, April 8, 2019

1. Let $K = \{(x,y)| -1 < x < 1, -1 < y < 1, \text{ or } x = 1, -1 \le y \le 0\} \subset \mathbb{R}^2$, $(x_1, y_1) = (1, \frac{1}{2})$, and $(x_2, y_2) = (1, 1)$.
   (a) Construct an explicit unique nonzero linear functional l satisfying $l(x_1, y_1) = 1$ and
   $$l(x,y) \le l(x_1, y_1) \quad \text{for all } (x,y) \in K.$$
   (b) Show that there are infinitely many linear functionals l satisfying $l(x_2, y_2) = 1$ and
   $$l(x,y) \le l(x_2, y_2) \quad \text{for all } (x,y) \in K.$$
2. Prove that any two norms on a finite dimensional linear space X are equivalent.
3. Prove the Hölder inequality for l^p ($p \in (1, \infty)$). More precisely, for $x = (x_1, x_2, \cdots) \in l^p$ ($p \in (1,\infty)$) and $y = (y_1, y_2, \cdots) \in l^{\frac{p}{p-1}}$, prove that
   $$\left|\sum_{i=1}^{\infty} x_i y_i\right| \le \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{\infty} |y_i|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}. \tag{1}$$
   Furthermore, the equality in (1) holds iff
   $$\arg x_j y_j \text{ and } \frac{|x_j|^p}{|y_j|^{\frac{p}{p-1}}} \text{ are independent of } j.$$
4. Given $f \in C((a,b))$, define its support as
   $$\text{supp} f = \overline{\{x | x \in (a,b), f(x) \neq 0\}},$$
   i.e., the closure of the set where f is not zero. Let X be the space of all real-valued, continuous functions f with compact support and the norm is defined as
   $$\|f\| = \max_{x \in (a,b)} |f(x)|.$$
   Prove that X is not a complete normed linear space.
5. If $(X, \|\cdot\|)$ is a uniformly convex normed linear space, then for any x, y satisfying $\|x\| = \|y\| = 1$, prove that
   $$l_{x,y} = \Big\{\theta x + (1-\theta) y \Big| \theta \in [0,1]\Big\} \cap \Big\{z \Big| \|z\| = 1, z \in X\Big\} = \{x, y\},$$
   which means X is strictly convex.
6. Exercises 1, 2 on page 22
7. Exercise 1 on page 37
8. Exercise 2 on page 38
9. Exercise 3 on page 39
10. Exercises 4, 5, 6 on page 44