SIAM J. NUMER. ANAL.
Vol. 62, No. 2, pp. 749–774

ON THE CONVERGENCE OF CONTINUOUS AND DISCRETE UNBALANCED OPTIMAL TRANSPORT MODELS FOR 1-WASSERSTEIN DISTANCE*

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Abstract. We consider a Beckmann formulation of an unbalanced optimal transport (UOT) problem. The Γ-convergence of this formulation of UOT to the corresponding optimal transport (OT) problem is established as the balancing parameter α goes to infinity. The discretization of the problem is further shown to be asymptotic preserving regarding the same limit, which ensures that a numerical method can be applied uniformly and the solutions converge to the one of the OT problem automatically. Particularly, there exists a critical value, which is independent of the mesh size, such that the discrete problem reduces to the discrete OT problem for α being larger than this critical value. The discrete problem is solved by a convergent primal-dual hybrid algorithm and the iterates for UOT are also shown to converge to that for OT. Finally, numerical experiments on shape deformation and partial color transfer are implemented to validate the theoretical convergence and the proposed numerical algorithm.

Key words. unbalanced optimal transport, asymptotic preserving, Γ-convergence, primal-dual hybrid algorithm

MSC codes. 65K15, 90C08

DOI. 10.1137/22M1520748

1. Introduction.

The concept of optimal transport (OT) was first put forward in 1781 by Monge [30] and was relaxed later by Kantorovich [24] as a convex linear program. OT has since been extensively applied in various fields, including image processing [18, 31], machine learning [34, 1, 21], PDE theory [36, 35], and noise sampling [13]. We refer readers to [35, 37, 32] for overviews of theoretic and computational OT. The OT models have been extended to the so-called unbalanced optimal transport or unnormalized optimal transport (UOT) problems [12, 20, 28, 22] for applications involving mass distributions with different masses. Moreover, the UOT models can take into account the weight change even for probability measures so that they can be used more flexibly [12, 33]. For example, the UOT distance is applied to deal with the full waveform inverse problem [26] and is used for waveform based earthquake location [38]. And in [17] a gradient method based on UOT is put forward, which is employed in the domain adaption problem.

*Received by the editors September 8, 2022; accepted for publication (in revised form) August 30, 2023; published electronically March 5, 2024.
https://doi.org/10.1137/22M1520748
Funding: The work of the authors was partially supported by the National Key R&D Program of China, grants 2020YFA0712000 and 2021YFA1002800, and by Shanghai Municipal Science and Technology Major project 2021SHZDZX0102. The work of the first, third, and fourth authors was also partially supported by National Natural Science Foundation of China (NSFC) grant 12090024. The work of the second author was also partially supported by NSFC grants 12371400 and 12031013 and Shanghai Science and Technology Commission grant 21JC1402900.

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Let us start with the introduction to the OT problems. Suppose X, Y are two topological spaces with probability measures ν_1, ν_2, respectively. Given a cost function $c : X \times Y \to \mathbb{R}^+$, the Kantorovich problem is to find a joint measure π (called a "transport plan") on the product space $X \times Y$ such that

$$\begin{aligned}&\min_{\pi} \int_{X\times Y} c(x,y)\mathrm{d}\pi(x,y)\\ &\text{s.t } \pi(A,Y)=\nu_1(A), \quad \pi(X,B)=\nu_2(B) \quad \forall A\subset X, B\subset Y.\end{aligned} \tag{1.1}$$

In later discussions, we only focus on the case $X = Y = \Omega \subset \mathbb{R}^d$ for a domain Ω. Let $\mathcal{P}(\Omega)$ denote the set of probability measures on Ω and define $\mathcal{W}_p(\Omega) := \{\mu \in \mathcal{P}(\Omega) : \int |x|^p d\mu < \infty\}$. Choosing $c(x,y) = |x-y|^p$ for $p \geq 1$, then (1.1) induces a widely used distance between two measures $\rho_0, \rho_1 \in \mathcal{W}_p(\Omega)$, which is the so-called p-Wasserstein distance W_p,

$$W_p(\rho_0,\rho_1) = \left(\inf_{\pi\in\Pi(\rho_0,\rho_1)} \int |x-y|^p d\pi\right)^{1/p}, \tag{1.2}$$

where $\Pi(\rho_0,\rho_1)$ is the set of all transport plans for ρ_0 and ρ_1. In the case of $p = 1$, simplifying the dual problem of the Kantorovich formulation can lead to the following characterization of the W_1 distance:

$$W_1(\rho_0,\rho_1) = \inf_{\varphi\in \mathrm{Lip}_1(X)} \int \varphi \, d(\rho_0-\rho_1). \tag{1.3}$$

This characterization has important applications in the generative models [5, 14]. The dual problem of (1.3) is given by the flow-minimization model introduced by Beckmann [3], [35, section 4.2]:

$$W_1(\rho_0,\rho_1) = \min\left\{\int_\Omega |\mathbf{m}|\mathrm{d}x : \ \mathbf{m} : \Omega \to \mathbb{R}^d, \ \nabla\cdot \boldsymbol{m} = \rho_0 - \rho_1\right\}. \tag{1.4}$$

The Kantorovich problem mentioned in (1.1) is often regarded as the static formulation. In [4], Benamou and Brenier proposed a dynamical version of OT which seeks a geodesic path between the two measures ρ_0 and ρ_1 when Ω is convex. Suppose $\mathcal{M}(\Omega)$ and $(\mathcal{M}(\Omega))^d$ are the spaces of Radon measures and vector measures on Ω, respectively. Let $\rho(\cdot,t)_{0\leq t\leq 1} \in \mathcal{W}_p(X)$ be an absolutely continuous curve connecting ρ_0 and ρ_1. Then according to [35, Theorem 5.14], there exists a field $\boldsymbol{w} : [0,1] \to (\mathcal{M}(\Omega))^d$ such that $\boldsymbol{w}(\cdot,t) \ll \rho(\cdot,t)$ (hence $\boldsymbol{w} = \rho\boldsymbol{v}$ for some vector field $\boldsymbol{v}$) and the following continuity equation holds:

$$\partial_t \rho(\boldsymbol{x},t) + \nabla\cdot \boldsymbol{w}(\boldsymbol{x},t) = 0 \text{ on } \Omega\times[0,1]. \tag{1.5}$$

Correspondingly, the W_p distance can be recovered by solving the following problem (see [35] for more details):

$$\begin{aligned}W_p^p(\rho_0,\rho_1) = \inf_{\rho,\boldsymbol{w}}\Bigg\{ \int_0^1 \left(\int_\Omega \frac{|\boldsymbol{w}(\boldsymbol{x},t)|^p}{\rho^{p-1}(\boldsymbol{x},t)}\mathrm{d}\boldsymbol{x}\right)\mathrm{d}t : \partial_t\rho(\boldsymbol{x},t) + \nabla\cdot\boldsymbol{w}(\boldsymbol{x},t) = 0 \ (\boldsymbol{x},t)\in\Omega\\ \times (0,1), \ \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \rho(\boldsymbol{x},1) = \rho_1(\boldsymbol{x})\Bigg\}.\end{aligned} \tag{1.6}$$

The Beckmann formulation of W_1 can also be derived from this dynamical formulation. In fact, by considering $\boldsymbol{m}(\boldsymbol{x}) = \int_0^1 \boldsymbol{w}(\boldsymbol{x},t)\mathrm{d}t$, one can obtain (1.4).

To take into account the mass change, several UOT problems have been proposed and they are connected in various ways [2, 9, 33, 19]. In particular, the Wasserstein–Fisher–Rao (or Kantorovich–Helliger) distance has been proposed in [12, 11, 25, 28] by adding a source into the dynamics. The static formulation as an extension of the classical Kantorovich is derived for UOT in [11, 12, 28], using either so-called semicouplings [11, 12] or the relaxation of the marginal constraints [28].

In this paper, we will focus on the generalization of the 1-Wasserstein distance given in (1.4). In particular, we focus on the following Beckmann formulation of an unbalanced OT problem:

$$W_1^\alpha(\rho_0,\rho_1) := \min\left\{\int_\Omega |\boldsymbol{m}| + \alpha|\eta|\mathrm{d}x :\ \boldsymbol{m}:\Omega\to\mathbb{R}^d,\ \nabla\cdot\boldsymbol{m} = \rho_0-\rho_1+\eta\right\} \tag{1.7}$$

with suitable boundary conditions. More details can be seen in section 3.1. One way to understand this is through the dynamic formulation of the UOT studied in [11], which is a generalization of the Benamou–Brenier formulation (1.6). The dynamic formulation is given by

$$\begin{aligned}
&\min_{\rho,\boldsymbol{w},\zeta} \int_0^1 \left[\frac{1}{p}\int_\Omega \frac{|\boldsymbol{w}(\boldsymbol{x},t)|^p}{\rho^{p-1}(\boldsymbol{x},t)}\,\mathrm{d}\boldsymbol{x} + \alpha^p \frac{1}{q}\int_\Omega \frac{|\zeta(\boldsymbol{x},t)|^q}{\rho^{q-1}(\boldsymbol{x},t)}\,\mathrm{d}\boldsymbol{x}\right]\mathrm{d}t\\
&\text{s.t.}\ \ \partial_t\rho(\boldsymbol{x},t) + \nabla\cdot\boldsymbol{w}(\boldsymbol{x},t) = \zeta(\boldsymbol{x},t) \text{ on } \Omega\times[0,1]\\
&\text{with } \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \rho(\boldsymbol{x},1) = \rho_1(\boldsymbol{x}),
\end{aligned} \tag{1.8}$$

where $p,q \geq 1$ and $\alpha > 0$ is the weight parameter of the source term. The functional in (1.8) penalizes the transportation with p-norm and the source change with q-norm, respectively. When $p = q \in [1,\infty)$, this dynamic formulation gives a distance. Taking $p = q = 1$, and similarly letting $\boldsymbol{m}(\boldsymbol{x}) = \int_0^1 \boldsymbol{w}(\boldsymbol{x},t)\mathrm{d}t$ and $\eta(\boldsymbol{x}) = \int_0^1 \zeta(\boldsymbol{x},t)\mathrm{d}t$, the corresponding Beckmann formulation (1.7) can then be derived. One may refer to Lemma 3.1 for more details. Clearly, in this UOT problem, ρ_0 and ρ_1 do not necessarily have the same mass and the parameter α in problem (1.8) controls the penalization of the source term. As a last comment, one often requires Ω to be convex for the dynamic formulation (1.6) to give the Wasserstein distances when $p > 1$. As can be seen, the Beckmann formulation (1.4) for $p = 1$ does not require the convexity of Ω and it is equivalent to (1.6). This means that the convexity of Ω is not required to study the Beckmann formulation and the dynamical formulation for $p = 1$. Analogously, we do not require the convexity of Ω in (1.7).

Our main focus in this paper is the connection between the Beckmann formulation for UOT (1.7) and the Beckmann formulation for OT (1.4) when ρ_0 and ρ_1 are probability measures, particularly when they are solved numerically using some optimization algorithms. Specifically, we aim to study whether the numerical solution of the UOT one can somehow converge to that for the corresponding OT problem under suitable optimization algorithms. Note that while we make the assumption that ρ_0 and ρ_1 are probability measures, the two measures are allowed to be nonnegative measures with equal total mass, for which one can simply scale to probability measures. Such a problem is closely related to the so-called Γ-convergence and some related results have been investigated in the literature already. In particular, [12] gives the corresponding result for some static problems of the unbalanced OT via the Γ-convergence, while [22] mentions some numerical evidence of the convergence for the Wasserstein–Fisher–Rao distance. We focus on the Beckmann formulation because it corresponds to the Earth mover distance and has been widely applied in data science [27, 29, 15], and more importantly, it is easier for computation and more suitable for optimization algorithms. Furthermore, with the theoretical guarantee of convergence from the UOT to the OT problem, unified numerical methods for UOT problems can also be used for OT problems, and the solutions to UOT converge toward the one to the OT problem automatically by setting sufficiently large α in (1.7).

Our contributions can be summarized as follows. First, we establish the Γ-convergence between the Beckmann problem (1.7) and (1.4). Then in discrete settings, we provide an estimate of lower bound of the parameter α for the solution of UOT being the same as the OT problem, not just the convergence of the optimal solution. Last, the discrete UOT problem can be solved by a primal-dual hybrid gradient method (a.k.a. the Chambolle–Pock algorithm) [16, 10] and we also give the corresponding condition of the parameter α for the reduction of the iterates for UOT to that for OT.

The rest of the paper is organized as follows. First in section 2, we provide the definitions of the usual Γ-convergence and the sequence Γ-convergence, and the relationship between them. Particularly in section 2.2, we summarize some useful lemmas and theorems for Γ-convergence which will be applied in later demonstrations. Then in section 3, we derive the equivalence between the Beckmann formulations to the dynamical ones for both UOT and OT at the beginning and prove the existence of minimizers of these two problems. After that we establish the Γ-convergence between the UOT and OT problems. Later in section 4, the finite convergence in discrete problems and the asymptotic preserving property are built, and we present the iterates of a primal-dual hybrid gradient method for both UOT and OT problems and show the similar convergence between them. At last, in section 5 some numerical experiments on shape deformation and partial color transfer are implemented to validate the theoretical results and the algorithm.

2. Background on Γ-convergence.

To investigate the convergence of the optimization problems and their optimizers, one often makes use of the theory of Γ-convergence [6]. Here we first recall the definitions of the usual Γ-convergence.

Definition 2.1. *Let (f_n) be a sequence of functionals on X. Define*

$$\begin{aligned} \Gamma\text{-}\limsup_{n\to\infty} f_n(x) &= \sup_{N_x} \limsup_{n\to\infty} \inf_{y\in N_x} f_n(y), \\ \Gamma\text{-}\liminf_{n\to\infty} f_n(x) &= \sup_{N_x} \liminf_{n\to\infty} \inf_{y\in N_x} f_n(y), \end{aligned} \tag{2.1}$$

where N_x ranges over all the neighborhoods of x. If there exists a functional f defined on X such that

$$\Gamma\text{-}\limsup_{n\to\infty} f_n = \Gamma\text{-}\liminf_{n\to\infty} f_n = f, \tag{2.2}$$

then we say the sequence (f_n) Γ-converges to f.

The benefit of Γ-convergence is that any cluster point of the minimizers of a Γ-convergent sequence (f_n) is a minimizer of the corresponding Γ-limit functional f. This result can be found in many references like [6] and one may also refer to Lemma 2.3 later.

The verification of Γ-convergence using Definition 2.1 of the optimization problems in this paper is not that straightforward. Instead, we will make use of the results of Γ_{seq}-convergence studied in [8] to get some sufficient conditions for Γ-convergence in Definition 2.1 on product spaces and we will utilize them in our problems.

2.1. Notation and definitions. We first introduce some definitions and notation for Γ_{seq}-convergence in [8]. Define the operators $\mathcal{L}(\cdot)$ and $\mathcal{G}(\cdot)$ as

$$\mathcal{L}(\epsilon) = \begin{cases} \sup & \epsilon = +1, \\ \inf & \epsilon = -1, \end{cases} \qquad \mathcal{G}(\epsilon) = \begin{cases} \limsup & \epsilon = +1, \\ \liminf & \epsilon = -1. \end{cases} \tag{2.3}$$

Let (f_n) be a sequence of functions defined on a topological space X and

$$\mathcal{S}(x_0) := \{\{x^n\} \subset X : x^n \to x_0\} \tag{2.4}$$

be the set of sequences that converge to x_0. Define the Γ_{seq}-limits of (f_n) at point x_0 as

$$\Gamma_{\text{seq}}(\mathbb{N}^{\epsilon_0}, X^{\epsilon_1}) \lim_n f_n(x_0) = \mathop{\mathcal{L}(\epsilon_1)}_{\{x^n\}\in\mathcal{S}(x_0)} \mathop{\mathcal{G}(\epsilon_0)}_n f_n(x^n), \tag{2.5}$$

where $\epsilon_i \in \{+1, -1\}$, $i = 0, 1$.

The relation between Γ_{seq}-convergence and the usual Γ-convergence is given as follows, and we include a short proof in the supplement (supplement.pdf [local/web 194KB]) for our presentation to be self-contained.

Proposition 2.2. *It holds that*

$$\begin{aligned} \Gamma_{\text{seq}}(\mathbb{N}^+, X^-) \lim_n f_n &= \Gamma\text{-}\limsup_{n\to\infty} f_n, \\ \Gamma_{\text{seq}}(\mathbb{N}^-, X^-) \lim_n f_n &= \Gamma\text{-}\liminf_{n\to\infty} f_n. \end{aligned} \tag{2.6}$$

Consequently, if $f := \Gamma_{\text{seq}}(\mathbb{N}, X^-)\lim_n f_n$ *exists, then* (f_n) Γ*-converges to* f*.*

Many functionals in practice are defined on some natural product space. For two topological spaces X and Y and (f_n) defined on the product space $X \times Y$, we can similarly define the Γ_{seq}-limits of (f_n) at point $(x_0, y_0) \in X \times Y$ for $\epsilon_i \in \{+1, -1\}$, $i = 0, 1, 2$, as

$$\Gamma_{\text{seq}}(\mathbb{N}^{\epsilon_0}, X^{\epsilon_1}, Y^{\epsilon_2}) \lim_n f_n(x_0, y_0) = \mathop{\mathcal{L}(\epsilon_1)}_{\{x^n\}\in\mathcal{S}(x_0)} \mathop{\mathcal{L}(\epsilon_2)}_{\{y^n\}\in\mathcal{S}(y_0)} \mathop{\mathcal{G}(\epsilon_0)}_n f_n(x^n, y^n). \tag{2.7}$$

Here we take the space $X \times Y$ as an example to clarify the notation. Suppose $\epsilon_0 = +1$, $\epsilon_1 = -1$, $\epsilon_2 = -1$; then we have

$$\Gamma_{\text{seq}}(\mathbb{N}^+, X^-, Y^-) \lim_n f_n(x_0, y_0) = \inf_{\{x^n\}\in\mathcal{S}(x_0)} \inf_{\{y^n\}\in\mathcal{S}(y_0)} \limsup_n f_n(x^n, y^n), \tag{2.8}$$

where for any given convergent sequence $\{x^n\} \in \mathcal{S}(x_0)$ and $\{y^n\} \in \mathcal{S}(y_0)$, the $\limsup$ operator (or $\liminf$) is taken over the functional value sequence $(f_n(x^n, y^n))$ and the inf (or sup) operator is taken over all the sequence $\{x^n\} \in \mathcal{S}(x_0)$ and $\{y^n\} \in \mathcal{S}(y_0)$ converging to x_0 and y_0, respectively. Moreover, if the Γ_{seq}-limit is independent of the value of ϵ, then we omit the sign in the Γ_{seq}-limit, i.e., if

$$\Gamma_{\text{seq}}(\mathbb{N}^+, X^-, Y^-) \lim_n f_n(x_0, y_0) = \Gamma_{\text{seq}}(\mathbb{N}^-, X^-, Y^-) \lim_n f_n(x_0, y_0), \tag{2.9}$$

we can write $\Gamma_{\text{seq}}(\mathbb{N}, X^-, Y^-)\lim_n f_n(x_0, y_0)$ for simplicity. The notation is similar for the spaces X and Y.

2.2. Useful results. In this subsection, we summarize several useful lemmas and theorems for $\Gamma_{\rm seq}$-convergence from [8]. In particular, these results provide tools to check Γ-convergence on product spaces. For completeness, in the supplement (supplement.pdf [local/web 194KB]) we give simplified proofs of the lemmas and the theorem appearing in this section.

The following lemma states that any cluster point of the minimizers of a Γ-convergent sequence is the minimizer of the corresponding Γ-limit functional.

LEMMA 2.3. *Let X be a topological space, and let (f_n) be a sequence of functionals mapping from X to $\bar{\mathbb{R}}=[-\infty,+\infty]$. If*

$$\Gamma_{\rm seq}(\mathbb{N},X^-)\lim_n f_n = f,$$

then

$$\inf_X f \geq \limsup_n [\inf_X f_n]. \tag{2.10}$$

Moreover, if there exists a sequence (x^n) converging to some $x_0\in X$, with

$$\liminf_n f_n(x^n) = \liminf_n \left[\inf_X f_n\right],$$

then

$$f(x_0) = \inf_X f = \lim_n \left[\inf_X f_n\right]. \tag{2.11}$$

The UOT problem in consideration is naturally defined on a product space and the functional is of the form $J+\mathbf{1}_E$. We introduce some related results in [8] in this regard.

LEMMA 2.4. *Let X,Y be two topological spaces and $(f_n),(g_n)$ be two sequences of functionals defined on the product space $X\times Y$ to $\bar{\mathbb{R}}^+=[0,+\infty]$, and let $(x_0,y_0)\in X\times Y$. Suppose there exists $a,b\in\bar{\mathbb{R}}^+$ such that*

$$\Gamma_{\rm seq}(\mathbb{N},X^-,Y)\lim_n f_n(x_0,y_0)=a,$$
$$\Gamma_{\rm seq}(\mathbb{N},X,Y^-)\lim_n g_n(x_0,y_0)=b.$$

Then it holds that

$$\Gamma_{\rm seq}(\mathbb{N},X^-,Y^-)\lim_n (f_n+g_n)(x_0,y_0)=a+b.$$

Suppose X is a topological space and E is a set in X, and the indicator function of E is defined as follows:

$$\mathbf{1}_E(x)=\begin{cases}0 & \text{if } x\in E,\\ +\infty & \text{otherwise.}\end{cases} \tag{2.12}$$

The following lemma gives a sufficient condition of a sequence of the set indicator functions to be $\Gamma_{\rm seq}$-convergence.

LEMMA 2.5. *Suppose $\{E_n\}$ is a sequence of sets in space $X\times Y$. If there exists a set $E_\infty\subset X\times Y$ that satisfies two conditions,*

- *if $x^n\to x$, $y^n\to y$ and $(x^n,y^n)\in E_n$ for infinitely many n, then $(x,y)\in E_\infty$;*
- *if $(x,y)\in E_\infty$ and $x^n\to x$, then there exists $y^n\to y$ such that $(x^n,y^n)\in E_n$ for n large enough,*

then $\mathbf{1}_{E_\infty}=\Gamma_{\rm seq}(\mathbb{N},X,Y^-){\lim}_n\mathbf{1}_{E_n}$.

The following theorem provides us the criterion to check Γ-convergence of the functional of the form $J+\mathbf{1}_E$ on product spaces.

THEOREM 2.6. *Suppose X and Y are two topological spaces. (J_n) is a sequence of functionals defined on the product space $X \times Y$ and (E_n) is a sequence of sets in $X \times Y$. Suppose that J_n and $\mathbf{1}_{E_n}$ are sequential Γ-convergent in the following sense:*

$$J_\infty = \Gamma_{\text{seq}}(\mathbb{N}, X^-, Y) \lim_n J_n,$$
$$\mathbf{1}_{E_\infty} = \Gamma_{\text{seq}}(\mathbb{N}, X, Y^-) \lim_n \mathbf{1}_{E_n};$$

then $J_n + \mathbf{1}_{E_n}$ is Γ-convergent to $J_\infty + \mathbf{1}_{E_\infty}$ in the sense of Definition 2.1.

Consequently, for every $n \in \mathbb{N}^+$, let (x_n, y_n) be an optimal pair of the optimization problem

$$\min_{X\times Y} (J_n + \mathbf{1}_{E_n}).$$

If $x_n \to x_\infty$ in X and $y_n \to y_\infty$ in Y, then (x_∞, y_∞) is an optimal pair of the problem

$$\min_{X\times Y} (J_\infty + \mathbf{1}_{E_\infty}).$$

3. Convergence from UOT to OT.

In this section, we establish the convergence of the Beckmann formulation of the UOT problem (1.7) to the corresponding OT problem (1.4) in the sense of Γ-convergence.

3.1. Problem descriptions.

Fix a bounded domain $\Omega \subset \mathbb{R}^d$ with smooth boundary which is not necessarily convex. Suppose that ρ_0 and ρ_1 are two probability measures defined on Ω. To obtain the full description of the mathematical problems, one needs to specify the no-flux boundary condition $\boldsymbol{m} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$ by the physical significance. Hence, the Beckmann formulation of the UOT problem is given by

$$\begin{aligned} \min_{\boldsymbol{m},\eta} \quad & \int_\Omega |\boldsymbol{m}(\boldsymbol{x})| + \alpha|\eta(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \\ \text{s.t.} \quad & \nabla \cdot \boldsymbol{m} + \rho_1 - \rho_0 = \eta \text{ in } \Omega, \\ & \boldsymbol{m} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.1}$$

Correspondingly, the Beckmann formulation of the traditional OT problem is analogously given by

$$\begin{aligned} \min_{\boldsymbol{m}} \quad & \int_\Omega |\boldsymbol{m}(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \\ \text{s.t.} \quad & \nabla \cdot \boldsymbol{m} + \rho_1 - \rho_0 = 0 \text{ in } \Omega, \\ & \boldsymbol{m} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.2}$$

The constraint in (3.1) is understood in the weak sense, i.e.,

$$-\int_\Omega \boldsymbol{m} \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x} + \int_\Omega (\rho_1 - \rho_0 - \eta)\varphi \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi \in C_b^1(\bar{\Omega}). \tag{3.3}$$

The constraint for (3.2) is understood similarly.

Before we start the analysis, let us clarify its connection to the dynamic formulation as announced in the introduction. Recall the UOT problem for the case $p = q = 1$ in (1.8):

$$\begin{aligned} & \min_{\rho,\boldsymbol{w},\zeta} \int_\Omega \int_0^1 |\boldsymbol{w}(\boldsymbol{x},t)| + \alpha|\zeta(\boldsymbol{x},t)| \mathrm{d}t\mathrm{d}\boldsymbol{x} \\ & \text{s.t.} \quad \partial_t \rho(\boldsymbol{x},t) + \nabla \cdot \boldsymbol{w}(\boldsymbol{x},t) = \zeta(\boldsymbol{x},t) \text{ in } \Omega \times [0,1] \\ & \text{with } \boldsymbol{w}(\boldsymbol{x},t) \cdot \boldsymbol{n} = 0 \text{on } \partial\Omega \times [0,1], \\ & \qquad \rho(\boldsymbol{x},0) = \rho_0(\boldsymbol{x}), \rho(\boldsymbol{x},1) = \rho_1(\boldsymbol{x}). \end{aligned} \tag{3.4}$$

Here, $\boldsymbol{w}(\cdot,t) \in (\mathcal{M}(\Omega))^d, 0 \leq t \leq 1$ is a d-dimensional vector field and $\zeta(\cdot,t) \in \mathcal{M}(\Omega), 0 \leq t \leq 1$, is a source term on Ω. Note that $\mathcal{M}(\Omega)$ is the set of Radon measures, the dual space of $C_b(\Omega)$. Also, $t \mapsto \rho(\cdot,t) \in \mathcal{M}(\Omega)$ is a path on $\mathcal{M}(\Omega)$. Define

$$\boldsymbol{m}(\boldsymbol{x}) = \int_0^1 \boldsymbol{w}(\boldsymbol{x},t)\mathrm{d}t, \quad \eta(\boldsymbol{x}) = \int_0^1 \zeta(\boldsymbol{x},t)\mathrm{d}t. \tag{3.5}$$

Then we have the following lemma.

LEMMA 3.1. *Under the settings above, the Beckmann formulation of UOT* (3.1) *is equivalent to the dynamical UOT problem* (3.4)*. The same conclusion is made for the OT case.*

Proof. On the one hand, for any feasible pair $(\boldsymbol{w},\zeta)$ in (3.4), it holds that

$$\begin{aligned}\int_\Omega |\boldsymbol{m}(\boldsymbol{x})| + \alpha|\eta(\boldsymbol{x})|\mathrm{d}\boldsymbol{x} &= \int_\Omega \left|\int_0^1 \boldsymbol{w}(\boldsymbol{x},t)\mathrm{d}t\right| + \alpha\left|\int_0^1 \zeta(\boldsymbol{x},t)\mathrm{d}t\right|\mathrm{d}\boldsymbol{x} \\ &\leq \int_\Omega\int_0^1 |\boldsymbol{w}(\boldsymbol{x},t)| + \alpha|\zeta(\boldsymbol{x},t)|\mathrm{d}t\mathrm{d}\boldsymbol{x};\end{aligned} \tag{3.6}$$

therefore one can obtain that

$$\min_{\boldsymbol{m},\eta}\int_\Omega |\boldsymbol{m}(\boldsymbol{x})| + \alpha|\eta(\boldsymbol{x})|\mathrm{d}\boldsymbol{x} \leq \min_{\boldsymbol{w},\zeta}\int_\Omega\int_0^1 |\boldsymbol{w}(\boldsymbol{x},t)| + \alpha|\zeta(\boldsymbol{x},t)|\mathrm{d}t\mathrm{d}\boldsymbol{x}. \tag{3.7}$$

On the other hand, for any $\alpha > 0$ suppose $(\boldsymbol{m}_\alpha, \eta_\alpha)$ is an optimal pair to problem (3.1). Then letting $\boldsymbol{w}_\alpha(\boldsymbol{x},t) \equiv \boldsymbol{m}_\alpha(\boldsymbol{x})$ and $\zeta_\alpha(\boldsymbol{x},t) \equiv \eta_\alpha(\boldsymbol{x})$ for all $t \in [0,1]$ and defining

$$\rho_\alpha(\boldsymbol{x},t) = t\rho_1(\boldsymbol{x}) + (1-t)\rho_0(\boldsymbol{x}),$$

one can get that the triad $(\rho_\alpha, w_\alpha, \zeta_\alpha)$ is a feasible solution to problem (3.4). Therefore,

$$\begin{aligned}\min_{\rho,\boldsymbol{w},\zeta}\int_\Omega\int_0^1 |\boldsymbol{w}(\boldsymbol{x},t)| + \alpha|\zeta(\boldsymbol{x},t)|\mathrm{d}t\mathrm{d}\boldsymbol{x} &\leq \int_\Omega\int_0^1 |\boldsymbol{w}_\alpha(\boldsymbol{x},t)| + \alpha|\zeta_\alpha(\boldsymbol{x},t)|\mathrm{d}t\mathrm{d}\boldsymbol{x} \\ &= \min_{\boldsymbol{m},\eta}\int_\Omega |\boldsymbol{m}(\boldsymbol{x})| + \alpha|\eta(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{3.8}$$

Hence, combining the two parts one can conclude that the problems (3.4) and (3.1) are equivalent. ∎

In the previous formulation, the form $|\boldsymbol{m}(x)|$ is the Euclidean norm (ℓ_2-norm) of $\boldsymbol{m}$, and from now on, we consider the general case of ℓ_q-norm for $\boldsymbol{m}(\boldsymbol{x})$ where $1 \leq q < +\infty$. In other words, we use

$$|\boldsymbol{m}(\boldsymbol{x})|_q := \left(\sum_{i=1}^d |m_i(\boldsymbol{x})|^q\right)^{1/q}, \text{ for } q \in [1,+\infty),$$

to replace the original ℓ_2-norm $|\boldsymbol{m}(x)|$. Moreover, for convenience in later analysis, taking $\xi := \alpha\eta$ in problem (3.1) and adding the term ξ to the objective function in problem (3.2) as a free variable, we obtain the two equivalent UOT and OT problems, respectively:

$$\begin{aligned}\min_{\boldsymbol{m},\xi}\quad & \int_\Omega |\boldsymbol{m}(\boldsymbol{x})|_q + |\xi(\boldsymbol{x})|\mathrm{d}\boldsymbol{x} \\ \text{s.t.}\quad & \nabla\cdot\boldsymbol{m} + \rho_1 - \rho_0 = \frac{1}{\alpha}\xi \text{ in } \Omega, \\ & \boldsymbol{m}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega,\end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \min_{\boldsymbol{m},\xi} \quad & \int_\Omega |\boldsymbol{m}(\boldsymbol{x})|_q + |\xi(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} \\ \text{s.t.} \quad & \nabla \cdot \boldsymbol{m} + \rho_1 - \rho_0 = 0 \text{ in } \Omega, \\ & \boldsymbol{m} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.10}$$

Then our goal becomes to build a connection between the UOT problem (3.9) and the OT problem (3.10), by establishing the convergence from problem (3.9) to (3.10) as $\alpha \to +\infty$ in the sense of Γ-convergence.

3.2. Existence of minimizers. In this subsection, we first show the existence of minimizers of the above UOT problems. For the convenience of the discussion, we define the total variation norm of the fields $\boldsymbol{m}$ and ξ as follows:

$$\|\boldsymbol{m}\| := \int_\Omega |\boldsymbol{m}(\boldsymbol{x})|_q \mathrm{d}\boldsymbol{x}, \qquad \|\xi\| := \int_\Omega |\xi(\boldsymbol{x})| \mathrm{d}\boldsymbol{x}. \tag{3.11}$$

It is well known that the total variation norm is in fact the dual norm against the bounded continuous functions since Ω is bounded.

We first note that the continuity equation constraint depends only on the gradient in $\boldsymbol{m}$. By Helmholtz decomposition, we obtain

$$\boldsymbol{m} = -\nabla\Phi + \boldsymbol{h}, \quad \nabla \cdot \boldsymbol{h} = 0, \quad \frac{\partial \Phi}{\partial \boldsymbol{n}} = 0, \quad \boldsymbol{h} \cdot \boldsymbol{n} = 0, \tag{3.12}$$

where Φ is a scalar field and $\boldsymbol{h}$ is a field without divergence and the constraint is imposed on Φ:

$$\int_\Omega \nabla\Phi \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} + \int_\Omega \left(\rho_1 - \rho_0 - \frac{1}{\alpha}\xi\right) \varphi \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi \in C_b^1(\bar{\Omega}). \tag{3.13}$$

In our setting, the divergence-free condition $\nabla \cdot \boldsymbol{h} = 0$ should be understood in the weak sense. Hence, we introduce the following space:

$$\mathcal{H} := \left\{ \boldsymbol{h} \in (\mathcal{M}(\Omega))^d : \int_\Omega \boldsymbol{h} \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} = 0 \, \forall \varphi \in C_b^1(\bar{\Omega}) \right\}.$$

We note that the Helmholtz decomposition always exists if ξ has a bounded total variation by the lemma below, from [7, Theorem 22, Lemma 23].

LEMMA 3.2. *For each $\xi \in \mathcal{M}(\Omega)$ with $\int |\xi| \, dx < \infty$, there exists a weak solution $\Phi \in W^{1,p_0}(\Omega)$ where $p_0 \in [1, d/(d-1))$ with $\int \Phi \, dx = 0$ to the Poisson equation*

$$-\Delta u = \xi. \tag{3.14}$$

The solution satisfies

$$\|\Phi\|_{W^{1,p_0}} \le C_q \|\xi\| \tag{3.15}$$

for some constant $C_q > 0$ depending on α and Ω only.

Note that the result in [7] is for $\xi \in L^1(\Omega)$ while we are considering Radon measures here, although there is no essential difference. By the inequality (3.15), it yields that the TV norm of $\nabla\Phi$ defined in (3.11) can be controlled as $\|\nabla\Phi\| \le C\|\xi\|$.

Then, for each $(\xi, \boldsymbol{m})$ satisfying the constraint (3.13), by Lemma 3.2 one can find a weak solution Φ with $\int \Phi \, \mathrm{d}\boldsymbol{x} = 0$ satisfying

$$\Delta \Phi = \rho_1 - \rho_0 - \frac{1}{\alpha}\xi.$$

Define

$$\boldsymbol{h} := \boldsymbol{m} + \nabla \Phi \in \mathcal{H},$$

and one can get that the Helmholtz decomposition of $\boldsymbol{m}$ exists and is stable.

Hence, the problem (3.9) is reduced to

$$\begin{aligned} &\min_{(\xi,\boldsymbol{h},\Phi)} \int_\Omega |\boldsymbol{h} - \nabla\Phi|_q + |\xi(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \\ &\text{s.t.} \int_\Omega \nabla\Phi \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} + \int_\Omega \left(\rho_1 - \rho_0 - \frac{1}{\alpha}\xi\right)\varphi \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi \in C_b^1(\bar{\Omega}), \\ &\boldsymbol{h} \in \mathcal{H}. \end{aligned} \tag{3.16}$$

Similarly, the OT problem (3.10) becomes

$$\begin{aligned} &\min_{(\xi,\boldsymbol{h},\Phi)} \int_\Omega |\boldsymbol{h} - \nabla\Phi|_q + |\xi(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} \\ &\text{s.t.} \int_\Omega \nabla\Phi \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\rho_1 - \rho_0)\varphi \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi \in C_b^1(\bar{\Omega}), \\ &\boldsymbol{h} \in \mathcal{H}. \end{aligned} \tag{3.17}$$

Using these two reduced problems, we can establish the following existence results.

Proposition 3.3. *Both* (3.9) *and* (3.10) *have global minimizers over* $\mathcal{M}(\Omega) \times (\mathcal{M}(\Omega))^d$.

Proof. By the reformulation above, we prove the existence results for (3.16) and (3.17). We will take (3.16) as an example.

First of all, we equip the set $\mathcal{M}(\Omega) \times \mathcal{H}$ for $(\xi, \boldsymbol{h})$ with the weak topology: $(\xi_n, \boldsymbol{h}_n) \Rightarrow (\xi, \boldsymbol{h})$ if

$$\int f d\xi_n + \int \boldsymbol{g} \cdot d\boldsymbol{h}_n \to \int f d\xi + \int \boldsymbol{g} \cdot d\boldsymbol{h} \, \forall f \in C_b^1(\bar{\Omega}; \mathbb{R}), \boldsymbol{g} \in C_b^1(\bar{\Omega}; \mathbb{R}^d). \tag{3.18}$$

Clearly, the space $\mathcal{H}$ is closed in $\mathcal{M}(\Omega)^d$ under the weak topology. Consider the functional

$$(\xi, \boldsymbol{h}) \mapsto F(\xi, \boldsymbol{h}) := \int_\Omega |\boldsymbol{h}(\boldsymbol{x}) - \nabla\Phi(\boldsymbol{x}; \xi)|_q + |\xi(\boldsymbol{x})| \, \mathrm{d}\boldsymbol{x} = \|\boldsymbol{h} - \nabla\Phi(\cdot; \xi)\| + \|\xi\|, \tag{3.19}$$

where $\Phi(\cdot; \xi)$ indicates that Φ is solved according to the Poisson equation with given ξ.

It is straightforward to verify that F is lower semicontinuous under the topology for $\mathcal{M}(\Omega) \times \mathcal{H}$. In fact, if $(\xi_n, \boldsymbol{h}_n) \Rightarrow (\xi, \boldsymbol{h})$, one has that

$$\boldsymbol{h}_n - \nabla\Phi(\cdot; \xi_n) \Rightarrow \boldsymbol{h} - \nabla\Phi(\cdot; \xi).$$

To see this, for any test vector field $\boldsymbol{g} \in C_b^1(\bar{\Omega}; \mathbb{R}^d)$, one can also decompose $\boldsymbol{g}$ as

$$\boldsymbol{g} = \nabla\phi + \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega. \tag{3.20}$$

Then,

$$\int_\Omega \nabla\Phi(\cdot;\xi_n)\cdot \boldsymbol{g}\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\Phi(\cdot;\xi_n)\cdot\nabla\phi\,\mathrm{d}\boldsymbol{x} = \int_\Omega\left(\frac{1}{\alpha}\xi_n+\rho_0-\rho_1\right)\phi\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_\Omega\left(\frac{1}{\alpha}\xi+\rho_0-\rho_1\right)\phi\,\mathrm{d}\boldsymbol{x} = \int_\Omega \nabla\Phi(\cdot;\xi)\cdot\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}. \tag{3.21}$$

Since bounded smooth functions are dense in the space of bounded continuous functions under the topology of uniform convergence (recall that $\bar\Omega$ is a bounded set), the above therefore holds for all $\boldsymbol{g}\in C_b^1(\bar\Omega;\mathbb{R}^d)$.

Consequently,

$$\|\boldsymbol{h}-\nabla\Phi(\cdot;\xi)\|+\|\xi\| \le \liminf_{n\to\infty}\|\boldsymbol{h}_n-\nabla\Phi(\cdot;\xi_n)\|+\|\xi_n\|. \tag{3.22}$$

Hence, the lower semicontinuity is established.

It is clear that

$$F_* := \inf_{(\xi,\boldsymbol{h})\in\mathcal{M}(\Omega)\times\mathcal{H}} F(\xi,\boldsymbol{h}) > -\infty.$$

Then, consider a minimizing sequence, $(\xi_n,\boldsymbol{h}_n)$ such that $F(\xi_n,\boldsymbol{h}_n)\to F_*$. Then for this minimizing sequence, one has

$$\sup_n \|\boldsymbol{h}_n-\nabla\Phi(\cdot;\xi_n)\|+\|\xi_n\| < +\infty.$$

According to (3.15), $\|\nabla\Phi(\cdot;\xi_n)\|$ is also uniformly bounded. Consequently,

$$\sup_n \|\boldsymbol{h}_n\|+\|\xi_n\| \le \sup_n \|\boldsymbol{h}_n-\nabla\Phi(\cdot;\xi_n)\|+\|\nabla\Phi(\cdot;\xi_n)\|+\|\xi_n\| < +\infty. \tag{3.23}$$

The Banach–Alaoglu theorem indicates that there must be a weakly convergent subsequence. Hence, together with the lower semicontinuity, the minimizer exists. ∎

3.3. Convergence. By noticing the conditions in Theorem 2.6, we will regard ξ and Φ as independent variables. Define the functional J for all $(\xi,\boldsymbol{h},\Phi)\in(\mathcal{M}(\Omega)\times\mathcal{H})\times W^{1,1}(\Omega)$ by

$$J(\xi,\boldsymbol{h},\Phi) = \int_\Omega |\boldsymbol{h}(\boldsymbol{x})-\nabla\Phi(\boldsymbol{x})|_q + |\xi(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}. \tag{3.24}$$

We equip the space for $(\xi,\boldsymbol{h})$ with the weak convergence of the measures defined in (3.18),

$$X := (\mathcal{M}(\Omega)\times\mathcal{H},\ \Rightarrow).$$

As mentioned above, the space $\mathcal{H}$ is closed in $\mathcal{M}(\Omega)^d$ under the weak topology.

Note that this weak topology for measures is closer to the weak* convergence in functional analysis. Moreover, the topology we choose for the space of $\nabla\Phi$ is the total variation norm, or the $W^{1,1}$ norm of Φ (assuming Φ has mean zero)

$$Y := W^{1,1}(\Omega).$$

Now, we introduce the set of constraints

$$E_\alpha := \Bigg\{ ((\xi,\boldsymbol{h}),\Phi)\in X\times Y : \int_\Omega \Phi\,\mathrm{d}\boldsymbol{x}=0, \int_\Omega \nabla\Phi\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x} + \int_\Omega\left(\rho_1-\rho_0-\frac{1}{\alpha}\xi\right)\varphi\,\mathrm{d}\boldsymbol{x} = 0 \quad \forall\varphi\in C_b^1(\bar\Omega)\Bigg\}. \tag{3.25}$$

Similarly,

$$E_\infty := \left\{ ((\xi, \boldsymbol{h}), \Phi) \in X \times Y : \int_\Omega \Phi \, \mathrm{d}\boldsymbol{x} = 0, \right.$$
$$\left. \int_\Omega \nabla\Phi \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\rho_1 - \rho_0)\varphi \, \mathrm{d}\boldsymbol{x} = 0 \quad \forall \varphi \in C_b^1(\bar{\Omega}) \right\}. \tag{3.26}$$

Problem (3.9) can be reformulated as

$$\min_{(\xi, \boldsymbol{h}) \in X, \ \Phi \in Y} J(\xi, \boldsymbol{h}, \Phi) + \mathbf{1}_{E_\alpha}. \tag{3.27}$$

Similarly, (3.10) is

$$\min_{(\xi, \boldsymbol{h}) \in X, \ \Phi \in Y} J(\xi, \boldsymbol{h}, \Phi) + \mathbf{1}_{E_\infty}. \tag{3.28}$$

The following theorem states the convergence from (3.9) to (3.10) as α goes to infinity.

THEOREM 3.4. *Suppose for any $\alpha > 0$, $(\boldsymbol{m}^\alpha, \xi^\alpha)$ is an optimal solution of the corresponding UOT problem* (3.9)*. Then*

(i) *with a decomposition*

$$\boldsymbol{m}^\alpha = \boldsymbol{h}^\alpha - \nabla\Phi^\alpha$$

such that $((\xi^\alpha, \boldsymbol{h}^\alpha), \Phi^\alpha) \in \mathbf{1}_{E_\alpha}$, there exists some constant $\alpha_0 > 0$, $M > 0$ such that for all $\alpha \geq \alpha_0$

$$\sup_\alpha \|\boldsymbol{h}^\alpha\| + \|\nabla\Phi^\alpha\| + \|\xi^\alpha\| \leq M;$$

(ii) *for any increasing sequence $\{\alpha_I\}$ going to infinity, where I is an index set, there exists a convergent subsequence $((\xi^{\alpha_k}, \boldsymbol{h}^{\alpha_k}), \Phi^{\alpha_k}) \in X \times Y$ with $\alpha_k \uparrow +\infty$ such that the limit $((\xi^\infty, \boldsymbol{h}^\infty), \Phi^\infty) \in \mathbf{1}_{E_\infty}$ and $(\xi^\infty, \boldsymbol{m}^\infty) = (\xi^\infty, \boldsymbol{h}^\infty - \nabla\Phi^\infty)$ is a solution of the OT problem* (3.10)*. Moreover, $\xi_\infty = 0$.*

To prove this theorem, we first show the Γ-convergence of $J + \mathbf{1}_{E_\alpha}$ to $J + \mathbf{1}_{E_\infty}$.

LEMMA 3.5. *With the above setup, $J + \mathbf{1}_{E_\alpha}$ is Γ-convergent to $J + \mathbf{1}_{E_\infty}$.*

Proof. Here, we verify the two conditions in Theorem 2.6.

We will first show that $\Gamma_{\text{seq}}(\mathbb{N}, X^-, Y)\lim\limits_\alpha J(\xi^\alpha, \boldsymbol{h}^\alpha, \Phi^\alpha) = J(\xi, \boldsymbol{h}, \Phi)$. It suffices to prove the following two results:

$$\inf_{(\xi^\alpha, \boldsymbol{h}^\alpha) \Rightarrow (\xi, \boldsymbol{h})} \sup_{\Phi^\alpha \to \Phi} \limsup_{\alpha \to +\infty} J(\xi^\alpha, \boldsymbol{h}^\alpha, \Phi^\alpha) \leq J(\xi, \boldsymbol{h}, \Phi),$$
$$\inf_{(\xi^\alpha, \boldsymbol{h}^\alpha) \Rightarrow (\xi, \boldsymbol{h})} \inf_{\Phi^\alpha \to \Phi} \liminf_{\alpha \to +\infty} J(\xi^\alpha, \boldsymbol{h}^\alpha, \Phi^\alpha) \geq J(\xi, \boldsymbol{h}, \Phi).$$

These two relations, by the sandwich theorem, can ensure that both both the signs of $\mathbb{N}$ and Y in the Γ_{seq}-limit can be omitted.

For any pair $((\xi, \boldsymbol{h}), \Phi)$ and for any convergent sequence $\Phi^\alpha \to \Phi$ in $W^{1,1}$, one can choose a particular weak convergent sequence $\{(\xi^\alpha, \boldsymbol{h}^\alpha)\}$ such that $(\xi^\alpha, \boldsymbol{h}^\alpha) \Rightarrow (\xi, \boldsymbol{h})$, $\|\xi^\alpha\| \to \|\xi\|$ and that $\|\boldsymbol{h}^\alpha - \nabla\Phi^\alpha\| \to \|\boldsymbol{h} - \nabla\Phi\|$. Such a sequence of $(\xi^\alpha, \boldsymbol{h}^\alpha)$ clearly exists (for example, one can choose the constant sequence $(\xi^\alpha, \boldsymbol{h}^\alpha) = (\xi, \boldsymbol{h})$). Then, one has

$$\begin{aligned}\limsup_{\alpha \to +\infty} J(\xi^\alpha, \boldsymbol{h}^\alpha, \Phi^\alpha) &= \limsup_{\alpha \to +\infty} \left(\int_\Omega |\xi^\alpha(\boldsymbol{x})| \mathrm{d}\boldsymbol{x} + \int_\Omega |\boldsymbol{h}^\alpha(\boldsymbol{x}) - \nabla\Phi^\alpha(\boldsymbol{x})|_q \mathrm{d}\boldsymbol{x} \right) \\ &= \|\xi\| + \|\boldsymbol{h} - \nabla\Phi\| = J(\xi, \boldsymbol{h}, \Phi).\end{aligned}$$

Hence, it holds that

$$\inf_{(\xi^\alpha,\boldsymbol{h}^\alpha)\Rightarrow(\xi,\boldsymbol{h})}\ \sup_{\Phi^\alpha\to\Phi}\ \limsup_{\alpha\to+\infty} J(\xi^\alpha,\boldsymbol{h}^\alpha,\Phi^\alpha)\le J(\xi,\boldsymbol{h},\Phi). \tag{3.29}$$

Remark 3.6. The strong convergence of Φ^α here is essential to obtain the limit $J(\xi,\boldsymbol{h},\Phi)$ as an upper bound. If there is only weak convergence of $\nabla\Phi$ as used in the proof of Proposition 3.3, such an upper bound cannot be established.

On the other hand, for any weak convergent sequence $(\xi^\alpha,\boldsymbol{h}^\alpha)\Rightarrow(\xi,\boldsymbol{h})$ in X and $\Phi^\alpha\to\Phi$ in Y, one has $\|\xi\|\le\liminf\|\xi^\alpha\|$ and $\|\boldsymbol{h}-\nabla\Phi\|\le\lim\|\boldsymbol{h}^\alpha-\nabla\Phi^\alpha\|$. Consequently,

$$\begin{aligned}\liminf_{\alpha\to+\infty} J(\xi^\alpha,\boldsymbol{h}^\alpha,\Phi^\alpha) &= \liminf_{\alpha\to+\infty}\left(\int_\Omega|\xi^\alpha(\boldsymbol{x})|\mathrm{d}\boldsymbol{x}+\int_\Omega|\boldsymbol{h}^\alpha(\boldsymbol{x})-\nabla\Phi^\alpha(\boldsymbol{x})|_q\mathrm{d}\boldsymbol{x}\right)\\ &\ge\|\xi\|+\|\boldsymbol{h}-\nabla\Phi\|=J(\xi,\boldsymbol{h},\Phi).\end{aligned}$$

It follows that

$$\inf_{(\xi^\alpha,\boldsymbol{h}^\alpha)\Rightarrow(\xi,\boldsymbol{h})}\ \inf_{\Phi^\alpha\to\Phi}\ \liminf_{\alpha\to+\infty} J(\xi^\alpha,\boldsymbol{h}^\alpha,\Phi^\alpha)\ge J(\xi,\boldsymbol{h},\Phi). \tag{3.30}$$

Combining the two formulas, one obtains

$$\Gamma_{\mathrm{seq}}(\mathbb{N},X^-,Y)\lim_\alpha J(\xi^\alpha,\boldsymbol{h}^\alpha,\Phi^\alpha)=J(\xi,\boldsymbol{h},\Phi). \tag{3.31}$$

Next, we will show $\mathbf{1}_{E_\infty}=\Gamma_{\mathrm{seq}}(\mathbb{N},X,Y^-)\lim_\alpha\mathbf{1}_{E_\alpha}$. Using Lemma 2.5, it suffices to show that

(i) if $(\xi^\alpha,\boldsymbol{h}^\alpha)\Rightarrow(\xi,\boldsymbol{h})$, $\Phi^\alpha\to\Phi$ in $W^{1,1}$, $((\xi^\alpha,\boldsymbol{h}^\alpha),\Phi^\alpha)\in E_\alpha$ for infinitely many α, then $((\xi,\boldsymbol{h}),\Phi)\in E_\infty$;

(ii) if $((\xi,\boldsymbol{h}),\Phi)\in E_\infty$ and $(\xi^\alpha,\boldsymbol{h}^\alpha)\Rightarrow(\xi,\boldsymbol{h})$, then there exists $\Phi^\alpha\to\Phi$ such that $((\xi^\alpha,\boldsymbol{h}^\alpha),\Phi^\alpha)\in E_\alpha$ for α large enough.

For (i), we consider the sequence α such that $((\xi^\alpha,\boldsymbol{h}^\alpha),\Phi^\alpha)\in E_\alpha$, then

$$\int_\Omega\Phi^\alpha\,\mathrm{d}\boldsymbol{x}=0,\quad \int_\Omega\nabla\Phi^\alpha\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}+\int_\Omega\left(\rho_1-\rho_0-\frac{1}{\alpha}\xi^\alpha\right)\varphi\,\mathrm{d}\boldsymbol{x}=0\quad\forall\varphi\in C_b^1(\bar\Omega). \tag{3.32}$$

Since $\varphi\in C_b$ and $\nabla\varphi\in C_b$, one clearly has

$$\int_\Omega\boldsymbol{m}^\alpha\cdot\nabla\varphi\to\int_\Omega\boldsymbol{m}\cdot\nabla\varphi,\quad 0=\int_\Omega\Phi^\alpha\,\mathrm{d}\boldsymbol{x}\to\int_\Omega\Phi\,\mathrm{d}\boldsymbol{x}.$$

As $\xi^\alpha\Rightarrow\xi$, it is uniformly bounded and

$$\lim_{\alpha\to+\infty}\int_\Omega\frac{1}{\alpha}\xi^\alpha\varphi\,\mathrm{d}\boldsymbol{x}=0. \tag{3.33}$$

Hence, it is easy to see that $(\xi,\boldsymbol{h},\Phi)\in E_\infty$. Note that here $\lim\int_\Omega\boldsymbol{h}^\alpha\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}=\int_\Omega\boldsymbol{h}\cdot\nabla\varphi\,\mathrm{d}\boldsymbol{x}$ using the fact that X is closed under the weak convergence of measures. For (ii) we consider the following Poisson equation:

$$\begin{cases}-\Delta u=\dfrac{1}{\alpha}\xi^\alpha\ \text{in}\ \Omega,\\ \dfrac{\partial u}{\partial\boldsymbol{n}}=0\ \text{on}\ \partial\Omega.\end{cases} \tag{3.34}$$

Here the sequence $\{\xi^\alpha\}$ is given in (ii) which weakly converges to ξ. Consequently, by Lemma 3.2, there exists ϕ^α with $\int \phi^\alpha \,\mathrm{d}\boldsymbol{x} = 0$ and

$$\lim_{\alpha\to+\infty} \|-\nabla\phi^\alpha\| \leq \lim_{\alpha\to+\infty} \frac{C}{\alpha}\|\xi^\alpha\| = 0.$$

Defining $\Phi^\alpha = \Phi + \phi^\alpha$, one clearly has $\int \Phi^\alpha \,\mathrm{d}\boldsymbol{x} = 0$, $\Phi^\alpha \to \Phi$ and by the definition of the weak solution of the Poisson equation that

$$\int_\Omega \nabla\Phi^\alpha \cdot \nabla\varphi \,\mathrm{d}\boldsymbol{x} + \int_\Omega \left(\rho_1 - \rho_0 - \frac{1}{\alpha}\xi^\alpha\right)\varphi \,\mathrm{d}\boldsymbol{x} = 0 \quad \forall\varphi \in C_b^1(\bar{\Omega}), \tag{3.35}$$

which implies that $(\xi^\alpha, \boldsymbol{h}^\alpha, \Phi^\alpha) \in E_\alpha$ for all α.

Now, we prove the main result in this section.

Proof of Theorem 3.4. Suppose $\boldsymbol{m}$ is a feasible solution to problem (3.10). It is obvious that $(0, \boldsymbol{m})$ is also a feasible solution to problem (3.9) for any $\alpha > 0$. Therefore,

$$\int_\Omega |\boldsymbol{m}^\alpha|_q + |\xi^\alpha| \mathrm{d}\boldsymbol{x} \leq \int_\Omega |\boldsymbol{m}|_q \mathrm{d}\boldsymbol{x} < +\infty.$$

where $(\xi^\alpha, \boldsymbol{m}^\alpha)$ is an optimal solution of problem (3.10). Then, by Lemma 3.2, there exists $\Phi^\alpha \in W^{1,1}$ with $\int \Phi^\alpha \,\mathrm{d}\boldsymbol{x} = 0$ that is a weak solution to

$$-\Delta\Phi^\alpha + \rho_1 - \rho_0 = \frac{1}{\alpha}\xi^\alpha, \quad \frac{\partial\Phi^\alpha}{\partial n} = 0$$

with

$$\|\Phi^\alpha\|_{W^{1,1}} \leq C\|\rho_0 - \rho_1 - \frac{1}{\alpha}\xi^\alpha\| \leq C\left(2 + \frac{1}{\alpha}\int |\boldsymbol{m}|_q \,\mathrm{d}\boldsymbol{x}\right).$$

Moreover, define

$$\boldsymbol{h}^\alpha = \boldsymbol{m}^\alpha + \nabla\Phi^\alpha.$$

It is easy to see that $\boldsymbol{h}^\alpha \in \mathcal{H}$ and consequently, $((\xi^\alpha, \boldsymbol{h}^\alpha), \Phi^\alpha) \in \mathbf{1}_{E_\alpha}$. Moreover,

$$\|\boldsymbol{h}^\alpha\| \leq \|\boldsymbol{m}^\alpha\| + \|\nabla\Phi^\alpha\| \leq C\left(2 + (1 + \alpha^{-1})\int |\boldsymbol{m}|_q \,\mathrm{d}\boldsymbol{x}\right).$$

The first claim follows if $\alpha \geq \alpha_0 > 0$.

We now show that for any optimal sequence $\{(\xi^\alpha, \boldsymbol{m}^\alpha)\}$ with $\boldsymbol{m}^\alpha = \boldsymbol{h}^\alpha - \nabla\Phi^\alpha$ as above, there exists a convergent subsequence $(\xi^{\alpha^k}, \boldsymbol{h}^{\alpha^k}) \Rightarrow (\xi, \boldsymbol{h})$ and $\Phi^{\alpha_k} \to \Phi$.

Using the Banach–Alaoglu theorem we have that any bounded set in X is precompact. Consequently, there is a subsequence $(\xi^{\alpha_k}, \boldsymbol{h}^{\alpha_k}) \Rightarrow (\xi, \boldsymbol{h}) \in X$. Moreover, let Φ with $\int \Phi \,\mathrm{d}\boldsymbol{x} = 0$ be the solution to

$$\begin{cases} -\Delta\Phi + \rho_1 - \rho_0 = 0 \text{ in } \Omega, \\ \dfrac{\partial\Phi}{\partial\boldsymbol{n}} = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.36}$$

Since $\xi^{\alpha_k} \Rightarrow \xi$, ξ^{α_k} is thus uniformly bounded. Then as $\alpha_k \to +\infty$, one has

$$\|\Phi^{\alpha_k} - \Phi\|_{W^{1,1}} \leq \frac{C}{\alpha_k}\|\xi^{\alpha_k}\| \to 0.$$

Using Lemma 3.5, one obtains Γ-convergence of the functional. Using Theorem 2.6 it follows that $(\xi, \boldsymbol{h}, \Phi)$ is a minimizer of $J + \mathbf{1}_{E_\infty}$. Hence, the conclusion follows. Moreover, since $(\xi^\infty, \boldsymbol{m}^\infty)$ is an optimal pair of problem (3.10), it is obvious that ξ^∞ should be 0. ∎

4. Convergence in a discrete setting and the asymptotic preserving property. In this section, we focus on the discretized problems of the Beckmann formulation for UOT and OT problems. We take Ω to be a bounded rectangular domain in $\mathbb{R}^d$. We will show that the convergence from UOT to OT is preserved in the discrete setting so that the discretization is asymptotic preserving, which guarantees that a numerical method for the optimization problems can be applied for the discrete problems in a uniform manner, and the optimizer of the UOT can converge to that for the OT problem along the limit. Moreover, we show that when α is larger than some critical value that is only related to the dimension d and the width of the domain L, and independent to the mesh size h, the minimizer of the discrete UOT problem is then reduced to that for OT, which we call finite convergence. We also present the algorithm proposed in [27] and applied to solve both UOT and OT problems and show that the iterates for UOT will reduce to that for OT as the penalty parameter $\alpha > M$ for some constant M dependant on the discrete problem merely.

4.1. Discretized problems. We first formulate the discrete UOT and OT problems. We use the same discrete scheme as [27, 29]. Let $\Omega = [0,1]^d$ be a d-dimensional rectangular domain, and let Ω_h be the discrete meshgrid of Ω with step size h, i.e.,

$$\Omega_h = \{h, 2h, \ldots, 1\}^d. \tag{4.1}$$

Let $N = 1/h$ be the grid size. For a more general case $\Omega = [0,L]^d$, it can be transformed to $\Omega = [0,1]^d$ by scaling. For all $x \in \Omega_h$, x is a d-dimensional vector, where the ith component x_i takes values from $\{h, 2h, 3h, \ldots, 1\}$. The discretized distributions $\rho_h^0 = \{\rho^0(x)\}_{x\in\Omega_h}, \rho_h^1 = \{\rho^1(x)\}_{x\in\Omega_h}$, and $\eta_h = \{\eta(x)\}_{x\in\Omega_h}$ are all N^d tensors. The discretized flux $\boldsymbol{m}_h = \{\boldsymbol{m}(x)\}_{x\in\Omega_h}$ is an $N^d \times d$ tensor, which can be regarded as a map from Ω_h to $\mathbb{R}^d$. Then the discretized problem for (3.1) is

$$\begin{aligned} &\min_{m_h,\eta_h} \sum_{x\in\Omega_h} \left(|\boldsymbol{m}_h(x)|_q h^d + \alpha|\eta_h(x)|h^d\right) \\ &\text{s.t.}\quad \operatorname{div}^h(\boldsymbol{m}_h(x)) - \eta_h(x) = \rho_h^0(x) - \rho_h^1(x) \qquad \forall x \in \Omega_h, \end{aligned} \tag{4.2}$$

where the discrete boundary conditions are given such that $\boldsymbol{m}_{h,i}(x_{-i}, x_i) = 0$ if $x_i = 1$ and $\boldsymbol{m}_{h,i}(x_{-i}, x_i - h) = 0$ if $x_i = h$ for all $i \in \{1, 2, \ldots, d\}$, and $\sum_{x\in\Omega_h} \eta_h(x) = 0$. Here the notion "-$i$" refers to all the components excluding i, i.e.,

$$x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d),$$

and for any $z \in \mathbb{R}$

$$\boldsymbol{m}_{h,i}(x_{-i}, z) = \boldsymbol{m}_{h,i}(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d).$$

Note that with this discretization, the boundary condition $\boldsymbol{m} \cdot \boldsymbol{n} = 0$ holds at points $(x_{-i}, x_i = 0)$ and $(x_{-i}, x_i = 1)$ for all $i \in 1, 2, \ldots, d$. And correspondingly, the discrete divergence operator $\operatorname{div}^h(\cdot)$ is defined as

$$\operatorname{div}^h(\boldsymbol{m}_h(x)) = \sum_{i=1}^d D_{h,i}\boldsymbol{m}_h(x),$$

and for all $i \in \{1, 2, \ldots, d\}$

$$D_{h,i}\boldsymbol{m}_h(x) = \begin{cases} (\boldsymbol{m}_{h,i}(x_{-i}, x_i))/h, & x_i = h, \\ (\boldsymbol{m}_{h,i}(x_{-i}, x_i) - \boldsymbol{m}_{h,i}(x_{-i}, x_i - h))/h, & h < x_i < 1, \\ (-\boldsymbol{m}_{h,i}(x_{-i}, x_i - h))/h, & x_i = 1, \end{cases} \tag{4.3}$$

which makes the discrete approximation consistent with the zero-flux boundary condition. In the above definition, $\boldsymbol{m}_h(x)\in\mathbb{R}^d$ denotes the flow at point x and $\boldsymbol{m}_{h,i}(x)\in\mathbb{R}$ denotes the ith component of $\boldsymbol{m}_h(x)$. Moreover, we define $f(\cdot)=\sum_{\Omega_h}|\cdot|_q h^d$ as a discrete $\ell_{q,1}$ norm on Ω_h; then the problem (4.2) can be reformulated as

$$\begin{aligned}&\min_{\boldsymbol{m}_h,\eta_h}\ f(\boldsymbol{m}_h)+\alpha f(\eta_h),\\ &\text{s.t.}\quad \operatorname{div}^h(\boldsymbol{m}_h)-\eta_h=\rho_h.\end{aligned}\tag{4.4}$$

Similarly, the discrete OT problem (3.2) is given as

$$\begin{aligned}&\min_{\boldsymbol{m}_h} f(\boldsymbol{m}_h)\\ &\text{s.t.}\quad \operatorname{div}^h(\boldsymbol{m}_h)=\rho_h\end{aligned}\tag{4.5}$$

with the zero-flux boundary condition in (4.2).

4.2. A primal-dual hybrid algorithm. With the discrete formulation, we can apply a primal-dual hybrid algorithm [16, 10] to solve both the UOT and OT problems. Note that this algorithm is also adopted in [27] for the OT problem. We first give some definitions on the discrete space Ω_h:

$$\langle \boldsymbol{m}_h,\boldsymbol{m}_h'\rangle_h=\sum_{x\in\Omega_h}\boldsymbol{m}_h(x)\boldsymbol{m}_h'(x)h^d,\quad \|\boldsymbol{m}_h\|_{h,2}^2=\sum_{x\in\Omega_h}|\boldsymbol{m}(x)|_2^2h^d.$$

Then for the OT problem (4.5), we solve the min-max reformulation

$$\min_{\boldsymbol{m}_h}\max_{\varphi_h} L(\boldsymbol{m}_h,\varphi_h)=f(\boldsymbol{m}_h)+\langle\varphi_h,\operatorname{div}^h\boldsymbol{m}_h-\rho_h\rangle_h\tag{4.6}$$

by the primal-dual hybrid algorithm, whose updating rule is given as

$$\begin{aligned}\boldsymbol{m}_h^{k+1}&=\arg\min_{\boldsymbol{m}_h} L(\boldsymbol{m}_h,\varphi_h^k)+\frac{1}{2\mu}\|\boldsymbol{m}_h-\boldsymbol{m}_h^k\|_{h,2}^2,\\ \tilde{\boldsymbol{m}}_h^{k+1}&=2\boldsymbol{m}_h^{k+1}-\boldsymbol{m}_h^k,\\ \varphi_h^{k+1}&=\arg\max_{\varphi_h} L(\tilde{\boldsymbol{m}}_h^{k+1},\varphi_h^k)-\frac{1}{2\tau}\|\varphi_h-\varphi_h^k\|_{h,2}^2.\end{aligned}\tag{4.7}$$

The update is equivalent to

$$\begin{aligned}\boldsymbol{m}_h^{k+1}&=\operatorname{Prox}_{\mu f}(\boldsymbol{m}_h^k-\mu(\operatorname{div}^h)^*(\varphi_h^k)),\\ \tilde{\boldsymbol{m}}_h^{k+1}&=2\boldsymbol{m}_h^{k+1}-\boldsymbol{m}_h^k,\\ \varphi_h^{k+1}&=\varphi_h^k+\tau(\operatorname{div}^h(\tilde{\boldsymbol{m}}_h^{k+1})-\rho_h),\end{aligned}\tag{4.8}$$

where $\operatorname{Prox}_{\mu f}$ is the proximity operator of function μf defined as

$$\operatorname{Prox}_{\mu f}(\boldsymbol{m}_h^k-\mu(\operatorname{div}^h)^*(\varphi_h^k))=\arg\min_{\boldsymbol{m}_h}\mu f(\boldsymbol{m}_h)+\frac{1}{2}\|\boldsymbol{m}_h-(\boldsymbol{m}_h^k-\mu(\operatorname{div}^h)^*(\varphi_h^k))\|_{h,2}^2,$$

and μ, τ are algorithmic parameters and $(\operatorname{div}^h)^*$ represents the conjugate operator of div^h.

By the definition of conjugate operator, for all $\boldsymbol{m}_h$ and $u_h\in\Omega_h$, we have

$$\langle\operatorname{div}^h(\boldsymbol{m}_h),u_h\rangle_h=\langle\boldsymbol{m}_h,(\operatorname{div}^h)^*(u_h)\rangle_h.$$

Then it is easy to check that $(\mathrm{div}^h)^* = -\nabla_h$ and

$$\nabla_h(u_h) = (\partial_{h,1}u_h, \partial_{h,2}u_h, \dots, \partial_{h,d}u),$$

where each $\partial_{h,i}u_h$ is

$$\partial_{h,i}u_h(x_1, x_2, \dots, x_d) = \begin{cases} (u(x_{-i}, x_i + h) - u(x_{-i}, x_i))/h, & h \le x_i < 1, \\ 0, \quad x_i = 1, \end{cases} \tag{4.9}$$

for $i = 1, 2, \dots, d$. According to [10], the algorithm is ensured to be convergent if $\mu\tau\|\mathrm{div}^h\|^2 < 1$.

Similarly, we solve the UOT problem (4.2) with the min-max reformulation

$$\min_{\boldsymbol{m}_h, \eta_h} \max_{\varphi_h} L(\boldsymbol{m}_h, \eta_h, \varphi_h) = f(\boldsymbol{m}_h) + \alpha f(\eta_h) + \langle \varphi_h, \mathrm{div}^h(\boldsymbol{m}_h) - \eta_h - \rho_h \rangle_h \tag{4.10}$$

and by the primal-dual hybrid algorithm as

$$\begin{aligned} (\boldsymbol{m}_h^{k+1}, \eta_h^{k+1}) &= \operatorname*{arg\,min}_{\boldsymbol{m}_h, \eta_h} L(\boldsymbol{m}_h, \eta_h, \varphi_h^k) + \frac{1}{2\mu}(\|\boldsymbol{m}_h - \boldsymbol{m}_h^k\|_{h,2}^2 + \|\eta_h - \eta_h^k\|_{h,2}^2), \\ \tilde{\boldsymbol{m}}_h^{k+1} &= 2\boldsymbol{m}_h^{k+1} - \boldsymbol{m}_h^k, \qquad \tilde{\eta}_h^{k+1} = 2\eta_h^{k+1} - \eta_h^k, \\ \varphi_h^{k+1} &= \operatorname*{arg\,max}_{\varphi_h} L(\tilde{\boldsymbol{m}}_h^{k+1}, \tilde{\eta}_h^{k+1}, \varphi_h^k) - \frac{1}{2\tau}\|\varphi_h - \varphi_h^k\|_{h,2}^2, \end{aligned} \tag{4.11}$$

which can be written as

$$\begin{aligned} \boldsymbol{m}_h^{k+1} &= \mathrm{Prox}_{\mu f}(\boldsymbol{m}_h^k - \mu(\mathrm{div}^h)^*(\varphi_h^k)), \\ \eta_h^{k+1} &= \mathrm{Prox}_{\alpha\mu f}(\eta_h^k + \mu\varphi_h^k), \\ \tilde{\boldsymbol{m}}_h^{k+1} &= 2\boldsymbol{m}_h^{k+1} - \boldsymbol{m}_h^k, \qquad \tilde{\eta}_h^{k+1} = 2\eta_h^{k+1} - \eta_h^k, \\ \varphi_h^{k+1} &= \varphi_h^k + \tau(\mathrm{div}^h(\tilde{\boldsymbol{m}}_h^{k+1}) - \tilde{\eta}_h^{k+1} - \rho_h). \end{aligned} \tag{4.12}$$

The algorithm is convergent if $\mu\tau\|[\mathrm{div}^h, -I]\|^2 < 1$ and we terminate the algorithm when the primal-dual gap R_n^k,

$$\begin{aligned} R_h^k = &\frac{1}{\mu}(\|\boldsymbol{m}_h^{k+1} - \boldsymbol{m}_h^k\|_{h,2}^2 + \|\eta_h^{k+1} - \eta_k^h\|_{h,2}^2) + \frac{1}{\tau}\|\varphi_h^{k+1} - \varphi_h^k\|_{h,2}^2 \\ &- 2\langle \varphi_h^{k+1} - \varphi_h^k, \mathrm{div}^h(\boldsymbol{m}_h^{k+1} - \boldsymbol{m}_h^k) - (\eta_h^{k+1} - \eta_h^k) \rangle_h, \end{aligned} \tag{4.13}$$

falls below a predefined threshold ϵ.

4.3. Convergence of the discrete UOT problem. In section 3 we showed the Γ-convergence from UOT to OT in the continuous case. For the discrete problem, we also have a similar proposition.

PROPOSITION 4.1. *Define* $X_h := \mathbb{R}^{N^d}$ *and* $Y_h := \mathbb{R}^{N^d \times d}$. *Taking* $\xi_h := \alpha\eta_h$ *in* (3.1) *we have that*

$$J_h(\xi_h, \boldsymbol{m}_h) := \sum_{x \in \Omega_h} \left(|\boldsymbol{m}_h(x)|_q h^d + |\xi_h(x)| h^d \right), \tag{4.14}$$

and we define

$$\begin{aligned} E_h^\alpha &:= \left\{ (\xi_h, \boldsymbol{m}_h) \in X_h \times Y_h : \mathrm{div}^h(\boldsymbol{m}_h) - \frac{1}{\alpha}\xi_h - \rho_h = 0 \right\}, \\ E_h^\infty &:= \left\{ (\xi_h, \boldsymbol{m}_h) \in X_h \times Y_h : \mathrm{div}^h(\boldsymbol{m}_h) - \rho_h = 0 \right\}. \end{aligned}$$

Then we have the Γ*-convergence from* $J_h + \mathbf{1}_{E_h^\alpha}$ *to* $J_h + \mathbf{1}_{E_h^\infty}$.

Proof. For the discrete case, the convergence in the space X_h and Y_h reduces to pointwise convergence, and it is obvious that

$$J_h(\xi_h, \boldsymbol{m}_h) = \Gamma_{\text{seq}}(\mathbb{N}, X_h^-, Y_h) \lim_{\alpha} J_h(\xi_h, \boldsymbol{m}_h). \tag{4.15}$$

Then our goal is to verify that E_h^α and E_h^∞ also satisfy the conditions given in Lemma 2.5:

(i) If $\xi_h^\alpha \to \xi_h, \boldsymbol{m}_h^\alpha \to \boldsymbol{m}_h$ and $(\xi_h^\alpha, \boldsymbol{m}_h^\alpha) \in E_h^\alpha$ for infinitely many α, then $(\xi_h, \boldsymbol{m}_h) \in E_h^\infty$.

(ii) If $(\xi_h, \boldsymbol{m}_h) \in E_h^\infty$ and $\xi_h^\alpha \to \xi_h$, then there exists $\boldsymbol{m}_h^\alpha \to \boldsymbol{m}_h$ such that $(\xi_h^\alpha, \boldsymbol{m}_h^\alpha) \in E_h^\alpha$ for α large enough.

For (i), due to the fact that $(\xi_h^\alpha, \boldsymbol{m}_h^\alpha) \in E_h^\alpha$ for infinitely many α, we have

$$\operatorname{div}^h(\boldsymbol{m}_h^\alpha) - \frac{1}{\alpha}\xi_h^\alpha - \rho_h = 0. \tag{4.16}$$

As $\xi_h^\alpha \to \xi_h$, the sequence $\{\xi_h^\alpha\}$ is uniformly bounded, and $\operatorname{div}^h(\boldsymbol{m}_h^\alpha)$ is a simple matrix-vector multiplication for the discrete problem, then letting α go to infinity in the equality (4.16) we obtain that

$$\operatorname{div}^h(\boldsymbol{m}_h) - \rho_h = 0, \tag{4.17}$$

which leads to $(\xi_h, \boldsymbol{m}_h) \in E_h^\infty$.

For (ii), as $(\xi_h, \boldsymbol{m}_h) \in E_h^\infty$, for any $\alpha > 0$ and the given sequence $\{\xi_h^\alpha\}$, it suffices to solve the equations directly:

$$\operatorname{div}^h(\delta \mathbf{m}_h^\alpha) = \frac{1}{\alpha}\xi_h^\alpha, \tag{4.18}$$

where $\delta\mathbf{m}_h^\alpha$ also satisfies the discrete boundary conditions $\delta\mathbf{m}_{h,i}(x_{-i}, x_i - h) = 0$ if $x_i = 1$ for all $i \in \{1, 2, \ldots, d\}$. For any $u \in \ker((\operatorname{div}^h)^*)$, $\nabla_h u = 0$ yields $u \equiv C$ for some constant C. Therefore, for all $u \in \ker((\operatorname{div}^h)^*)$, we have

$$\langle \xi_h^\alpha, u\rangle_h = C \sum_{x\in\Omega_h} \xi_h^\alpha(x) h^d = 0,$$

which implies $\xi_h^\alpha \perp \ker((\operatorname{div}^h)^*)$. Therefore the linear system (4.18) is soluble for any $\alpha > 0$. Moreover, with the condition $\xi_h^\alpha \to \xi_h$, it is obvious that the right side in (4.18) $\frac{1}{\alpha}\xi_h^\alpha \to 0$ as α goes to infinity. Hence for each α, we choose the solution that has the least norm $\delta\mathbf{m}_h^\alpha$ (which should be perpendicular to the kernel space of div^h) such that $\delta\mathbf{m}_h^\alpha \to 0$ as $\alpha \to +\infty$. Then define

$$\boldsymbol{m}_h^\alpha = \boldsymbol{m}_h + \delta\mathbf{m}_h^\alpha$$

and correspondingly $\boldsymbol{m}_h^\alpha \to \boldsymbol{m}_h$ and $(\xi_h^\alpha, \boldsymbol{m}_h^\alpha) \in E_h^\alpha$ for any $\alpha > 0$.
Using Lemma 2.5 we obtain that $\mathbf{1}_{E_h^\infty} = \Gamma_{\text{seq}}(\mathbb{N}, X_h, Y_h^-) \lim_{\alpha} \mathbf{1}_{E_h^\alpha}$, and using Theorem 2.6 one can conclude that $J_h + \mathbf{1}_{E_h^\alpha}$ is Γ-convergent to $J_h + \mathbf{1}_{E_h^\infty}$. ∎

The Γ-convergence for discrete problems indicates that the minimizers, which are bounded obviously, would have a convergent subsequence with the limit being the minimizer of the discrete OT problem.

In fact, for the discrete problem, we can show a stronger convergence for the minimizers of the two problems, as shown in the following theorem.

THEOREM 4.2. *Suppose $\Omega = [0, L]^d$ where d is the dimension of the space and L is the length of the interval in each dimension. Let $\Omega_h = [h, L]_h^d = \{h, 2h, \ldots, L\}^d$ be the discrete meshgrid of Ω defined in* (4.1) *and $Nh = L$. If $\alpha > \frac{dL}{2}$, then the optimal η_h^* of the discrete UOT problem equals to* 0*. Consequently, the minimizer reduces to that for OT.*

Proof. Different from before, here we will use the optimal conditions to show the result. Using the Lagrangian multiplier and omitting the scaling h^d, the discrete UOT problem (4.4) is equivalent to the following min-max problem:

$$\min_{\boldsymbol{m}_h, \eta_h} \max_{\varphi_h} \sum_{x \in \Omega} (|\boldsymbol{m}_h(x)|_q + \alpha |\eta_h(x)|) + \langle \varphi_h, \operatorname{div}^h(\boldsymbol{m}_h) - \eta_h - \rho_h \rangle, \tag{4.19}$$

where the inner product $\langle \cdot, \cdot \rangle = \frac{1}{h^d} \langle \cdot, \cdot \rangle_h$. Suppose $(\boldsymbol{m}_h^*, \eta_h^*, \varphi_h^*)$ is an optimal solution to (4.19); then by the first-order optimal conditions, for each point $x \in \Omega_h$ we have that

$$\begin{cases} 0 \in \partial |\boldsymbol{m}_h^*(x)|_q - \nabla_h(\varphi_h^*(x)), \\ 0 \in \alpha \partial |(\eta_h^*(x)| - \varphi_h^*(x), \\ 0 = \operatorname{div}^h(\boldsymbol{m}_h^*(x)) - \eta_h^*(x) - \rho_h(x). \end{cases} \tag{4.20}$$

Here $\partial |\boldsymbol{m}_h^*(x)|_q$ and $\partial |(\eta_h^*(x)|$ represent the subgradients of ℓ_q-norm of vector $\boldsymbol{m}_h^*(x)$ and the absolute value of $\eta_h^*(x)$, respectively. From the second condition we obtain that for any point $(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) \in (h, L]_h^d$,

$$-\alpha \le \varphi_h^*(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) \le \alpha.$$

To show $\eta_h^* = 0$, without loss of generality, we suppose that there exists a point $(x_{i_1}, x_{i_2}, \ldots, x_{i_d})$ in $[h, L]_h^d$ such that $\eta_h^*(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) > 0$; then by the second condition we have

$$\varphi_h^*(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) = \alpha.$$

Notice that for any $q \in [1, +\infty)$, the subgradient of $\partial |\boldsymbol{m}_h^*(x)|_q$ is defined as

$$\partial |\boldsymbol{m}_h^*(x)|_q = \left\{ \mathbf{v} \in \mathbb{R}^d : |\mathbf{v}|_{q'} \le 1,\ \mathbf{v}^\top \boldsymbol{m}_h^*(x) = |\boldsymbol{m}_h^*(x)|_q \right\},$$

where $1/q + 1/q' = 1$ and for $q = 1$, $|\mathbf{v}|_{q'}$ implies L_∞-norm of vector $\mathbf{v}$. Then by the first condition in (4.20), we have that for each $x \in \Omega_h$,

$$|\nabla \varphi_h^*(x)|_{q'} \le 1,$$

which indicates that the absolute value of each component of $\nabla \varphi_h^*(x)$ is also less than 1. More precisely, we have the following different situations:

- For $h < x_{i_k} < L$, at the point (x_{-i_k}, x_{i_k}) and $(x_{-i_k}, x_{i_k} - h)$ we can get that

$$\begin{cases} \nabla_h(\varphi_h^*(x_{-i_k}, x_{i_k})) \in \partial |(\boldsymbol{m}_h^*(x_{-i_k}, x_{i_k})|_q, \\ \nabla_h(\varphi_h^*(x_{-i_k}, x_{i_k} - h)) \in \partial |\boldsymbol{m}_h^*(x_{-i_k}, x_{i_k} - h)|_q, \end{cases} \tag{4.21}$$

which leads to

$$\begin{cases} \dfrac{\varphi_h^*(x_{-i_k}, x_{i_k} + h) - \varphi_h^*(x_{-i_k}, x_{i_k})}{h} \in \partial |\boldsymbol{m}_{h,k}^*(x_{-i_k}, x_{i_k})|, \\ \dfrac{\varphi_h^*(x_{-i_k}, x_{i_k}) - \varphi_h^*(x_{-i_k}, x_{i_k} - h)}{h} \in \partial |\boldsymbol{m}_{h,k}^*(x_{-i_k}, x_{i_k} - h)|. \end{cases} \tag{4.22}$$

- For $x_{i_k} = L$, at the point $(x_{-i_k}, x_{i_k} - h)$ we can get that

$$\nabla_h(\varphi_h^*(x_{-i_k}, x_{i_k} - h)) \in \partial|\boldsymbol{m}_h^*(x_{-i_k}, x_{i_k} - h)|_q,$$

i.e.,

$$\frac{\varphi_h^*(x_{-i_k}, x_{i_k}) - \varphi_h^*(x_{-i_k}, x_{i_k} - h)}{h} \in \partial|\boldsymbol{m}_{h,k}^*(x_{-i_k}, x_{i_k} - h)|. \tag{4.23}$$

- For $x_{i_k} = h$, at the point (x_{-i_k}, x_{i_k}) we can get that

$$\nabla_h(\varphi_h^*(x_{-i_k}, x_{i_k})) \in \partial|\boldsymbol{m}_h^*(x_{-i_k}, x_{i_k})|_q,$$

i.e.,

$$\frac{\varphi_h^*(x_{-i_k}, x_{i_k} + h) - \varphi_h^*(x_{-i_k}, x_{i_k})}{h} \in \partial|\boldsymbol{m}_{h,k}^*(x_{-i_k}, x_{i_k})|. \tag{4.24}$$

For all the cases we get that (if it exists)

$$\begin{cases} -1 \le \dfrac{\varphi_h^*(x_{-i_k}, x_{i_k}) - \varphi_h^*(x_{-i_k}, x_{i_k} - h)}{h} \le 1, & k = 1, 2, \ldots, d, \\ -1 \le \dfrac{\varphi_h^*(x_{-i_k}, x_{i_k} + h) - \varphi_h^*(x_{-i_k}, x_{i_k})}{h} \le 1, & k = 1, 2, \ldots, d. \end{cases} \tag{4.25}$$

Similarly, we can continue to use the first condition at all the points $(x_{-i_k}, x_{i_k} - h)$ and $(x_{-i_k}, x_{i_k} + h)$ for $k = 1, 2, \ldots, d$ (if it exists) and get the restriction of φ_h^* at any point on the meshgrid $[h, L]_h^d$:

$$\alpha - dL = \alpha - dNh \le \varphi_h^*(x_{\tilde{i}_1}, x_{\tilde{i}_2}, \ldots, x_{\tilde{i}_d}) \le \alpha \quad \forall\, (x_{\tilde{i}_1}, x_{\tilde{i}_2}, \ldots, x_{\tilde{i}_d}) \in [h, L]_h^d. \tag{4.26}$$

On the other hand, as ρ_h^0 and ρ_h^1 are equal mass, from the third condition we get

$$\sum_{x \in \Omega_h} \eta_h^*(x) = \sum_{x \in \Omega_h} (A_h \boldsymbol{m}_h^*) - \sum_{x \in \Omega_h} \rho_h = 0.$$

For $\eta_h^*(x_{i_1}, x_{i_2}, \ldots, x_{i_d}) > 0$, there must exist another point $(x_{j_1}, x_{j_2}, \ldots, x_{j_d})$ in the space Ω_h such that $\eta_h^*(x_{j_1}, x_{j_2}, \ldots, x_{j_d}) < 0$. Then by the second condition, we get $\varphi_h^*(x_{j_1}, x_{j_2}, \ldots, x_{j_d}) = -\alpha$. Then for $\alpha > dL/2$, we have

$$\varphi_h^*(x_{j_1}, x_{j_2}, \ldots, x_{j_d}) = -\alpha < \alpha - dL,$$

which is contradiction to the condition (4.26). Therefore the optimal $\eta_h^* \equiv 0$ on the space Ω_h and $\boldsymbol{m}_h^*$ is a solution to the OT problem. ∎

Remark 4.3. Note that the condition given in the theorem is only sufficient; in practice the exact threshold of $\eta_h^* \equiv 0$ tends to be smaller than $dL/2$. Moreover, though we only prove for $\Omega = [0, L]^d$, the theorem holds for the general triangular case $\Omega_h = [a_1, b_1] \times [a_2, b_2] \times \cdots [a_d, b_d]$ and the corresponding condition of α should be changed as $\alpha > d\max_i\{b_i - a_i\}/2$.

The above result indicates that the Beckmann formulation is advantageous as the usual discretization is asymptotic preserving, which means that the convergence of UOT to OT can be preserved as $\alpha \to \infty$. This property is beneficial in the sense that when a numerical method is applied to the discretization problems, the numerical method can reduce to the one for OT as $\alpha \to \infty$ automatically.

In fact, as for the iterations (4.8) and (4.12) in the primal dual algorithm, with some restriction to the parameter α, we can get the connection between them, which is stated as the following theorem.

THEOREM 4.4. *Suppose in* (4.8) *and* (4.12), μ *and* τ *are set the same and satisfy the convergence condition* $\mu\tau\|[\mathrm{div}^h, -I]\|^2 < 1$. *Then there exists a constant* $M > 0$ *such that for all* $\alpha > M$, *we have that* $\eta_h^k \equiv 0$ *on* Ω_h *for* k *large enough. Correspondingly, the subsequent iterates of UOT* (4.12) *reduce to those of OT* (4.8).

Proof. By the equivalence of norms in finite dimensional space and the convergence analysis in [23] it leads to

$$\lim_{k\to+\infty} \sum_{x\in\Omega_h} \left(|\eta_h^k(x) - \eta_h^*(x)| + |\varphi_h^k(x) - \varphi_h^*(x)|\right) = 0,$$

where $(\boldsymbol{m}_h^*, \eta_h^*, \varphi_h^*)$ is a group of optimal solutions to the UOT problem (4.4). Therefore combining Theorem 4.2, for any $\alpha > \frac{dL}{2}$ we have $\eta_h^* \equiv 0$ and $(\boldsymbol{m}_h^*, \varphi_h^*)$ is a pair of optimal solutions to the OT problem (4.5). Correspondingly,

$$\lim_{k\to+\infty} \sum_{x\in\Omega_h} \left(|\eta_h^k(x)| + |\varphi_h^k(x) - \varphi_h^*(x)|\right) = 0, \tag{4.27}$$

which indicates that

$$\lim_{k\to+\infty} \|\eta_h^k\|_{h,\infty} + \|\varphi_h^k - \varphi_h^*\|_{h,\infty} = 0, \tag{4.28}$$

where the norm $\|u\|_{h,\infty}$ is defined as

$$\|u\|_{h,\infty} = \max_{x\in\Omega_h} |u(x)|, \quad \text{for } \forall u \in \mathbb{R}^{(N+1)^d}.$$

Equivalently, one has that for any $\epsilon > 0$, there exists an integer $N > 0$ such that for any $k > N$,

$$\|\eta_h^{k-1}\|_{h,\infty} + \|\varphi_h^{k-1} - \varphi_h^*\|_{h,\infty} < \epsilon. \tag{4.29}$$

Note that for the step η_h^k in (4.12), at each point $x \in \Omega_h$ we have

$$\begin{aligned}
\eta_h^k(x) &= \mathrm{Prox}_{\alpha\mu f}(\eta_h^{k-1}(x) + \mu\varphi_h^{k-1}(x)) \\
&= \arg\min_{\eta_h} |\eta_h| + \frac{1}{2\alpha\mu}\|\eta_h - (\eta_h^{k-1}(x) + \mu\varphi_h^{k-1}(x))\|_{h,2}^2 \\
&= \mathrm{sign}(\eta_h^{k-1}(x) + \mu\varphi_h^{k-1}(x)) \cdot \max\left\{|\eta_h^{k-1}(x) + \mu\varphi_h^{k-1}(x)| - \alpha\mu, 0\right\}.
\end{aligned} \tag{4.30}$$

Define

$$M := \max\left\{\frac{dL}{2}, \|\varphi_h^*\|_{h,\infty}\right\}$$

and for any $\alpha > M$, let $\epsilon \le \min\{1, \mu\}(\alpha - M)$ in (4.29). Then for any $k > N$ we obtain that

$$\begin{aligned}
|\eta_h^{k-1}(x) + \mu\varphi_h^{k-1}(x)| &\le \|\eta_h^{k-1}\|_{h,\infty} + \mu\|\varphi_h^{k-1}\|_{h,\infty} \\
&\le \|\eta_h^{k-1}\|_{h,\infty} + \mu\|\varphi_h^{k-1} - \varphi_h^*\|_{h,\infty} + \mu\|\varphi_h^*\|_{h,\infty} \\
&\le \max\{1, \mu\}\left(\|\eta_h^{k-1}\|_{h,\infty} + \|\varphi_h^{k-1} - \varphi_h^*\|_{h,\infty}\right) + \mu\|\varphi_h^*\|_{h,\infty} \\
&\le \max\{1, \mu\}\,\epsilon + \mu\|\varphi_h^*\|_{h,\infty} \le \max\{1, \mu\}\,\epsilon + \mu M \le \alpha\mu,
\end{aligned} \tag{4.31}$$

which indicates that $\eta_h^k(x) \equiv 0$ for any $x \in \Omega_h$. And correspondingly, the iterates of UOT (4.12) reduce to those of OT (4.8) for k large enough. ∎

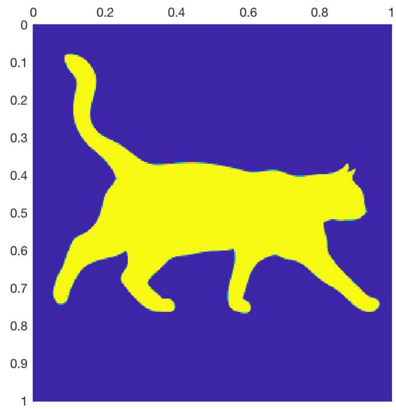

FIG. 1. ρ_h^0.

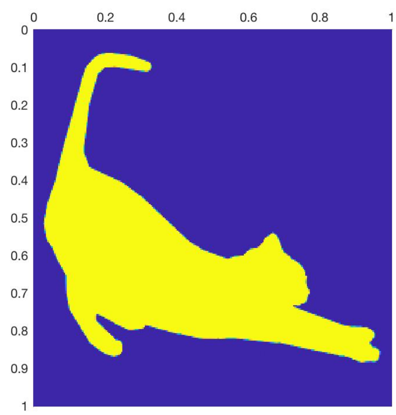

FIG. 2. ρ_h^1.

5. Numerical experiments. In this section we use two examples, shape deformation and color transfer problems, to illustrate the application of UOT and OT problems using the primal-dual hybrid algorithm discussed above.

I. Shape deformation The first example is used to illustrate the convergence of the UOT problem to the OT problem. Particularly, we take $d = 2$, $\Omega_h = [0,1]^2$ and the discrete distributions ρ_h^0, ρ_h^1 are the silhouettes of cat images of the same mass [27], as shown in Figures 1 and 2. The size of both images is 256×256 and the algorithm is terminated when the primal-dual gap $R_h^k < 10^{-6}$ or the iteration number reaches 300000.

- **Convergence with $\boldsymbol{\alpha}$.** We tune the value of α from 0.01 to 1 with an interval of 0.01 and get an optimal solution $(\boldsymbol{m}_\alpha, \eta_\alpha)$ for each α in the UOT problem. Also we can obtain an optimal solution $\boldsymbol{m}_{\mathrm{ot}}$ of the OT problem. As Theorem 4.2 states, the solution of the discrete UOT problem (4.4) converges to the solution of the discrete OT problem (4.5) as α gets larger, i.e., $\boldsymbol{m}_\alpha \to \boldsymbol{m}_{\mathrm{ot}}$ and $\eta_\alpha \to 0$. Figure 3 shows the difference in $m_{\mathrm{dif}} = |\boldsymbol{m}_\alpha - \boldsymbol{m}_{\mathrm{ot}}|_{h,2}$ and

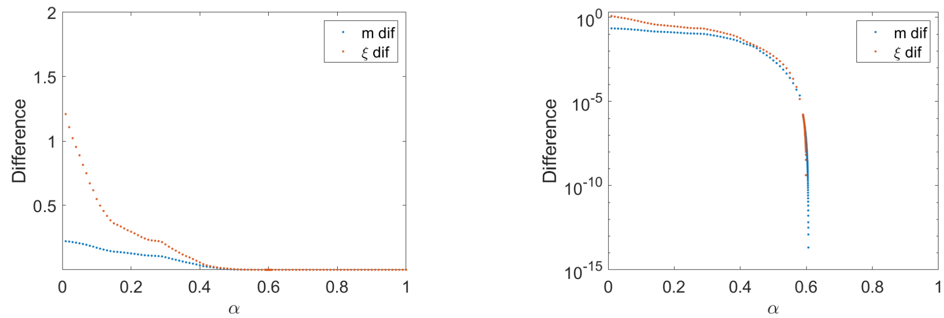

FIG. 3. *The figure shows the difference in normal y-axis (left) and* log *y-axis (right). Here we choose $N = 256$. It can be seen that both m_{dif} and η_{dif} go to 0 as α gets sufficiently large. In particular when $\alpha = 0.6$, $m_{\text{dif}} = 3.2011 \times 10^{-8}$ and $\eta_{\text{dif}} = 0$, and when α is larger than* 0.6061*, $\eta_{dif} = 0$ and $m_{dif} = 0$, which is consistent to the results proved in Theorems* 4.2 *and* 4.4.

TABLE 1
The value of $|\eta^*_{\text{dif}}|_{h,2}$ with different α and different N (or h).

	$N=128$	$N=256$	$N=512$	$N=1024$
$\alpha=0.1$	0.3621	0.3583	0.3570	0.3563
$\alpha=0.4$	0.0296	0.0288	0.0285	0.0283
$\alpha=0.6$	0.1301×10^{-4}	0.1048×10^{-4}	0.1369×10^{-4}	0.1515×10^{-4}
$\alpha=0.63$	0.1248×10^{-7}	0	0	0
$\alpha=1.0$	0	0	0	0

$\eta_{\text{dif}} = |\eta_\alpha|_{h,2}$ with different α. For $\alpha = 0.6$, $m_{\text{dif}} = 3.2011 \times 10^{-8}$ and $\eta_{\text{dif}} = 0$, and when α is larger than some constant between 0.6 and 0.7, $\eta_{\text{dif}} = 0$ and $m_{\text{dif}} = 0$, which is consistent with the results proved in Theorems 4.2 and 4.4. In particular, we fine tune α from 0.59 to 0.61 with interval equal to 10^{-4}, and when α overrides 0.6061, both $\boldsymbol{m}_{\text{dif}}$ and η_{dif} come to 0, which indicates that $\eta_\alpha = 0$ and the UOT degenerates to OT with $\boldsymbol{m}_\alpha = \boldsymbol{m}_{\text{ot}}$.

- **Independence to the size of meshgrid $\boldsymbol{h}$.** Notice that in Theorem 4.2, the convergence of UOT to OT is unrelated to the choice of grid size h for $\alpha > dL/2$; the optimal solution of UOT is always convergent to OT. We note that here $dL/2 = 1$. In the experiments we choose four different sizes of images with 128×128, 256×256, 512×512, 1024×1024, and the length is unified to 1, i.e., $Nh = L = 1$. For different $\alpha = 0.1, 0.4, 0.6, 0.63, 1.0$, the results are listed in Table 1. As we can see, for any fixed α, $|\eta^*_{\text{dif}}|_{h,2}$ remains almost the same and when α is larger than 0.63, which is smaller than $dL/2 = 1$, all $|\eta^*_{\text{dif}}|_{h,2}$ equal to 0 for different N.

II. Color transfer Besides the transformation between shape images, we also provide an application of the UOT model for color transfer between three-channel images.

The given target and source images are first transferred to the CIE-lab space (l, a, b), where the l-space represents the luminance of the image, and a and b are chromaticity coordinates. We fix the l-space as it is related to the lightness and normalize a- and b-components into $[0, 1]$. Then both a- and b-components are divided into 32 intervals and the color histograms are obtained on these intervals respectively.

For both a- and b-components, we solve both UOT and OT problems to get the optimal flux m_h^*. Then we can compute the velocity field with a value in a-space and b-space through the connection in Lemma 3.1 as follows:

$$v(t;x) = \frac{m_h^*(x)}{t\rho_1(x) + (1-t)\rho_0(x)} := \frac{m_h^*(x)}{\mu_t(x)}, \tag{5.1}$$

where t is virtual time and $x \in [0,1]$ is the partition position. In practice, the supports of ρ_0 and ρ_1 are not always the same. To ensure the existence of $v(t;x)$ and eliminate the singularity of $v(t;x)$ caused by small μ_t, we add a small perturbation ϵ on μ_t, i.e.,

$$v_\epsilon(t;x) = \frac{m_h^*(x)}{t\rho_1(x) + (1-t)\rho_0(x) + \epsilon} = \frac{m_h^*(x)}{\mu_t(x) + \epsilon}. \tag{5.2}$$

For each x_0 sampled from ρ_0, the corresponding trasported x_1 in ρ_1 can be obtained by solving the following ODE:

$$\begin{cases} \dot{x}(t) = v_\epsilon(t; x(t)), \quad t \in (0,1), \\ x(0) = x_0, \end{cases} \tag{5.3}$$

and $x(1)$ is the target distribution.

By the one-step forward Euler method, we obtain the formula

$$x(t_k) = x(t_{k-1}) + \frac{m_h^*(x(t_{k-1}))}{\mu_{t_{k-1}}(x(t_{k-1})) + \epsilon} \text{ for } k = 1, 2, \ldots, N, \tag{5.4}$$

where $t_k = k\Delta t$ and $N\Delta t = 1$.

Following this procedure, we transport the a and b components of every pixel in the target image to the corresponding one in the new image. Figure 4 shows the results of color transfer for three pair of images of size $512 * 512$.

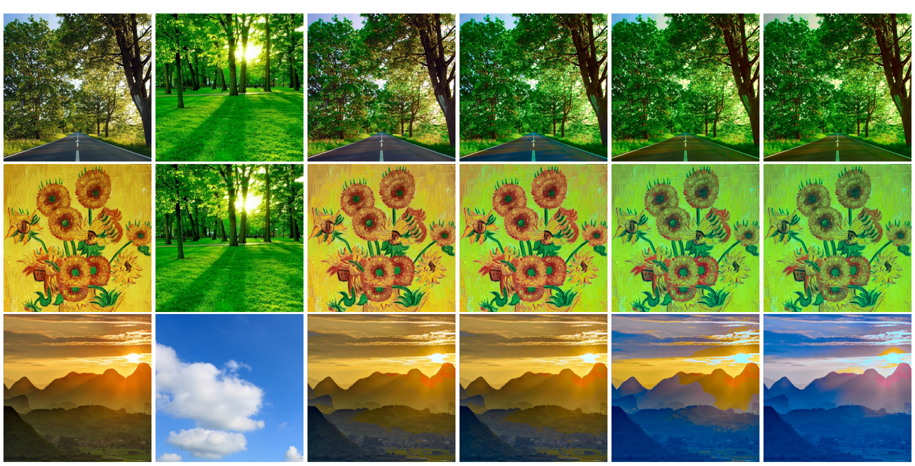

FIG. 4. *In each row, the first two columns are the target and source images, respectively. The last four are the results by color transferring with different* $\alpha = 0.05, 0.1, 0.2, 0.5$, *respectively. As the value of* α *increases, the color distribution of the target image is partially transferred.*

6. Conclusions. In this paper, we established the convergence from the Beckmann formulation of UOT to that of OT in both continuous and discrete settings. We proposed to apply a primal-dual hybrid algorithm for solving the UOT problem, and we provided a lower bound for the regularization parameter α of UOT for its solution reducing to the one of OT problem. Finally, we provided some applications of the UOT model and illustrated the convergence numerically.

Acknowledgment. We thank the Student Innovation Center at Shanghai Jiao Tong University for providing us with the computing services.

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