Homework 5

May 25, 2017

Problem 1. Braess book, Chapter 1, 4.8, 4.9

4.8

Proof. Consider the function $W(x)$ *on* Ω_h.

$$W(x) := \begin{cases} h^2, \ x \in \Omega_h \\ 0, \ x \ on \ \partial\Omega_h \end{cases} \tag{1}$$

It can be verified that

$$L_h W(x) := \begin{cases} \geq 1, \ x \in \Omega_h \backslash \Omega_{h,0} \\ 0, \ x \ on \ \Omega_{h,0} \end{cases} \tag{2}$$

Hence, $L_h(V - W) \leq 0 \ in \ \Omega_h$, and $V \leq W \ on \ \partial\Omega_h$. The discrete comparision principle implys that $V \leq W \ on \ \Omega_h$. □

4.9

Proof. To verify: $L_h v = \lambda v$

$$\frac{1}{h^2} 4v_{k,l} - v_{k,l+1} - v_{k,l-1} - v_{k+1,l} - v_{k-1,l} = \lambda_{\alpha,\beta}$$
$$\frac{1}{h^2} 4\sin(\alpha\pi kh)\sin(\beta\pi lh) - \sin(\alpha\pi(k+1)h)\sin(\beta\pi lh)\sin(\alpha\pi(k-1)h)\sin(\beta\pi lh)$$
$$- \sin(\alpha\pi kh)\sin(\beta\pi(l+1)h) - \sin(\alpha\pi kh)\sin(\beta\pi(l-1)h)$$
$$= \frac{1}{h^2}\sin(\alpha\pi kh)\sin(\beta\pi lh)4(\sin^2(\frac{\alpha\pi h}{2}) + \sin^2(\frac{\beta\pi h}{2}) \tag{3}$$

Hence, $\lambda_{\alpha,\beta} = \frac{4}{h^2}(\sin^2(\frac{\alpha\pi h}{2}) + \sin^2(\frac{\beta\pi h}{2}))$.

Compared with the eigenvalues of $-\nabla \ (\alpha^2 + \beta^2)\pi^2$, It's obvious that the small ones are better approximated for the property of sin around $x = 0$. □

Problem 2. To solve the boundary value problem

$$-u_{xx}(x) = f(x), \quad x \in (0, 1). \tag{4}$$

with boundary condition $u(0) = u(1) = 0$, $f \in C^0$, we subdivide interval $[0, 1]$ into n equal subintervals with $h = 1/n$. Let $x_j = jh$, $j = 0, \ldots, n$, we are looking for u_j, the approximations to the exact solution $u(x_j)$ at x_j.

(a) (Formulation) If we use central differences to approximate u_{xx},

$$u_{xx}(x_i) \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{5}$$

write down the resulting finite difference scheme (including boundary condition), and the associated linear system $Au = F$ for the unknowns, specify A, u and F.

(b) (Existence and Uniqueness) Prove that A is nonsingular, therefore the finite difference scheme has a unique solution. *hint: there are many ways to do this, one way is to show that* $v^T Av > 0$ *for any* $v \neq 0$*, namely,* A *is symmetrically positive definite.*

(c) (Numerics) A is a tri-diagonal matrix, $Au = F$ can be efficiently solved by Gaussian elimination method which will be introduced later. In this homework, suppose that $f = (3x + x^2)e^x$, implement the numerical scheme. Take $n = 4$, plot the solution you obtain.

(d) (Maximum Principle) For $v \in \mathbb{R}^m$, we say that $v \geq 0$ if $v_i \geq 0$ for $1 \leq i \leq m$. Show that if $Aw = v$ and $v \geq 0$, then $w \geq 0$. Furthermore, this implies that $\alpha_{ij} \geq 0$, where α_{ij} are the entries of A^{-1}. Use this property to show that if $f \geq 0$, then $u_j \geq 0$, for $j = 0, \ldots, n$.

(e) (Discrete Stability) The function $v(x) = \frac{x(1-x)}{2}$ satisfies

$$-\frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{h^2} = 1. \tag{6}$$

Use this to show that the entries α_{ij} of A^{-1} satisfies

$$0 \leq \sum_{j=1}^{n-1} \alpha_{ij} \leq \frac{1}{8}. \tag{7}$$

Prove that

$$\max_{1 \leq i \leq n-1} |u_i| \leq \frac{1}{8} \max_{1 \leq i \leq n-1} |f(x_i)| \tag{8}$$

(f) (Truncation Error) Like ODE, we can define truncation error,

$$T_j = -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - f(x_j) \tag{9}$$

Calculate the leading order term of T_j.

(g) (Error Equation) Let $e_j = u(x_j) - u_j$ be the discretization error. Show that e_j satisfies the equation $Ae = T$, where $e = (e_1, \ldots, e_{n-1})^T$ and $T = (T_1, \ldots, T_{n-1})^T$. Using (8) to prove the convergence result

$$\max_{1 \leq i \leq n-1} |u(x_i) - u_i| \leq \frac{h^2}{96} \max_{0 \leq x \leq 1} |u^{(4)}(x)|. \tag{10}$$

(h) (Justification) When $f = (3x + x^2)e^x$, the exact solution is $u(x) = x(1-x)e^x$. Take $n = 4, 8, 16, 32, 64, 128, 256$, and compute numerical solutions u^n. Calculate $\|u - u^n\|_\infty := \max_{1\le i\le n-1} |u(x_i) - u_i^n|$, plot (log-log) the convergence with resepct to n. Numeically estimate the prefactor in the estimate $\|u - u^n\|_\infty \simeq Ch^\alpha$, compare it with the result in (10)

Solution (a):

$$A = \begin{pmatrix} \frac{2}{h^2} & -\frac{1}{h^2} & & & \\ -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} \\ & & -\frac{1}{h^2} & \frac{2}{h^2} \end{pmatrix}, \ u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix}, \ F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{pmatrix} \tag{11}$$

Solution (b):

Matrix that satisfies 1 weakly diagonally dominant, 2 symmetric, 3 positive diagonal elements are positive definite.

Solution (d): Analogy with Discrete Maximum Principle.

Solution (e):

Proof. $v(x)$ attains its maximum $\frac{1}{8}$ at $x = \frac{1}{2}$. Hence $v(x_j) \le \frac{1}{8}$ for $j = 1, ..., n$. Moreover, $v = A^{-1}\mathbf{1}$. (7) is proved then. And noticing that $a_{i,j} \ge 0$, it implies $\|A^{-1}\|_\infty \le \frac{1}{8}$. (8) is proved then.

Solution (f): omitted

Solution (g): It can be proved based on truncation error (f) combined with discrete stability (e) □