Homework 3 solution

March 30, 2017

Problem 1. Problem 1.12 at page 33

1.12 A variant of Friedrichs' inequality. Let Ω be a domain which satisfies the hypothesis of Theorem 1.9. Then there is a constant $c = c(\Omega)$ such that

$$\|v\|_0 \leq c(|\overline{v}| + |v|_1) \quad for\ all\ v \in H^1(\Omega)$$

with $\overline{v} = \frac{1}{\mu(\Omega)} \int_\Omega v(x)dx$.

Hint: This variant of Friedrichs' inequality can be established using the technique from the proof the inequalty 1.5 only under restrictive conditions on the domain. Use the compactness of $H^1(\Omega) \to L_2(\Omega)$ in the same way as in the proof of Lemma 6.2 below.

Proof 1.

For one dimension case.

Based on mean value theorem, there exists $x_0 \in \Omega$, such that

$$v(x_0) = \overline{v}.$$

Now, we have

$$v(x) = v(x_0) + \int_\Omega v'(y)dy = \overline{v} + \int_\Omega v'(y)dy$$

and by Cauchy-Schwartz inequality and Poincare Friedrichs' inequality we get

$$\begin{aligned}
|v(x)| &= |\overline{v} + \int_\Omega v'(y)dy| \\
&\leq |\overline{v}| + \int_\Omega |v'(y)|dy \\
&\leq |\overline{v}| + (\int_\Omega 1dy)^{\frac{1}{2}}(\int_\Omega |v'(y)|^2dy)^{\frac{1}{2}} \\
&\leq |\overline{v}| + |\mu(\Omega)||v|_{H^1}
\end{aligned}$$

Now we can take L^2 norm on both sides, note the term on the right is a constant, and we also recall $\|\lambda u\|_{L^2} = \lambda\|u\|_{L^2}$,

So, we have

$$\begin{aligned}
\|v\|_0 &\leq (|\overline{v}| + |\mu(\Omega)||v|_1)(\int_\Omega 1dx)^{\frac{1}{2}} \\
&= |\mu(\Omega)||\overline{v}| + |\mu(\Omega)|^2|v|_1 \\
&\leq C(|\overline{v}| + |v|_1)
\end{aligned}$$

where $C = max(|\mu(\Omega)|, |\mu(\Omega)|^2)$ which is bounded and positive. $\square$

Proof 2.

Suppose that the inequality

$$\|v\|_0 \leq c(|\overline{v}| + |v|_1) \quad for\ all\ v \in H^1(\Omega)$$

fails for every positive c. Then there exits a sequence (v_k) in $H^1(\Omega)$ with

$$\|v_k\|_0 = 1, \quad |\overline{v}| + |v|_1 \leq \frac{1}{k}, \quad k = 1, 2, 3...$$

By the compactness of $H^1(\Omega) \hookrightarrow L^2(\Omega)$, a subsequence of v_k converges in $L^2(\Omega)$. Without loss of generality, we can assume the sequence v_k converges. Then v_k is a Cauchy sequence in L^2. From $|v|_1 \to 0$, we conclude that v_k is also a Cauchy sequence in $H^1(\Omega)$. Because of the completness of $H^1(\Omega)$, v_k converges to a element $v^\star$ in the sense of H^1. By continuity, we have

$$\|v^\star\|_0 = 1, \quad |v^\star|_1 = 0, \quad \overline{v} = 0$$

This implys that $v^\star$ equals some constant c in the sense of H^1. And $c \neq 0$ because that $\|v^\star\|_0 = 1 = c^2\,\mu(\Omega)$, which is a contradiction, since $\overline{v} = 0$

Problem 2. Problem 2.12 at page 42

2.12. Consider the elliptic, but not uniformly elliptic, bilinear form

$$a(u, v) := \int_0^1 x^2 u' v' dx$$

on the interval $[0, 1]$. Show that the problem $\frac{1}{2}a(u, v) - \int_0^1 udx \to min!$ does not have a solution in $H_0^1(0, 1)$. What is the associated (ordinary) differential equation?

Proof.

Firstly, we find the associated ordinary differential equation. By bilinear form, we have

$$a(u, v) = \int_0^1 x^2 u' v' dx \tag{1}$$

$$\frac{1}{2}a(u, v) - \int_0^1 vdx \to min\ ! \tag{2}$$

$$\Leftrightarrow \frac{1}{2}a(u, v) - \int_0^1 v \cdot 1dx \to min\ ! \tag{3}$$

We have the associated ordinary differential equation form: $Lu = f$

So, we have $Lu = 1$,

For the solution in H_0^1, $x^2 u' v|_0^1 = 0$

$$(Lu, v) = a(u, v) = \int_0^1 x^2 u' v' dx = \int_0^1 \frac{d}{dx}(x^2 u') v dx$$

Correspondingly, $Lu = -\frac{d}{dx}(x^2u') = 1 \Rightarrow -2xu' - x^2u'' = 1$ is the associated ordinary differential equation.
Secondly, we prove this problem does not have a solution in $H_0^1(0,1)$. By $Lu = -\frac{d}{dx}(x^2u') = 1$, we integral this equation in $H_0^1(0,1)$, we get $x^2u' = -x$. By ODE, we can solve it, $u = -lnx$, but u is not in $H_0^1(0,1)$, so the solution is not existence. □

Problem 3. Problem 2.15 at page 43
2.15. Show that

$$\int_\Omega \phi\ div\ v\ dx = -\int_\Omega grad\ \phi \cdot vdx + \int_{\partial\Omega} \phi v \cdot vds$$

for all sufficiently smooth functions v and ϕ with values in R^n and R, respectively.
Here

$$div\ v := \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}.$$

Proof.
Obviously, we have

$$\begin{aligned}
&\nabla \cdot (\phi v) \\
=&\phi(\nabla \cdot v) + (\nabla\phi) \cdot v \\
=&\phi \cdot div\ v + grad\ \phi \cdot v
\end{aligned}$$

We integral the both sides on Ω and get

$$\int_\Omega \nabla \cdot \phi vdx = \int_\Omega \phi\ div\ vdx + \int_\Omega grad\ \phi \cdot vdx$$
$$\int_\Omega \phi\ div\ vdx = \int_\Omega \nabla \cdot \phi vdx - \int_\Omega grad\ \phi \cdot vdx$$

by $\int_\Omega \nabla \cdot \phi vdx = \int_{\partial\Omega} \phi v \cdot vds$ we get

$$\int_\Omega \phi\ div\ v\ dx = -\int_\Omega grad\ \phi \cdot vdx + \int_{\partial\Omega} \phi v \cdot vds$$

□

Problem 4. Problem on the pdf. For the following equation with Robin boundary condition

$$\begin{cases} -u'' = f, \quad x \in (0,1) \\ u'(0) + \gamma_0 u(0) = \alpha_0 \\ u'(1) + \gamma_1 u(1) = \alpha_1 \end{cases} \tag{4}$$

show that the Galerkin form is:
determine $u \in H^1$ satisfying

$$a(u,v) = (f,v) + (\alpha_1 - \gamma_1 u(1))v(1) - (\alpha_0 - \gamma_0 u(0))v(0), \quad \forall v \in H^1 \tag{5}$$

also show that the function $\omega \in H^1$ that minimizes

$$J[\omega] = a(\omega, \omega) - 2(f, \omega) - 2\alpha_1\omega(1) + \gamma_1\omega(1)^2 + 2\alpha_0\omega(0) - \gamma_0\omega(0)^2 \tag{6}$$

is u, the solution of the Galerkin problem.

Proof.

Firstly, By bilinear form,

Let $a(u, v) = \int_0^1 u'v'dx \quad (f, v) = \int 0^1 fvdx$

Mutiply the test function v on both sides of $-u'' = f$ and integral on $(0, 1)$, we get

$$\begin{aligned} -u'' &= f \\ \Rightarrow \int_0^1 -u''vdx &= \int_0^1 fvdx \\ \Rightarrow -u'v|_0^1 + \int_0^1 u'v'dx &= \int_0^1 fvdx \\ \Rightarrow -u'(1)v(1) + u'(0)v(0) + a(u, v) &= (f, v) \end{aligned}$$

Plug the Robin boundary condition in the fomulation above

$$\begin{cases} u'(0) + \gamma_0 u(0) = \alpha_0 \\ u'(1) + \gamma_1 u(1) = \alpha_1 \end{cases}$$

$$a(u, v) = (f, v) + (\alpha_1 - \gamma_1 u(1))v(1) - (\alpha_0 - \gamma_0 u(0))v(0), \quad \forall v \in H^1$$

The second equation is to testify that for all v, $J[\omega + \lambda v]$ attains its minmum at $\lambda = 0$ if and only if 6 is ture. □