



A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

OFFPRINT

Propagating stress-pulses and wiggling transition revealed in string dynamics

ZHENWEI YAO

EPL, **121** (2018) 44002

Please visit the website
www.epljournal.org

Note that the author(s) has the following rights:

- immediately after publication, to use all or part of the article without revision or modification, **including the EPLA-formatted version**, for personal compilations and use only;
- no sooner than 12 months from the date of first publication, to include the accepted manuscript (all or part), **but not the EPLA-formatted version**, on institute repositories or third-party websites provided a link to the online EPL abstract or EPL homepage is included.

For complete copyright details see: <https://authors.eplletters.net/documents/copyright.pdf>.



epl

A LETTERS JOURNAL EXPLORING
THE FRONTIERS OF PHYSICS

AN INVITATION TO SUBMIT YOUR WORK

epljournal.org

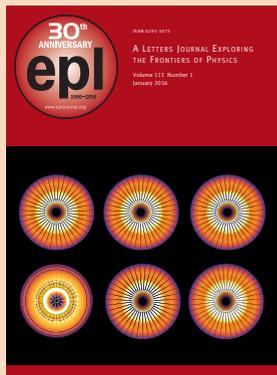
The Editorial Board invites you to submit your letters to EPL

EPL is a leading international journal publishing original, innovative Letters in all areas of physics, ranging from condensed matter topics and interdisciplinary research to astrophysics, geophysics, plasma and fusion sciences, including those with application potential.

The high profile of the journal combined with the excellent scientific quality of the articles ensures that EPL is an essential resource for its worldwide audience. EPL offers authors global visibility and a great opportunity to share their work with others across the whole of the physics community.

Run by active scientists, for scientists

EPL is reviewed by scientists for scientists, to serve and support the international scientific community. The Editorial Board is a team of active research scientists with an expert understanding of the needs of both authors and researchers.



epljournal.org

OVER

568,000

full text downloads in 2015

18 DAYS

average accept to online
publication in 2015

20,300

citations in 2015

*"We greatly appreciate
the efficient, professional
and rapid processing of
our paper by your team."*

Cong Lin
Shanghai University

Six good reasons to publish with EPL

We want to work with you to gain recognition for your research through worldwide visibility and high citations. As an EPL author, you will benefit from:

- 1 Quality** – The 60+ Co-editors, who are experts in their field, oversee the entire peer-review process, from selection of the referees to making all final acceptance decisions.
- 2 Convenience** – Easy to access compilations of recent articles in specific narrow fields available on the website.
- 3 Speed of processing** – We aim to provide you with a quick and efficient service; the median time from submission to online publication is under 100 days.
- 4 High visibility** – Strong promotion and visibility through material available at over 300 events annually, distributed via e-mail, and targeted mailshot newsletters.
- 5 International reach** – Over 3200 institutions have access to EPL, enabling your work to be read by your peers in 100 countries.
- 6 Open access** – Articles are offered open access for a one-off author payment; green open access on all others with a 12-month embargo.

Details on preparing, submitting and tracking the progress of your manuscript from submission to acceptance are available on the EPL submission website epletters.net.

If you would like further information about our author service or EPL in general, please visit epijournal.org or e-mail us at info@epijournal.org.

EPL is published in partnership with:



European Physical Society



Società Italiana
di Fisica

 **edp sciences** **IOP Publishing**

EDP Sciences

IOP Publishing

Propagating stress-pulses and wiggling transition revealed in string dynamics

ZHENWEI YAO

*School of Physics and Astronomy, and Institute of Natural Sciences, Shanghai Jiao Tong University
Shanghai 200240 China*

received 2 January 2018; accepted in final form 27 March 2018

published online 17 April 2018

PACS 46.70.Hg – Membranes, rods, and strings

PACS 45.50.Jf – Few- and many-body systems

PACS 45.80.+r – Control of mechanical systems

Abstract – Understanding string dynamics yields insights into the intricate dynamic behaviors of various filamentary thin structures in nature and industry covering multiple length scales. In this work, we investigate the planar dynamics of a flexible string where one end is free and the other end is subject to transverse and longitudinal motions. Under transverse harmonic motion, we reveal the propagating pulse structure in the stress profile over the string, and analyze its role in bringing the system into a chaotic state. For a string where one end is under longitudinal uniform acceleration, we identify the wiggling transition, derive the analytical wiggling solution from the string equations, and present the phase diagram.

Copyright © EPLA, 2018

Introduction. – An inextensible flexible string is the backbone of many complicated quasi-one-dimensional thin objects, and represents one of the simplest organization of matter [1–3]. Interest in inextensible flexible strings can be traced back to the beginnings of the calculus [4]. When in motion, a flexible string can exhibit a number of counterintuitive dynamic behaviors, ranging from the acceleration of a string when striking a table [5–7], the formation of the chain fountain structure [8–11], to the spontaneous rise-up and lift-off of a pulled string in the plane [12,13] and on a pulley [14,15]. Understanding the intricate dynamics of the filamentary string structure is important as they are ubiquitous in nature and industry covering length scales of several orders of magnitude [3,16–18]. Much has been learnt about the string dynamics by analyzing the equations of motion of the string [4,15,19–22]. However, the analytical solution to the coupled, nonlinear string equations is only limited to some special cases [13,23,24]. Particle simulation based on the spring-bead model has proven to be a powerful tool to study the dynamic states of the string [3,12,25,26].

The goal of this work is to explore the planar dynamics of the string where one end is free and the other end is in transverse and longitudinal motion, respectively. This model system provides the opportunity to clarify a host of questions with broader implications, such as: How will the motion at one end of the string propagate to the other

end? Will any dynamic instability occur in the string? What kinds of featured dynamic states will emerge? To address these questions, we resort to the combination of particle simulations and theoretical analysis of the string equations. The main results of this work are presented below. When shaking the string at one end in harmonic motion, we find that the propagation of stress is realized by the oscillating stress-pulse structure across the string. The back-and-forth movement of the stress-pulse induces more pulses and ultimately leads the whole string to a chaotic state. For a traveling string in uniform acceleration, we find a new dynamic state of the string in which it starts to wiggle and deviate from the straight shape after finite duration. We derive an analytical wiggling solution from the string equations which can substantiate the numerical observation. We further characterize the wiggling transition, and present the phase diagram.

Model and method. – An inextensible flexible string can be modeled by the geometric curve $\vec{X}(s, t)$, where s is the natural parameter of the curve and t is the time. The inextensibility condition is $\partial_s \vec{X}(s, t) \cdot \partial_s \vec{X}(s, t) = 1$. The dynamics of the flexible, inextensible string with uniform mass density μ is governed by the following equation of motion [4,19]:

$$\mu \partial_t^2 \vec{X}(s, t) = \partial_s [\sigma(s, t) \partial_s \vec{X}(s, t)], \quad (1)$$

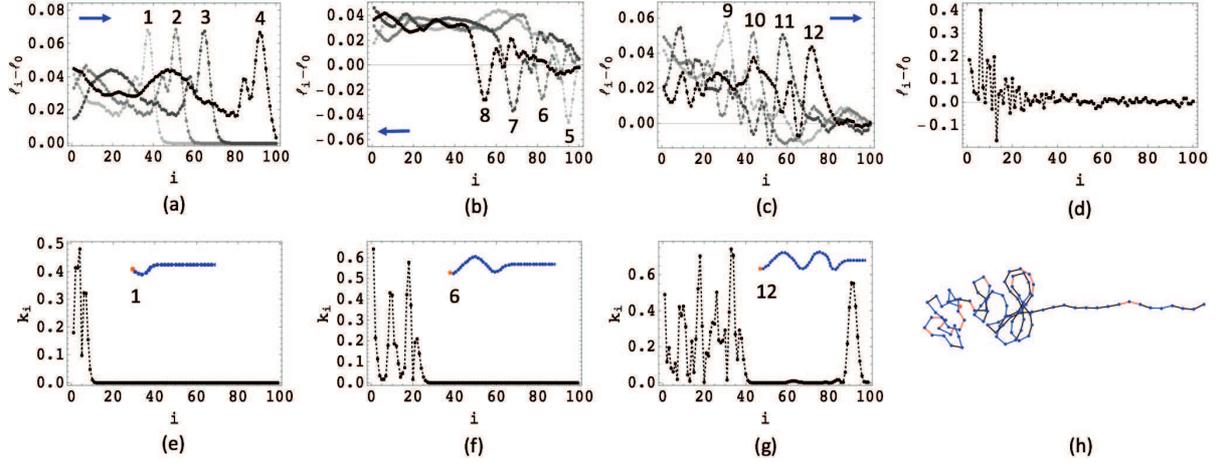


Fig. 1: (Colour online) The propagating stress-pulse structure in the string when one end is under transverse harmonic oscillation with amplitude $A = 2.5\ell_0$ and period $T = 100\tau_0$. Panels (a)–(d) show the distribution of the bond length. The labeled numbers indicate the temporal sequence: from 1 to 12, $t/\tau_0 = 42, 56, 70, 98, 112, 126, 140, 154, 182, 196, 210, 224$. $t/\tau_0 = 700$ in the rightmost figures. Panels (e)–(g) show the distribution of curvature κ in the typical conformations of the string. Note that only a small part of the straight segment is shown in the insets for visual convenience. $A_{\text{noise}} = 10^{-5}\ell_0$.

where the stress σ arises as a Lagrangian parameter to keep neighbouring parts of the string at a fixed distance [4]. By projecting eq. (1) along the tangent and normal vectors, we obtain the following string equations [24]:

$$\sigma\kappa^2 - \partial_s^2\sigma = \mu\partial_t\hat{t} \cdot \partial_t\hat{t}, \quad (2)$$

$$2\kappa\partial_s\sigma + \sigma\partial_t\kappa = \mu\partial_t^2\hat{t} \cdot \hat{n}, \quad (3)$$

where (\hat{t}, \hat{n}) is the dyad of unit tangent and normal vectors on a planar curve, and κ is the curvature. It is a challenge to analytically solve the coupled nonlinear differential equations [4]. Furthermore, due to its flexibility, the string may exhibit shapes that are beyond the functional space of $\vec{X}(s, t) \in C^2(Q_T)$ and $\sigma(s, t) \in C^1(Q_T)$, where $Q_T = J_s \times J_t$, $s \in J_s = [0, L]$, and $t \in J_t = [0, T]$ [4]. The above string equations lay the foundation for the theoretical analysis of relevant simulation results.

In our simulations, the string is modeled by $N + 1$ massive beads connected by high stiffness linear springs lying on the plane. The balance length of each spring is $\ell_0 \equiv 1$. The mass of each bead is $m_0 \equiv 1$. In the initial state, the string is free of stress, and lies along the x -axis. The position of each bead is subject to a small quantity of noise $\delta\vec{x}$ whose x - and y -components conform to the uniform distribution in the interval $[-A_{\text{noise}}, A_{\text{noise}}]$. The introduction of this noise is to trigger the possible instability of a string in longitudinal motion, and also reflects the small fluctuation of the string under various noise sources in reality. We implement the Verlet integration to construct the trajectory of each bead in the discretized string (see the Supplemental Material [SupplementaryMaterial.pdf](#) (SM)) [26]. We work in the regime of highly inextensible string with large k_0 . Specifically, $\tilde{T} \equiv T/\tau_0 = T\sqrt{k_0}/\sqrt{m_0} \gg 1$ in the transverse harmonic oscillation

of period T , and $\tilde{a} \equiv a/a_0 = am_0/(\ell_0k_0) \ll 1$ in the case of longitudinal uniform acceleration. $\tau_0 = \sqrt{m_0/k_0} = 1$, $a_0 = \ell_0/\tau_0^2 = 1$.

Transverse harmonic oscillation. – In this section, we present the main results about the planar dynamics of the string when one end is under transverse harmonic oscillation. The motion of the shaking end (labeled as $i = 0$) is $\{x_0(t) = 0, y_0(t) = A \sin(2\pi t/T)\}$.

We tune the amplitude of the noise in the position of each bead to be a very small fraction of the balance length ℓ_0 of the spring, and work in the regime of large T (*i.e.*, highly inextensible string). Simulations with varying shaking amplitude A from ℓ_0 to $5\ell_0$ show that in general the harmonic motion at the head of the string can propagate in the form of a cosine-like wave by only a few wavelengths. In fig. 1(e)–(g), we present the typical case of $A = 2.5\ell_0$ and $T = 100\tau_0$. The entire string consists of the straight and the wavy segments. The horizontal orientation of the tangent vector at the connection of the straight and the wavy parts of the string seems crucial for maintaining the straight segment of the string. Continuously shaking the string finally leads to the chaotic state as shown in fig. 1(h), which is characterized by the large deformation of the waves in the head part and the growing transverse fluctuation in the remaining part of the string.

The formation of the wave structure near the shaking end reduces the longitudinal length of the string due to the rigidity of the spring. The realization of the geometric shrinking of the string relies on the propagation of stress. The question of how the stress propagates across the string naturally arises. In the following, we analyze the evolution of the stress profile over the string in this process. The results are summarized in fig. 1.

Figures 1(a)–(d) show the variation of the stress distribution over the string as it evolves towards the chaotic state. The shaking bead is at $i = 0$. The labeled numbers at the peaks indicate the temporal sequence, some of which correspond to the labeled shapes in fig. 1(e)–(g). Simulations reveal the peak structure in the stress profile. It indicates that the stress propagates in the manner of pulses. The peak structures in the stress profile are named as stress-pulses. The stress-pulse region, where $l - l_0 > 0$, is stretched much more than the remaining part of the string. From fig. 1(a), we see that the region from the location of the stress-pulse to the free end is free of stress. The stress-pulse sharply separates the stretched and the stress-free regions. Here, we emphasize that the word “pulse” specifically refers to the peak structure in the stress profile in fig. 1(a)–(d), but not the wave structure in the string shape as shown in fig. 1(e)–(g). Simulations show the steady propagation of the stress-pulse across the string in a rate that is much faster than the propagation of the wave in the string shape, as shown in fig. 1(a). From fig. 1(a), we obtain the value of the pulse speed to be the characteristic speed of the string ℓ_0/τ_0 , which is proportional to $\sqrt{k_0}$. Further simulations for the case of $T/\tau_0 = 1000$, which is ten times the value for T/τ_0 in fig. 1, confirm that the pulse speed is the same as that in fig. 1 (see SM). Therefore, the pulse propagates infinitely fast in the inextensible limit.

Figure 1(b) shows that when the stress-pulse labeled 4 in fig. 1(a) reaches the free end of the string, it is bounced back, becoming the pulse labeled 5. Remarkably, the stress-pulse is inverted in this process. In other words, the pulse region that is originally stretched becomes compressed. Consequently, the stress distribution over the string is divided into a number of compressed and stretched regions; the shaking end is always stretched. In contrast, for a pulse whose dynamics is governed by the wave equation, no inversion occurs when reflecting off a free end of the medium [27]. Here, the behavior of the stress-pulse in the highly inextensible string system is governed by the coupled string equations in eqs. (2) and (3) rather than the law of the wave equation. Figure 1(b) shows that the negative stress-pulse continues propagating towards the shaking end. It is finally reflected back, and becomes inverted again (see the pulse labeled 9 in fig. 1(c)). It is of interest to note that propagation and reflection of small waves along a hanging chain subject to an initial velocity have been studied [21] and an interesting pattern of kicks at the free end has been revealed [28].

With the back-and-forth movement of the stress-pulse, we numerically observe the continuous retreat of the free end towards the shaking end along the axis of the string. More information about the axial retreat of the free end is provided in the SM. Furthermore, the oscillation of the stress-pulse across the string induces more pulses as shown in fig. 1(c). Repeating this process ultimately destroys the wavy shape near the shaking end, and the shape of the

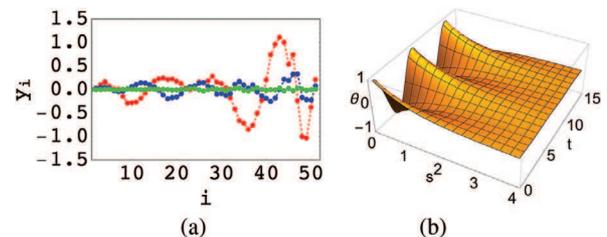


Fig. 2: (Colour online) Wiggling of the string when one end is under longitudinal uniform acceleration. (a) y_i is the transverse displacement of the beads. $a/a_0 = 10^{-4}$ (green), 10^{-3} (blue), and 10^{-2} (red). $t = 1000\tau_0$. $A_{\text{noise}} = 10^{-3}\ell_0$. (b) Plot of the wiggling solution derived from the string equations. $\theta(s, t) = \theta_0 \exp(-qs) \cos(wt)$, where $q = 1$, $w = 1$, and $\theta_0 = 1$.

string becomes chaotic as shown in fig. 1(h). The transition to the chaotic state is also reflected in the stress profile. From fig. 1(d), we see that the stress is highly concentrated in the chaotic segment of the string, and the stress level at the straight segment is significantly reduced. The screening of the stress by the highly curved segment in the chaotic string can be rationalized by the first term in eq. (2). Equation (2) is recognized as the screened Poisson’s equation $(\frac{d^2}{ds^2} - k^2)\psi = f(x)$ for constant κ ; the source term is the temporally varying tangent vector [27]. The corresponding Green’s function is $G(x_1, x_2) = \frac{1}{2k} e^{-k|x_1 - x_2|}$ under the boundary condition that the Green’s function vanishes for $x \rightarrow \pm\infty$. Therefore, the effect of curvature in the string is to screen the stress.

Longitudinal uniform acceleration. – We proceed to discuss the planar dynamics of the string under longitudinal uniform acceleration based on simulations and theoretical analysis. The head bead of the string is pulled and maintained in uniform acceleration along the x -axis: $x_0(t) = \frac{1}{2}at^2$. For a straight string in longitudinal uniform acceleration, the string equations show that the stress is linear with s , increasing from zero to μaL from the tail ($s = 0$) to the head ($s = L$) of the string. However, simulations reveal that the traveling string will suddenly deviate from the straight shape after finite duration, and the shape fluctuation persists thereafter. We name such a dynamic transition as the wiggling transition.

In fig. 2, we present typical snapshots of wiggling strings. y_i is the transverse displacement of each bead. The head bead is labeled as $i = 0$. The magnitude of acceleration increases from the green to the red lines. From fig. 2, we see that the tail of the string is generally subject to a stronger shape fluctuation than the head part. Increasing the acceleration enhances the strength of string wiggling. Wiggling transition still occurs by reducing the noise level to as low as $A_{\text{noise}} = 10^{-5}\ell_0$.

Considering that the string in simulations is not strictly inextensible, is it possible that the wiggling of the string is caused by the extensibility of the string? To clarify this

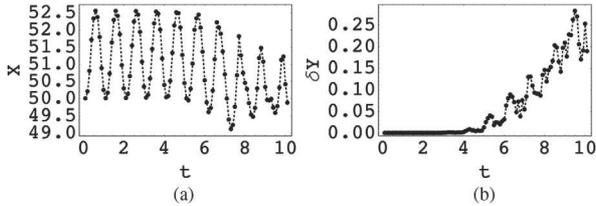


Fig. 3: Characterization of the wiggling phenomenon of the traveling string in uniform acceleration. (a) and (b): plots of the longitudinal length X and the averaged transverse displacement δy of the string. $X(t) = |x_N(t) - x_0(t)|$. $\delta y = \sqrt{\sum_{i=0}^N y_i^2}/N$. t is measured in the unit of $200\tau_0$. $A_{\text{noise}} = 10^{-5}\ell_0$. $a/a_0 = 10^{-3}$. $N = 50$.

question, we perform theoretical analysis based on eqs. (2) and (3) for inextensible strings. Furthermore, theoretical analysis based on the string equations allows us to explore the inextensible regime which is beyond the applicability of our numerical simulations. Here, we emphasize that our numerical simulations are based on the spring-bead model with large spring constant (but not strictly inextensible), and the string equations are for inextensible strings.

We focus on the behavior of the string at the onset of wiggling transition when the shape fluctuation is small and varies slowly over the string. This justifies a continuum description of the string based on the equations of motion. The main results are presented below. The detailed information is provided in the SM. The stress distribution can be written as $\sigma(s, t) = f(t)s + \alpha(t)$. The requirement of a stress-free end at $s = 0$ sets α to be zero. The shape of the string is represented by the orientation of the tangent vector which makes angle θ with respect to the x -axis. The shape of the string is represented by θ . $\theta(s, t) = \theta_1(s)\theta_2(t)$, where $\theta_1(s) = \theta_{10} \exp(-qs)$, and $\theta_2(t)$ satisfies

$$\ddot{\theta}_2(t) + qg(t)\theta_2(t) = 0, \quad (4)$$

where $g(t) = 2f(t)/\mu$ and q is a constant. It is of interest to note that eq. (4) has the same mathematical form as the Schrödinger equation; $g(t)$, the time-dependent part of the stress $\sigma(s, t)$, corresponds to the physical potential in the Schrödinger equation. Equation (4) suggests the rich dynamics of the string even in the perturbation regime.

Now, consider the case of interest: $g(t) = g_0$. g_0 is a constant, and $g_0 > 0$ without loss of generality. Such a distribution of stress is identical to that over a straight string in uniform acceleration $a = g_0/2$. By inserting $\theta_2(t) = \theta_{20} \exp(iwt)$ into eq. (4), we obtain the dispersion relation: $(iw)^2 = -qg_0$. For real positive q , $w = \sqrt{qg_0}$. Such a solution is plotted in fig. 2(b). The tail of the string wiggles, and the spatial extension q^{-1} of the wiggling segment is linear with the magnitude of acceleration at fixed frequency w . Therefore, in addition to the trivial straight-string solution, the tail of a uniformly accelerating, inextensible string can wiggle. This analytical result and the preceding simulation results suggest

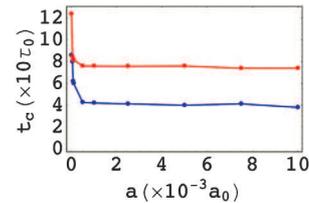


Fig. 4: (Colour online) Phase diagram of the string in uniform acceleration. The curves of $N = 50$ (bottom, blue) and $N = 100$ (top, red) indicate the transition of the dynamic state of the string from the straight to the wiggling state. $A_{\text{noise}} = 10^{-5}\ell_0$.

that the extensibility of the string is not a necessary condition for the occurrence of the string wiggling, but it may contribute to the propagation of the wiggling deformation to the entire string. Here, it is of interest to point out that the wiggling transition is an intrinsic property of the string without dependence on any external transverse force.

In the following, we further characterize the wiggling transition by the evolution of its longitudinal length X and the averaged transverse displacement δy . $\delta y(t) = \sqrt{\sum_{i=0}^N y_i^2(t)}/N$. Figure 3 shows that the transition from the straight to the wiggling state is well signified by the entire decline of the oscillations in X , and the simultaneously occurring take-off of the $\delta y(t)$ curve from the zero line. Long-time observation up to $t = 10$ millions simulation steps shows the convergence of the string wiggling; the strength of wiggling remains in the interval of $\delta y \in [0.12, 0.20]$, and $X \in [50.2, 51.0]$.

In fig. 4, we present the phase diagram of the dynamic state of the string under longitudinal uniform acceleration. The lower (blue) and upper (red) curves are for the cases of $N = 50$ and $N = 100$, respectively. The state of the string is characterized by the averaged transverse displacement δy . The string is regarded to be in the wiggling state when δy exceeds ten times the initially introduced noise A_{noise} . Figure 4 shows that a longer string can stay in the straight state for a longer time. The straight-to-wiggling transition becomes insensitive to the magnitude of acceleration when it exceeds about $10^{-3}a_0$.

Conclusions. – To summarize, in this work we have investigated the planar dynamics of a flexible string that is subject to transverse and longitudinal motions at one end. We revealed the pulse structure in the propagation of stress when one end of the string is under transverse harmonic motion, and identified the wiggling transition in a traveling string in uniform acceleration. These results may find applications in the remote control of various filamentary thin structures by manipulating the end.

This work was supported by NSFC Grant No. 16Z103010253, the SJTU startup fund under

Grant No. WF220441904, and the award of the Chinese Thousand Talents Program for Distinguished Young Scholars under Grants No. 16Z127060004 and No. 17Z127060032. The author thanks KAI YAO for stimulating discussions.

REFERENCES

- [1] WHITTAKER E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press) 1988.
- [2] AUDOLY B. and POMEAU Y., *Elasticity and Geometry* (Oxford University Press) 2010.
- [3] O'REILLY O. M., *Modeling Nonlinear Problems in the Mechanics of Strings and Rods* (Springer) 2017.
- [4] REEKEN M., *Math. Z.*, **155** (1977) 219.
- [5] GREWAL A., JOHNSON P. and RUINA A., *Am. J. Phys.*, **79** (2011) 723.
- [6] GREWAL A., JOHNSON P. and RUINA A., *Am. J. Phys.*, **79** (2011) 981.
- [7] CORBIN N. A., HANNA J. A., ROYSTON W. R., SINGH H. and WARNER R. B., arXiv preprint, arXiv:1712.05778 (2017).
- [8] BIGGINS J. S. and WARNER M., *Proc. R. Soc. A*, **470** (2014) 20130689.
- [9] BIGGINS J. S., *EPL*, **106** (2014) 44001.
- [10] VIRGA E. G., *Phys. Rev. E*, **89** (2014) 053201.
- [11] PANTALEONE J., *Am. J. Phys.*, **85** (2017) 414.
- [12] HANNA J. and KING H., arXiv preprint, arXiv:1110.2360 (2011).
- [13] HANNA J. and SANTANGELO C., *Phys. Rev. Lett.*, **109** (2012) 134301.
- [14] CAMBOU A., GAMARI B., HAMM E., HANNA J., MENON N., SANTANGELO C. and WALSH L., arXiv preprint, arXiv:1209.0481 (2012).
- [15] BRUN P.-T., AUDOLY B., GORIELY A. and VELLA D., *Proc. R. Soc. A*, **472** (2016) 20160187.
- [16] COSTELLO G. A., *Theory of Wire Rope* (Springer Science & Business Media) 1997.
- [17] NEUKIRCH S., *Phys. Rev. Lett.*, **93** (2004) 198107.
- [18] CARTER L., *Submarine Cables and the Oceans* (UNEP/Earthprint) 2009.
- [19] BROER L., *J. Eng. Math.*, **4** (1970) 195.
- [20] EDWARDS S. and GOODYEAR A., *J. Phys. A: Gen. Phys.*, **5** (1972) 965.
- [21] SCHAGERL M. and BERGER A., *Wave Motion*, **35** (2002) 339.
- [22] ŞENGÜL Y. and VOROTNIKOV D., *J. Differ. Equ.*, **262** (2017) 3610.
- [23] FETTER A. L. and WALECKA J. D., *Theoretical Mechanics of Particles and Continua* (Courier Corporation) 2003.
- [24] HANNA J. and SANTANGELO C., arXiv preprint, arXiv:1209.1332 (2012).
- [25] MILCHEV A. and BINDER K., *Macromolecules*, **29** (1996) 343.
- [26] RAPAPORT D., *The Art of Molecular Dynamics Simulation* (Cambridge University Press) 2004.
- [27] ARFKEN G. B. and WEBER H. J., *Mathematical Methods for Physicists* (AAPT) 1999.
- [28] BAILEY H., *Am. J. Phys.*, **68** (2000) 764.