QUIZ

May 21, 2017

Problem 1. (6 pts) Consider the two-point Dirichlet problem

$$\begin{cases} -(p(x)u')' + q(x) = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

with $p(x) > 0$, $q(x) \geq 0$, and $f(x)$ being smooth functions on $0 \leq x \geq 1$.

1. Wright down the Galerkin(weak) formulation for the solution u in $V = H_0^1(0,1)$, and the finite element formulation for a finite dimensional subspace $V_h \subset V$.
2. Wright down the Cea's lemma for the problem, and give the proof.
3. Let the $x_k = kh,\ h = 1/N$, those nodes introduce a discretization of $\mathcal{T}_h$ of [0,1]. Define the piecewise linear finite element space $V_h \subset V$ on $\mathcal{T}_h$. What are the triple (T, Π, Σ) for V_h
4. Let the $I_h u$ be the interpolation of $u \in H^2(0,1)$ in V_h. Write down the Bramble-Hilbert lemma.
5. For uniform discretization $\mathcal{T}_h$, prove that $\|u - I_h u\|_1 \leq ch|u|_2$ using scaling argument and Bramble-Hilbert lemma, for example, you can choose a reference interval [-1,1].
6. Write down the H_1 error estimate for V_h.

Proof. 1. the Galerkin formulation reads

$$\int_0^1 p(x)u'v' + q(x)uv\,dx \;=\; \int_0^1 fv\,dx \quad for\ all\ v \in H_0^1(0,1)$$

and the formulation for a finite dimensional subspace $V_h \in V$ reads

$$\int_0^1 p(x)u_h'v_h' + q(x)u_h v_h\,dx \;=\; \int_0^1 fv_h\,dx, \quad for\ all\ v_h \in V_h$$

Let $\{\phi_i\}_{i=1}^N$ be the basis for subspace V_h. Assuming u_h has the form $u_h = \sum_{k=1}^N c_k\phi_k$, we have the system of equations

$$\sum_{k=1}^{N} c_k a(\phi_k, \phi_i) = \langle \ell, \phi_i \rangle, for\ i = 1, 2..., N$$

where $a(u,v) = \int_0^1 p(x)u'v' + q(x)uv\,dx$, and $\langle \ell, v\rangle = \int_0^1 fv\,dx$

2. To prove cea's lemma for the problem. We need to prove the bilinear form a is elliptic in $H_0^1(0,1)$. The continuous is obvious. Noting that $p(x) > 0$ and $q(x) \geq 0$, there exist some positive constant α, such that $a(v,v) \geq \alpha|v|_1$ for all $v \in H_0^1(0,1)$. Combined with Poincare Inequality on $H_0^1(0,1)$, we can get the ellipticity. Here we probably need the condition that $p(x), q(x) \in C^\infty[0,1]$. The rest of the proof is the same as textbook

3. T is interval $[x_k, x_{k+1}]$. Π is the linear function space on T. Σ are two functionals which involve point evaluation of functions on x_k, x_{k+1}. (T, Π, Σ)

4. Refer to Braess book lemma 6.3

5. Refer to Braess book remark 6.5

6. By the convexity of Ω, the problem is $H^2(0,1)$ regular, and $\|u\|_2 \leq c\|f\|_0$. By the conclusion of, there exist $v_h \in V_h$, such that $\|u - v_h\|_1 \leq ch|u|_2$. Combining these facts with cea's lemma gives $\|u - u_h\|_1 \leq c\|f\|_0$

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Problem 2. (4pts)The bilinear basis for reference region $[0,1] \times [0,1]$ is

$$\begin{aligned} N_{0,0} &= (1-\xi)(1-\eta) \\ N_{0,1} &= (1-\xi)\eta \\ N_{1,0} &= \xi(1-\eta) \\ N_{1,1} &= \xi\eta \end{aligned}$$

1. (2pt) Calculate the 4×4 local stiffness matrix for the reference region $[0,1] \times [0,1]$.

2. (2pt) To solve the Poisson equation

$$\begin{cases} -\triangle u = f \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega \end{cases} \tag{2}$$

using bilinear basis on $\Omega = [0,1] \times [0,1]$. Suppose the discretization is a regular lattice with $h = \dfrac{1}{n+1}$. Assemble the global stiffness matrix A using natural order.

Proof. 1. $a(u,v) = \int_0^1 \int_0^1 \nabla u \nabla v \, dxdy$. For the nodal basis given in the problem, we have

$$a(N_{0,0}, N_{0,0}) = \int_0^1 \int_0^1 (-1+x, -1+y)' \cdot (-1+x, -1+y) \, dxdy = 2/3.$$
$$a(N_{0,0}, N_{0,1}) = \int_0^1 \int_0^1 (-1+x, -1+y)' \cdot (-x, 1-y) \, dxdy = -1/6.$$
$$a(N_{0,0}, N_{1,1}) = \int_0^1 \int_0^1 (-1+x, -1+y)' \cdot (x, y) \, dxdy = -1/3$$

By the symmetry property, we have all the other items in local stiffness matrix.

$$\begin{pmatrix} 2/3 & -1/6 & -1/6 & -1/3 \\ -1/6 & 2/3 & -1/3 & -1/6 \\ -1/6 & -1/3 & 2/3 & -1/6 \\ -1/3 & -1/6 & -1/6 & 2/3 \end{pmatrix}$$

2. Let

$$A = \begin{pmatrix} 8/3 & -1/3 & & & \\ -1/3 & 8/3 & -1/3 & & \\ & \ddots & \ddots & \ddots & \\ & & -1/3 & 8/3 & -1/3 \\ & & & -1/3 & 8/3 \end{pmatrix}, B = \begin{pmatrix} -1/6 & -1/3 & & & \\ -1/3 & -1/3 & -1/3 & & \\ & \ddots & \ddots & \ddots & \\ & & -1/3 & -1/3 & -1/3 \\ & & & -1/3 & -1/6 \end{pmatrix}$$

Let

$$C = \begin{pmatrix} A & B & & & \\ B & A & B & & \\ & \ddots & \ddots & \ddots & \\ & & B & A & B \\ & & & B & A \end{pmatrix}$$

Then, $h^2 C$ is the global stiffness matrix

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