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## Analytical solution for laterally loaded long piles based on Fourier–Laplace integral



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## ABSTRACT

Piles are frequently used to support lateral loads. Elastic solutions based on the Winkler foundation model are widely used to design laterally loaded piles at working load. This paper reports a simplified analytical solution for laterally loaded long piles in a soil with stiffness linearly increasing with depth. Based on a Fourier–Laplace integral, a power series solution for small depth and a Wentzel–Kramers–Brillouin (WKB) asymptotic solution for large depth are derived. By using this analytical solution, the deflection and bending moment profiles of a laterally loaded pile can be obtained through simple calculation. The proposed power series solution agrees well with other existing methods on predicting the deflection and bending moment of laterally loaded piles. The WKB asymptotic solution developed in this study provides a better approach for engineers to analyze the responses and design of laterally loaded long piles.

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#### 1. Introduction

Piles are widely used to support laterally loaded structures, such as bridges, buildings, tanks, and wind turbines. Some analytical methods have been developed for analyzing such piles, including the elastic subgrade reaction approach by Matlock and Reese [1], Davisson and Gill [2], Shen and Teh [3] or the elastic continuum approach by Poulos and Davis [4], Zhang and Small [5], Shen and Teh [6]. Of these methods, the subgrade reaction approach based on Winkler foundation model is most widely used for its clear concept and simple mathematical treatment. Terzaghi [7] proposed that the modulus of subgrade reaction should be a constant with the depth of clay, whereas this modulus should increase linearly with depth from a value of zero at the ground surface for sand. By using the beam on elastic foundation model, Chang [8] derived an analytic solution for a laterally loaded pile in clay by assuming the coefficient of the subgrade reaction is a constant and the pile is sufficiently long. Several methods have been developed for the analysis of laterally loaded piles in sand, including the finite difference method by Gleser [9], Matlock and Reese [1], Reese and Matlock [10] and power series solution by Hetenyi [11], Rowe [12,13]. Finite difference solutions can be very close to the actual solution if sufficient segments are used.

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To analyze laterally loaded long piles with higher accuracy and simplicity, this paper proposes an analytical solution based on the Fourier–Laplace integral method, which recovers power series solutions for small depth and Wentzel–Kramers–Brillouin (WKB) asymptotic solutions for large depth. The power series solution is used to analyze the small depth part of the pile with high accuracy by use of only a few terms; while the WKB approximation is employed to analyze the large depth part of the pile with much less work but acceptable accuracy compared with the power series. For infinitely long piles, the proposed power series solution is an exact solution as the boundary conditions at the tip of the pile are satisfied exactly. Furthermore, the simplified analytical solutions to deflection and bending moment of laterally loaded long piles are obtained, which can be conveniently used by engineers to facilitate analysis and design of piles. In addition, the present method can also be extended to analyze laterally loaded long piles in soil with the modulus of subgrade reaction in some other functions of the depth. The method proposed in this article is also available to analyze a short pile which is addressed in Appendix B.

## 2. Definition of the problem

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According to Winkler foundation model, the flexural equation of a pile on the elastic subgrade can be written as

$$E_p I_p \frac{d^4 y}{dz^4} + K \cdot y = 0, \tag{1}$$

where  $E_p$  is the Young's modulus of the pile,  $I_p$  is the inertia moment of the pile, y is the pile deflection, z is the pile depth, K is the modulus of subgrade reaction.

The modulus of subgrade reaction increases linearly with the depth from a value of zero at the ground surface for sand and can be written as after Poulos and Davis [4].

$$K = n_h z, \tag{2}$$

where  $n_h$  is the constant of horizontal subgrade reaction.

Substituting Eq. (2) into Eq. (1) yields

$$E_p I_p \frac{d^4 y}{dz^4} + n_h z y = 0.$$
<sup>(3)</sup>

## 3. Solution procedure

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The relative stiffness factor, *T*, proposed by Reese and Matlock [10] is given by

$$T = \left(\frac{E_p I_p}{n_h}\right)^{\frac{1}{5}}.$$
(4)

By defining a dimensionless variable x = z/T, Eq. (3) can be reduced to

$$\frac{E_p I_p}{n_b T^5} \frac{d^4 y}{dx^4} + xy = 0.$$
(5)

Plugging Eq. (4) into Eq. (5) gives

$$\frac{d^4y}{dx^4} + xy = 0. ag{6}$$

Eq. (6) is the fundamental equation explored in this paper.

## 3.1. Fourier-Laplace integral representation for the solutions

In Shen and Teh [6], the solution procedure for the Airy Equation y''(x) = xy is discussed using a Fourier–Laplace Integral representation. Here, we apply this representation to solve our equation.

Consider the Fourier–Laplace representation of y(x) (after White [14]):

$$y(x) = \int_{C} e^{xt} f(t) dt, \tag{7}$$

where *C* is the contour in the complex plane with endpoints *a* and *b*.

Eq. (6) can be reduced to

$$\int_{C} t^{4} e^{xt} f(t) dt - \int_{C} e^{xt} f'(t) dt + e^{xt} f(t) \Big|_{a}^{b} = 0.$$
(8)

Choose a contour such that the last term in Eq. (8) vanishes, leading to

$$t^{4}f(t) - f'(t) = 0 \tag{9}$$

which gives

$$f(t) = e^{t^5/5}.$$
 (10)

To ensure that the last term vanishes in Eq. (8),  $t^5/5$  should go to negative infinity when approaching the endpoints. Therefore, *a* and *b* can be chosen as  $\infty e^{(i\pi+2k\pi)/5}$ . Denote these points as *A*:  $-\infty$ , *B*:  $\infty e^{i3\pi/5}$ , *C*:  $\infty e^{i\pi/5}$ , *D*:  $\infty e^{-i\pi/5}$  and *E*:  $\infty e^{-i3\pi/5}$ . We thus have four independent contours and four independent solutions correspondingly:

$$y_k(x) = \int_{C_k} e^{t^5/5 + xt} dt \quad (k = 1, 2, 3, 4).$$
(11)

Franklin and Scott [15] also obtained the contour integral solutions in their work. However, in their work, the convenient expressions to compute the four basis functions directly are not given and a numerical method is still adopted to calculate them. They also derive the leading order asymptotic solutions but these solutions fail to be effective near the soil surface and cannot be used to derive any useful formulas. Hence, their solutions cannot cover the whole pile.

Let

$$\phi(\mathbf{x},t) = \frac{t^5}{5} + \mathbf{x}t. \tag{12}$$

The integration contour of Eq. (11) is chosen as the steepest decent curves (after Bender and Orszag [16]) emerging from the saddle points of  $\varphi(x, t)$ . These saddle points are given by  $t_0 = \sqrt[4]{x}\omega$  where  $\omega = e^{\pm i\pi/4}$ ,  $e^{\pm i3\pi/4}$  can be obtained by  $\frac{\partial \phi}{\partial t} = 0$ . Then the four contours in Eq. (11) are given by the contour emerging from  $e^{3i\pi/4}$  with endpoints *A*, *B* (*C*<sub>1</sub>), the one emerging from  $e^{-3i\pi/4}$  with endpoints *A*, *E* (*C*<sub>2</sub>), the one emerging from  $e^{i\pi/4}$  with endpoints *B*, *C* (*C*<sub>3</sub>), and the one emerging from  $e^{-i\pi/4}$  with endpoints *E*, *D* (*C*<sub>4</sub>).

If *x* is real, it is easy to see that  $\bar{y}_1 = y_2$  and  $\bar{y}_3 = y_4$ . Let

$$y_1 = g_1 + ig_2, \quad y_3 = g_3 + ig_4,$$
 (13)

where  $g_i$  are real functions and are also four independent solutions. In order to get the solutions to laterally loaded piles including long piles and short piles, we need to calculate  $y_1$  and  $y_3$ .

Eq. (11) is the Fourier–Laplace Integral representation of the solutions. Since this form is not convenient for engineers to use, we change it into a WKB asymptotic solution when x is large and the power series when x is small. The results for the solutions are summarized as the following three theorems:

**Theorem 1.** As x approaches infinity, both the real and imaginary parts of  $y_3$  are highly oscillatory with rapidly increasing amplitudes and frequency, while both the real and imaginary parts of  $y_1$  decrease to zero. In particular, if we choose the direction of  $C_1$  to be from A to B and the direction of  $C_3$  to be from C to B, and then to leading order, we have:

$$y_k(x) \sim \frac{\sqrt{\pi\omega}}{\sqrt{2}} x^{-3/8} e^{4x^{5/4}\omega/5}.$$
 (14)

For  $y_1$ ,  $\omega = e^{i3\pi/4}$  and for  $y_3$ ,  $\omega = e^{i\pi/4}$ .

The proof of this theorem is included in Appendix A. The WKB asymptotic solution to  $y^{(n)} = Q(x)y$  is given by

$$y(x) \sim |Q(x)|^{-(n-1)/2n} \exp\left(\int^x Q(s)^{1/n} ds\right).$$
 (15)

Plugging in Q(x) = -x and n = 4, one can derive the same expression except that the coefficients are unknown. The asymptotic solution is thus of WKB type.

In this article, we are interested in infinitely long piles. For infinitely long piles, we have boundary conditions at infinity, which excludes  $g_3$  and  $g_4$ . Therefore, there is no need to calculate  $y_3$ . If one considers short piles, the solutions should be linear combinations of  $g_i$  (i = 1, 2, 3, 4) where  $g_3$  and  $g_4$  are real and imaginary parts of  $y_3$ . The methods to obtain the formulas would be similar. Besides,  $y_3$  is already calculated in Appendix B. Now, we give the expressions for  $g_1$  and  $g_2$  which we use in this paper.

**Theorem 2.** We have the following asymptotic results for  $g_1$  and  $g_2$  as x approaches positive infinity: *The first two terms in the asymptotic expression for*  $g_1$  *and*  $g_2$  *are:* 

$$g_{1}(x) \sim \sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ x^{-3/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) + \frac{9}{32}x^{-13/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) \right],$$

$$g_{2}(x) \sim \sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ x^{-3/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) - \frac{9}{32}x^{-13/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) \right].$$
(16)

The first two terms in the asymptotic expression for  $g_1'$  and  $g_2'$  are:

$$g_{1}'(x) \sim -\sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ x^{-1/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) - \frac{3}{32}x^{-11/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) \right],$$

$$g_{2}'(x) \sim -\sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ x^{-1/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) - \frac{3}{32}x^{-11/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) \right].$$
(17)

The first three terms in the asymptotic expression for  $g_1''$  and  $g_2''$  are:

$$g_{1}''(x) \sim \sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ x^{1/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) + \frac{7}{32}x^{-9/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) - \frac{231}{2048}x^{-19/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) \right],$$

$$g_{2}''(x) \sim -\sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ x^{1/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) + \frac{7}{32}x^{-9/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) - \frac{231}{2048}x^{-19/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) \right],$$

$$(18)$$

$$g_{1}^{\prime\prime\prime}(x) \sim \sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ -x^{3/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{3\pi}{8}\right) - \frac{3}{32}x^{-7/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{\pi}{8}\right) + \frac{273}{2048}\sqrt{\frac{\pi}{2}}x^{-17/8} \cos\left(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{3\pi}{8}\right) \right] \\ g_{2}^{\prime\prime\prime}(x) \sim -\sqrt{\frac{\pi}{2}} e^{-2\sqrt{2}x^{5/4}/5} \left[ -x^{3/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{3\pi}{8}\right) - \frac{3}{32}x^{-7/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{\pi}{8}\right) + \frac{273}{2048}x^{-17/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} - \frac{3\pi}{8}\right) \right].$$

$$(19)$$

One can refer to the Appendix A for the derivation.

The above expression works well for large x but fails around x = 0. For  $y_1$ , we deform the contour to the left half x-axis and the ray with polar angle  $\frac{3\pi}{5}$ . This new contour is convenient for us to get the expressions that works well for small x. Integration over this new contour gives us the following results.

**Theorem 3.**  $g_1$ ,  $g_2$ ,  $g'_1$ ,  $g'_2$ ,  $g''_1$  and  $g''_2$  have the following power series expressions which work quite well for small x:

$$g_{1}(x) = (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} \Gamma(n+1/5)}{(5n)!} x^{5n} - (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} \Gamma(n+2/5)}{(5n+1)!} x^{5n+1} + (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} \Gamma(n+3/5)}{(5n+2)!} x^{5n+2} - (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+3)!} x^{5n+3},$$
(20)

$$g_{2}(x) = \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} \Gamma(n+1/5)}{(5n)!} x^{5n} - \sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} \Gamma(n+2/5)}{(5n+1)!} x^{5n+1} \\ -\sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} \Gamma(n+3/5)}{(5n+2)!} x^{5n+2} + \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+3)!} x^{5n+3},$$
(21)

$$g_{1}'(x) = -(1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} \Gamma(n+2/5)}{(5n)!} x^{5n} - (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} \Gamma(n+3/5)}{(5n+1)!} x^{5n+1} + (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+2)!} x^{5n+2} - (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} (5n+1) \Gamma(n+1/5)}{(5n+4)!} x^{5n+4},$$
(22)

$$g_{2}'(x) = -\sin(\pi/5)\sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} \Gamma(n+2/5)}{(5n)!} x^{5n} - \sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} \Gamma(n+3/5)}{(5n+1)!} x^{5n+1} + \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+2)!} x^{5n+2} - \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} (5n+1) \Gamma(n+1/5)}{(5n+4)!} x^{5n+4},$$
(23)

$$g_{1}''(x) = (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} \Gamma(n+3/5)}{5n!} x^{5n} - (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+1)!} x^{5n+1} - (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} (5n+1) \Gamma(n+1/5)}{(5n+3)!} x^{5n+3} + (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} (5n+2) \Gamma(n+2/5)}{(5n+4)!} x^{5n+4},$$
(24)

$$g_{2}''(x) = -\sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} \Gamma(n+3/5)}{5n!} x^{5n} + \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+1)!} x^{5n+1} \\ -\sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} (5n+1) \Gamma(n+1/5)}{(5n+3)!} x^{5n+3} + \sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} (5n+2) \Gamma(n+2/5)}{(5n+4)!} x^{5n+4}, \quad (25)$$

$$g_{1}^{\prime\prime\prime}(x) = -(1 - \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{5n!} x^{5n} - (1 - \cos(2\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} (5n+1) \Gamma(n+1/5)}{(5n+2)!} x^{5n+2} + (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} (5n+2) \Gamma(n+2/5)}{(5n+3)!} x^{5n+3} - (1 + \cos(\pi/5)) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} (5n+3) \Gamma(n+3/5)}{(5n+4)!} x^{5n+4},$$
(26)

$$g_{2}^{\prime\prime\prime}(x) = \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{5n!} x^{5n} - \sin(2\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} (5n+1) \Gamma(n+1/5)}{(5n+2)!} x^{5n+2} + \sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} (5n+3) \Gamma(n+3/5)}{(5n+3)!} x^{5n+4} + \sin(\pi/5) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-2/5} (5n+3) \Gamma(n+3/5)}{(5n+4)!} x^{5n+4}.$$
 (27)

The derivation is put in the Appendix A.

## 3.2. Laterally loaded infinitely long piles

## 3.2.1. Expression for deflection

The general solution to Eq. (6) is  $y(x) = C_1g_1(x) + C_2g_2(x) + C_3g_3(x) + C_4g_4$ . However, as x approaches infinity,  $g_1$ ,  $g_2$  and their derivatives decay to 0 very fast (when x > 4, they are almost 0) while  $g_3$  and  $g_4$  keep oscillating with increasing amplitudes. We usually impose the following conditions:

$$y(+\infty) = 0, \quad y'(+\infty) = 0, \quad y''(+\infty) = 0, \quad y'''(+\infty) = 0$$

which require $C_3 = 0$  and  $C_4 = 0$ . Thus, for infinitely long piles, we have:

$$y(x) = C_1 g_1(x) + C_2 g_2(x).$$
(28)

Let us consider two boundary conditions of the pile head.

## ■ Free-head pile:

The boundary conditions are given by

$$y'''(0) = \frac{HT^3}{E_p I_p}, \quad y''(0) = \frac{MT^2}{E_p I_p},$$
(29)

where *H* is the horizontal load on the pile head and *M* is the bending moment on the pile head.

Therefore, two integration coefficients  $C_1$  and  $C_2$  can be easily obtained as



Fig. 1. The calculation schematic diagram.

$$\begin{bmatrix} C_1\\ C_2 \end{bmatrix} = \frac{T^2 5^{1/5}}{(2\sin(2\pi/5))^{5/5} \Gamma(4/5)M + \sin(\pi/5)\Gamma(3/5)HT} (1 - \cos(2\pi/5))5^{1/5} \Gamma(4/5)M + (1 + \cos(\pi/5))\Gamma(3/5)HT}{(2\sin(2\pi/5) - \sin(\pi/5))\Gamma(3/5)\Gamma(4/5)E_p I_p}.$$
(30)

## ■ Fixed-head pile:

The boundary conditions are given by

$$y'''(0) = \frac{HT^3}{E_p I_p}, \quad y'(0) = 0.$$
 (31)

Similarly, we can get:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{HT^3 5^{1/5}}{2\Gamma(4/5)E_p I_p} \begin{bmatrix} \frac{1}{1 + \cos(\pi/5)} \\ \frac{1}{\sin(\pi/5)} \end{bmatrix}.$$
(32)

Combining Eqs. (16), (20), (21), (28), (30) (or Eq. (32)) and x = z/T, we can get y(z) which is the pile deflection distribution with depth. The calculation schematic diagram is shown in Fig. 1 and the dimensionless constant  $\alpha$  will be discussed later. When  $x < \alpha$  ( $z < \alpha T$ ),  $g_1$  and  $g_2$  are determined by Eqs. (20) and (21); While  $x > \alpha$  ( $z > \alpha T$ ),  $g_1$  and  $g_2$  are given by Eq. (16). Furthermore, we can get an accurate expression for the deflection of the pile head for infinitely long piles:

$$y(0) = C_1(1 - \cos(2\pi/5))5^{-4/5}\Gamma(1/5) + C_2\sin(2\pi/5)5^{-4/5}\Gamma(1/5).$$
(33)

#### 3.2.2. Expressions for rotation, bending moment and shear force It is quite clear that

$$\begin{cases} y' = C_1 g'_1 + C_2 g'_2 \\ y'' = C_1 g''_1 + C_2 g''_2 \\ y''' = C_1 g''_1 + C_2 g''_2. \end{cases}$$
(34)

Thus we can get

$$\begin{cases} \theta(z) = \frac{dy}{dz} = \frac{1}{T} (C_1 g'_1(z/T) + C_2 g'_2(z/T)) \\ M(z) = E_p I_p \frac{d^2 y}{dz^2} = \frac{E_p I_p}{T^2} (C_1 g''_1(z/T) + C_2 g''_2(z/T)) \\ Q(z) = E_p I_p \frac{d^2 y}{dz^2} = \frac{E_p I_p}{T^3} (C_1 g''_1(z/T) + C_2 g''_2(z/T)), \end{cases}$$
(35)

where  $\theta$  is pile rotation, *M* is pile bending moment and *Q* is pile shear.

In the analysis of laterally loaded piles, pile deflection and bending moment are of greatest interest. Thus, only pile deflection and bending moment are considered in the sections below.

#### 4. Discussion of $\alpha$

The value of  $\alpha$  can be determined by evaluating the errors of y and M using a truncated power series and WKB asymptotic solutions at the depth of  $z = \alpha T$ . Since y and M are linear combinations of  $g_1$ ,  $g_2$  and  $g_1''$ ,  $g_2''$  respectively, we can just assess the errors of the basis functions,  $g_1$ ,  $g_2$ ,  $g_1''$  and  $g_2''$ . The following procedure is followed to obtain the value of  $\alpha$ :



**Fig. 2.** Variation of  $\Delta g1$ ,  $\Delta g2$ ,  $\Delta g''1$  and  $\Delta g''2$  with  $\alpha$ .

- (1) Obtain the accurate results of  $g_1$ ,  $g_2$ ,  $g''_1$  and  $g''_2$  by calculating sufficient terms (n = 50) with the proposed power series solution;
- (2) Choose n = 1 in Eqs. (20), (21), (24), and (25), namely, keep 8 terms in the power series, and the calculate the errors  $e_1(g_1)$ ,  $e_1(g_2)$  with  $\alpha$  ranging from 1 to 3 by Eq. (36);
- (3) Obtain the errors  $e^{2}(g_{1})$ ,  $e^{2}(g_{2})$ ,  $e^{2}(g_{1}'')$ ,  $e^{2}(g_{2}'')$  for the WKB asymptotic solution with  $\alpha$  varying in the same range by Eq. (37);
- (4) Obtain the difference between four pairs of errors with Eq. (38), and the results are shown in Fig. 2.

 $\begin{cases} e1(g_1) = |g_{1m} - g_1| \\ e1(g_2) = |g_{2m} - g_2| \\ e1(g_1') = |g_{1m}' - g_1''| \\ e1(g_2'') = |g_{2m}'' - g_2''|, \end{cases}$ 



Fig. 3. Error condition when  $\alpha$  exceeds 2.0–2.2.

 $(\mathbf{36})$ 

$$\begin{cases} e^{2}(g_{1}) = |g_{1w} - g_{1}| \\ e^{2}(g_{2}) = |g_{2w} - g_{2}| \\ e^{2}(g_{1}'') = |g_{1w}'' - g_{1}''| \\ e^{2}(g_{2}'') = |g_{2w}'' - g_{2}''|, \end{cases}$$

$$\begin{cases} \Delta g^{1} = |e^{2}(g_{1}) - e^{1}(g_{1})| \\ \Delta g^{2} = |e^{2}(g_{2}) - e^{1}(g_{2})| \\ \Delta g''^{1} = |e^{2}(g_{1}'') - e^{1}(g_{1}'')| \\ \Delta g''^{2} = |e^{2}(g_{2}'') - e^{1}(g_{2}'')|. \end{cases}$$

$$(37)$$

According to the property of the solutions, the errors by power series solutions should increase with *x*; while the errors by WKB asymptotic solutions should decrease with *x*. Therefore the selecting of  $\alpha$  is to find the point at which the errors of two solutions are very close, taking into account the calculation accuracy of the both parts. In Fig. 2,  $\Delta g1$  and  $\Delta g2$  are close to zero with  $\alpha$  ranging from 2.0 to 2.3; while  $\Delta g''1$  and  $\Delta g''2$  are close to zero with  $\alpha$  ranging from 1.3 to 2.2. Clearly, the errors of these four basis functions by WKB asymptotic solutions are very close to the ones by power series solutions when  $\alpha$  ranges from 2.0 to 2.2. If  $\alpha$  is less than than 2.0, the error of WKB asymptotic solution is larger for *x* between  $\alpha$  and 2.0 (namely *z* is between  $\alpha T$  and 2.0*T*); while if  $\alpha$  is greater than 2.2, the error of the power series solution is larger for *x* between 2.2 and  $\alpha$ . The schematic diagram of error condition when  $\alpha$  exceeds 2.0~2.2 is shown in Fig. 3. Hence we take the value of  $\alpha$  from 2.0 to 2.2. Similarly, the suggested value of  $\alpha$  can be taken from 2.5 to 3.0 if we choose n = 2 in step (2) mentioned above while the calculation terms increase to 12.

## 5. Simplified analytical solutions

Based on the discussion of  $\alpha$ , the simplified solutions to laterally loaded long piles are derived by setting n = 1 and  $\alpha = 2$ . With the simplified solutions, engineers can calculate deflection and bending moment of laterally loaded long piles at any depth easily. The simplified solutions are given as: (Note:  $x = z/T = z/(\frac{E_p I_p}{n_h})^{1/5}$ )

## For free-head piles:

Deflection:

$$y_{x \leq 2}(x) = (2.4292 - 1.6194x + 0.1667x^3 - 0.0202x^5 + 0.0045x^6 - 0.0001x^8) \frac{H}{(E_p I_p)^{2/5} n_h^{3/5}} + (1.6194 - 1.7468x + 0.5x^2 - 0.0135x^5 + 0.0049x^6 - 0.0006x^7) \frac{M}{(E_p I_p)^{3/5} n_h^{2/5}},$$
(39)

$$y_{x>2}(x) = e^{-2\sqrt{2}x^{5/4}/5} \left( \frac{0.6642H}{(E_p I_p)^{2/5} n_h^{3/5}} + \frac{1.1593M}{(E_p I_p)^{3/5} n_h^{2/5}} \right) \left\{ x^{-3/8} \cos(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{3\pi}{8}) + \frac{9}{32} x^{-13/8} \sin\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{\pi}{8}\right) \right\} \\ + e^{-2\sqrt{2}x^{5/4}/5} \left( \frac{2.0443H}{(E_p I_p)^{2/5} n_h^{3/5}} + \frac{0.8423M}{(E_p I_p)^{3/5} n_h^{2/5}} \right) \left\{ x^{-3/8} \sin(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{3\pi}{8}) - \frac{9}{32} x^{-13/8} \cos(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{\pi}{8}) \right\}.$$
(40)

Bending moment:

$$\begin{split} M_{x\leqslant 2}(x) &= (x - 0.4049x^3 + 0.1349x^4 - 0.0056x^6 + 0.0004x^8 - 0.0001x^9)H(E_pI_p/n_h)^{1/5} + (1 - 0.2699x^3 + 0.1456x^4 - 0.025x^5 + 0.0002x^8 - 0.0001x^9)M, \end{split}$$

$$M_{x>2}(x) = e^{-2\sqrt{2}x^{5/4}/5} (0.6642H(E_pI_p/n_h)^{1/5} + 1.1593M) \left[ x^{1/8} \sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) + \frac{7}{32}x^{-9/8}\cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) - \frac{231}{2048}x^{-19/8}\cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) \right] + e^{-2\sqrt{2}x^{5/4}/5} (2.0443H(E_pI_p/n_h)^{1/5} + 0.8423M) \left[ -x^{1/8}\cos\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) + \frac{7}{32}x^{-9/8}\sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{\pi}{8}\right) - \frac{231}{2048}x^{-19/8}\sin\left(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}\right) \right].$$

$$(42)$$

# **For fixed-head piles:** Deflection:

$$y_{x\leq2}(x) = (0.9279 - 0.4635x^{2} + 0.1667x^{3} - 0.0077x^{5} + 0.0006x^{7} - 0.0001x^{8}) \frac{H}{(E_{p}I_{p})^{2/5}n_{h}^{3/5}},$$

$$y_{x>2}(x) = \left[0.8628\sin\left(\frac{2\sqrt{2}}{5}x^{5/4}\right) + 1.0102\cos\left(\frac{2\sqrt{2}}{5}x^{5/4}\right) + \left(0.0293\sin\left(\frac{2\sqrt{2}}{5}x^{5/4}\right)\right) - 0.3725\cos\left(\frac{2\sqrt{2}}{5}x^{5/4}\right)\right]x^{-5/4}\right] \frac{x^{-3/8}e^{-2\sqrt{2}x^{5/4}/5}H}{(E_{p}I_{p})^{2/5}n_{h}^{3/5}}.$$
(43)
$$(43)$$

Bending moment:

$$M_{x \leq 2}(x) = (-0.9271 + x - 0.1546x^3 + 0.0232x^5 - 0.0056x^6 + 0.0001x^8)H(E_p I_p / n_h)^{1/5},$$
(45)



Fig. 4. Comparison of deflection and bending moment profiles.

$$M_{x>2}(x) = \left\{ \left[ 1.0102 \sin\left(\frac{2\sqrt{2}}{5} x^{5/4}\right) - 0.8628 \cos\left(\frac{2\sqrt{2}}{5} x^{5/4}\right) \right] + \left[ 0.2897 \sin\left(\frac{2\sqrt{2}}{5} x^{5/4}\right) + 0.0228 \cos\left(\frac{2\sqrt{2}}{5} x^{5/4}\right) \right] x^{-5/4} - \left[ 0.1139 \sin\left(\frac{2\sqrt{2}}{5} x^{5/4}\right) - 0.0973 \cos\left(\frac{2\sqrt{2}}{5} x^{5/4}\right) \right] x^{-5/2} \right\} x^{1/8} e^{-2\sqrt{2}x^{5/4}/5} H(E_p I_p / n_h)^{1/5}.$$

$$(46)$$

#### 6. Validations with existing solutions

There is a critical length of a pile beyond which the pile is considered to be a long pile or flexible pile, and its behavior is similar to that of an infinitely long pile (after Matlock and Reese [1], Randolph [17]). Accordingly, the solutions of an infinitely long pile can be applied to the laterally loaded long pile. The critical length of a pile for the subgrade reaction reported by Randolph [17], Tomlinson [18] and Fleming et al. [19] is given as

$$l_c = 4T = 4\left(\frac{E_p I_p}{n_h}\right)^{\frac{1}{5}}.$$
(47)

This critical length has also been confirmed by comparing the results of finite difference method and our analytical solution. Thus our analytical solution applies for the piles with  $L > l_c$  instead of just infinitely long piles.

In the following case studies, the simplified solutions are adopted.

## 6.1. Case 1

Shen and Teh [3] adopted a variational solution to calculate the displacement and bending moment profiles of one instrumented pile reported by Mohan and Shrivastava [20] out of a series of field tests on laterally loaded piles. The instrumented pile (Pile IN1) at a working load level of 4.90 kN is selected for analysis. The length of the pile is l = 5.25 m with a diameter d = 0.1 m and a bending rigidity  $E_p I_p = 320$  kN m<sup>2</sup>. The constant of horizontal subgrade reaction is  $n_h = 3.57$  MN/m<sup>3</sup>.

Firstly, we calculate the critical length of pile as:

$$l_c = 4T = 4\left(\frac{E_p I_p}{n_h}\right)^{\frac{1}{5}} = 2.47(m).$$
(48)

Since  $l = 5.25 > l_c = 2.47$ , this pile is regarded as a long pile, to which the approach proposed in this paper is applicable. The computed displacement and bending moment distributions are plotted and compared with the results of measurement and Shen and Teh's [3] solution in Fig. 4. It can be seen that excellent agreement with Shen and Teh's [3] solution has been achieved by the present method. However, comparing to Shen and Teh's [3] solution which needs complex matrix operation, the present simplified analytical solutions are more convenient to be used in analysis and design of laterally loaded long piles.



Fig. 5. Comparison of bending moment profiles.



Fig. 6. Comparison of bending moment profiles.

**Table 1** 

 Values of  $n_h$  (MN/m<sup>3</sup>) for sand [after Liang [21]].

N value (SPT)	$n_{\rm h}$ (Above water)	<i>n</i> <sub>h</sub> (Below water)
2-4	5.4-6.8	4.1-5.4
4-10	6.8-16.3	5.4-10.9
10-20	16.3-24.4	10.9-16.3
20-30	24.4-43.4	16.3-24.4
30–50	43.4-65.1	24.4-35.3
50-60	65.1-70.6	35.3-40.7



Fig. 7. Comparison between the measured and predicted pile head deflections.

#### 6.2. Case 2

Reese and Matlock [10] used a non-dimensional curve to obtain a moment profile for a free-head pile loaded with a horizontal force of 155.68kN and a bending moment of 395.43kN·m and the same pile with head fixed under a horizontal force of 155.68kN. The length of the pile is l = 24.38 m and a bending rigidity  $E_p I_p = 1.434836$  kN m<sup>2</sup>. The constant of horizontal subgrade reaction is  $n_h = 1357.17$  kN/m<sup>3</sup>. The critical length of pile is 16.10 which is less than the length of the pile.

Figs. 5 and 6 compare the computed bending moment distributions obtained by the present method and Reese and Matlock's method. Once again, excellent agreement has been achieved between these two methods. The present simplified solutions can directly calculate the deflection and bending moment of laterally loaded long piles at any depth, and therefore eliminates the errors of dimensionless coefficients determined by forms or curves.

## 6.3. Case 3

This case study is based on a lateral load test reported by Cox et al. [26]. The test was performed on a free-head steel tube pile with a diameter of 0.61 m and an embedment length of 21 m. The pile was embedded in sand at Mustang Island, Texas.

According to Cox et al. [26], the average standard penetration test (SPT) *N* value of the sand within 10*D* (where *D* is the pile diameter) below the ground surface is 18 blows per 30 cm. The stiffness of the sand is assumed to increase linearly with depth, and the constant of subgrade reaction for the sand layer is estimated to be 15MN/m<sup>3</sup> using Table 1. Other required input parameters are given by Cox et al. [26] as follows:  $E_p I_p = 163\,000$  kN m<sup>2</sup>;  $\gamma = 10.4$  kN/m<sup>3</sup>;  $\phi = 39^\circ$ . The distance between the loading point and the top of the soil surface was 0.305 m. The critical length of pile is 6.45 which is less than the length of the pile.

A comparison between the measured and predicted pile head deflections is shown in Fig. 7. It can be seen that the present solution reaches a good agreement with the measured data.

In this study, the simplified solutions to pile head deflection are given by Eqs. (49) and (50). Barber [22] proposed similar solutions based on numerical method. Different from Barber's [22] method, the coefficients in the present solution are calculated using exact formulas (Eq. (30) (or Eq. (32)) and Eq. (33)).

#### ■ Free-head pile:

$$y_0 = \frac{2.4292H}{(n_h)^{3/5} (E_p I_p)^{2/5}} + \frac{1.6194M}{(n_h)^{2/5} (E_p I_p)^{3/5}}.$$
(49)

#### ■ Fixed-head pile:

$$y_0 = \frac{0.9279H}{(n_h)^{3/5} (E_p I_p)^{2/5}}.$$
(50)

## 7. Conclusions

A Fourier–Laplace integral has been introduced to obtain the analytical solution for laterally loaded piles in soils with stiffness linearly increasing with depth. The deflection and bending moment profiles of laterally loaded piles can be evaluated using a simple analytical expression. High accuracy can be achieved with a small amount of calculation. The proposed power series solution is an accurate solution, based on which and the accurate deflection of the pile head is derived for infinitely long piles. The WKB asymptotic solution is novel for analyzing long piles. The simplified analytical solutions to laterally loaded long piles obtained in this study can be applied to engineering design conveniently. This method can also be easily extended to analyze laterally loaded long piles in soil with the coefficient of subgrade reaction varying with the depth in other forms such as one shown in Appendix B.

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#### Appendix A

## Proof. of Theorem 1

Provided  $\sqrt{-2\phi_{tt}}dt$  is positive on the steepest decent curve  $C_k$ , as *x* approaches infinity, one can have the following by Laplace's method (after Bender and Orszag [16]):

$$y_{k}(x) \sim e^{\phi(x,t_{0})} \int_{C_{k}} e^{\frac{1}{2}\phi_{tr}(x,t_{0})(t-t_{0})^{2}} dt$$

$$= \frac{2}{\sqrt{-2 \times 4t_{0}^{3}}} e^{t_{0}^{5}/5+xt_{0}} \int_{0}^{+\infty} \frac{e^{-u}}{\sqrt{u}} du$$

$$= \frac{2\sqrt{\pi}}{\sqrt{-2 \times 4t_{0}^{3}}} e^{t_{0}^{5}/5+xt_{0}},$$
(A-1)
$$\sum_{n=1}^{\infty} \frac{e^{-u}}{\sqrt{-2 \times 4t_{0}^{3}}} dt = \frac{4}{\sqrt{2}} (t_{0}) \text{ into the Eq. (A-1) gives:}$$

$$\int_0^{+\infty} \frac{e^{-u}}{\sqrt{u}} du = \Gamma(1/2) = \sqrt{\pi}, \text{ plugging } t_0 = \sqrt[4]{x}\omega \text{ into the Eq. (A-1) gives:}$$

$$y_k(x) \sim \frac{\sqrt{\pi\omega}}{\sqrt{2}} x^{-3/8} e^{4x^{5/4}\omega/5}.$$
 (A-2)

Equation (A-2) shows that  $y_1$  and  $y_2$  are decaying as x approaches infinity while  $y_3$  and  $y_4$  are highly oscillatory with increasing amplitudes. Note that  $\sqrt{-2\phi_{tt}}dt$ , equivalently  $\sqrt{-\omega^3}dt$ , should be positive. For  $C_1$ , if we choose  $1/\sqrt{-e^{i9\pi/4}} = \sqrt{e^{i3\pi/4}} = e^{i3\pi/8}$ , then  $e^{-i3\pi/8}dt > 0$ . Therefore, dt has phase  $e^{i3\pi/8}$ , which implies that the direction of  $C_1$  is from A to B. Similar analysis gives the direction  $C_3$ .

Proof of Theorem 1 is thus completed. When *x* approaches infinity, we have

$$y_1(x) \sim \sqrt{\frac{\pi}{2}} x^{-3/8} e^{-2\sqrt{2}x^{5/4}/5} \left( \cos\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{3\pi}{8}\right) + i \sin\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{3\pi}{8}\right) \right).$$
(A-3)

It is convenient to define  $u(x) = \sqrt{\frac{\pi}{2}} x^{-3/8} e^{-2\sqrt{2}x^{5/4}/5} (\cos(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8}) + i\sin(\frac{2\sqrt{2}}{5}x^{5/4} + \frac{3\pi}{8})).$ 

Lemma 1. One has the following asymptotic expression:

$$\int_{C_1} (t - t_0)^{2m} \exp\left(\phi_0 + \frac{1}{2!}\phi_{tt}(t - t_0)^2\right) dt \sim \left(\frac{-2}{\phi_{tt}}\right)^m \frac{\Gamma(m + \frac{1}{2})}{\Gamma(\frac{1}{2})} u(x),\tag{A-4}$$

While  $\int_{C_1} (t-t_0)^{2m+1} \exp(\phi_0 + \frac{1}{2!}\phi_{tt}(t-t_0)^2) dt \sim 0.$ 

This is again the corollary of Lapalace's method. To make this intuitive, one can do the substitution:  $\frac{1}{2!}\phi_{tt}(t-t_0)^2 = -u$ . Using this lemma, we can show the proof of Theorem 2.

## Proof. of Theorem 2

We Taylor expand  $\phi$  around  $t_0$  and apply the lemma above:

$$\begin{split} y_{1}(x) &= \int_{C_{1}} \exp\left(\frac{1}{3!}\phi_{ttt}(t-t_{0})^{3} + \frac{1}{4!}\phi_{t}^{(4)}(t-t_{0})^{4} + \frac{1}{5!}\phi_{t}^{(5)}(t-t_{0})^{5}\right)\exp(\phi_{0} + \frac{1}{2!}\phi_{tt}(t-t_{0})^{2})dt \\ &\sim \int_{C_{1}} \left(1 + \frac{1}{4!}\phi_{t}^{(4)}(t-t_{0})^{4} + \frac{1}{2!}(\frac{\phi_{ttt}}{3!})^{2}(t-t_{0})^{6} + \frac{1}{2!}\left(2\frac{\phi_{ttt}\phi_{t}^{(5)}}{3!5!}(t-t_{0})^{8}\right) + \cdots\right)\exp(\phi_{0} + \frac{1}{2!}\phi_{tt}(t-t_{0})^{2})dt \\ &\sim \left(1 + \frac{\phi_{t}^{(4)}}{4!}\frac{3}{\phi_{tt}^{2}} - \frac{\phi_{ttt}^{2}}{2\times6^{2}\phi_{tt}^{3}} \times 15\right)u(x) \\ &= \left(1 - \frac{9}{32}x^{-5/4}e^{i\pi/4}\right)u(x), \end{split}$$
(A-5)

where  $\phi_0$  represents  $\phi(x, t_0)$ ,  $\phi_t$  represents  $\frac{\partial \phi}{\partial t}$ ,  $\phi_{ttt}$  represents  $\frac{\partial^3 \phi}{\partial t^3}$ ,  $\phi^{(4)}$  represents  $\frac{\partial^4 \phi}{\partial t^4}$ ,  $\phi^{(5)}$  represents  $\frac{\partial^5 \phi}{\partial t^5}$ . Fixing  $C_1$  and taking first order derivative of  $y_1$  and also Taylor expanding the integrand, we have:

$$y'_{1}(x) = \int_{C_{1}} t \exp(t^{5}/5 + xt) dt,$$
 (A-6)

$$(t_{0} + (t - t_{0})) \exp\left(\frac{1}{3!}\phi_{ttt}(t - t_{0})^{3} + \frac{1}{4!}\phi_{t}^{(4)}(t - t_{0})^{4} + \frac{1}{5!}\phi_{t}^{(5)}(t - t_{0})^{5}\right)$$
  

$$\approx t_{0} + \left(\frac{1}{3!}\phi_{ttt} + \frac{1}{4!}\phi_{t}^{(4)}\right)(t - t_{0})^{4} + \left(\frac{1}{5!}\phi_{t}^{(5)} + \frac{t_{0}}{72}(\phi_{t}^{(3)})^{2}\right)(t - t_{0})^{6} + \dots + \text{odd powers.}$$
(A-7)

Applying the lemma once again, we have:

$$\left[t_0 + (2t_0^2 + t_0^2) \left(\frac{2}{4t_0^3}\right)^2 \frac{\Gamma(5/2)}{\Gamma(1/2)} - \left(\frac{1}{5!} \times 24 + \frac{t_0^5}{72} \times 144\right) \frac{8}{4^3 t_0^9} \frac{\Gamma(7/2)}{\Gamma(1/2)}\right] u(x).$$
(A-8)

We just keep the first two orders and have:

$$\begin{split} [t_0 + (9/16)t_0^{-4} - (15/32)t_0^{-4}]u(x) &= [t_0 + (3/32)t_0^{-4}]u(x) = (x^{1/4}e^{i3\pi/4} - (3/32)x^{-1})u(x) \\ &= -\sqrt{\frac{\pi}{2}}x^{-1/8}\exp(-2\sqrt{2}x^{5/4}/5 + i(2\sqrt{2}x^{5/4}/5 + \pi/8)) - \frac{3}{32}\sqrt{\frac{\pi}{2}}x^{-11/8}\exp(-2x^{-1/8}) \\ &\times \sqrt{2}x^{5/4}/5 + i(2\sqrt{2}x^{5/4}/5 + 3\pi/8)). \end{split}$$

For the second order derivatives, the procedure is similar, we can get:

$$g_{1}''(x) + ig_{2}''(x) = \int_{C_{1}} t^{2} e^{t^{5}/5 + xt} dt = \int_{C_{1}} (t_{0}^{2} + 2t_{0}(t - t_{0}) + (t - t_{0})^{2}) \\ \times \exp\left(\frac{1}{3!}\phi_{ttt}(t - t_{0})^{3} + \frac{1}{4!}\phi_{t}^{(4)}(t - t_{0})^{4} + \frac{1}{5!}\phi_{t}^{(5)}(t - t_{0})^{5}\right) \exp(\phi(t_{0}) + \frac{1}{2!}\phi_{tt}(t - t_{0})^{2}) dt.$$
(A-10)

Firstly, expand  $\exp(\frac{1}{3!}\phi_{ttt}(t-t_0)^3 + \frac{1}{4!}\phi_t^{(4)}(t-t_0)^4 + \frac{1}{5!}\phi_t^{(5)}(t-t_0)^5)$  in Talyor series and keeps even powers of  $(t-t_0)$ :

$$t_{0}^{2} + (t - t_{0})^{2} + \left(\frac{2t_{0}}{3!}\phi_{ttt} + \frac{t_{0}^{2}}{4!}\phi_{tttt}\right)(t - t_{0})^{4} + \frac{t_{0}^{2}(\phi_{ttt})^{2}}{2!(3!)^{2}}(t - t_{0})^{6} + \frac{2t_{0}\phi_{t}^{(5)}}{5!}(t - t_{0})^{6} + \frac{1}{4!}\phi_{tttt}(t - t_{0})^{6}.$$
(A-11)

Substituting Eq. (A-11) into Eq. (A-10) and after integration using the lemma, we have the first two terms:

$$y_1''(\mathbf{x}) \sim \left(t_0^2 - \frac{1}{\phi_{tt}} + \left(\frac{2t_0}{3!}\phi_{ttt} + \frac{t_0^2}{4!}\phi_{tttt}\right)\frac{3}{\phi_{tt}^2} - \left(\frac{t_0^2(\phi_{ttt})^2}{2!(3!)^2} + \frac{2t_0\phi_t^{(5)}}{5!} + \frac{\phi_{tttt}}{4!}\right)\frac{15}{\phi_{tt}^3}\right)u(\mathbf{x})$$

$$\sim \left(-i\mathbf{x}^{1/2} + \frac{7}{32}\mathbf{x}^{-3/4}e^{-i\pi/4}\right)u(\mathbf{x}).$$
(A-12)

To get the third order, one should calculate up to  $(t - t_0)^{12}$ . We omit the detail here since there is no new idea but the derivation is tedious.

Taking the real and imaginary parts, we obtain the results in Theorem 2.

## Proof. of Theorem 3

For the first part of the new contour  $(A \rightarrow 0)$ , substitute t = -v; while for the second part of the new contour  $(0 \rightarrow B)$ , substitute  $t = ve^{i3\pi/5}$ :

$$y_1(x) = -\int_0^{+\infty} e^{-\nu^5/5 - x\nu} (-d\nu) + \int_0^{+\infty} e^{-\nu^5/5 - x\nu e^{i3\pi/5}} e^{i3\pi/5} d\nu.$$
(A-13)

Then, v is a real variable.

Using the substitution  $u = v^5/5$  in Eq. (A-13), we have

$$y_1(x) = \int_0^{+\infty} e^{-u} \left( \sum_{k=0}^{+\infty} \frac{x^k e^{i3\pi(k+1)/5} (5u)^{(k-4)/5}}{k!} \right) du + \int_0^{+\infty} e^{-u} \left( \sum_{k=0}^{+\infty} \frac{(-1)^k x^k (5u)^{(k-4)/5}}{k!} \right) du.$$
(A-14)

Set *k* = 5*n* + (0, 1, 2, 3, 4):

$$y_{1}(x) = (e^{3i\pi/5} + 1) \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{5n}}{5n!} \int_{0}^{+\infty} e^{-u} (5u)^{n-4/5} du - (e^{i\pi/5} + 1) \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{5n+1}}{(5n+1)!} \int_{0}^{+\infty} e^{-u} (5u)^{n-3/5} du + (1 - e^{4i\pi/5}) \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{5n+2}}{(5n+2)!} \int_{0}^{+\infty} e^{-u} (5u)^{n-2/5} du + (e^{2i\pi/5} - 1) \sum_{n=0}^{\infty} \frac{(-1)^{n} x^{5n+3}}{(5n+3)!} \int_{0}^{+\infty} e^{-u} (5u)^{n-1/5} du.$$
(A-15)

Taking the real and imaginary parts of Eq. (A-15) and after simplifying, we obtain the results in Theorem 3. To get the expressions for the derivatives, we just need to differentiate these expressions term by term.

Proof of Theorem 3 is thus completed.

## Appendix **B**

#### B.1. Extension and application

## B.1.1. Subgrade reaction model 1

The solutions proposed in this paper are based on the elastic subgrade model, which are applicable for the pile under working load. This solution can be to the pile under higher load level using the ideal elastic-plastic subgrade model.



Fig. B-1. The ideal elastic-plastic subgrade model.



Fig. B-2. Schematic diagram of the soil-pile system.

Madhav et al. [23] proposed an ideal elastic-plastic subgrade model to simulate the relationship between soil horizontal resistance and pile deflection. This model is shown in Fig. B-1 in which K is the modulus of subgrade reaction,  $p_c$  is the yielding soil horizontal resistance, and  $y_c$  is the yielding soil displacement.

The soil near the surface can yield firstly while laterally loaded piles are under lower load level (after Yokoyama [24], Poulos and Davis [4]). The yield zone propagates downward as the applied loads increase (after Hsiung [27]). The soil-pile system can be divided to two parts: the plastic zone above and the elastic zone below (after Guo [28]). The schematic of the soil-pile system is shown in Fig. B-2 in which *H* is the horizontal load applied to the pile head,  $p_z$  is the soil horizontal resistance at depth *z*, *L* and *D* are the length and diameter of the pile, respectively,  $L_s$  is the thickness of the yield zone of soil, *K* is the modulus of subgrade reaction assumed to increase linearly with depth from a value of zero at the ground surface as Eq. (2) shown.

In summary, the ideal elastic-plastic model mentioned above can be described as Eq. (B-1).

$$p(z,y) = \begin{cases} K(z)y = n_h z y & y \leq y_c, \\ K(z)y_c = n_h z y_c & y > y_c. \end{cases}$$
(B-1)

According to the Winkler foundation model, another form of the flexural equation of a pile on the elastic subgrade can be written as

$$E_p I_p \frac{d^4 y}{dz^4} + p(z, y) = \mathbf{0},\tag{B-2}$$

where p(z, y) is soil horizontal resistance around the pile. Plugging Eq. (B-1) into Eq. (B-2):

$$E_p I_p \frac{d \cdot y}{dz^4} + n_h z I(y) = \mathbf{0},$$

$$I(y) = \begin{cases} y & y \leq y_c, \\ y_c & y > y_c. \end{cases}$$
(B-3)

Finite difference method is ordinarily adopted to solve Eq. (B-3). Based on the analytical solutions which we have already obtained, an analytical method is proposed to solve Eq. (B-3).

Similarly we can get:

 $d^4 v$   $d^4 v$   $d^4 v$ 

$$\frac{dx^4}{dx^4} + XI(y) = 0,$$

$$I(y) = \begin{cases} y & y \leq y_c, \\ y_c & y > y_c. \end{cases}$$
(B-4)

In Eq. (B-4), x = z/T where *T* can be calculated by Eq. (4).

It is obvious that there exists a critical point  $x_c$  such that  $y \leq y_c$  if  $x \geq x_c$  and  $y \geq y_c$  if  $x < x_c$  for those boundary conditions we are interested in.

Eq. (B-4) can be reduced to

$$\begin{cases} y^{(4)} + xy_c = 0 & x < x_c \\ y^{(4)} + xy = 0 & x \ge x_c. \end{cases}$$
(B-5)

We can solve Eq. (B-5) on both intervals:



Fig. B-3. Comparison between the measured and predicted pile head deflections.

$$y(x) = \begin{cases} -\frac{1}{5!} y_c x^5 + \frac{f_1}{3!} x^3 + \frac{f_2}{2!} x^2 + c_1 x + c_0 & x < x_c \\ c_2 g_1(x) + c_3 g_2(x) & x \ge x_c, \end{cases}$$
(B-6)

where  $f_1 = y''(0) = \frac{HT^3}{E_p l_p}$  and  $f_2 = y''(0) = \frac{MT^2}{E_p l_p}$ . Therefore we need to figure out  $c_i$ ,  $0 \le i \le 3$  and  $x_c$ . Of course, here, on  $x \ge x_c$ , we should use the WKB form for  $g_i$  calculated by Eq. (16).

Besides, y''', y'', y'', y'' and y should be continuous at  $x_c$ . Thus we have the following equations:

$$\begin{cases} -\frac{1}{5!}y_c x_c^5 + \frac{f_1}{3!} x_c^3 + \frac{f_2}{2!} x_c^2 + c_1 x_c + c_0 = y_c \\ c_2 g_1(x_c) + c_3 g_2(x_c) = y_c \\ -\frac{1}{4!} y_c x_c^4 + \frac{f_1}{2!} x_c^2 + f_2 x_c + c_1 = c_2 g_1'(x_c) + c_3 g_2'(x_c) \\ -\frac{1}{3!} y_c x_c^3 + f_1 x_c + f_2 = c_2 g_1''(x_c) + c_3 g_2''(x_c) \\ -\frac{1}{2!} y_c x_c^2 + f_1 = c_2 g_1'''(x_c) + c_3 g_2'''(x_c). \end{cases}$$
(B-7)

The function values of  $g_i^{(k)}(x_c)$  can be evaluated approximately using the WKB forms. This system can be written as:

$$M(\mathbf{x}_c) \cdot \vec{a} = \vec{b}. \tag{B-8}$$



Fig. B-4. Comparison between the measured and predicted maximum bending moment.



Fig. B-5. Variation of the modulus of subgrade reaction with soil depth.

 $M(x_c)$  is a 5 × 4 matrix and  $\vec{a} = [c_0, c_1, c_2, c_3]^T$ . Since this system is consistent, we have:

$$f(x_c) = \det([M(x_c), b]) = 0.$$
(B-9)

Eq. (B-9) is an algebraic equation for  $x_c$  which is much simpler than the differential equation. For example, using dichotomy or Newton iteration, we can obtain  $x_c$ . Plugging this back, we could solve the coefficients and thus get the solution.

To demonstrate the application of this extended solution, one free-head pile example by Rollins et al. [25] is examined. The following input parameters were used: D = 0.324 m, L = 11.5 m,  $E_p I_p = 28\,600$  kN m<sup>2</sup>;  $\gamma = 10.3$  kN/m<sup>3</sup>,  $\phi = 35.3^\circ$ , the average standard penetration test (SPT) N value of the sand within 10D (where D is the pile diameter) below the ground surface is 10 blows per 30 cm. The distance between the loading point and the top of the soil surface was 0.69 m and the constant of subgrade reaction for the sand layer is estimated to be 10 MN/m<sup>3</sup> using Table 1.

A comparison between the measured and predicted pile head deflections and the maximum bending moment of pile is shown in Figs. B-3 and B-4 respectively. It can be seen that the present solution reaches a good agreement with the measured data.

#### B.1.2. Subgrade reaction model 2

In this subgrade reaction model shown in Fig. B-5, the modulus of subgrade reaction increases linearly with the depth from a value of zero at the ground surface when soil depth is less than a critical depth  $z_{cr}$  and stays constant when soil depth beyond the critical depth.

Firstly, we can concern about the basic equation:

$$(\mathbf{B}-\mathbf{10})$$

where h(x) is a smooth function around 0 and satisfies h(0) = 0, h'(0) > 0 and h(x) > 0 when x > 0. The idea is to use the WKB approximation solution when  $x \gg 0$ :

$$y(x) \sim C(h(x))^{-3/8} \exp\left(\omega \int^x h(s)^{1/4} ds\right),$$
 (B-11)

where  $\omega^4 = -1$ .

Around 0, Eq. (B-10) can be approximated by

$$\frac{d^4y}{dx^4} + h'(0)xy = 0.$$
 (B-12)

This equation has been solved already as Eq. (6). Let us consider this calculation model:

$$\begin{cases} h(x) = x, \quad x < D\\ h(x) = D, \quad x \ge D. \end{cases}$$
(B-13)

When x < D, we have to calculate power series solutions of  $y_1$  and  $y_3$ . Power series solutions of  $y_1$  are obtained as Eqs. (20) and (21). Similarly to "Proof of Theorem 3", we can get power series solutions of  $y_3$ .

$$y_{3} = (-e^{i3\pi/5} + e^{i\pi/5}) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-4/5} \Gamma(n+1/5)}{(5n)!} x^{5n} + (e^{i\pi/5} + e^{i2\pi/5}) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-3/5} \Gamma(n+2/5)}{(5n+1)!} x^{5n+1} + (e^{i4\pi/5} + e^{i3\pi/5}) \sum_{n=0}^{\infty} \frac{(-1)^{n} 5^{n-1/5} \Gamma(n+4/5)}{(5n+3)!} x^{5n+3}.$$
(B-14)

As Eq. (13) shown,  $g_3$  and  $g_4$  is real part and imaginary part of  $y_3$ , respectively. When  $x \ge D$ , we have to calculate WKB approximation solutions of  $y_1$  and  $y_3$ . We know that

$$\int^{x} h(s)^{1/4} ds = D^{1/4} x - \frac{1}{5} D^{5/4}, \quad x \ge D.$$
(B-15)

According to this, we have the catenating formulas:

$$y_{1}(x) \sim \sqrt{\frac{\pi}{2}} x^{-3/8} e^{-2\sqrt{2}x^{5/4}/5} \left( \cos\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{3\pi}{8}\right) + i \sin\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{3\pi}{8}\right) \right) \rightarrow \sqrt{\frac{\pi}{2}} D^{-3/8} \exp\left(-\frac{\sqrt{2}}{2} D^{1/4} x + \frac{\sqrt{2}}{10} D^{5/4}\right) \\ \times \left( \cos\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{3\pi}{8}\right) + i \sin\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{3\pi}{8}\right) \right), \tag{B-16}$$

$$y_{3}(x) \sim \sqrt{\frac{\pi}{2}} x^{-3/8} e^{-2\sqrt{2}x^{5/4}/5} \left( \cos\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{\pi}{8}\right) + i \sin\left(\frac{2\sqrt{2}}{5} x^{5/4} + \frac{\pi}{8}\right) \right) \rightarrow \sqrt{\frac{\pi}{2}} D^{-3/8} \exp\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4}\right) \\ \times \left( \cos\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{\pi}{8}\right) + i \sin\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{\pi}{8}\right) \right).$$
(B-17)

Then we can catenate as following:

$$\begin{cases} g_1(x) \to p_1(x) = \sqrt{\frac{\pi}{2}} D^{-3/8} \exp\left(-\frac{\sqrt{2}}{2} D^{1/4} x + \frac{\sqrt{2}}{10} D^{5/4}\right) \cos\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{3\pi}{8}\right) \\ g_2(x) \to p_2(x) = \sqrt{\frac{\pi}{2}} D^{-3/8} \exp\left(-\frac{\sqrt{2}}{2} D^{1/4} x + \frac{\sqrt{2}}{10} D^{5/4}\right) \sin\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{3\pi}{8}\right) \\ g_3(x) \to p_3(x) = \sqrt{\frac{\pi}{2}} D^{-3/8} \exp\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4}\right) \cos\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{3\pi}{8}\right) \\ g_4(x) \to p_4(x) = \sqrt{\frac{\pi}{2}} D^{-3/8} \exp\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4}\right) \sin\left(\frac{\sqrt{2}}{2} D^{1/4} x - \frac{\sqrt{2}}{10} D^{5/4} + \frac{3\pi}{8}\right). \end{cases}$$
(B-18)

Thus, for the problem:

$$E_p I_p \frac{d^4 y}{dz^4} + K(z) y = 0,$$

$$K(z) = \begin{cases} n_h z & z \leq A/n_h \\ A & z \geq A/n_h, \end{cases}$$
(B-19)

where K(z) is the modulus of subgrade reaction changing with soil depth and A is a constant depending on the properties of soil.

Scaling x = z/T, we have:

$$\frac{d^4y}{dx^4} + \frac{T^4}{E_p I_p} K(Tx)y = 0.$$
(B-20)

Knowing  $n_h T^5/E_p I_p = 1$  and letting  $D = AT^4/E_p I_p$ , this model can be replaced as above. For x < D, we use the power series expression of  $g_i$  and for  $x \ge D$ , we use  $p_i$ .

$$\begin{cases} y(z) = \sum_{i=1}^{4} c_i g_i(Tz) & z < A/n_h, \\ y(z) = \sum_{i=1}^{4} c_i p_i(Tz) & z \ge A/n_h. \end{cases}$$
(B-21)

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