A Harnack Inequality Approach to the Regularity of Free Boundaries
Part II: Flat Free Boundaries are Lipschitz

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In this second paper on the regularity of free boundaries we develop a technique that yields the following type of result:

Suppose that u is a weak solution of a free boundary problem of the type $\Delta u = 0$ on $\Omega^+ = \{u > 0\}$ and on $\Omega^- = \{u \leqq 0\}^0$ and that along the free boundary, $F = \partial\Omega^+$, the relation $u_\nu^+ = G(u_\nu^-, X, \nu)$ is satisfied in some weak sense.

Then, if F is close to being the graph of a Lipschitz function in, say, the unit ball B_1 of $\mathbb{R}^n$, F is indeed the graph of a Lipschitz function in, say, $B_{1/2}$.

This, together with Part I of this theory, allows us to conclude that F is a $C^{1,\alpha}$ surface.

In Part III, we shall present a general existence theory of weak solutions with free boundaries of finite $(n-1)$-dimensional Hausdorff measure, where in particular, free boundaries will be flat almost everywhere.

In the important case of variational solutions, Alt, Friedman and the author have already shown the existence of such solutions (see [1] and [2]).

This paper is therefore of interest by itself, providing the existence of "as classical as possible" solutions to those free boundary problems.

These techniques can be applied also to obtain Liouville-type theorems concerning the fact that global solutions to these problems have to be one-dimensional.

We shall discuss this matter in a wider context.

1. Basic Definition and Main Theorems

We shall follow the notation of Part I, and use freely the results and techniques employed there.

We recall the definition of a weak solution

DEFINITION 1. In the unit cylinder $C_1 = B_1 \times [-1, 1]$ of $\mathbb{R}^{n+1}$, we are given a continuous function u satisfying:

(i) $\Delta u = 0$ on $\Omega^+(u) = \{u > 0\}$.

(ii) $\Delta u = 0$ on $\Omega^- = \{u \leqq 0\}^0$.

Communications on Pure and Applied Mathematics, Vol. XLII 55–78 (1989)
 CCC 0010-3640/89/010055-24$04.00

(iii) (The weak free-boundary condition) Along $F = \partial\{u > 0\}$, u satisfies the free-boundary condition

$$u_\nu^+ = G(u_\nu^-, X, \nu)$$

in the following sense.

If $X_0 \in F$ and F has a one-sided tangent ball at X_0 (i.e., there is a $B_p(Y)$ such that $X_0 \in \partial B_p(Y)$ and $B_p(Y)$ is contained either in Ω^+ or Ω^-), *then*

$$u(X) = \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

and

$$\alpha = G(\beta, X_0, \nu).$$

The basic requirements on G will be strict monotonicity in u_ν and Lipschitz continuity in all its arguments.

In fact, we may substitute condition (iii) by the weaker condition:

(iii)* (The weak free-boundary condition) Along $F = \partial\{u > 0\}$, u satisfies the free boundary condition

$$u_\nu^+ = G(u_\nu^-, X, \nu)$$

in the following sense.

If $X_0 \in F$, F has a tangent ball at X_0 from Ω^- (i.e., there is a $B_\rho(Y) \subset \Omega^-$, such that $X_0 \in \partial B_\rho(Y)$), and, on $B_\rho(Y)$,

$$u(X) \leqq -\beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|),$$

then $u(X) \geqq \alpha\langle X - X_0, \nu\rangle^+ + o(|X - X_0|)$ on $\mathscr{C}B_\rho$ for any α such that

$$\alpha < G(\beta, X_0, \nu).$$

If $X_0 \in F$, F has a tangent ball at X_0 from Ω^+ (i.e., there is a $B_\rho(Y) \subset \Omega^+$, such that $X_0 \in \partial B_\rho(Y)$) and

$$u^+(X) \geqq \alpha\langle X - X_0, \nu\rangle^+ + o(|X - X_0|),$$

then

$$u^-(X) > \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

for any β such that

$$\alpha > G(\beta, X_0, \nu).$$

This new definition says, basically, that u is a supersolution at those points where we have a tangent ball from Ω^+ and a subsolution where we have a tangent ball from Ω^-.

Going back to Part I, we point out that in constructing our family of subsolutions v_φ (Lemma 11 of Part I) the free boundary asymptotic behavior (Lemma 11 (c)) is derived from a point where u has a tangent ball from $\Omega^-(u)$ and therefore the inequality remains valid (see also Lemma A1 below, about the behavior of u at such point).

Also, when proving a comparison lemma between u and v_φ, the point of comparison is one for which $F(u)$ has a tangent ball from Ω^+ (see Lemma 6 of Part I).

Therefore, the main comparison lemma and hence the regularity theory of Part I is valid with this definition.

The advantage of this definition is that it remains closed under uniform limits, as we shall see later.

The disadvantage is that it produces an undesirable solution of our free boundary problem, namely

$$u = \alpha_1 x_1^+ + \alpha_2 x_1^-$$

with any

$$\alpha_1, \alpha_2 \leqq \inf_X G(0, X, e_1).$$

Extra care will be therefore necessary to construct, in Part III, weak solutions with the desired geometric measure theoretic properties to ensure that they are satisfactory ones.

Definition 2. We shall say that u is ε-monotone in the direction τ if $u(x + \lambda\tau) \geqq u(x)$ for any $\lambda \geqq \varepsilon$.

Our simplest theorem is then as follows.

Theorem 1. *Let $\frac{1}{4}\pi < \theta_0 < \frac{1}{2}\pi$ be given, let u be a solution of our free boundary problem with $G = G(u_\nu^-)$: $R^+ \to R^+$ satisfying: G is strictly monotone and, for some large C, $s^{-C}G(s)$ is decreasing. Then, there is an $\varepsilon(\theta_0, G)$ such that if u is ε-monotone for any unit direction τ in the cone*

$$\Gamma(\theta_0, e_{n+1}) = \{\tau: \alpha(\tau, e_{n+1}) \leqq \theta_0\},$$

on C_1, then, on $C_{1/2} = B_{1/2} \times [-\frac{1}{2}, \frac{1}{2}]$, u is fully monotone in any direction τ in the cone $\Gamma(\theta_1, e_{n+1})$, with $\theta_1 = \theta_1(\theta_0, \varepsilon)$.

THEOREM 2. *Let u be a solution of our free boundary problem with the two extra properties*

(a) $G(0) > 0$, *G bounded in compact sets,*

(b) $\alpha_0 \leqq u^+(X)/d(X, F) \leqq \alpha_1$.

Then there exists

(i) *a* $\theta_0 < \frac{1}{2}\pi$

and

(ii) *an* ε_0,

such that if, in C_1, u^+ is ε-monotone in $\Gamma(\theta_0, e_{n+1})$ for some $\varepsilon < \varepsilon_0$, then in $\Gamma_{1/2}$, u^+ is (fully) monotone in $\Gamma(\theta_1, e_{n+1})$ for some $\theta_1(\theta_0, \varepsilon_0)$.

In particular, F is the graph of a Lipschitz function, Part I applies, and F is the graph of a $C^{1,\alpha}$ function.

THEOREM 2′. *Replace the ε-monotonicity hypothesis by:*

F is in an ε-neighborhood of the graph of a Lipschitz function $y = f(x)$, with Lipschitz norm

$$\|f\|_{\mathrm{Lip}} \leqq \tan\left(\tfrac{1}{2}\pi - \theta_0\right).$$

Then, the same conclusion holds.

THEOREM 3. *In Theorem 2, allow G to depend Lipschitz continuously in x and ν, uniformly on bounded values of u_ν, then, the same conclusion holds. (Condition* (a) *now becomes $G(0, x, \nu) \geqq G_0 > 0$ for any ν in S, any x in C_1.)*

2. ε-Monotonicity and Full Monotonicity

The proof of our theorems, as those of Part I, will be based on an iterative technique that will consist, after an appropriate normalization to the cylinder C_1, in reducing the ε, in the condition of ε-monotonicity, by giving up some ε^α factor on the aperture θ of our cone of directions of monotonicity.

The basic technique will be that of Part I, of constructing a continuous family of subsolutions from point-dependent families of translations of u.

The first lemma shows that if we have ε-monotonicity in a solution τ, we enjoy full monotonicity as we go ε-away from the free boundary.

LEMMA 1. *Let u be a 1-monotone harmonic function in the direction e, in the ball B_M of $\mathbb{R}^n$. Then, if $M = M(n)$ is large enough, $D_e u(0) \geqq C[u(e) - u(0)] \geqq 0$.*

Proof: Since, for any $1 < \lambda < \frac{1}{2}M$,

$$w_\lambda(x) = u(x + \lambda e) - u(x)$$

is harmonic and non-negative on $B_{M/2}$, by Harnack's inequality,

$$0 \leqq C_1 \leqq \frac{w_\lambda(x)}{w_\lambda(y)} \leqq C_2$$

for any X, Y on $B_{M/4}$.

In particular, for λ an integer, $\lambda < \frac{1}{8}M$,

$$0 \leqq C_1 \leqq \frac{w_\lambda(x)}{\lambda w_1(y)} \leqq C_2$$

and, since for λ real, $\lambda \geqq 2$,

$$w_{[\lambda]-1} \leqq w_\lambda \leqq w_{[\lambda]+2},$$

the same inequality holds for any λ with (say) $2 \leqq \lambda \leqq \frac{1}{8}M$ on $B_{M/8}$. (In particular we may assume that $w_1(0) > 0$, since the equality case is now trivial.)

Finally, on $B_{M/8}$, we also have

$$|D_e w_\lambda| \leqq \frac{C}{M} w_\lambda(0).$$

But, for $2 < \lambda < 3$,

$$D_e w_\lambda(0) = D_e u(\lambda e) - D_e(u)(0).$$

Therefore,

$$\begin{aligned} C_1 w_1(0) \leqq w_1(2e) = u(3e) - u(2e) &= \int_2^3 D_e w_\lambda \, d\lambda + D_e u(0) \\ &\leqq \frac{C}{M}\int_2^3 w_\lambda(0)\, d\lambda + D_e u(0) \leqq \frac{C}{M} w_1(0) + D_e u(0). \end{aligned}$$

If M is large enough our lemma is complete.

Given our comparison techniques we will like to write our basic ε-monotonicity assumption on a given cone of directions $\Gamma(\theta, e)$ in the following form, that differs slightly from the obvious one.

DEFINITION 2. u is ε_0-monotone in the cone $\Gamma(\theta, e)$ if, for any $\varepsilon > \varepsilon_0$,

$$u(X) \geqq \sup_{|Y| < \sin\theta} u(X - \varepsilon(e + Y)).$$

COROLLARY 1. *Let u be a weak solution of our free boundary problem and assume that u is ε_0-monotone in $\Gamma(\theta, e)$.*

Then, outside an $M\varepsilon_0$-neighborhood of F, u is actually monotone for any direction τ in $\Gamma(\theta, e)$.

3. The Level Surfaces of ε-Monotone Functions and their Normal Perturbations

We next study the level surfaces of u and of

$$v(x) = \sup_{B_{\varphi(x)}(x)} u(y).$$

Remark. Let u be ε-monotone in the cone $\Gamma(\theta, e)$, then the level surfaces of u, $\partial\{u > \alpha\}$, are contained in a $(1 - \sin\theta)\varepsilon$-neighborhood of the graph of a Lipschitz function, with Lipschitz norm $\cot\theta$.

Proof: We construct the graph of such functions as the union of cones with vertices on $\{u > \alpha\}$.

The next lemma studies the level surfaces of v.

Lemma 2. *Let u be ε-monotone in the cone $\Gamma(\theta, e)$. Let*

$$v(x) = \sup_{B_{\varphi(x)}(x)} u(y).$$

Assume that, for every X under consideration,

$$\sin\bar{\theta} \leqq \frac{1}{1 + |\nabla\varphi|}\left(\sin\theta - \frac{\varepsilon}{2\varphi}\cos^2\theta - |\nabla\varphi|\right).$$

Then, v is monotone in the cone $\Gamma(\bar{\theta}, e)$.

Proof: Let

$$v(X_0) = u(Y_0)$$

with $|Y_0 - X_0| = \varphi(X_0)$. We first estimate $\alpha = \alpha(Y_0 - X_0, e)$. The worst case is when the ball $B_{\varphi(X_0)}(X_0)$ is tangent to $B_{\varepsilon\sin\theta}(Y_0 + \varepsilon e_n)$, at a point Z_0.

Then,

$$|\varepsilon\sin\theta + \varphi(X_0)|^2 = |Y_0 + \varepsilon e - X_0|^2 = |Y_0 - X_0|^2 + \varepsilon^2 + 2\langle Y_0 - X_0, \varepsilon e\rangle$$

$$= \varepsilon^2 + \varphi^2(X_0) + 2\varepsilon\varphi(X_0)\cos\alpha.$$

That is,

$$\cos\alpha = \sin\theta - \frac{\varepsilon}{2\varphi}\cos^2\theta.$$

Following now the discussion in Lemma 10 of Part I, the set

$$\{v(X) \geqq v(X_0)\}$$

contains the domain

$$\varphi(x)^2 \geqq |X - Y_0|^2$$

that has unit normal

$$\bar{\nu} \| Y_0 - X_0 + \varphi(X_0) \nabla \varphi$$

at X_0. Therefore, if $\bar{\theta} < \frac{1}{2}\pi - \alpha(\bar{\nu}, e)$, the set

$$\{v(x) \geqq v(x_0)\}$$

contains a neighborhood of the vertex of the cone $X_0 + \Gamma(\bar{\theta}, e)$.

By applying sin this requirement translates into

$$\sin \bar{\theta} \leqq \cos \alpha(\bar{\nu}, e) = \langle \bar{\nu}, e \rangle.$$

But

$$\langle \bar{\nu}, e \rangle \geqq \left\langle \frac{\nu + \nabla \varphi}{1 + |\nabla \varphi|}, e \right\rangle \geqq \frac{1}{1 + |\nabla \varphi|}\left(\sin \theta - \frac{\varepsilon}{2\varphi}\cos^2 \theta - |\nabla \varphi|\right)$$

with $\nu = (Y_0 - X_0)/|Y_0 - X_0|$. Therefore, it is enough to require that

$$\sin \bar{\theta} \leqq \frac{1}{1 + |\nabla \varphi|}\left(\sin \theta - \frac{\varepsilon}{2\varphi}\cos^2 \theta - |\nabla \varphi|\right).$$

In the next few sections we shall assume that θ_0 is large enough so that, if φ is kept between $\frac{1}{2}\varepsilon \sin \theta_0$ and ε, and $|\nabla \varphi| < \varepsilon < \varepsilon_0$, v will be monotone in a cone $\Gamma(\theta_1, e)$ with θ_1 strictly positive.

4. The Auxiliary Perturbation Function

Our family of perturbations will be basically the one we denote by $\bar{v}_t$ in the proof of Lemma 14, Part I, with a somewhat different φ.

LEMMA 3. *Let* $\{y = f(x)\} = A$ *be the graph of a Lipschitz function, with Lipschitz norm*

$$\|f\|_{\Lambda^1} \leqq M$$

in the cylinder $\mathscr{C} = B_1 \times [-2M, 2M]$.

Then, given $\delta > 0$, *there exist a family of functions* φ_t, $0 \leqq t \leqq 1$, *and a* μ (*independent of* δ) *such that*

(i) $1 \leqq \varphi_t \leqq 1 + t\mu$,
(ii) $\varphi_t \Delta\varphi_t \geqq C|\nabla\varphi_t|^2$,
(iii) $\varphi_t \equiv 1$ *on*

$$S_\delta = \{X\colon d(X, A \cap \partial\mathscr{C}) < \delta\},$$

(iv) $\varphi_t \geqq 1 + \mu t\,(1 - C\delta(d(X, \partial\mathscr{C}) + \delta)/[d(X, A \cap \partial\mathscr{C}) + \delta]^2)$,
(v) $|\nabla\varphi_t| \leqq Ct/\delta$.

Proof: Let ψ_0 be the auxiliary function,
(i) $\Delta\psi_0 = 0$ on $\mathscr{C}$,
(ii) $\psi_0|_{\partial\mathscr{C}}$ be a δ-smoothing of $\chi_{S_{\sigma\delta}}$. In particular,

$$\psi_0|_{\partial\mathscr{C}\cap S_{(\sigma-1)\delta}} \equiv 1,$$

$$\psi_0|_{\partial\mathscr{C}\cap\mathscr{C}S_{(\sigma+1)\delta}} \equiv 0.$$

From the behavior of the Poisson kernel near a smooth part of $\partial\Gamma$,

$$P(X, Y) = \frac{d(X, \partial\mathscr{C})}{|X - Y|^{n+1}},$$

and the fact that, under the Lipschitz transformation $(x, y) \to (x, y - f(x))$, S_δ becomes equivalent to a δ-strip, one can verify that
(i) $\psi_0 \leqq C(\delta(d(X, \partial\mathscr{C}) + \delta))/[d(X, A \cap \partial\mathscr{C}) + \delta]^2$ in $\mathscr{C}$
and
(ii) $C \leqq \psi_0$ in $S_{3\delta}$.
Indeed, the above mentioned Lipschitz transformation preserves the behavior of the Poisson kernel, reducing the problem mainly to estimating, in a two-dimensional half-plane ($y > 0$), the behavior of the harmonic function with smooth boundary values

$$h|_{y=0} = \text{a } \delta\text{-mollification of } \chi_{|x| \leqq C\delta}.$$

Then

$$h \leqq \frac{C\delta(y + \delta)}{x^2 + (y + \delta)^2}$$

which becomes the above estimate after the inverse Lipschitz transformation.
In particular,
(iii) $|\nabla\psi_0| \leqq C/\delta$, outside a δ neighborhood of $\partial\mathscr{C}$.
We next expand, truncate, and mollify of order δ,

$$\psi_1 = \min(C\psi_0((1 - \delta)X), 1) * \xi_\delta.$$

If C is chosen large enough, ψ_1 satisfies

(i) ψ_1 is defined and superharmonic in $\mathscr{C}$,

(ii) $|\nabla\psi_1| \leqq C/\delta$,

(iii) $\psi_1|_{\partial\mathscr{C}} \cap S_\delta \equiv 1$,

(iv) $\psi_1 \leqq C\delta(d(X, \partial\mathscr{C}) + \delta)/[d(X, A \cap \partial\mathscr{C}) + \delta]^2$.

We next construct

$$\psi_2 = \left(\frac{\psi_1 + 1}{2}\right)^{1/(1-2C)},$$

with C the constant from Lemma 1.3 (ii), Part I. Then,

(i) $0 \geqq \Delta\psi_2^{1-2C} = (1 - 2C)\psi_2^{-2C}\Delta\psi_2 + (1 - 2C)(-2C)\psi_2^{-2C-1}(\nabla\psi_2)^2$

or

$$\psi_2\Delta\psi_2 \geqq 2C|\nabla\psi_2|^2,$$

(ii) $1 \leqq \psi_2 \leqq 2^{1/(2C-1)}$, $\psi_2|S_\delta \equiv 1$, and

$$\psi_2 \geqq 2^{1/(2C-1)} - C\delta\big(d(X, \mathscr{C}_1) + \delta\big)/\big[d(X, A \cap \partial\mathscr{C}_1) + \delta\big]^2.$$

Finally we construct

$$\varphi_t = 1 + t\frac{(\psi_2 - 1)}{2^{1/(2C-1)} - 1}.$$

5. A Continuous Family of Subsolutions

In this section we construct our basic family of subsolutions.

LEMMA 4. *In the cylinder $\mathscr{C}_1$, let u be a solution of our free boundary problem, ε-monotone on $\Gamma(\theta, e_{n+1})$. Let*

(i) *φ_t be the function constructed in Lemma 3 (with A the graph constructed in the remark at the beginning of Section 3);*

(ii) *$v_t(X) = \sup_{B_{\sigma\varphi_t}(X)} u$ with $\frac{1}{2}\varepsilon < \sigma < 2\varepsilon$;*

(iii) *w the auxiliary harmonic function, defined on*

$$\Omega^+(v_t) \cap N_{CM\varepsilon}(A)\big(N_\delta(A) = \{X: d(X, A) \leqq \delta\}\big)$$

with boundary values

$$w = \begin{cases} u & \text{on} \quad (\partial N_{CM\varepsilon}(A)) \cap \Omega^+(v_t), \\ 0 & \text{otherwise}; \end{cases}$$

(iv) *$\bar{v}_t = v_t + \eta w$.*

Then, if

(i) $\theta \geqq \frac{1}{4}\pi$,

(ii) $\eta \geqq C\sigma/\delta$,

(iii) *σ/δ small,*

(δ *from the construction of* φ_t *in Lemma* 3), $\bar{v}_t$ *is a subsolution of our free boundary problem on* $\mathscr{C}_{1-C\varepsilon}$.

Proof: We first make sure that $\partial\Omega^+(v_t)$ is uniformly Lipschitz. According to Lemma 2,

$$\frac{1}{1+|\nabla\sigma\varphi_t|}\left(\sin\theta - \frac{\varepsilon}{2\sigma\varphi}\cos^2\theta - |\nabla\sigma\varphi_t|\right)$$

must be strictly positive, that is

$$\frac{1}{1+C\sigma/\delta}\left(\sin\theta - \frac{\varepsilon}{2\sigma}\cos^2\theta - C\frac{\sigma}{\delta}\right)$$

must be strictly positive. Hence, if say $\theta \geqq \frac{1}{4}\pi$ and $\frac{1}{2}\varepsilon < \sigma < \sigma_0$, and σ_0/δ is small, this is ensured. And second we want the free boundary inequality to hold at any point in $\partial\Omega^+(v_t)$.

According to the calculations in Lemma 10 of Section 2, Part I, v_t has at any free boundary point the development

$$v_t \geqq \alpha\langle X - X_0, \nu + \nabla\varphi\rangle^+ - \beta\langle X - X_0, \nu + \nabla\varphi\rangle^- + o(|X - X_0|)$$

with $\alpha = G(\beta)$.

Concerning the correction term, we can say, in view of Dahlberg's theorem (Lemma 1 of Part I), that if we stay $C\varepsilon$-away from ∂C_1 then, near the free boundary of v, $F(v) = \partial\Omega^+(v)$,

$$\frac{w}{v} \geqq C$$

for some constant C depending only on the Lipschitz norm of F (and the C in $N_{CM\varepsilon}$).

Hence, $C\varepsilon$-away from ∂C_1,

$$\bar{v}_t \geqq \tilde{\alpha}\langle X - X_0, \tilde{\nu}\rangle - \tilde{\beta}\langle X - X_0, \tilde{\nu}\rangle + o(|X - X_0|),$$

where

$$\tilde{\alpha} \geqq (1 + C\eta)(1 - C\sigma|\nabla\varphi_t|)\alpha, \qquad \tilde{\beta} \leqq (1 + C\sigma|\nabla\varphi|)\beta.$$

For $\bar{v}_t$ to be a subsolution we want that

$$\bar{\alpha} \geqq G(\bar{\beta}),$$

that is

$$(1 + C\eta)(1 - C\sigma|\nabla\varphi_t|)G(\beta) \geqq G((1 + C\sigma|\nabla\varphi_t|)\beta).$$

Since we have

$$|\nabla \varphi_t| \leqq \frac{t}{\delta},$$

we want η large enough so that

$$(1 + C\eta)\left(1 - C\frac{\sigma t}{\delta}\right)G(\beta) \geqq G\left(\left(1 + C\frac{\sigma t}{\delta}\right)\beta\right).$$

Since $s^{-C}G(s)$ is decreasing this is fulfilled if

$$(1 + C\eta C)\left(1 - C\frac{\sigma t}{\delta}\right) \geqq \left(1 + C\frac{\sigma t}{\delta}\right)^C.$$

That is, if $\sigma t/\delta$ is small we need

$$C\eta \geqq C\frac{\sigma t}{\delta}.$$

Remark. If our free boundary condition were to depend on X and ν, in order to obtain a subsolution we would have to require

$$(1 + C\eta)(1 - C\sigma|\nabla\varphi_t|)G(\beta, X_1, \nu_1) \geqq G((1 + C\sigma|\nabla\varphi_t|)\beta, X_2, \nu_2)$$

with $|X_1 - X_2| \leqq \sigma\varphi_t$ and $|\nu_1 - \nu_2| < \sigma|\nabla\varphi_t|$. Hence, if the Lipschitz norm of G on X and ν is bounded, the dependence on these variables offers no change in our argument.

6. A Basic Inductive Lemma

Our basic inductive lemma will be the following.

LEMMA 5. *Let $\frac{1}{4}\pi < \theta_0 \leqq \theta \leqq \theta_1 \leqq \frac{1}{2}\pi$, let u be a solution of our free boundary problem in C_1, such that u is ε-monotone in the cone $\Gamma(\theta, e_{n+1})$ for some $\varepsilon < \varepsilon_0$.*

Then there exists an $\varepsilon_0 > 0$, $\lambda = \lambda(\varepsilon_0, \theta_0, \lambda, \theta_1) < 1$, such that u is $\lambda\varepsilon$-monotone in the cone of directions $\Gamma(\theta - C\varepsilon^{1/4}, e_{n+1})$ in the domain $B_{1-\varepsilon^{1/8}} \times [-4M, 4M]$.

Proof: Consider, for $\lambda < 1$, the function $u_1(X) = u(X - \lambda\varepsilon e_n)$. Then, from the ε-monotonicity hypothesis (for $1 - \lambda < \sin\frac{1}{4}\pi$),

$$\sup_{B_{\varepsilon(\sin\theta - (1-\lambda))}(X)} u_1(X) \leqq u(X) = u_2(X)$$

since

$$B_{\varepsilon(\sin\theta-(1-\lambda))}(X - \lambda\varepsilon e_n) \subset B_{\varepsilon\sin\theta}(X - \varepsilon e_n).$$

Also, from the full monotonicity lemma (Lemma 1) u is (fully) monotone in the cone $\Gamma(\theta, e_{n+1})$ outside an $M\varepsilon$-neighborhood of $\{y = f(x)\}$.

Therefore,

$$\sup_{B_{\lambda\varepsilon\sin\theta}(X)} u_1(X) \leqq u_2(X)$$

for any X outside such neighborhood. We shall now obtain an intermediate radius,

$$\sigma\varphi_t,$$

in the remaining domain

$$C_{1-\varepsilon^{1/4}} \cap N_{M\varepsilon}(\{y = f(x)\})$$

by means of the family v_t.

Here, we shall use

$$\sigma = \varepsilon(\sin\theta - (1-\lambda)), \quad \lambda \geqq \tfrac{3}{2} - \sin\tfrac{1}{4}\pi = \tfrac{3}{2} - 1/\sqrt{2}, \quad \eta = \varepsilon^{1/4}, \quad \delta = \varepsilon^{1/2}.$$

We shall also limit t in order to make sure that

$$\sigma\varphi_t \leqq \lambda\varepsilon\sin\theta - C\varepsilon^{1+1/4},$$

that is,

$$(\sin\theta - (1-\lambda))(1+t) \leqq \lambda\sin\theta - C\varepsilon^{1/4}$$

or

$$1 + t \leqq \frac{\lambda\sin\theta - C\varepsilon^{1/4}}{\sin\theta - (1-\lambda)}.$$

To be able to attain equality for some $t \leqq 1$, we choose λ close enough to one so that

$$\frac{\lambda\sin\frac{1}{4}\pi}{\sin\frac{1}{4}\pi - (1-\lambda)} \leqq 2.$$

Finally, we verify that the family

$$\bar{v}_t$$

so chosen satisfies

$$\bar{v}_t \leqq u_2$$

on

$$C_{1-C\varepsilon} \cap N_{CM\varepsilon}\{y = f(x)\}.$$

Indeed, along

$$[\partial N_{CM\varepsilon}(\text{graph } f)] \cap C_{1-CM\varepsilon},$$

we have, for $l_1 < l_2 < \lambda\varepsilon \sin\theta$,

$$\sup_{B_{l_1}(X)} u_1 \leqq \sup_{B_{l_2}(X)} u_1(X) - (l_2 - l_1)|\nabla u_1|(X) \leqq \left[1 - \frac{(l_2 - l_1)}{C\varepsilon}\right] u_2(X)$$

(by Lemma 4 of Part I, properly scaled, applied to $u - \bar{C}M\varepsilon$). Note that the level surfaces of u become Lipschitz graphs $M\varepsilon$-away from the free boundary, and that choosing $1 \ll \bar{C} \ll C$ the level surface $u = \bar{C}M\varepsilon$ falls in between $\partial N_{M\varepsilon}$ and $\partial N_{CM\varepsilon}$. This choice of $\bar{C}$ and C depends only on the non-degeneracy hypothesis on u^+.

Therefore, for $0 \leqq t$, up to

$$[\sin\theta - (1 - \lambda)](1 + t) = \lambda \sin\theta - C\varepsilon^{1/4},$$

we get that v_t remains smaller than $(1 - C\varepsilon^{1/4})u_2$ along

$$\partial N_{CM\varepsilon} \cap \mathscr{C}_{1-C\varepsilon}$$

and hence $\bar{v}_t \leqq u_2$.

Along $\partial\mathscr{C}_{1-C\varepsilon} \cap N_{CM\varepsilon}$, we use Dahlberg's theorem and the estimates above. Indeed, since $\delta \gg \varepsilon$, $\varphi_t = \varphi_1 = 1$ in an ε-neighborhood of such a domain,

$$v_t = v_1(X) \leqq u_2(X)$$

and

$$v_t = v_1 \leqq (1 - C\varepsilon^{1/4})u_2(X)$$

along $\partial N_{CM\varepsilon}$ near $\partial\mathscr{C}_{1-C\varepsilon}$, it follows from Dahlberg's theorem that

$$\bar{v}_t \leqq u_2(X)$$

on $\partial N_{CM\varepsilon} \cap \mathscr{C}_{1-C\varepsilon}$ and hence on

$$\mathscr{C}_{1-4\varepsilon} \cap N_{M\varepsilon}$$

for such range of t.

Now, on $\mathscr{C}_{1-C\varepsilon^{1/8}}$,

$$\bar{v}_t \geqq v_t.$$

Also, on such a domain, φ_t can be estimated from below by

$$1 + t(1 - \delta/d^2) = 1 + t(1 - C\varepsilon^{1/4}), \qquad \delta = \varepsilon^{1/2}.$$

For the maximum possible t, the restrictions in Lemma 5 imply that

$$\sigma\varphi_t \geqq \varepsilon\lambda \sin\theta - C\varepsilon^{1+1/4},$$

that is,

$$\sup_{B_{\varepsilon(\lambda \sin\theta - C\varepsilon^{1/4})}(X)} u_1 \leqq u_2(X).$$

Since

$$u_1(X) = u(X - \varepsilon\lambda e_{n+1}),$$

$$u_2(X) = u(X),$$

the lemma is complete.

Remark. From the discussion after Lemma 4, it follows that this lemma remains true if G is allowed to depend on X and ν with bounded Lipschitz norm.

7. Proofs of Theorems 1 and 2

Theorem 1 now follows from the requirement that $\varepsilon = \varepsilon(\theta_0)$ be so small that all domains in the sequence

$$C^k = B_{1 - \sum_{p=1}^{k} (C\varepsilon\lambda^p)^{1/8}} \times [-2M, 2M]$$

contain $B_{1/2} \times [-2M, 2M]$ and the cones

$$\Gamma\left(\theta_0 - \sum_{p=1}^{k} (C\varepsilon\lambda^p)^{1/4}, e_{n+1}\right)$$

all contain

$$\Gamma\left(\tfrac{1}{2}\left(\theta_0 + \tfrac{1}{4}\pi\right), e_n\right),$$

applying inductively Lemma 5 to u, which basically becomes $\varepsilon\lambda^k$-monotone in the k-th cone and k-th cylinder described above.

We also state

THEOREM 1′. *Theorem* 1 *holds if* G *is allowed to depend in a Lipschitz continuous fashion on* X *and* ν.

Proof of Theorem 2: The difficulty in reducing Theorem 2 to Theorem 1 stems from the fact that u^- could be degenerate, that is, very close to zero. For instance, u^- could be identically zero below the graph $\{y = f(x)\}$ and strictly positive somewhere in between $\{y = f(x)\}$ and $\Omega^+(u)$. Therefore, it could not be ε-monotone for any ε. On the other hand, if $u^- \equiv 0$ in $\Omega^-(u)$, Theorem 1 would apply.

What we shall do, is to balance both situations through the following dichotomy.

LEMMA 6. *Let* u *be as in Theorem* 2. *Fix* $\frac{1}{4}\pi < \theta_1 < \frac{5}{16}\pi$, *and let* δ_0 *be as in Lemma* 5 *of Part* I, *with* $M = \operatorname{arctg}(\frac{15}{16}\pi)$. *Call* $u_0 = (\frac{1}{2}e_{n+1})$.

Then, there exists a θ_0 (*close to* $\frac{1}{2}\pi$) *and an* ε_0 (*small*) *such that, if* $\theta \geqq \theta_0$ *and* $\varepsilon \leqq \varepsilon_0$, *we have the following alternative: There is a large constant* C, *such that*

(a) *if* $u^-(-\frac{1}{2}e_n) \geqq C\varepsilon u_0$, *then* u *is* $C\varepsilon^{1/8}$-*monotone in* $\Gamma(\theta_1, e_{n+1})$ *in a* δ_0-*neighborhood of* $A = \{y = f(x)\}$ *in* $\Gamma_{1/2} = B_{1/2} \times [-M, M]$ (*and we fall under Theorem* 1, *for* ε *small enough*);

(b) *if* $u^-(-\frac{1}{2}e_n) \leqq C\varepsilon u_0$, *then* u^+ *is* $\lambda\varepsilon$-*monotone in the cone*

$$\Gamma(\theta - \varepsilon^{\bar{\tau}_1}, e_{n+1})$$

in the domain

$$\mathscr{C}_{1-\varepsilon^{\bar{\tau}_2}}$$

(*and we can iterate as in Theorem* 1, *until we fall under* (a) (*if ever*)).

Proof: From the monotonicity formula (Lemma A3), on, say, $\mathscr{C}_{1-\rho}$,

$$(u_\nu^-)^2(u_\nu^+)^2 \leqq C\left(\frac{\sup|u|}{\rho}\right)^4$$

at any point where $F(u)$ has a tangent ball from u^- and hence (since $u_\nu^+ \geqq Cu_\nu^-$), in view of Lemma A2,

$$\|u^-\|_{\mathrm{Lip}(\mathscr{C}_{1-2\rho}) \cap N_\rho(A)} \leqq \frac{C}{\rho}\left[u\left(\tfrac{1}{2}e_{n+1}\right)\right] = \frac{Cu^0}{\rho}.$$

In particular,

$$u^- \leqq \frac{C\varepsilon u^0}{\rho}$$

above the graph

$$A = \{y = f(x)\}, \qquad \rho \gg \varepsilon.$$

Let v be the harmonic function on $\mathscr{C}_{1-2\rho} \cap \{y \leqq f(x)\}$ with boundary values

$$v = \begin{cases} 0 & \text{on } A, \\ u^- & \text{otherwise.} \end{cases}$$

Then $v \leqq u^- \leqq v + C\varepsilon u_0/\rho$ and Section 1 of Part I applies to v. It follows that for some k large, depending only on $\|f\|_{\mathrm{Lip}}$,

$$v|_{\mathscr{C}_{1-2\rho}} \leqq C\frac{v\left(-\frac{1}{2}e_{n+1}\right)}{\rho^k} \leqq C\frac{u^-\left(-\frac{1}{2}e_{n+1}\right)}{\rho^k},$$

and we may refine our monotonicity formula to

$$C(u_\nu^-)^4 \leqq (u_\nu^-)^2(u_\nu^+)^2 \leqq \frac{C}{\rho^4}(\sup u^-)^2(\sup u^+)^2$$

$$\leqq \frac{C}{\rho^4}\left(\frac{\varepsilon}{\rho}u_0 + \frac{u^-\left(-\frac{1}{2}e_{n+1}\right)^2}{\rho^k}\right)^2\left(\tfrac{1}{2}u_0\right)^2$$

or

$$(u_\nu^-) \leqq \frac{C}{\rho}(u_0)^{1/2}\left(\frac{\varepsilon}{\rho}u_0 + u^-\frac{\left(-\frac{1}{2}e_{n+1}\right)}{\rho^k}\right)^{1/2}$$

on $\mathscr{C}_{1-4\rho}$.

Assume now we are under (a). Choose $\rho = \frac{1}{20}$ and on $\mathscr{C}_{3/5}$ we have

$$\|u^-\|_{\mathrm{Lip}} \leqq u_0^{1/2}\left[u^-\left(-\tfrac{1}{2}e_{n+1}\right)\right]^{1/2}.$$

In particular,

$$u^-(X) \leqq C u_0^{1/2}\left[u^-\left(-\tfrac{1}{2}e_{n+1}\right)\right]^{1/2} d(X, \partial\Omega^+).$$

On the other hand, below $\{y = f(x)\}$ in the δ_0-neighborhood of the free boundary, v is fully monotonic in the cone of directions

$$\Gamma\left(\theta_0 - \tfrac{1}{16}\pi, e_{n+1}\right).$$

Furthermore, let $h(Z)$ be the auxiliary harmonic function of the form

$$h(Z) = r^{\alpha} g(\sigma)$$

with $g(\sigma)$ the first eigenfunction of the Laplacian in the polar cap

$$\{\alpha(\sigma, -e_n) \leqq \theta_0\}.$$

Then, for X_0 in $\{y = f(x)\} \cap \Gamma_{1-5\rho}$,

$$v(X) \geqq Cu(-\tfrac{1}{2}e_{n+1})h(X - X_0)$$

for any $X - X_0$ in the cone $\Gamma(\theta_0, -e_n)$.

In particular, for $\theta_2 < \theta_0$,

$$v(X) \geqq C|X - X_0|^{\alpha} u(-\tfrac{1}{2}, e_{n+1}),$$

whenever X_0 is in $\{y = f(x)\}$ and $X - X_0$ in $\Gamma(\theta_1, -e_{n+1})$.

We are now ready to prove the $C\varepsilon$-monotonicity of u^- in the cone $\Gamma(\theta_1, -e_{n+1})$ in the $\delta_{0/2}$ neighborhood of $\{y = f(x)\}$. Let X_1, X_2 satisfy

$$C_1\varepsilon^{1/8} \leqq |X_1 - X_2| \leqq C_2\varepsilon^{1/8}$$

and

$$X_1 - X_2 \in \Gamma(\theta_1, e_{n+1}).$$

Choose θ_0 so that, say,

$$\theta_1 \leqq \theta_2 - \tfrac{1}{16}\pi = \theta_0 - \tfrac{1}{8}\pi.$$

Then, since for $X_1 \in \Omega^+(u)$ the inequality is trivial, we restrict ourselves to X_1 below $\{y = f(x) + \varepsilon\}$. Then from the choice of δ_0 and θ_2, for

$$\tau = \frac{X_2 - X_1}{|X_2 - X_1|},$$

we have (see Lemmas 4 and 5 of Part I)

(a) $D_\tau v \geqq 0$,

(b) $D_\tau v(X) \geqq C(v(X_2))/\delta_0$ for

$$X - X_1 = \lambda(X_2 - X_1) \quad \text{with} \quad \tfrac{1}{2} \leqq \lambda \leqq 1.$$

It follows that

$$v(X_2) - v(X_1) \geqq C\frac{v(X_2)}{\delta_0}\varepsilon^{1/8}$$

and, since

$$v(X_2) \geqq Cu\left(-\tfrac{1}{2}e_{n+1}\right)\varepsilon^{\alpha/8},$$

we obtain

$$v(X_2) - v(X_1) \geqq \frac{C}{\delta_0}u\left(-\tfrac{1}{2}e_{n+1}\right)\varepsilon^{(\alpha+1)/8}.$$

On the other hand, from the sharper estimate

$$\|u^-\|_{\mathrm{Lip}} \leqq Cu_0^{1/2}u^{1/2}\left(-\tfrac{1}{2}e_{n+1}\right),$$

we have

$$v(X) \leqq u^-(X) \leqq v(X) + Cu_0^{1/2}u^{1/2}\left(-\tfrac{1}{2}e_{n+1}\right)\varepsilon.$$

Thus we get

$$u(X_2) \geqq u(X_1),$$

provided that

$$u^{1/2}\left(-\tfrac{1}{2}e_n\right) \geqq Cu_0^{1/2}\varepsilon^{(7-\alpha)/8}.$$

But when θ_0 goes to $\frac{1}{2}\pi$, α goes to one. Hence we may assume $\alpha < 3$, and that proves alternative (a).

Alternative (b). The proof of alternative (b) is the same as that of Lemma 5, but we must replace u by u^+. That is

$$u_1(X) = u(X - \lambda\varepsilon e_n),$$

$$u_2(X) = u(X).$$

We now repeat the Lipschitz estimates for u_ν^-, but choose ρ so that ρ^k is a small power of ε:

$$\rho^k = \varepsilon^\mu.$$

Then, along the free boundary, on $\Gamma_{1-5\rho} \subseteq \Gamma_{1-5\varepsilon^\mu}$,

$$0 \leqq u_\nu^- \leqq Cu_0^{1/2}\varepsilon^{(1-\mu)/2} \leqq C\varepsilon^{(1-\mu)/2}.$$

Therefore, in our construction of $\bar{v}_t$ we must obtain, along the free boundary

$$(1 + C\eta)\left(1 - C\frac{\sigma t}{\delta}\right)G(0) \geqq G\left(C\varepsilon^{(1-\mu)/8}\right).$$

Here $\delta = \varepsilon^{\mu}$ and $G(C\varepsilon^{(1-\mu)/2}) \leq G(0) + C\varepsilon^{(1-\mu)/2}$. Thus we must have

$$\eta \geqq C\left(\frac{\sigma t}{\varepsilon^{\mu}} + \varepsilon^{(1-\mu)/2}\right) \geqq C\varepsilon^{(1-\mu)/2}$$

and the proof proceeds as that of Lemma 5.

Remark. As before a Lipschitz dependence in X and ν introduces no variation in this argument.

The proof of Theorem 2 now follows by iterating alternative (b) infinitely many times or until we reach alternative (a).

Proof of Theorem 2′: Theorem 2′ reduces to Theorem 2, with another value of θ_0 and ε_0. Indeed, let θ be θ_0 of Theorem 2 and $\bar{\theta} > \theta$. Let u satisfy the hypothesis of Theorem 2′ with $\theta = \bar{\theta}$ and $\varepsilon < \bar{\varepsilon}$ to be chosen. Let v be the harmonic function vanishing in $A = \{y = f(x) - C_0\varepsilon\}$ and $v \equiv u^+$ on the part of $\partial\Gamma_1$ above it. Then $v(X) \geqq u^+(X) \geqq C_1 d(X, A) - C_2\varepsilon$; and, for any $\bar{\bar{\theta}}$, $\theta < \bar{\bar{\theta}} < \bar{\theta}$, v is (fully) monotonic in the cone $\Gamma(\bar{\bar{\theta}}, e_{n+1})$ in a $\delta_0(\bar{\theta}, \bar{\bar{\theta}})$-neighborhood of A, in the domain $\Gamma_{3/4}$.

In particular, if $\delta_0 \geqq d(X, A) > C_3\varepsilon$ we get $D_\tau v \geqq C(\bar{\theta}, \bar{\bar{\theta}})$ for any unit τ in $\Gamma(\bar{\bar{\theta}}, e_{n+1})$ (from Lemma 4 of Part I properly scaled).

For ε, small, we therefore have, for X in $\Omega^+(u)$,

$$\begin{aligned} u^+(X + C_4\varepsilon\tau) - u^+(X) &\geqq v(X + C_4\varepsilon\tau) - v(X) - 2C_0\varepsilon \\ &\geqq (C_4 - CC_3)\varepsilon - 2C_0\varepsilon \\ &\geqq 0, \end{aligned}$$

provided C_4 is large enough.

This completes our proof.

APPLICATION. An important application of Theorem 2 (or 2′) is to the variational problem treated by H. W. Alt, A. Friedman and the author in [2]. In this context, Theorem 2 says:

THEOREM 4. *Let u be a local minimizer of the variational integral*

$$J(v) = \int (\nabla v)^2 + \lambda^2(v)\, dx,$$

with

$$\lambda(v) = \lambda_1\chi_{(v>0)} + \lambda_2\chi_{(v\leqq 0)}$$

and

$$\lambda_1 > \lambda_2.$$

Then, the free boundary $\partial\{v > 0\}$ is locally a $C^{1,\alpha}$ surface except for a closed set of $(n-1)$-dimensional Hausdorf measure zero.

In order to prove Theorem 4, we show that the solution to the variational problem is one of our weak solutions to certain free boundary problems, and that the hypotheses of Theorem 2′ are fulfilled almost everywhere along the free boundary.

The regularity and non-degeneracy hypotheses on u and u^+ are the subjects of Sections 3 and 5 of [2] (see Theorem 3.1 and Theorem 5.3 there).

We next prove that, at any admissible point, according to our definition, u satisfies the required asymptotic inequalities with

$$\alpha = \left(\lambda_1^2 - \lambda_2^2 + \beta^2\right)^{1/2}.$$

LEMMA 7. *Assume now that $X_0 \in F$, that it has a tangent ball $B_r(Y_0)$ from say $\Omega^+(u)$, and that u^+ has, on $B_r(Y_0)$, near X_0, linear behavior*

$$u^+ = \alpha\langle X - X_0, \nu\rangle^+ + o(|X - X_0|),$$

then u^- has on $\mathscr{C}B_r(Y_0)$, near X_0, linear behavior

$$u^- \geqq \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with $\alpha^2 + \beta^2 = \lambda_1^2 - \lambda_2^2$.

Proof: We may assume that $\beta > 0$, $\Omega^-(u)$ is tangent to ∂B_r at X_0 and hence that

$$u = \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|).$$

Furthermore, by blowing up (see Section 6 of [2]), it follows that

$$\alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^-$$

is a global minimizer of our functional. The classical Hadamard variational formulas then tell us that

$$\alpha^2 - \beta^2 = \lambda_1^2 - \lambda_2^2.$$

This proves that variational solutions are weak solutions of an admissible free boundary problem $(G(\beta) = [\lambda_1^2 - \lambda_2^2 + \beta^2]^{1/2})$.

Finally, we recall that from the fact that u^+ was Lipschitz and non-degenerate and from the positive density of $\Omega^+(u)$ and $\Omega^-(u)$ (Theorems 3.1, 5.3, and

7.1 of [2]) we deduce that

(a) F has finite $(n-1)$-dimensional Hausdorff measure (Theorem 7.3 of [2]) and

(b) u is a.e. in Hausdorff measure on F a two plane solution (Theorem 7.4 of [2]).

In particular, at any such point, a suitable dilation

$$u_\lambda = \frac{u(\lambda X)}{\lambda} \quad \text{for } \lambda \text{ small,}$$

falls under the hypotheses of Theorem 1 (case (i) of Theorem 7.4 in [2]) or Theorem 2 (case (ii)).

Appendix

LEMMA A1 (Linear behavior of a harmonic function at a regular boundary point). *Let Ω_1 (respectively Ω_2) be such that*

$$\Omega_1 \cap B_1 \supset \{y > 0\} \cap B_1$$

$$(\textit{respectively } \Omega_2 \cap B_1 \subset \{y > 0\} \cap B_1).$$

Assume that u is a Lipschitz positive harmonic function in Ω_1 (respectively Ω_2) vanishing in $\partial\Omega_1$ (respectively Ω_2) and assume that

$$\bar{B}_1 \cap \partial\Omega_i \cap \{y = 0\} = \{0\}.$$

Then, near zero, u has the asymptotic development

$$u(X) = \alpha y + o(|X|)$$

on $\{y > 0\}$. Furthermore,

$$\alpha > 0 \quad \textit{for} \quad \Omega_1$$

and if

$$\alpha > 0 \quad \textit{for} \quad \Omega_2,$$

then Ω_2 is tangent to $\{y = 0\}$ at zero.

Proof: For Ω_1, let

$$\varepsilon_0 = \sup \varepsilon \colon u \geqq \varepsilon y \quad \text{on} \quad B_1^+$$

and

$$\varepsilon_k = \sup \varepsilon \colon u \geqq \varepsilon_k y \quad \text{on} \quad B_{2^{-k}}^+.$$

Then ε_k is an increasing sequence bounded from above by the Lipschitz norm of u.

Let $\alpha = \varepsilon_\infty = \lim \varepsilon_k$. From its definition,

$$u \geqq \alpha y + o(|X|)$$

in B_1^+.

Assume that there is a sequence of points X_k, with $|X_k| = r_k \to 0$,

$$u(X_k) - \alpha y(X_k) > \delta_0 |X_k|.$$

Since u is Lipschitz, this implies

$$u(X) - \alpha y(X) \geqq \tfrac{1}{2}\delta_0 r_k$$

in a fixed portion of the sphere

$$S_k = \{|X| = |X_k| = r_k\}$$

or, normalizing

$$\frac{u(r_k X)}{r_k} - \alpha y \geqq \tfrac{1}{2}\delta_0$$

in a fixed portion of the unit sphere. Since

$$\frac{u(r_k X)}{r_k} - \alpha y \geqq o(1)$$

in the remainder of the sphere (by the definition of α), we get from the Poisson representation formula,

$$\frac{u(r_k X)}{r_k} - \alpha y \geqq \varepsilon y$$

near zero, for k large enough. This contradicts the definition of α.

To prove the second case, we extend u (to $\tilde{u}$) from Ω_2 to B_1^+ by zero. Now $\tilde{u}$ is subharmonic, and we define

$$\varepsilon_0 = \inf \varepsilon\colon\ \varepsilon y \geqq \tilde{u} \quad \text{in} \quad B_1^+,$$

$$\varepsilon_k = \inf \varepsilon\colon\ \varepsilon y \geqq \tilde{u} \quad \text{in} \quad B_{2^{-k}}^+.$$

This sequence ε_k is decreasing and we proceed as before.

LEMMA A2 (Lipschitz regularity of harmonic functions with control on u_ν). *Let u be a non-negative harmonic function in $\Omega \cap B_1$. Assume that u vanishes on $(\partial\Omega) \cap B_1$, that $0 \in \partial\Omega$, and that whenever $X_0 \in (\partial\Omega) \cap B_1$ has a tangent ball from inside, i.e., ($X_0 \in \partial B_\rho(Y)$, $B_\rho(Y) \subset \Omega$)*

$$\liminf_{\varepsilon \to 0^+} \frac{u(X_0 + \varepsilon\nu)}{\varepsilon} \leqq \alpha_0 \qquad \nu = \frac{Y - X_0}{|Y - X_0|}.$$

Then u is Lipschitz in $B_{1/2} \cap \Omega$ and

$$\|u\|_{\mathrm{Lip}(B_{1/2})} \leqq C\alpha_0.$$

(*Note*: This type of lemma can be found in [1].)

Proof: We assume $\alpha_0 = 1$. It is enough to show that $u(X) < Cd(X, \partial\Omega)$. Let $d(X, \partial\Omega) = h(< \frac{1}{2})$. Then $B_h(X)$ is tangent to $\partial\Omega$ at some point X_0.

By Harnack's inequality, $\inf u|_{\Gamma B_{h/2}(X)} \geqq C_0 u(X)$. In particular, if v is the auxiliary harmonic function in $B_h(X) \setminus B_{h/2}(X)$ defined by

$$v|_{\Gamma \partial B_h(X)} = 0,$$

$$v|_{\Gamma \partial B_{1/2}(X)} = C_0 u(X),$$

we get $v \leqq u$.

But v is an explicit radically symmetric harmonic function, for which

$$\lim_{\varepsilon \to 0^+} \frac{v(X_0 + \varepsilon\nu)}{\varepsilon} = C_1 \frac{u(X)}{h}.$$

Therefore $C_1 u(X)/h \leqq \alpha$ and the proof is complete.

The following monotonicity formula is due to Alt, Friedman and myself (see [2]).

LEMMA A3 (A monotonicity formula and its consequences). *Let u_1, u_2 be two continuous non-negative subharmonic functions in B_1, with disjoint support and both vanishing at the origin.*

Then, the following function is finite and monotone increasing in B_r:

$$\Phi(R) = \frac{\int_{\substack{r<R \\ \sigma \in S_1}} (\nabla u_1)^2 r\,dr\,d\sigma \int_{\substack{r<R \\ \sigma \in S_1}} (\nabla u_2)^2 r\,dr\,d\sigma}{R^4}.$$

If, at 0,

$$u_1 \geqq \alpha y^+ + o(|X|),$$

$$u_2 \geqq \beta y^- + o(|X|),$$

then

$$\Phi(0) \geqq C\alpha^2\beta^2.$$

Note: The lemma and its proof are a minor variant of [2], Lemma 5.1.

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Received March, 1988.