Vibrations of high-contrast media: asymptotics for spectra and long-time approaches from quasimodes

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CLASSICAL SPECTRAL BOUNDARY HOMOGENIZATION PROBLEMS

Asymptotics for \((\lambda^\varepsilon, u^\varepsilon), \varepsilon \to 0\)

\[ P^\varepsilon \begin{cases} -\Delta u^\varepsilon = \lambda^\varepsilon u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \Gamma_\Omega \cup \bigcup T^\varepsilon \\ \frac{\partial u^\varepsilon}{\partial n} = 0 & \text{on } \Sigma \setminus \bigcup T^\varepsilon \end{cases} \]

\(\Omega \subset \mathbb{R}^n, n = 2 \text{ or } 3, \quad \partial \Omega = \overline{\Gamma_\Omega \cup \Sigma} \)

\(\eta = \eta(\varepsilon) \to 0, \quad \varepsilon \ll \eta \text{ or } \eta = O(\varepsilon)\)

Fixed \(\varepsilon: 0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \lambda_i^\varepsilon \leq \cdots \xrightarrow{i \to \infty} \infty\)

Fixed \(i: \lambda_i^\varepsilon \xrightarrow{\varepsilon \to 0} \lambda_i^h \) with conservation of the multiplicity \((u_i^\varepsilon \approx u_i^h)\)

\[ HP \begin{cases} -\Delta u^h = \lambda^h u^h & \text{in } \Omega, \\ u^h = 0 & \text{on } \Gamma_\Omega \\ + \text{ homogenized b.c. on } \Sigma \end{cases} \]
Some variants of the spectral problem:

- Other elliptic operators
- Other strongly alternating b.c.

- $\Gamma_\Omega = \emptyset$: $\Omega \subset \mathbb{R}^2$.

- Non-periodic structures:
  restrictions on the total number of $T^\varepsilon$

- Solid holes near the boundary $\Omega$

- “Light” concentrated masses:
  the $T^\varepsilon$ are also a part of the boundary of
  not very heavy small inclusions in $\Omega$
Formulation of the problem $P^\varepsilon$:

Let $V^\varepsilon, H^\varepsilon$ be two Hilbert spaces $V^\varepsilon \subset H^\varepsilon$ with a dense and compact imbedding $a^\varepsilon$ a sesquilinear, hermitian, continuous, coercive form on $V^\varepsilon$

To find $(\lambda^\varepsilon, u^\varepsilon), u^\varepsilon \neq 0, u^\varepsilon \in V^\varepsilon$, satisfying:

$$P^\varepsilon \quad a^\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon(u^\varepsilon, v)_{H^\varepsilon}, \quad \forall v \in V^\varepsilon$$

Convergence results:

- Asymptotic expansions for $(\lambda^\varepsilon, u^\varepsilon)$
- Bounds for convergence rates of $(\lambda^\varepsilon, u^\varepsilon)$
- For fixed $i$, $\lambda_i^\varepsilon = O(1)$
- Convergence of the eigenelements of $P^\varepsilon$ towards those of $HP$
  with conservation of the multiplicity: $\lambda_i^\varepsilon \xrightarrow{\varepsilon \to 0} \lambda_i^h$
- $\{u_i^\varepsilon\}_{i=1}^{\infty}$ and $\{u_i^h\}_{i=1}^{\infty}$ form a basis of $H^\varepsilon \equiv H$
Some authors:

Weaker convergence results:

• Stronger perturbation of the physical characteristics of the medium: **Vibrating systems with many concentrated masses near the boundary**

• The spectral parameter appearing in the strongly alternating b.c.: **Steklov type eigenvalue problems** \( (\varepsilon = O(\eta)) \)
We deal with:

- Asymptotics for eigenfunctions via quasimodes
- Approaches to solutions via standing waves for long times

SHORT SUMMARY OF THE TALK


II.- On proofs and the general framework for spectral perturbation problems where quasimodes arise

I.- THE HOMOGENIZATION OF THE STEKLOV TYPE EIGENVALUE PROBLEM:

the stationary problem
and
the evolution problem
(2006–2011)

We deal with:

• Low frequencies
• Asymptotics for eigenfunctions via quasimodes
• Approaches to solutions via standing waves for long times
• The very low frequencies: approaches to true eigenfunctions
STRONGLY ALTERNATING BOUNDARY CONDITIONS OF STEKLOV TYPE

For fixed $i$: $\beta_i^\varepsilon = O(\varepsilon^{-1})$, $C_\varepsilon^{-1} \leq \beta_i^\varepsilon \leq C_i \varepsilon^{-1}$

Very large low frequencies!!

Asymptotics for $(\varepsilon \beta_i^\varepsilon, u_i^\varepsilon)$??

$\varepsilon \rightarrow 0$

\[ P^\varepsilon \left\{ \begin{array}{l}
-\Delta u^\varepsilon = 0 \text{ in } \Omega, \\
\quad u^\varepsilon = 0 \text{ on } \partial \Omega \setminus \bigcup T^\varepsilon \\
\quad \frac{\partial u^\varepsilon}{\partial n} = \beta^\varepsilon u^\varepsilon \text{ on } \bigcup T^\varepsilon 
\end{array} \right. 
\]

$\Omega \subset \mathbb{R}^2$, $\eta(\varepsilon) = O(\varepsilon)$

$\Sigma \subset \{x_2 = 0\}$, $\beta^\varepsilon \equiv \lambda^\varepsilon$
On the Low Frequencies

**Theorem:** For fixed $i = 1, 2, 3 \cdots$, $\beta_1^0$ the first eigenvalue of $LP$

$$\varepsilon \beta_i^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \beta_1^0$$

**The Spectral Local Problem** $LP$:

$$\left\{ \begin{array}{l}
-\Delta_y V^0 = 0 \text{ in } G^1, \\
V^0 = 0 \text{ on } \Sigma^1 \setminus T^1, \\
-\frac{\partial V^0}{\partial y_2} = \beta^0 V^0 \text{ on } T^1, \\
V^0 \text{ is } y_1 - \text{periodic}, \\
V^0(y) \rightarrow c_{V^0} \text{ as } y_2 \rightarrow +\infty,
\end{array} \right.$$ 

$y = \frac{x}{\varepsilon}$ the local variable

For fixed $i$, asymptotics for $u_i^\varepsilon$ as $\varepsilon \rightarrow 0$?? $u_i^\varepsilon$??

Normalization: $\| \nabla u_i^\varepsilon \|_{L^2(\Omega)} = 1 \Rightarrow \lim_{\varepsilon \rightarrow 0} u_i^\varepsilon = 0$ in $H^1(\Omega)$-weak,
**Theorem.** (No conservation of the multiplicity! E. Pérez, DCDS-B 2007)

1. For fixed $i$, $\varepsilon \beta_i^\varepsilon \xrightarrow{\varepsilon \to 0} \beta_1^0$

2. For each eigenvalue $\beta^0$ of the local problem $LP$, $\exists \beta_i(\varepsilon) /$

$\varepsilon \beta_{i(\varepsilon)}^\varepsilon \xrightarrow{\varepsilon \to 0} \beta^0$ (now $i(\varepsilon) \to \infty$ if $\beta^0 \neq \beta_1^0$).

3. For each eigenvalue $\beta^0$ of $LP$ the total multiplicity of the eigenvalues $\beta_{i(\varepsilon)}^\varepsilon / \varepsilon \beta_{i(\varepsilon)}^\varepsilon \xrightarrow{\varepsilon \to 0} \beta^0$, converge towards $\infty$.

4. For each eigenfunction $V^0$ associated with an eigenvalue $\beta^0$ of $LP$ and for $\varepsilon \beta_{i(\varepsilon)}^\varepsilon \approx \beta^0$, the associated eigenfunctions $u_{i(\varepsilon)}^\varepsilon$ are approached by certain boundary layer functions $W^\varepsilon$ concentrating asymptotically their support in a thin layer around the part of the boundary $\Sigma$ with s.o.b.c.

**Proof:** $(2) + (3) + (4) + (\lim_{\varepsilon \to 0}(\varepsilon \beta_{i}^\varepsilon) = \beta_1^0) \implies \lim_{\varepsilon \to 0}(\varepsilon \beta_{i}^\varepsilon) = \beta_1^0$

$w_{i(\varepsilon)}^\varepsilon$??
Theorem (cont.):
4.- For each eigenelement \((\beta^0, V^0)\) of \(LP, \|\nabla_y V^0\|_{L^2(G^1)} = 1\),
the function \(W^\varepsilon(x) = \alpha^\varepsilon(w^\varepsilon(x)\eta^\varepsilon(x_2)\psi(x_1))\) can be approached
in \(H^1(\Omega)\) by \(\tilde{w}^\varepsilon\), \(\tilde{w}^\varepsilon\) being a linear combination of eigenfunctions
of \(P^\varepsilon\) associated with all the eigenvalues \(\beta^\varepsilon\) such that
\(\varepsilon\beta^\varepsilon \in [\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon]\), for a certain \(d^\varepsilon \xrightarrow{\varepsilon \to 0} 0\).

\(w^\varepsilon(x_1, x_2) = V^0(y_1, y_2)\) in \(G^1\), extended by periodicity,
\(\eta^\varepsilon(x_2)\) and \(\psi(x_1)\) cut-off functions with supports in the lined band,
\(\eta^\varepsilon\psi \equiv 1\) in \([a, b] \times [0, 2^{-1}\varepsilon|ln\varepsilon|]\), \(\alpha^\varepsilon\) constant / \(\|\nabla W^\varepsilon\|_{L^2(\Omega)} = 1\)

\[W^\varepsilon = \alpha^\varepsilon(w^\varepsilon\eta^\varepsilon\psi)\] “quasimode” of \(P^\varepsilon\)
Theorem (cont.):
4.- For each eigenelement \((\beta^0, V^0)\) of \(LP / \|\nabla_y V^0\|_{L^2(G^1)} = 1\), the function \(W^\varepsilon(x) = \alpha^\varepsilon(w^\varepsilon(x)\eta^\varepsilon(x_2)\psi(x_1))\) can be approached in \(H^1(\Omega)\) by \(\tilde{u}^\varepsilon\), \(\tilde{u}^\varepsilon\) being a linear combination of eigenfunctions of \(P^\varepsilon\) associated with all the eigenvalues \(\beta^\varepsilon\) such that 
\[
\varepsilon\beta^\varepsilon \in [\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon], \quad \text{for a certain } d^\varepsilon \rightarrow 0.
\]

\(w^\varepsilon(x_1, x_2) = V^0(y_1, y_2)\) in \(G^1\), extended by periodicity, \(\eta^\varepsilon(x_2)\) and \(\psi(x_1)\) cut-off functions with supports in the lined band, \(\eta^\varepsilon\psi \equiv 1\) in \([a, b] \times [0, 2^{-1}\varepsilon|\ln \varepsilon|]\), \(\alpha^\varepsilon\) constant / \(\|\nabla W^\varepsilon\|_{L^2(\Omega)} = 1\)

\[W^\varepsilon = \alpha^\varepsilon(w^\varepsilon\eta^\varepsilon\psi)\] “quasimode” of \(P^\varepsilon\)
Theorem (cont.):
4.- Bounds for discrepancies between quasimodes and eigenfunctions:

\[ \| W^\varepsilon - \tilde{u}^\varepsilon \|_{H^1(\Omega)} \leq C |\ln \varepsilon|^{-1/4} \]

for a certain choice of \( d^\varepsilon \) (\( d^\varepsilon = O(|\ln \varepsilon|^{-1/4}) \))

\[ \varepsilon \beta^\varepsilon \in [\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon] \]

Other approaches for \( \tilde{u}^\varepsilon \) in different spaces can exist: (spaces of traces!)

Namely, for \( \tilde{w}^\varepsilon = W^\varepsilon |_\Sigma \),

\[ \| \tilde{u}^\varepsilon - \tilde{w}^\varepsilon \|_{L^2(\Sigma)} \leq C \sqrt{\varepsilon} |\ln \varepsilon|^{-1/4} \quad \text{and} \quad \| \tilde{u}^\varepsilon - \tilde{w}^\varepsilon \|_{H^{1/2}(\Sigma)} \leq C |\ln \varepsilon|^{-1/4} \]

Larger \( d^\varepsilon \) \( \implies \) improved bounds for the discrepancies \( \| W^\varepsilon - \tilde{u}^\varepsilon \| \)

Real Statements of Theorems?

EVP
On the quasimodes and the evolution problem \( \text{EVP}^{\varepsilon} \)

\[
\mathcal{V}^{\varepsilon} = \{ u|_\Sigma / u \in H^1(\Omega), \ \Delta u = 0 \text{ in } \Omega, \ u|_{\partial \Omega \setminus \bigcup T^{\varepsilon}} = 0 \} \subset H^{1/2}(\Sigma)
\]

\( \mathcal{H}^{\varepsilon} \) completion of \( \mathcal{V}^{\varepsilon} \) in \( L^2(\Sigma) \)

Given \((\varphi^{\varepsilon}, \psi^{\varepsilon}) \in \mathcal{V}^{\varepsilon} \times \mathcal{H}^{\varepsilon}\), consider \( u^{\varepsilon}(t) \) the solution of

\[
\begin{aligned}
\text{EVP}^{\varepsilon} \quad \begin{cases}
-\Delta u^{\varepsilon} &= 0 \quad \text{in } \Omega \\
u^{\varepsilon} &= 0 \quad \text{on } \partial \Omega \setminus \bigcup T^{\varepsilon} \\
\frac{\partial^2 u^{\varepsilon}}{\partial t^2} &= \frac{\partial u^{\varepsilon}}{\partial x_2} \quad \text{on } \bigcup T^{\varepsilon} \\
u^{\varepsilon}(0) &= \varphi^{\varepsilon}, \quad \frac{\partial u^{\varepsilon}}{\partial t}(0) = \psi^{\varepsilon}
\end{cases}
\end{aligned}
\]

For \((\varphi^{\varepsilon}, \psi^{\varepsilon}) = (u^{\varepsilon}_{i(\varepsilon)}, 0)\), \((\beta^{\varepsilon}_{i(\varepsilon)}, u^{\varepsilon}_{i(\varepsilon)})\) being an eigenelement of \( P^{\varepsilon} \), the solution of \( \text{EVP}^{\varepsilon} \):

\[
u^{\varepsilon}(t) = \cos \left( \sqrt{\beta^{\varepsilon}_{i(\varepsilon)}} t \right) u^{\varepsilon}_{i(\varepsilon)}
\]

What happens if \( \varphi^{\varepsilon} \) is a quasimode of \( P^{\varepsilon} \)??
On the quasimodes and $\text{EVP}^\varepsilon$ (cont.)

For $(\varphi^\varepsilon, \psi^\varepsilon) = (\tilde{w}^\varepsilon, 0)$, where $(\beta^0, V^0)$ is an eigenelement of $LP$, $W^\varepsilon$ an associated quasimode, $W^\varepsilon(x) = \alpha^\varepsilon V^0(y_1, y_2)\eta^\varepsilon(x_2)\psi(x_1)$.

\[ \tilde{w}^\varepsilon = W^\varepsilon|_\Sigma = \alpha^\varepsilon \psi(x_1) V^0(y_1, 0), \quad \psi \text{ smooth} \]

Then, the solution $u^\varepsilon(t)$ of $\text{EVP}^\varepsilon$ satisfies

\[
\begin{align*}
\left\| \cos \left( \sqrt{\frac{\beta^0}{\varepsilon}} t \right) \tilde{w}^\varepsilon - u^\varepsilon(t) \right\|_{H^{1/2}(\Sigma)} &\leq C_1 |\ln \varepsilon|^{-1/4} \max (1, \sqrt{\varepsilon} t), \\
\left\| \sqrt{\frac{\beta^0}{\varepsilon}} \sin \left( \sqrt{\frac{\beta^0}{\varepsilon}} t \right) \tilde{w}^\varepsilon + d\frac{d u^\varepsilon}{dt}(t) \right\|_{L^2(\Sigma)} &\leq C_2 |\ln \varepsilon|^{-1/4} \max (1, \sqrt{\varepsilon} t + \varepsilon)
\end{align*}
\]

where $C_1, C_2$ are constants independent of $t$ and $\varepsilon$.

for the choice of $d^\varepsilon = O(|\ln \varepsilon|^{-1/4})$. / 2$d^\varepsilon$ is the amplitude of the interval $[\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon]$ containing the eigenvalues $\varepsilon \beta^\varepsilon$.

We can write:

\[
\left\| \cos \left( \sqrt{\varepsilon^{-1} \beta^0} t \right) \tilde{w}^\varepsilon - u^\varepsilon(t) \right\|_{H^{1/2}(\Sigma)} \leq C |\ln \varepsilon|^{-\frac{1}{4}},
\]

\[ \forall t \in [0, |\ln \varepsilon|^{-\frac{1}{2}} \varepsilon^{-\frac{3}{2}}], \quad 0 < \alpha < 1 \]
On the quasimodes and $EVP^\varepsilon$ (cont.)

Given $(\beta^0, V^0)$ any eigenelement of the Local Problem LP

The quasimodes $W^\varepsilon = \alpha^\varepsilon(w^\varepsilon \eta^\varepsilon \psi)$ for the spectral problem $P^\varepsilon$, that we have constructed from $V^0$, taken as initial data in the second order evolution problem $EVP^\varepsilon$, provide solutions $u^\varepsilon(t)$ which are approached by standing waves for long period of times. These standing waves on $\Sigma$ are of the type:

$$
\cos \left( \sqrt{\frac{\beta^0}{\varepsilon}} t \right) \times V^0 \left( \frac{x_1}{\varepsilon}, 0 \right) \times \psi(x_1)
$$

where $V^0|_{T^\varepsilon}$ is extended by periodicity along $\Sigma$ and $\psi$ are cut-off functions with support contained in the part of $\Sigma$ where the strongly oscillating boundary condition are imposed. Times for approaches depend on $\varepsilon$

M. Lobo, E. Pérez, M2AS 2010

s.o.f. on $\Sigma / L.F.O.$
Remarks on proofs

We use and provide general results that can be applied to many problems from spectral perturbation theory

- Minimax principle
- Fourier method
- Almost eigenvalues and almost eigenfunctions
- Almost orthogonality for the eigenfunctions
- Conservation of the energy
- ....

We provide a general abstract framework for time dependent problems...
Remarks: Limitations & Extensions

- In the theorems we have approaches through quasimodes! More precise results on the structure the eigenfunctions individually ⇒ more information on the distance between the eigenvalues.

- To choose a suitable normalization of the eigenfunctions is important in order to localize certain classes of vibrations: V.S.C.M!

- The high frequencies can give rise to other kinds of vibrations! \( \beta^\varepsilon = O(\varepsilon^{-\beta}) \) with \( \beta > 1 \) V.S.C.M!

- In terms of the second order evolution problem: quasimodes ⇒
  - Approaches to solutions via standing waves for long times: Localization of certain kinds of vibrations!
  - Approaches to the eigenfunctions via quasimodes amounts approaches to true eigenfunctions: Classical Spectral BHP!

Limitations & Extensions (cont.)

- Extension to the linear elasticity operator
  D. Gómez, M. Lobo, S.A. Nazarov, E. Pérez, to be continued

- Extension to the case of vibrating systems with many concentrated masses along lines
  E. Pérez, M2AS 2005; S.A. Nazarov, E. Pérez, C.R. Mecanique, 2009

\[-\Delta u^\varepsilon = \rho^\varepsilon(x) \lambda^\varepsilon u^\varepsilon \text{ in } \Omega \quad m > 2\]

\[\rho^\varepsilon(x) = \begin{cases} 
\frac{1}{\varepsilon^{m}}, & \text{if } x \in \bigcup B^\varepsilon \\
1, & \text{if } x \in \Omega - \overline{\bigcup B^\varepsilon}
\end{cases}\]

- Information for the structure of the eigenfunctions individually!?
The asymptotic expansions:

For the eigenelements \((\beta^\varepsilon, u^\varepsilon)\) of \(P^\varepsilon\) consider the asymptotics:

\[
\beta^\varepsilon = \frac{\beta^0}{\varepsilon} + \beta^1 + \varepsilon\beta^2 + \cdots
\]

\[
u^\varepsilon(x) = w^0(x) + \varepsilon w^1(x) + \cdots \quad \text{in } \Omega, \quad \text{for } x_2 > 0
\]

\[
u^\varepsilon(x) = U^0(x, y) + \varepsilon U^1(x, y) + \cdots \quad \text{near } x_2 = 0
\]

where \(U^0(x, y)\) are \(y_1\)-periodic functions in \(G^1\)

The matching conditions at the second order provide

\[
U^0(x, y) = (C_{V^0})^{-1} V^0(y) w^0(x_1, 0)
\]

where \((\beta^0, V^0)\) is an eigenelement of the local problem \(LP\)

while \((\beta^1(\beta^0 C_{V^0}^2)^{-1}, w^0)\) is an eigenelement of the Limiting Steklov Problem:

\[
LSP \quad \begin{cases}
-\Delta u = 0 \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega \setminus \Sigma \\
\frac{\partial u}{\partial n} = \mu u \quad \text{on } \Sigma
\end{cases}
\]
On asymptotic expansions (cont.)

Taking in the expansions \((\beta^0, V^0)\) the first eigenelement of \(LP\):

\[(\beta^0, V^0) = (\beta^0_1, V^0_1) \quad \text{with} \quad V^0_1(y) > 0 \quad \text{in} \quad G^1, \quad C_{V^0_1} > 0\]

we have the splitting

\[
\beta^\varepsilon_i = \frac{\beta^0_1}{\varepsilon} + \mu^0_i (C_{V^0_1}^2 \beta^0_1) + \cdots
\]

and, in \(\Omega\):

\[
u^\varepsilon_i(x) \approx (C_{V^0_i}^0)^{-1} w^\varepsilon_i(x) w^0_i(x)
\]

\[\{ (\mu^0_i, w^0_i) \}_{i=1}^\infty \] the eigenelements of \(LSP\)

...and, the technique in S.A. Nazarov & E. Pérez, *C.R. Mecanique, 2009*

⇒ explaining, e.g., the shape on \(\Sigma\) of eigenfunctions computed in M. Campillo, C. Dascalu, I. Ionescu (Geophys. J.Int., 2004), H. Perfettini, M. Campillo, I. Ionescu (J. Geophysical Research, 2003), C. Dascalu, I. Ionescu (M3AS, 2004)

**Justifications ⇒ Reformulating the Problem** \(P^\varepsilon\)

- Introducing new spectral parameters and eigenfunctions
- Weighted Sobolev spaces

S.A. Nazarov & E. Pérez, *under way*

II.- THE GENERAL FRAMEWORK
AND
GENERAL RESULTS FOR PROOFS
II.- THE GENERAL FRAMEWORK FOR SPECTRAL PERTURBATION PROBLEMS

• Formulation of the problem $P^\varepsilon$:
  \hspace{1cm} $V^\varepsilon \subset H^\varepsilon$ Hilbert spaces
  \hspace{1cm} $a^\varepsilon$ a sesquilinear, hermitian, continuous, coercive form on $V^\varepsilon$
  \hspace{1cm} $P^\varepsilon \quad a^\varepsilon(u^\varepsilon, v) = \lambda^\varepsilon (u^\varepsilon, v)_{H^\varepsilon}, \quad \forall v \in V^\varepsilon$

• Discrete spectrum:
  For fixed $\varepsilon$, the eigenvalues of $P^\varepsilon$, $\{\lambda_i^\varepsilon\}_{i=1}^\infty$, satisfy
  \hspace{1cm} $0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \cdots \leq \lambda_n^\varepsilon \leq \cdots \xrightarrow{n \to \infty} \infty$
  The eigenfunctions of $P^\varepsilon$ (suitably normalized)
  \hspace{1cm} $\{u_i^\varepsilon\}_{i=1}^\infty$ form a basis of $H^\varepsilon \equiv H$
II.- THE GENERAL FRAMEWORK (cont.)

- Estimates for the eigenvalues:
  \[ \exists \text{ constants } \alpha, C, C_i / \]
  \[ C \varepsilon^\alpha \leq \lambda_i^\varepsilon \leq C_i \varepsilon^\alpha, \quad i = 1, 2, 3, \ldots \]

- Low Frequencies (LF): \( \lambda_i^\varepsilon = O(\varepsilon^\alpha) \)
- High Frequencies (HF): \( \lambda_i(\varepsilon) = O(\varepsilon^\beta), \quad \beta < \alpha \)

Asymptotics for \((\lambda^\varepsilon, u^\varepsilon)\) as \(\varepsilon \to 0\) ?

The best situation: Convergence with conservation of the multiplicity:
\[ (\lambda_i^\varepsilon^{-\alpha}, u_i^\varepsilon) \xrightarrow{\varepsilon \to 0} (\lambda_i^0, u_i^0) \text{ in } \mathbb{R} \times \mathbb{H}, \]
\[ \{(\lambda_i^0, u_i^0)\}_{i=1}^\infty \text{ being the eigenelements of certain limiting problem LP,} \]
and \(\{u_i^0\}_{i=1}^\infty\) form a basis of \(\mathbb{H}\) ??
Some restrictions:

1. \( \{u_i^0\}_{i=1}^\infty \) do not form a basis of \( \mathbf{H} \)

The asymptotic behavior of the HF, which accumulate asymptotically in the whole positive real axis, and of the associated eigenfunctions can be important!!

2. All the re-scaled LF have a common limit point:

\[
\lambda_i^\varepsilon \varepsilon^{-\alpha} \xrightarrow{\varepsilon \to 0} \lambda^* > 0, \quad i = 1, 2, 3, \ldots
\]

but, asymptotics for \( u_i^\varepsilon \)?

3. Convergences of \( \lambda_i^\varepsilon \varepsilon_n^{-\alpha} \) for \( \varepsilon_n \to 0 \),

\[
\lim_{\varepsilon \to 0} \lambda_1^\varepsilon \varepsilon^{-\alpha}, \quad u_1^\varepsilon
\]

Distance between consecutive eigenvalues!?
On remarks & definitions & examples

Characterizing the eigenfunctions \( u_i \) individually involves either precise information on the distances between eigenvalues or different “suitable” normalizations for eigenfunctions.

We may obtain approaches to “groups of eigenfunctions” via quasimodes!

Quasimodes: “functions \( W^{\varepsilon} \), \( \|W^{\varepsilon}\|_H = 1 \), \( W^{\varepsilon} \) approaching linear combination of eigenfunctions \( u^{\varepsilon} \) associated with the eigenvalues or re-scaled eigenvalues in small intervals: \( [\lambda^* - \delta^{\varepsilon}, \lambda^* + \delta^{\varepsilon}] \), \( \delta^{\varepsilon} \to 0 \). ( \( \lambda^* \equiv \lambda^*(\varepsilon) \)?)

[Visik and Lusternik (1957); Lazutkin (1999)]

The quasimode: the pair \( (W^{\varepsilon}, \lambda^*) \) where \( \lambda^* \equiv \lambda^*(\varepsilon) \) is the almost eigenfrequency. The definition also involves a remainder \( r^{\varepsilon} \) somehow related to \( \delta^{\varepsilon} \).

Vibrating systems with concentrated masses: example on why constructing quasimodes, approaching eigenfunctions associated with LF and HF, in singularly perturbed spectral problems, can be of interest!
II.- PROOFS: GENERAL RESULTS

...and more on proofs, convergence results, real statements of our theorems, reformulations, references...
The real statement of the Theorem:

Theorem:
Let \((\beta^0, V^0)\) be any eigenelement of \(LP, V^0\) satisfying \(\int_{\Omega^1} |\nabla_y V^0|^2 dy = 1\). There exists a sequence \(d^\varepsilon, \tilde{d}^\varepsilon \to 0\), as \(\varepsilon \to 0\), such that there are eigenvalues \(\beta^\varepsilon\) of \(P^\varepsilon\) with \(\varepsilon \beta^\varepsilon \in [\beta^0 - d^\varepsilon, \beta^0 + d^\varepsilon]\) (or equivalently, such that \((\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - \tilde{r}^\varepsilon, \varepsilon(\beta^0)^{-1} + \tilde{r}^\varepsilon]\) for \(\tilde{r}^\varepsilon = O(d^\varepsilon)\)). In addition, there are \(\tilde{u}^\varepsilon\), with \(\int_{\Omega} |\nabla \tilde{u}^\varepsilon|^2 dx = 1\), \(\tilde{u}^\varepsilon\) in the eigenspace of all the eigenfunctions \(u^\varepsilon\) of \((P^\varepsilon)\) associated with the eigenvalues \(\beta^\varepsilon\) such that \(\varepsilon \beta^\varepsilon \in [\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]\) (or equivalently, such that \((\beta^\varepsilon)^{-1} \in [\varepsilon(\beta^0)^{-1} - \tilde{r}^\varepsilon, \varepsilon(\beta^0)^{-1} + \tilde{r}^\varepsilon]\) for \(\tilde{r}^\varepsilon = O(d^\varepsilon)\) with \(\tilde{d}^\varepsilon \to 0\) and \(d^\varepsilon/\tilde{d}^\varepsilon \to 0\) as \(\varepsilon \to 0\), (or equivalently, \(\tilde{r}^\varepsilon \to 0\) and \(r^\varepsilon/\tilde{r}^\varepsilon \to 0\) as \(\varepsilon \to 0\)), and \(\tilde{u}^\varepsilon\) satisfying:

\[
\int_{\Omega} |\nabla (\tilde{u}^\varepsilon - \alpha^\varepsilon w^\varepsilon \eta^\varepsilon \psi)|^2 dx \leq C r^\varepsilon \tilde{r}^\varepsilon,
\]

where \(\alpha^\varepsilon\) is the constant \(\alpha^\varepsilon = (\int_{\Omega} |\nabla (w^\varepsilon \eta^\varepsilon \psi)|^2 dx)^{-1/2}\), \(C\) is a constant independent of \(\varepsilon\). Sequences \(d^\varepsilon\) and \(r^\varepsilon\) can be taken as follows:

\[
d^\varepsilon = K_1 |\ln \varepsilon|^{-1/2}, \quad \text{and} \quad r^\varepsilon = K_2 |\ln \varepsilon|^{-1/2}
\]

where \(K_1, K_2\) are certain constants independent of \(\varepsilon\). Also, sequences \(\tilde{d}^\varepsilon\) and \(r^\varepsilon/\tilde{r}^\varepsilon = d^\varepsilon/\tilde{d}^\varepsilon\) can be chosen in order to get either smaller intervals \([\beta^0 - \tilde{d}^\varepsilon, \beta^0 + \tilde{d}^\varepsilon]\) or improved bounds, \(d^\varepsilon/\tilde{d}^\varepsilon = r^\varepsilon/\tilde{r}^\varepsilon = O(|\ln \varepsilon|^{-1/4})\) being one of these possible choices (\(\tilde{r}^\varepsilon = K_2 |\ln \varepsilon|^{-\beta}\) for \(0 < \beta < 1/2\)).

In addition, for \(\tilde{w}^\varepsilon = w^\varepsilon \eta^\varepsilon \psi\) we also have:

\[
\|\tilde{u}^\varepsilon\|_{H^\varepsilon} \leq \tilde{c}_1 \sqrt{\varepsilon} \quad \text{and} \quad \|\tilde{u}^\varepsilon - \alpha^\varepsilon \tilde{w}^\varepsilon\|_{H^\varepsilon} \leq \tilde{c}_1 \sqrt{\varepsilon} \frac{r^\varepsilon}{\tilde{r}^\varepsilon}
\]

\[
\|\tilde{u}^\varepsilon\|_{V^\varepsilon} \leq \tilde{c}_1 \quad \text{and} \quad \|\tilde{u}^\varepsilon - \alpha^\varepsilon \tilde{w}^\varepsilon\|_{V^\varepsilon} \leq \tilde{c}_2 \frac{r^\varepsilon}{\tilde{r}^\varepsilon}
\]
The real statement of the Theorem (cont.)

Let $K$ be any natural number and \{a_k\}_{k=0}^{K-1}$, \{b_k\}_{k=0}^{K-1}$, \{α_k\}_{k=0}^{K-1}$, \{β_k\}_{k=0}^{K-1}$ increasing sequences of numbers in $Σ$, such that the interval $(α_k, β_k)$ is strictly contained in $(a_k, b_k) ⊂ Σ$ (i.e., $(α_k, β_k) ⊂ Σ$), for any fixed $k = 0, 1, 2, \ldots, K-1$, \bigcup_{k=0}^{K-1} (a_k, b_k) ⊂ Σ$ and $(a_p, b_p) ∩ (a_k, b_k) = ∅$ if $p \neq k$. For each $k = 0, 1, 2, \ldots, K-1$, let $ψ_k(x)$ be a function satisfying

\[ ψ_k \in C_0^∞(ℝ), \quad 0 ≤ ψ_k ≤ 1, \quad ψ_k(x_1) = 1 \text{ if } x_1 ∈ [α_k, β_k), \quad ψ_k(x_1) = 0 \text{ if } x_1 /∈ [a_k, b_k]. \]

Theorem:
Let us consider an eigenvalue $β^0$ of $LP$ and $V^0$ any associated eigenfunction of norm 1 in $V^1$. For any $K > 0$ there is a sequence $d^ε → 0$ as $ε → 0$, and $ε^*(K)$ such that, for $ε < ε^*(K)$, the interval $[β^0 - d^ε, β^0 + d^ε]$ contains re-scaled eigenvalues of $P^ε$, $εβ^ε_i$, with total multiplicity greater than or equal to $K$; namely
\[ \{εβ^ε_i+1\}_{i=0}^{K(ε)} ∈ [β^0 - d^ε, β^0 + d^ε] \quad \text{for a certain } K(ε) ≥ K - 1, \quad \text{and } d^ε = C_K |ln ε|^{-1/2}. \]
In addition, there are $K$ linearly independent functions, \{υ^ε_i+1\}_{j=0}^{K(ε)}$, each $υ^ε_i+1 ∈ V^ε$, such that $||υ^ε_i+1||_{V^ε} = 1$, $υ^ε_i+1$ belongs to the eigenspace associated with all the eigenvalues in $[β^0 - d^ε, β^0 + d^ε]$, and it satisfies
\[ \int_Ω |∇(υ^ε_i+1 - α^ε_j(w^ε η^ε ψ_j))|^2 dx ≤ C_K \frac{d^ε}{d^ε}, \quad \text{for } j = 0, 2, \ldots, K - 1. \]

where $d^ε → 0$ and $d^ε / d^ε → 0$ as $ε → 0$, $α^ε_j$ is the constant $α^ε_j = (\int_Ω |∇(w^ε η^ε ψ_j)|^2 dx)^{-1/2}$, $C_K$ is a constant independent of $ε$. 

SOME REFERENCES: on parts I-II


III.- VIBRATING SYSTEMS WITH CONCENTRATED MASSES:

the low frequency vibrations
and
the high frequency vibrations
(1993–2008)

...a revisit and other related issues
VIBRATING SYSTEMS
WITH CONCENTRATED MASSES

Some authors:

- On one single concentrated mass:

- On many concentrated masses:
ONE SINGLE CONCENTRATED MASS


\[ \varepsilon \to 0, \quad (\lambda^\varepsilon, u^\varepsilon) \]

\[
P^\varepsilon \begin{cases} 
-\Delta u^\varepsilon = \rho^\varepsilon(x) \lambda^\varepsilon u^\varepsilon \text{ in } \Omega, \\
u^\varepsilon = 0 \text{ on } \partial\Omega \end{cases}
\]

\[
\rho^\varepsilon(x) = \begin{cases} 
\frac{1}{\varepsilon^m}, & \text{if } x \in B^\varepsilon \\
1, & \text{if } x \in \Omega - B^\varepsilon. 
\end{cases}
\]

\[ \Omega \subset \mathbb{R}^n, n = 2 \text{ or } 3, \quad m > 2 \]

\[ \lambda_i^\varepsilon = O(\varepsilon^{m-2}) / \text{Low frequencies/Local vibrations } u_i^\varepsilon \]

\[ \lambda^\varepsilon = O(1) / \text{High frequencies/Global vibrations} \]

\[ \frac{\lambda_i^\varepsilon}{\varepsilon^{m-2}} \xrightarrow{\varepsilon \to 0} \mu_i / \text{Eigenvalues of a Local Problem } LP \text{ in } \mathbb{R}^n \]

(conservation of the multiplicity)

HF??
General results for LF and HF

For fixed $i = 1, 2, \cdots$, 

$$\lambda_i^\varepsilon = O(\varepsilon^{m-2})$$

The low frequencies concentrate at 0!

$$\lambda^\varepsilon = O(1): \text{The high frequencies accumulate on } (0, \infty)!$$

But, only the eigenfunctions $u^\varepsilon$ associated with the eigenvalues $\lambda^\varepsilon$ which are asymptotically near an eigenvalue of the Dirichlet Problem DP in $\Omega$ could be asymptotically different from 0 in $H^1(\Omega)!$ Approaches to eigenfunctions through quasimodes!!
Two spectral problems related to $P^\varepsilon$:

The global problem $\leftrightarrow$ High frequency vibrations: the Dirichlet problem in $\Omega$

$$DP \left\{ \begin{array}{l} -\Delta u = \lambda u \text{ in } \Omega, \\ u = 0 \text{ on } \partial \Omega. \end{array} \right.$$ 

The local problem $\leftrightarrow$ Low frequency vibrations

$$LP \left\{ \begin{array}{l} -\Delta_y U = \lambda U \text{ in } B, \\ -\Delta_y U = 0 \text{ in } \mathbb{R}^n - \overline{B}, \\ [U] = [\frac{\partial U}{\partial n_y}] = 0 \text{ on } \partial B, \\ U(y) \rightarrow c, \text{ when } |y| \rightarrow \infty, \text{ for } n = 2 \\ U(y) \rightarrow 0, \text{ when } |y| \rightarrow \infty, \text{ for } n = 3 \end{array} \right.$$ 

$$y = \frac{x}{\varepsilon} \text{ local variable, } \quad B^\varepsilon = \varepsilon B$$
On the structure of the eigenfunctions associated with the HF

$LP \longrightarrow$ correcting terms for the eigenfunctions associated with HF:

For fixed $\lambda > 0$ and $\lambda^\varepsilon \approx \lambda$, the correcting terms are computed for

$$\varepsilon \in \{\varepsilon_l\}_{l=1}^\infty, \quad \varepsilon_l = \left( \frac{\lambda}{\mu_l} \right)^{\frac{1}{2(m-2)}}$$

with $\mu_l$ eigenvalues of $LP$, $\mu_l \to \infty$ as $l \to \infty$,

from the eigenfunctions of $LP$ associated with very large $\mu_l$

Some restrictions:
- On $\varepsilon$: $\varepsilon$ ranges in certain subsequences
- On the geometry of $B$: Obtaining precise bounds for convergence rates implies explicit computations of the eigenfunctions of $LP$

Gómez&Lobo&Pérez (1999–2008) to be continued...

http://grupos.unican.es/vecham/
http://personales.unican.es/meperez/
First eigenmode for a vibrating membrane with a concentrated mass: $m=3$

$\varepsilon = 0.1$

$\varepsilon = 0.025$

$m$ larger and/or $\varepsilon$ smaller imply smaller frequencies of vibration
If $\lambda^\varepsilon \approx \lambda$

If $\lambda$ is an eigenvalue of the Dirichlet P.

In the macroscopic variable

If $\lambda$ is not an eigenvalue of the Dirichlet P.

In the microscopic variable
Some remarks

- As regards the low frequencies: Normalization condition:
  \[
  \int_{\varepsilon^{-1}\Omega} |\nabla_y u^\varepsilon|^2 \, dy + \int_{\varepsilon^{-1}\Omega} \beta^\varepsilon(y)|u^\varepsilon|^2 \, dy = 1
  \]
  where \( \beta^\varepsilon(y) = 1 \) if \( y \in B \), \( \beta^\varepsilon(y) = \varepsilon^m \) if \( y \in \varepsilon^{-1}\Omega \setminus B \)

- As regards the high frequencies: Normalization condition:
  \[
  \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx + \int_{\Omega} \rho^\varepsilon(x)|u^\varepsilon|^2 \, dx = 1
  \]

- On other convergence sequences:
  \[
  \nexists \frac{\lambda_{i(\varepsilon)}^\varepsilon}{\varepsilon^{\beta}} \to \lambda > 0, \quad \beta \neq 0 \text{ or } \beta \neq m - 2
  \]
  giving rise to local or global vibrations:
  \[!\]
  (very important to locate local or global vibrations or...!?)

- In the case of many concentrated masses, the quasi-modes also appear when approaching low frequency vibrations:
  \[
  \frac{\lambda_{i}^\varepsilon}{\varepsilon^{m-2}} \xrightarrow{\varepsilon \to 0} \lambda_1^*\]
Many concentrated masses: High frequency vibrations (HP ↔ DP)

\[ P^\varepsilon \] \[
\begin{align*}
-\Delta u^\varepsilon &= \rho^\varepsilon(x) \lambda^\varepsilon u^\varepsilon \text{ in } \Omega, \\
u^\varepsilon &= 0 \text{ on } \Gamma_\Omega \cup \bigcup T^\varepsilon, \\
\frac{\partial u^\varepsilon}{\partial n} &= 0 \text{ on } \Sigma - \bigcup T^\varepsilon,
\end{align*}
\]

“High contrast homogenization problems”
Many concentrate masses: strongly oscillating b.c.

High frequency vibrations

$$\eta = O(|\log \varepsilon|^{-1})$$
But models of vibrating systems with concentrated masses (at points or along lines) appear, e.g., in thin structures, in reinforcement problems,…

Effects on localization of eigenmodes of vibration associated to LF, MF or HF have been detected in the vibration of certain structures depending on the geometry, physical characteristics, boundary conditions, …


…outlined in “Asymptotology”, 1962, by Martin D. Kruskal:

D. Gómez, S. Nazarov, E. Pérez: *Networks & Heterog. Media*, 2011
“...By asymptotology I mean something much broader than asymptotics, but including it; pending further elaboration. I would briefly define asymptotology as the art of dealing with applied mathematical systems in limit cases.”

“...Frequent in asymptotology analyses is the occurrence of phenomena on different scales of distance or time....”

One of the principles in the process: “no term should be neglected without a good reason”

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ASYMPTOTOLOGY

Martin D. Kruskal

MATT-160 December, 1962

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An example in

ASYMPTOTOLOGY—A CAUTIONARY TALE

R. L. DEWAR1 ANZIAM J. 44(2002), 33-40
SOME REFERENCES: on parts II-III


References on parts II-III (cont.)


