Functional Analysis **Homework 6** **Spring 2019**

Due: Monday, May 20, 2018

1. Given a sequence $\{x_n\}_{n=1}^{\infty}$ in a Hilbert space H, show that the strong convergence $\|x_n - x\| \to 0$ holds if and only if

$$\|x_n\| \to \|x\| \quad \text{and} \quad x_n \rightharpoonup x \text{ (weak convergence).}$$

2. Consider a bounded sequence of functions $f_n \in L^2([0,T])$. As $n \to \infty$, show that the weak convergence $f_n \rightharpoonup f$ holds if and only if

$$\lim_{n\to\infty} \int_0^b f_n(x)dx = \int_0^b f(x)dx \quad \text{for every } b \in [0,T].$$

3. Suppose that Ω is a Lebesgue measurable set and $p \in (1,\infty)$. If $f_n \rightharpoonup f$ in $L^p(\Omega)$ and $\|f_n\|_{L^p(\Omega)} \to \|f\|_{L^p(\Omega)}$, then prove that $f_n \to f$ strongly in $L^p(\Omega)$. How about the case in $L^1(\Omega)$?

4. Let H be an infinite-dimensional Hilbert space and let any vector $x \in H$ be given, with $\|x\| \leq 1$. Construct a sequence of vectors x_n with $\|x_n\| = 1$ for every $n \geq 1$, such that the weak convergence holds: $x_n \rightharpoonup x$.

5. Exercises 1, 2, 3 on page 101

6. Exercises 4, 5 on page 104

7. Exercise 6 on page 106