Quiz

May 17, 2017

Problem 1. (6 pts) Consider the two-point Dirichlet problem

$$\begin{cases} -(p(x)u')' + q(x)u = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0. \end{cases}$$

with $p(x) > 0$, $q(x) \geq 0$, and $f(x)$ being smooth functions on $0 \leq x \leq 1$.

1. Write down the Galerkin(weak) formulation for the solution u in $V = H_0^1(0,1)$, and the finite element formulation for a finite dimensional subspace $V_h \subset V$.
2. Write down the Céa's lemma for this problem, and give the proof.
3. Let $x_k = kh$, $h = 1/N$, those nodes introduce a discretization $\mathcal{T}_h$ of $[0,1]$. Define the piecewise linear finite element space $V_h \subset V$ on $\mathcal{T}_h$. What are the triple (T, Π, Σ) for V_h?
4. Let $I_h u$ be the interpolation of $u \in H^2(0,1)$ in V_h. Write down the Bramble-Hilbert lemma.
5. For uniform discretization $\mathcal{T}_h$, prove that $\|u - I_h u\|_1 \leq ch|u|_2$ using scaling argument and Bramble-Hilbert lemma, for example, you can choose a reference interval $[-1, 1]$.
6. Write down the H^1 error estimate for V_h.

Problem 2. (4 pts) The bilinear basis for reference region $[0,1] \times [0,1]$ is

$$\begin{aligned} N_{0,0} &= (1-\xi)(1-\eta) \\ N_{0,1} &= (1-\xi)\eta \\ N_{1,0} &= \xi(1-\eta) \\ N_{1,1} &= \xi\eta \end{aligned}$$

1. Calculate the 4×4 local stiffness matrix for the reference region $[0,1] \times [0,1]$.

2. Consider solving the Poisson equation

$$\begin{cases} -\triangle u = f \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega \end{cases}$$

using bilinear basis on $\Omega = [0,1] \times [0,1]$. Suppose the discretization is a regular lattice with $h = \dfrac{1}{n+1}$. Assemble the global stiffness matrix A using natural order.