Luis A. Caffarelli

**A Harnack inequality approach to the regularity of free boundaries.
Part III : existence theory, compactness, and dependence on X**

A Harnack Inequality Approach to the
Regularity of Free Boundaries
Part III: Existence Theory, Compactness, and Dependence on X

LUIS A. CAFFARELLI

1. - Introduction

In this third part of our work on regularity of free boundaries we will show the existence of weak solutions to the Dirichlet problem.

In the meantime, we will have to develop compactness theorems, for solutions (or supersolutions) of our F.B. problem, that will allow us to show regularity results for equations and free boundary conditions exhibiting dependence on X.

We recall our definition of weak solution.

DEFINITION 1. In the unit cylinder $C_1 = B_1 \times [-1,1]$ of R^{n+1}, we are given a continuous function u satisfying (L a uniformly elliptic operator $\partial_i(a_{ij}\partial_j u) = 0$ with C^α coefficients):

(i) $\mathbf{L}u = 0$ on $\Omega^+(u) = \{u > 0\}$,

(ii) $\mathbf{L}u = 0$ on $\Omega^- = \{u \leq 0\}^0$,

(iii) (The weak free-boundary condition) Along $F = \partial\{u > 0\}$, u satisfies the free boundary condition

$$u_\nu^+ = G(u_\nu^-, X, \nu)$$

in the following sense.
If $X_0 \in F$, F has a tangent ball at X_0 from Ω^- (resp. Ω^+) (i.e., there exists $B_\rho(Y) \subset \Omega^-$, such that $X_0 \in \partial B_\rho(Y)$), and on $B_\rho(Y)$

$$u(X) \leq -\beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|),$$

$$(\text{resp. } u \geq \alpha < X - X_0, \nu >^+ + o(|X - X_0|))$$

then $u(X) \geq \alpha \langle X - X_0, \nu \rangle^+ + o(|X - X_0|)$ on $C B_\rho$ for any α such that

$$\alpha \leq G(\beta, X_0, \nu).$$

(resp. $u(X) \leq -\beta < X - X_0, \nu >^- + o(|X - X_0|)$).

The basic requirements on G will be strict monotonicity in u_ν and Lipschitz continuity in all its arguments.

We will basically construct our solution as the infimum of a family of restricted supersolutions.

DEFINITION 2. We will say that w belongs to the class $\mathcal{F}$ if w is a continuous function in D and satisfies

(a) $Lw \leq 0$ on $\Omega^+(w)$, $\Omega^-(w)$;

(b)   The set $\Omega^+(w) = \{w > 0\}$ has a tangent ball from outside at every point of $F(w) = \partial \Omega^+$;

(c)   If $X_0 \in F$ and

$$w^+ \geq \alpha \langle X - X_0, \nu \rangle + o(|X - X_0|),$$

then, for some $\varepsilon = \varepsilon(X_0, w) > 0$,

$$w^- \geq \beta \langle X - X_0, \nu \rangle + o(|X - X_0|)$$

for any β such that

$$\alpha + \varepsilon \geq G(\beta, X_0, \nu).$$

(This is a non-uniform strict supersolution condition).

Finally, we will say that $\underline{u}$ is a strict minorant if $\underline{u}$ is locally Lipschitz and for any X_0 in $\partial \Omega^+(\underline{u})$, we have a tangent ball from Ω^+ and whenever

$$\underline{u}^- \geq \beta \langle X - X_0, \nu \rangle^- + o(|X - X_0|),$$

then, for some $\varepsilon = \varepsilon(X_0, \underline{u})$,

$$\underline{u}^+ \geq (\alpha + \varepsilon) \langle X - X_0, \nu \rangle^+ + o(|X - X_0|)$$

for any $\alpha \leq G(\beta, X_0, \nu)$.

THEOREM 1. *Let Ω be a domain with Lipschitz boundary, ϕ a continuous function on $\partial\Omega$. If $\underline{u}$ is a minorant of our F.B.P., with boundary data ϕ, then*

$$u = \inf_{\substack{w \in \mathcal{F} \\ \underline{u} < w}} w$$

is a weak solution of our free boundary problem, provided that the family on the right is non-empty.

THEOREM 2. *The free boundary* $F = \partial\Omega(u^+)$ *has finite* $(n-1)$-*dimensional Hausdorf measure. Further*

$$H^{n-1}(F \cap B(x)) \leq r^{n-1}$$

for any r, *any* x, *and at* H^{n-1} *almost any point of* $F, \Omega^+(u)$ *has a normal vector* ν *in the measure theoretic sense. That is*

$$\lim_{r\to 0} \frac{|\Omega^+ \cap B_r \cap \{\langle X - X_0, \nu \rangle \geq 0\}|}{B_r} = \frac{1}{2}$$

and

$$\lim_{r\to 0} \frac{|\Omega^- \cap B_r \cap \{\langle X - X_0, \nu \rangle \leq 0\}|}{B_r} = \frac{1}{2}.$$

2. - A monotonicity formula

In this section we modify slightly the monotonicity formula from [A-C-F] by letting the operator L depend on X.

LEMMA 1. *Let* u_1, u_2 *be two non-negative continuous subsolutions of* $Lu = D_i(a_{ij}D_ju) = 0$, *in* B_1. *Assume further that* $u_1u_2 = 0$ *and that* $u_1(0) = u_2(0) = 0$. *About* a_{ij} *assume that* a_{ij} *is of class* C^α *and* $a_{ij}(0) = \delta_{ij}$. *Then, the following function is monotone in* R

$$\tilde{\Phi}(R) = \frac{\int\limits_{B_R} (\nabla u_1)^2 r dr d\sigma \int\limits_{B} (\nabla u_2)^2 r dr d\sigma}{g(R)},$$

where

$$g(R) = R^4 e^{-CR^\alpha}.$$

PROOF. We may assume $u_i \equiv 0$ near zero, by replacing u_i by $\max\{(u_i - \varepsilon), 0\}$.

We recall, (see [G-T]) that the fundamental solution V, for $L^*u = 0$ satisfies

$$V = C_0 r^{2-n} + \mathcal{O}(r^{2-n+\alpha}),$$
$$\nabla V = (2 - n)C_0 r^{-n} X + \mathcal{O}(r^{1-n+\alpha}).$$

Then

$$\int\limits_{B_R} (\nabla u_i)^2 \rho \mathrm{d}\rho \mathrm{d}\sigma \le (1 + CR^\alpha) \int \mathbf{L}\left(\frac{1}{2}u_i^2\right) V \mathrm{d}X$$

$$\le (1 + CR^\alpha) \left[(n-2)R^{-n+1}(1 + CR^\alpha) \int\limits_{\partial B_R} \frac{1}{2}u^2 a_{ij}\nu_i\nu_j \mathrm{d}s \right.$$

$$\left. + R^{-n+2}(1 + CR^\alpha) \int\limits_{\partial B_R} \frac{1}{2}uu_j a_{ij}\nu_i \mathrm{d}s \right]$$

$$\le (1 + CR^\alpha) \left\{ \int\limits_{\partial B_R} \left[\frac{n-2}{2} u^2 R^{-n+1} + \frac{1}{2}uu_r R^{-n+2} \right. \right.$$

$$\left. \left. + CR^\alpha u |\nabla_T u| R^{-n+2} \right] \mathrm{d}s \right\}.$$

The proof now follows that of Lemma 5.1 in [A-C-F].

The main difference appears in formula 5.3 (in [A-C-F]), since now the r-factor in $(\tilde{\Phi})'$ satisfies

$$R \frac{g'}{g} = -4 + \alpha C_0 R^\alpha.$$

If this last constant C_0 is chosen large, it controls all of the CR^α error terms in the formulas above.

3. - u^+ is Lipschitz

In this section we show that u^+ is Lipschitz.

LEMMA 2. *Given w in $\mathcal{F}$, we may substitute it by another $\tilde{w} \in \mathcal{F}$, so that $\tilde{w}^+ \le w^+$ and $(\tilde{w})^- = (w)^-$, $\mathbf{L}\tilde{w} = 0$ in $\Omega^+(\tilde{w}) \subseteq \Omega^+(w)$, and furthermore $F(\tilde{w}) \subset F(w)$.*

PROOF. We solve the Dirichlet problem $\tilde{w} = w$ on $\partial\Omega^+(w)$, $\mathbf{L}\tilde{w} = 0$ on $\Omega^+(w)$, for instance by the method of supersolutions. The continuity of $\tilde{w}$ is assured by the fact that w and $\max(0, \mathbf{u})$ are upper and lower barriers.

Note that $\Omega^+(w)$ does not necessarily coincide with $\Omega^+(\tilde{w})$ since $\tilde{w}$ may become identically zero in some connected components of $\Omega^+(w)$. Nevertheless, $F(\tilde{w}) \subset F(w)$ and, if $\tilde{w} \ge \alpha\langle X - X_0, \nu\rangle + o(|X - X_0|)$, since $w \ge \tilde{w}$, the upper linear bound, required in Definition 2 for $\tilde{w}$, follows.

We next prove

LEMMA 3. *Let* $w \in \mathcal{F}$ *and* $Lw^+ = 0$ *on* $\Omega^+(w)$. *Assume that at* $X_0 \in \partial\Omega^+(w)$, w^+ *has the asymptotic development*

$$w^+ \geq \alpha\langle X - X_0, \nu\rangle + o(|X - X_0|).$$

Then, denoting by $G^{-1}(\alpha) = \inf_{X,\nu} G^{-1}(\alpha, X, \nu)$,

(i) $$\alpha G^{-1}(\alpha) \leq \frac{C}{h^2}(\|w\|_{L^\infty})^2,$$

with $h = \mathrm{d}(X_0, \partial C)$;

(ii) *for any domain* D, *compactly contained in* C_1, w^+ *is locally Lipschitz in* D *with*

$$\|w^+\|_{\mathrm{Lip}(\overline{D})} G^{-1}(\|w^+\|_{\mathrm{Lip}(\overline{D})}) \leq C \left(\frac{\|w\|_{L^\infty(D)}}{\mathrm{d}(\overline{D}, \partial C_1)}\right)^2$$

(C *depending only on* diam $\overline{D}$);

(iii) *in particular* $\Omega^+(u)$ *is open.*

PROOF. We recall that from Definition 2, w has the asymptotic development

$$w^- \geq \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|).$$

If $G^{-1}(\alpha, \cdot, \cdot) = 0$, then α is bounded (nothing to prove).
If not, we next prove that

$$\lim_{R \to 0} \tilde{\Phi}(R) \geq C(n)\alpha^2 \cdot \beta^2,$$

with $C(n)$ chosen to give equality if w were smooth on both sides of $w = 0$ and $w_\nu^+ = \alpha$, $w_\nu^- = \beta$. We first estimate by below

$$\int_{B_R} (\nabla w^+)^2 \, dX.$$

Let $\nu = e_{n+1}$, $X = (x, y)$. Then

$$\int_{B_R} (\nabla w^+)^2 \geq \int_{|x| \leq R} dx \int_{y^2 + |x|^2 \leq R^2} (w_y^+)^2 dy$$

$$\geq \int_{|x| \leq R} dx \frac{1}{\ell(x)} \left(\int_{y \leq \sqrt{R^2 - |x|^2}} w_y^+ dy\right)^2$$

$$\geq \int_{|x| \leq R} dx \frac{1}{\ell(x)} \left(\max_{y \leq \sqrt{R^2 - |x|^2}} w^+\right)^2.$$

But the above sequence of inequalities are equalities for $w^+ = \alpha y$. Therefore

$$\int_{B_R} (\nabla w^+)^2 \; dx \geq \int_{B_R} [\alpha^2 - o(1)] \; dx.$$

We now integrate by parts in r,

$$\tilde{\Phi}(R) = C(n) \; \frac{\displaystyle\int_0^R \left(\int_{B_r} (\nabla w^+)^2 \; dx \right) r^{1-n} \; dr \int_0^R \left(\int_{B_r} (\nabla w^-)^2 \; dx \right) r^{1-n} dr}{g(R)}$$

$$\geq C(n) \; \frac{\displaystyle\int_0^R [\alpha^2 - o(1)] \; r dr \int_0^R [\beta^2 - o(1)] \; r dr}{g(R)}$$

and the inequality

$$C(n)\alpha^2\beta^2 \leq \lim_{R \to 0} \tilde{\Phi}(R)$$

is proved.

But

$$\Phi(h) \leq \frac{C}{h^4} \int_{B_{2h}} (w^+)^2$$

$$\int_{B_{2h}} (w^-)^2 \leq \frac{C}{h^4} \|w\|_{L^\infty}^4.$$

This completes the proof of part (i).

To prove part (ii), let now $X_0 \in \Omega^+(w)$, with

$$d(X_0, F) = r < h \leq d(X_0, \partial D).$$

By standard *a priori* estimates

$$|\nabla w(X_0)| \leq \frac{C}{r} \; w(X_0).$$

We now introduce a family of barriers that we will use rather often.

$$h_{X_0,r} \; \text{DEF} \; \begin{cases} h_{X_0,r}|_{\partial B_r(X_0)} = 0 \\ h_{X_0,r}|_{\partial B_{r/2}(X_0)} = 1 \\ L(h_{X_0,r}) = 0, \end{cases}$$

L always normalized so that

$$a_{ij}(X_0) = \delta_{ij}.$$

Note that, for the proper choice of $C(n)$,

$$\left| \nabla h - C(n) r^{n-2} \frac{X}{|X|^n} \right| \leq C r^{\alpha-1}$$

(from the C^α-nature of a_{ij} and the $C^{1,\alpha}$ (normalized) *a priori* estimates).

Since $w|_{B_{r/2}(X_0)} \sim w(X_0)$, for C small we get (on $B_r(X_0) \backslash B_{r/2}(X_0)$)

$$C w(X_0) h_{X_0, r} \leq w.$$

In particular, if Y_0 is the contact point of $B_r(X_0)$ and $F(w)$,

$$w(Y) \geq C \frac{w(X_0)}{r} \langle Y - Y_0, \nu \rangle^+,$$

with $\nu = \dfrac{X_0 - Y_0}{|X_0 - Y_0|}$. Therefore, from Lemma 2,

$$C \frac{w(X_0)}{r} G^{-1} \left(\frac{w(X_0)}{r}, Y_0, \nu \right) \leq \frac{C}{h^2} \|w\|_{L^\infty}^2$$

or

$$|\nabla w(X_0)| G^{-1} \left(C |\nabla w(X_0)| \right) \leq \frac{C}{h^2} \|w\|_{L^\infty}^2.$$

COROLLARY 1. $u = \inf\limits_{w \in \mathcal{F}}$, u satisfies u^+ is Lipschitz, with

$$\|u^+\|_{\mathrm{Lip}(\overline{D})} \, G^{-1} \left(C \|u^+\|_{\mathrm{Lip}(\overline{D})} \right) \leq \frac{C}{h^2} \|u\|_{L^\infty}^2.$$

4. - u is Lipschitz

We next prove that the function u is continuous and in fact u^- is also Lipschitz, in particular u is Lipschitz. We first point out:

LEMMA 4. *If* $w_1, w_2 \in \mathcal{F}$, $w^* = \min\{w_1, w_2\}$ *also belongs to* $\mathcal{F}$.

Next, the main lemma:

LEMMA 5. *Given a point* X_0, *where* $u(X_0) = \sigma < 0$, *the sequence* $w_n \searrow u$ *at* X_0 *may be taken so that, in a* σ-*neighbourhood of* X_0,

(i) w_n *are equilipschitz*,

(ii) $\mathbf{L} w_n = 0$.

PROOF. Let n be large so that

$$w_n(X_0) \leq \frac{u(X_0)}{2} < 0.$$

On the ball $B_{\varepsilon\sigma}$ (ε to be chosen small) we will replace the boundary values of w and construct a new $\tilde{w}$ as follows:

(i) On $\Omega^{+}(w)\backslash\overline{B}_{\varepsilon\sigma}(X_0)$, $\mathbf{L}\tilde{w} = 0$ with boundary data $\tilde{w} = w$, except on $\partial B_{\varepsilon\sigma}$, where $\tilde{w} = 0$;

(ii) On $B_{\varepsilon\sigma}$, $\mathbf{L}\tilde{w} = 0$, with boundary data $\tilde{w} = -w^{-}$;

(iii) $\tilde{w} = w$ otherwise.

We will use this construction often and call it

$$\tilde{w} = R(w, B) \quad \text{(the replacement of } w \text{ on } B\text{)}.$$

(Note that $\underline{u}$, being locally Lipschitz, is strictly negative in $B_{\varepsilon\sigma}$ for ε small, since $\underline{u}(X_0) \leq \sigma$. Hence $\tilde{w} \geq \underline{u}$). Let us see that $\tilde{w} \in \mathcal{F}$.

(a) If $X_1 \in F(\tilde{w})$, for some $w(X_1) = 0$, then $X_1 \in F(w)$ and since $\tilde{w} \leq w$, the asymptotic behavior inequalities follow.

(b) We now study a point $X_1 \in F(\tilde{w})$ where $\tilde{w} = 0$ and $w > 0$. Hence $X_1 \in \partial B_{\varepsilon\sigma} \cap F(\tilde{w})$. Since w^{+} is Lipschitz and vanishes at X_0,

$$\tilde{w} \leq w \leq C\varepsilon\sigma \quad \text{on} \quad B_{2\varepsilon\sigma}(X_0).$$

Therefore in the ring $B_{2\varepsilon\sigma}(X_0)\backslash B_{\varepsilon\sigma}(X_0)$

$$\tilde{w} \leq C\varepsilon\sigma \left(1 - h_{X_0, 2\varepsilon\sigma}\right).$$

In particular, if

$$\tilde{w}(X) \geq \alpha\langle X - X_1, \nu\rangle^{+},$$

we are forced to have

$$\nu = \frac{X_1 - X_0}{|X_1 - X_0|}$$

and

$$\alpha \leq C.$$

On the other hand, coming from inside $B_{\varepsilon\sigma}(X_0)$, we have

$$\tilde{w}(X_0) \leq w(X_0) \leq -\frac{\sigma}{2}.$$

Since

$$\tilde{w}|_{B_{\varepsilon\sigma}} \leq 0$$

and

$$\mathbf{L}\tilde{w} = 0,$$

we get, from Harnack's inequality,

$$\tilde{w}|_{B_{\varepsilon\sigma/2}} \sim -\frac{\sigma}{2}.$$

Hence

$$\tilde{w}\big|_{B_{\varepsilon\sigma}} \leq -C\sigma h_{X_0.\varepsilon\sigma}.$$

Therefore, near X_1, coming from $B_{\varepsilon\sigma}(X_0)$,

$$\tilde{w} \leq -\frac{C\sigma}{\varepsilon\sigma} \langle X - X_1, \nu \rangle$$

(same ν as before). That is

$$\beta \geq \frac{C}{\varepsilon}.$$

This makes of $\tilde{w}$ a supersolution if ε is chosen small.

Finally, from *a priori* estimates and Harnack's inequality,

$$\left|\nabla\tilde{w}\right|_{B_{\varepsilon\sigma/2}(X_0)} \leq \frac{C}{\varepsilon\sigma}\left|\tilde{w}(X_0)\right| \sim \frac{C}{\varepsilon}.$$

The equilipschitzianity now follows.

COROLLARY 2. $Lu = 0$ *in* $\left[\mathcal{C}\Omega^+(u)\right]^0 = \Omega^-(u)$, *and* u *is Lipschitz everywhere.*

PROOF. By standard replacement techniques, it follows that $u = \lim \tilde{w}_k$ on say $B_{\varepsilon\sigma/4}$. (By definition $u \leq \lim \tilde{w}_k$). If $u(X_2) < \lim \tilde{w}_k$, we consider a new sequence $\overset{\approx}{w}_k$ converging to u at X_2, and replace $\min\left(\tilde{w}_k, \overset{\approx}{w}_k\right)$ on $B_{\varepsilon\sigma/2}$ (everything is non-positive on $B_{3\varepsilon\sigma/4}$ and hence the free boundary inequality is not affected).

Note that it also follows that if $u(X) = 0$ for a point of $(\mathcal{C}(\Omega^+))^0$, then $u(X) \equiv 0$ on the corresponding connected component of $(\mathcal{C}(\Omega^+))^0$.

COROLLARY 3. *If K is compactly contained in D, then u is in K the uniform limit of a sequence of functions $w_k \in \mathcal{F}$.*

If $\overline{K} \subset\subset \left[\mathcal{C}(\Omega^+(u))\right]^0$, w_k may be taken non-positive on $\overline{K}$.

PROOF. The first part follows from the fact that w^+ are equilipschitz and the previous substitution lemma. For the second part, let

$$\overline{K} \subset\subset \overline{\overline{K}} \subset\subset \left[\mathcal{C}(\Omega^+)\right]^0$$

and assume $\overline{K}$ and $\overline{\overline{K}}$ have smooth boundary. Let

$$h = \begin{cases} \mathbf{L}h = 0 & \text{on } \overline{\overline{K}}\backslash\overline{K} \\ h \equiv 0 & \text{on } \overline{K} \\ h \equiv 1 & \text{on } \partial\overline{\overline{K}}. \end{cases}$$

If ε is small

$$\varepsilon \left.\frac{\partial h}{\partial\nu}\right|_{\partial\overline{K}} \leq G(0) = \inf G(0, X, \nu).$$

Let k be large so that $w_k \leq \frac{\varepsilon}{2}$ on $\overline{\overline{K}}$, and consider

$$\overline{w}_k = \min (w_k, \varepsilon h).$$

5. - u^+ is non-degenerate

In this section we use the fact that u is the least supersolution to show that u^+ is non-degenerate.

LEMMA 6. (a) *If* $X_0 \in \Omega^+(u)$ *and* $d(X_0, \partial\Omega^+(u)) = r$, *then* $u(X_0) \geq Cr$.
(b) *If* $X_1 \in F$ *and* Σ *is a connected component of* $\Omega^+(u) \cap [B_r(X_1) \setminus B_{r/2}(X_1)]$
with

$$\begin{cases} \overline{\Sigma} \cap \partial B_{r/2} \neq \emptyset \\ \overline{\Sigma} \cap \partial B_r \neq \emptyset, \end{cases}$$

then $\sup_{\Sigma} u \geq Cr$.

(c) $\dfrac{|\Sigma \cap B_r|}{|B_r|} \geq C > 0$.

Note: (a), (b) and (c) are *interior estimates*. That is, they are valid in any compact subset K of D, and the constants C depend on K.

PROOF. Since by Hopf principle $\underline{u}$ grows linearly away from $F(\overline{u})$, part (a) is clear if

$$d(X_0, \Omega^+(\overline{u})) \leq \frac{r}{2}.$$

Indeed, in that case,

$$\sup_{B_{3r/4}(X_0)} \underline{u} \geq Cr,$$

hence

$$\sup_{B_{3r/4}} \underline{u} \geq Cr$$

and by Harnack inequality

$$u(X_0) \geq Cr.$$

We may assume therefore that $\overline{u} \leq 0$ on $B_{r/2}(X_0)$.

Let $u(X_0) = \sigma \ll r$. By Harnack's inequality on u, and the uniform convergence of the w's, we may choose $w \in \mathcal{F}$, $w \leq C\sigma$ on $B_{r/2}$.

Let

$$\tilde{w} = \begin{cases} 0 & \text{on } B_{r/4}(X_0) \\ \min(w, M\sigma(1 - h_{X_0, r/w})) & \text{on } B_{r/2} \setminus B_{r/4} \\ w & \text{otherwise.} \end{cases}$$

Then, for M large enough (depending only on the C of Harnack's inequality) $\tilde{w}$ is continuous along $\partial B_{r/2}(X_0)$. On the other hand, for $M\sigma \ll r$, $\tilde{w}$ is a

supersolution along $\partial B_{r/4}$, since $|(h_{X_0,r/2})_\nu| \sim \frac{C}{r}$. That completes the proof of part (a).

Part (c) will follow from part (b) and the fact that u^+ is Lipschitz.

Part (b) is a general property of non-degenerate Lipschitz solutions of $Lu = 0$. We write it as an auxiliary lemma.

LEMMA 7. *Let u be a Lipschitz non-degenerate function in $\overline{\Omega} \cap B_1$ satisfying* $Lu = 0$, $u|_{\partial\Omega \cap B_1} = 0$ *(with, as usual, a_{ij} uniformly elliptic). Assume further that if $X_0 \in B_{1/2}$,*

$$u(X_0) \geq C d(X_0, \partial\Omega), \qquad 0 \in \partial\Omega.$$

Then, for $r \leq \frac{1}{4}$,

$$\sup_{B_r(0)} u \geq Cr.$$

PROOF. The proof relies on the following observation.

Let $d(X_0, \partial\Omega) = \varepsilon$, then $u(X_0) \sim C\varepsilon$. By the mean value theorem

$$u(X_0) = \int_{\partial B_\varepsilon(X_0)} u \ f \ d\sigma,$$

where f, the Poisson kernel, satisfies

(a)
$$\int_{\partial B_\varepsilon(X_0)} f \ d\sigma = 1$$

(since constants are solutions of $Lu = 0$).

(b) For Y on $\partial B_\varepsilon(X_0)$

$$\int_{\partial B_\varepsilon(X_0)} \chi_{B_{\delta\varepsilon}(Y)} f \ d\sigma \geq C(\delta)$$

(C depending only on the ellipticity of a_{ij}).

Now, by the Lipschitz continuity of u,

$$u \leq \frac{u(X_0)}{2} \sim C\varepsilon$$

in an $\sim \sigma$-neighbourhood of $Y_0 \in \partial B_\varepsilon \cap \partial\Omega$. Therefore

$$u(X_0) \leq \int_{\partial B_\varepsilon(X_0)} \chi_{B_{\varepsilon\delta}(Y_0)} \frac{u(X_0)}{2} \ f \ d\sigma$$

$$+ \sup_{\partial B_\varepsilon(X_0)} u \int_{\partial B_\varepsilon(X_0)} \left(1 - \chi_{B_{\varepsilon\delta}(Y_0)}\right) \ f \ d\sigma.$$

That is,

$$\left(1 - \frac{\lambda}{2}\right) u(X_0) \leq \sup_{\partial B_\epsilon} u(1 - \lambda)$$

for some $\lambda \geq C(\delta)$. That is

$$\sup_{\partial B_\epsilon} u \geq \left[1 + \tilde{C}(\delta)\right] u(X_0)$$

or, for some point X_1 in ∂B_ϵ, the following three quantities are comparable:

$$\begin{cases} u(X_1) - u(X_0) \\ u(X_0) \\ |X_1 - X_0|. \end{cases}$$

If we repeat this argument n-times, we find points X_n so that

(i) $u(X_n) - u(X_0) \geq C|X_n - X_0|,$

(ii) $u(X_n) \geq (1 + \tilde{C})^n u(X_0),$

(iii) $|X_n - X_{n-1}| = d(X_{n-1}, \partial\Omega).$

Let us start with X_0, a point very close to 0 (since $0 \in \partial\Omega$). It follows from (ii) that there is a last $X_n \in B_r$ (since $u(X_n) \to +\infty$). Such an $X_n \notin B_{r/2}$. (This would imply $X_{n+1} \in B_r$, by (iii)). Therefore, by (i),

$$u(X_n) \geq u(X_0) + C|X_n - X_0| \geq C\left(\frac{r}{2} - \varepsilon\right).$$

The proof is complete.

Part (b) of Lemma 6 follows by taking origin at a point of $\partial\Sigma$ in the annulus

$$B_{3/4}(X_1)\backslash B_{2/3}(X_1).$$

(If such a point does not exist (b) is trivial).

6. - u is a supersolution

In this section we will prove that u is a supersolution (not necessarily in $\mathcal{F}$).

LEMMA 8. *Assume that* $u(X_0) = 0$ *and*

$$u(X)^+ = \alpha\langle X - X_0, \nu\rangle^+ + o(|X - X_0|).$$

Then, if

$$u^- = \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|),$$

we must have

$$\alpha \leq G(\beta, X_0, \nu).$$

PROOF. Let $w_k \searrow u$ uniformly. Then w_k cannot remain strictly positive on a neighbourhood of X_0 (if not, u would be a non-negative solution of $Lu = 0$). For each w_k, let

$$B_{m.k} = B_{\lambda_{m.k}} \left(X_0 + \frac{1}{m} \, \nu \right)$$

be the largest ball $\left(\text{with center } X_0 + \frac{1}{m} \, \nu \text{ and radius } \lambda_{m,k}\right)$ contained in $\Omega^+(w_k)$ tangent to $\partial\Omega^+(w_k')$ at $X_{m,k}$, with normal $\nu_{m,k}$ at such points. Then, for adequate subsequences,

$$\lim_{k \to \infty} \lambda_{m,k} = \lambda_m,$$

$$\lim_{k \to \infty} X_{m,k} = X_m,$$

$$\lim_{k \to \infty} \nu_{m,k} = \nu_m,$$

where $B_{\lambda_m} \left(X_0 + \frac{1}{m} \, \nu \right)$ is tangent to $\partial\Omega^+(u)$ at X_m, with normal ν_m.
From the behavior of u^+,

$$|X_m - X_0| = o\left(\frac{1}{m}\right),$$

$$\frac{1}{m} - o\left(\frac{1}{m}\right) \leq \lambda_m \leq \frac{1}{m},$$

$$|\nu_m - \nu_0| = o(1).$$

Now, by definition of supersolution, there exist at $X_{m,k}$ an $\alpha_{m.k}$ and a $\beta_{m,k}$, with

$$\alpha_{m,k} \leq G(\beta_{m,k}, X_{m,k}, \nu_{m,k}),$$

that provide an asymptotic upper barrier for w_k.
That is, on $B_{m.k}$,

$$w_k^+ \leq \alpha_{m,k} \langle X - X_{m,k}, \nu_{m,k} \rangle^+ + o(|X - X_{m,k}|),$$

$$w_k^- \geq G^{-1}(\alpha_{m,k}, X_{m,k}, \nu_{m,k}) \ \langle X - X_{m.k}, \nu_{m,k} \rangle.$$

See Lemma A5 of Part II, adapted to $Lu = 0$ in the obvious manner.
Since $w_k^+ \geq u^+ \geq \alpha\langle X - X_0, \nu\rangle^+ + o(|X - X_0|)$,

$$\underline{\alpha}_m = \lim_{k \to \infty} \alpha_{m,k} \geq \alpha - o(1)$$

($o(1)$ refers of course to the m-variable). Therefore it will suffice to prove that

$$\underline{\beta}_m = \varlimsup_k \beta_{m,k} \leq \beta + o(1).$$

If $\beta_{m,k} = 0$ there is nothing to prove. If $\beta_{m,k} \neq 0$, for any fixed r, from the proof of Lemma 3,

$$\tilde{\Phi}_{m,k}(r) \geq C(n)\alpha_{m,k}^2\beta_{m,k}^2.$$

Also

$$\lim_{k \to \infty} \tilde{\Phi}_{m,k}(r) = \tilde{\Phi}_{X_m;u}(r),$$

since

$$X_{m,k} \to X_m$$

and w_k converges uniformly to u. For the same reason

$$\lim_m \tilde{\Phi}_{X_m;u}(r) = \tilde{\Phi}_{X_0;u}(r).$$

Finally,

$$\lim_{r \to 0} \tilde{\Phi}_{X_0;u}(r) = C(n)\alpha^2\beta^2,$$

(also from the proof of Lemma 3). Hence, given ε, there exists r so that

$$\tilde{\Phi}_{X_0;u}(r) \leq C(n)\alpha^2\beta^2 + \varepsilon,$$

there exist some m and some k so that

$$\tilde{\Phi}_{m,k} = \tilde{\Phi}_{X_{m,k};w_k} \leq C(n)\alpha^2\beta^2 + 2\varepsilon,$$

from where

$$C(n)\alpha_{m,k}^2\beta_{m,k}^2 \leq C(n)\alpha^2\beta^2 + 2\varepsilon.$$

Since $\lim_{m,k} \alpha_{m,k} \geq \alpha$, it follows that $\lim_{m,k} \beta_{m,k} \leq \beta$. The proof is now complete.

7. - u is a subsolution

We are now ready to prove that u is a subsolution.

LEMMA 9. *Assume that u has the asymptotic behavior*

$$u(X) = \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|),$$

with $\alpha > 0$, $\beta \geq 0$. Then $\alpha \geq G(\beta, X_0, \nu)$.

PROOF. The proof of this lemma requires, as usual, a perturbation argument, showing that, if $\alpha < G(\beta, X_0, \nu)$ we may construct a function w in $\mathcal{F}$ smaller than u, contradicting its minimality.

We first note that, even when $\beta = 0$, both domains $\Omega^+(u)$ and $\Omega^-(u)$ are tangent to $\{\langle X - X_0, \nu \rangle = 0\}$ at X_0 from the non-degeneracy of u^+. Our argument will be centered on

$$u_0(X) = \lim_{\lambda \to 0} \frac{1}{\lambda}\, u(X_0 + \lambda X)$$
$$= \alpha \langle X, \nu \rangle^+ - \beta \langle X, \nu \rangle^-.$$

Assume

$$\alpha \leq G(\beta, X_0, \nu) - \delta_0,$$

with $\delta_0 > 0$ and choose $w_k \to u$, uniformly, so that w_k becomes non-positive on any compact set of $\Omega^-(u)$. Then for any $\gamma > 0$ ($\gamma(\delta)$ to be chosen) we may find λ and k so that $w_{k,\lambda}(X) = \frac{1}{\lambda} w_k(X_0 + \lambda X)$ satisfies

(a) If $\beta > 0$, $w_{k,\lambda}(X)\big|_{\partial B_1} \leq u_0 + \gamma \min(\alpha, \beta)$

(b) If $\beta = 0$, $w_{k,\lambda}(X)\big|_{\partial B_1} \leq u_0 + \alpha\gamma$ and $w_{k,\lambda}(X) \leq 0$ on $\{\langle X, \nu \rangle < -\gamma\} \cap B_1$.
Or equivalently

$$w_{k,\lambda}(X) \leq u_0(X + \gamma\nu), \qquad \text{on } \partial B_1.$$

We now make a standard perturbation on u_0 by changing its free boundary,

$$\begin{cases} u_1 = u_0(X + \gamma\nu) & \text{on } \partial B_1, \\ u_1 = 0 & \text{on } \langle X, \nu \rangle = -\gamma + \varepsilon\varphi(X), \\ \mathbf{L}_\lambda u_1 = 0 & \text{on each side of } u_1 = 0, \end{cases}$$

where $\mathbf{L}_\lambda = a_{ij}(X_0 + \lambda X)$ and φ is a C_0^∞ function with $\varphi \equiv 0$ outside $B_{1/2}$, $\varphi \equiv 1$ inside $B_{1/4}$.

From standard perturbation theory, along the new free boundary,

$$\partial\Omega^+(u_1) = \{\langle X - X_0, \nu \rangle = -\gamma + \varepsilon\varphi(X)\}$$
$$|u_{1,\nu}^- - \beta|, \ |u_{1,\nu}^+ - \alpha| \leq \mathcal{O}(\varepsilon + \lambda);$$

hence, if ε, λ are chosen small, depending only on δ_0,

$$\overline{w}_k = \begin{cases} \min\left(w_k, \lambda u_1\left(\frac{1}{\lambda}(X - X_0)\right)\right) & \text{on } B_\lambda(X_0), \\ w_k & \text{otherwise}, \end{cases}$$

becomes an element of $\mathcal{F}$.

Now, if $\gamma \ll \varepsilon$, the set

$$\{\langle X, \nu \rangle \leq -\gamma + \varepsilon\varphi(X)\}$$

contains a neighbourhood of the origin, or equivalently X_0 belongs to $\Omega^-(w_k)$, a contradiction.

PROOF OF THEOREM 1. The proof of Theorem 1 is now complete since whenever $X_0 \in F$ has a tangent ball B_ρ from Ω^+, both

$$u^+ \big|_{B_\rho}$$

and

$$u^- \big|_{CB_\rho}$$

have a linear asymptotic behavior (see Lemma A1 of Part II and the comments after Definition 1) and when X_0 has a tangent ball from Ω^-, u has a full asymptotic development as follows from the non-degeneracy of u^+.

REMARK. We have not used yet the fact that w_k belongs to the restricted subfamily $\mathcal{F}$.

We therefore may now state a compactness theorem for weak solutions of our free boundary problems.

THEOREM 4. *Let u_k be a sequence of minimal weak solutions to free boundary problems*

$$L_k u_k = 0,$$

$$(u_k)^+_\nu = G_k((u_k)^-_\nu, X, \nu),$$

$$\underline{u}_k \leq u_k \leq \overline{u}_k.$$

Assume that $L_k \to L$, $G_k \to G$ uniformly in all of its variables, $\underline{u}_k \to \underline{u}$ uniformly. Assume that the assumptions in L_k, G_k and $\underline{u}_k$ are satisfied uniformly (in particular the uniform one side regularity of the free boundary of $\underline{u}_k$).

Then, if $u_k \to u$ uniformly in a domain D, u is a weak solution of the limiting free boundary problem in D.

PROOF. The proof follows those of Lemmas 8 and 9.

We now start discussing Theorem 2. The proof follows closely the theory developed by Alt and the author (see [A-C]).

LEMMA 10. *Let $X_0 \in \partial\Omega^+$, then, given $\varepsilon < \delta$, the following four quantities are comparable:*

(a) $\frac{1}{\varepsilon} |\{u^+ < \varepsilon\} \cap B_\delta|,$

(b) $Area (\partial B_\delta) = C\delta^{n-1},$

(c) $H^{n-1}(\partial\Omega^+(u) \cap B_\delta),$

(d) $N\varepsilon^{n-1}$, *where N is the number of any family of balls of radius ε with finite overlapping covering $\partial\Omega^+(u) \cap B_\delta$.*

(We recall for the reader that given a compact set in R^n, that has associated to each of its points a ball centred at it, we can always extract a countable subcovering such that the balls of the covering overlap a finite number of times m, depending only on the dimension, that is $\Sigma \chi_{B_{\rho k}(X_k)} \leq m(n)$).

PROOF. Let $u_{\varepsilon.s} = \max(s, \min(u, \varepsilon))$. Then $(0 < s < \varepsilon)$

$$0 = - \int_{B_1(X_0)} u_{\varepsilon.s} L u^+$$

$$= \int_{0 < s < u < \varepsilon} a_{ij} D_i u_{\varepsilon.s} D_j u + \int_{\partial B_1(X_0)} u_{\varepsilon.s} a_{ij} D_j u \nu_i \, dA.$$

But $D_i u_{\varepsilon.s} = D_i u \chi_{s < u < \varepsilon}$. Hence

$$\int_{0 < s < u < \varepsilon} (\nabla u)^2 \leq \int_{\partial B_1(X_0)} C\varepsilon \, dA.$$

Letting s go to zero

$$\int_{0 < u < \varepsilon} (\nabla u)^2 \leq C\varepsilon \ \text{Area} \ (B_1).$$

Now, on any ball B_σ centred at the free boundary we have $\sup u^+ \sim \sigma$, $\inf u^+ = 0$. Hence (u^+ being Lipschitz)

$$\int_{B_\sigma} (\nabla u)^2 \sim \text{Vol} \ (B_\sigma).$$

Consider now a finite overlapping covering of $\partial \Omega^+ \cap B_{1-\varepsilon}$ by balls $B_\varepsilon(X_j)$. Then,

$$\int_{\cup B_\varepsilon(X_j)} |\nabla u|^2 \sim \sum \int_{B_\varepsilon(X_j)} |\nabla u|^2$$

$$\sim \sum \text{Vol} \ (B_\varepsilon(X_j)) \sim N\varepsilon^n \sim \sum \text{Vol} \ B_{2\varepsilon}(X_j)$$

$$\geq \text{Vol} \ \mathcal{N}_\varepsilon(\partial \Omega^+) \cap B_1 \geq C \int_{\cup B_\varepsilon(X_j)} (\nabla u)^2.$$

Hence, all of these quantities are comparable and controlled by

$$\varepsilon \ \text{Area} \ (B_1).$$

Since u is Lipschitz and non-degenerate we may add

$$\left| \{0 < u^+ < \varepsilon\} \cap B_1 \right| \sim \text{Vol} \ \mathcal{N}_\varepsilon(\partial \Omega^+) \cap B_1$$

since, up to constants, one is contained in the others.

Next, let V be the solution of

$$\begin{cases} LV = -\dfrac{1}{|B_\sigma(X_0)|}\, \chi_{B_\sigma(X_0)} & \text{in } B_1(X_0), \\ V = 0 & \text{on } \partial B_1(X_0). \end{cases}$$

From the behavior of the Green function for L (see [L-S-W]), $V \leq C\sigma^{2-n}$ outiside of B_σ and $V_\nu \sim C$ on ∂B_1.

We next write

$$\int\limits_{\partial B_1} V_\nu \, \frac{u\ u_\varepsilon}{\varepsilon} = \int\limits_{B_\sigma} (LV)\, \frac{u\ u_\varepsilon}{\varepsilon} - \int\limits_{B_1} VL\, \frac{u\ u_\varepsilon}{\varepsilon}.$$

The integral on the left is of order C independently of σ. On the right,

$$\left| \int\limits_{B_\sigma} (LV)\, \frac{u\ u_\varepsilon}{\varepsilon} \right| \leq \int\limits_{B_\sigma} u \leq \overline{C}\sigma < \frac{C}{2}$$

for σ small. Hence

$$\int VL \left(\frac{u\ u_\varepsilon}{\varepsilon} \right) \geq \frac{C}{2}.$$

That is

$$C\sigma^{2-n} \frac{1}{\varepsilon} \int\limits_{0<u<\varepsilon} |\nabla u|^2 \geq \frac{C}{2}.$$

Therefore $\varepsilon\, \mathrm{Area}(B_1)$ is actually comparable to all other quantities.

Finally, we compare with $H^{n-1}(\partial\Omega^+ \cap B_1)$. Let B_{φ_k} be a finite covering of $\partial\Omega^+ \cap \overline{B_{1-\varepsilon}}$ by balls, with $\varphi_k < \varepsilon$, that approximates $H^{n-1}(\partial\Omega \cap B_{1-\varepsilon})$.

Let $\varphi < \inf \varphi_k$ and $B_\varphi(X_j^k)$ a finite overlapping covering for $\partial\Omega^+ \cap B_{\varphi_k}$. Then, on one hand

$$\sum A(B_\varphi) \leq C\ A(B_1),$$

by the discussion above with $\varepsilon = \varphi$. On the other hand

$$\sum_j A(B_\varphi(X_j^k)) \geq A(B_{\varphi_k}),$$

also by the discussion above after a φ dilatation of k. Hence the last equivalence is established.

The next observation is that for any w, ∇w is a continuous vector field in $\Omega^+(u)$, since $\Omega^+(u)$ is locally compactly contained in $\Omega^+(w)$.

COROLLARY 4. *The reduced boundary of Ω^+ in the sense of De Giorgi (see [G]) has uniformly positive density in H^{n-1}-measure at any point of $\partial\Omega^+$.*

PROOF. We will prove that, if $X_0 \in F(u)$, $H^{n-1}(\partial_{\mathrm{red}}\Omega \cap B_r) \geq C\ r^{n-1}$.

By scaling

$$u_r = \frac{1}{r} u(r(X - X_0)),$$

it is enough to prove it for $r = 1$. We use the same auxiliary function V, and since ∇w is a continuous vector field in $[\overline{\Omega^+(u)}]$, we may write

$$\int_{\Omega^+ \cap B_1} V \mathbf{L}w - w \mathbf{L}V = \int_{\partial_{\mathrm{red}}\Omega^+ \cap B_1} [V a_{ij}\partial_i w \nu_j - w\, a_{ij}\partial_j V \nu_i]\mathrm{d}A$$

$$- \int_{\partial B_1} w\, a_{ij}\partial_j \nu_i \mathrm{d}A.$$

That is,

$$\left| \int_{B_\sigma} w + \int_{\partial B_1} w\, V_{\nu^*}\mathrm{d}A \right| \le C\sigma^{1-n}H^{n-1}(\partial_{\mathrm{red}}\Omega^+).$$

We let w converge uniformly to u, and choose σ small so that

$$\int_{B_\sigma} u \sim C\sigma$$

cannot compete with

$$\int_{\partial B_1} u\, V_{\nu^*} \sim - \int_{\partial B_1} u \sim -C$$

and the corollary is proved.

We finally point out the regularity of the free boundary.

LEMMA 11. *If* $X_0 \in \partial_{\mathrm{red}}\Omega^+(u)$, u *has at* X_0 *the asymptotic development*

$$u = \alpha\langle X - X_0, \nu\rangle^+ - \beta\langle X - X_0, \nu\rangle^- + o(|X - X_0|)$$

with

$$\alpha = G(\beta, X_0, \nu).$$

PROOF. From the compactness theorem (Theorem 4), the sequence $u_\lambda = \frac{1}{\lambda} u(\lambda(X - X_0))$ converges to a solution u_0 of the free boundary problem

$$\Delta u^0 = 0 \qquad \text{in } \Omega^+(u^0), \ \Omega^-(u^0)$$

and

$$\alpha = G(\beta, X_0, \nu).$$

Furthermore, since $X_0 \in \partial_{\mathrm{red}}\Omega^+(u^0)$, it follows from the non-degeneracy of u^+ that

$$\Omega^-(u^0) \supset \{\langle \nu, X\rangle < 0\}.$$

We now refer to Parts I and II (see [C,I] and [C,II]).

Since u_0^+ is Lipschitz and non-degenerate, it follows from Lemma A1 of Part II that at infinity u_0^+ behaves non-tangentially like $\alpha\langle X, \nu\rangle + o(|X|)$.

Shrinking back,

$$u_{00} = \lim_{\lambda \to \infty} \frac{1}{\lambda}\, u(\lambda X) = \alpha\langle X, \nu\rangle^+ - \beta\langle X, \nu\rangle^-$$

with

$$\alpha = G(\beta, \nu).$$

Therefore u_0 falls under the hypothesis of Theorem 3 of Part II and Theorem 2 of Part I for large balls B_M $(M \to \infty)$. Hence

$$u_0 = \alpha\langle X, \nu\rangle^+ - \beta\langle X, \nu\rangle^-.$$

In turn, this says that, if $L = \Delta$, u itself falls under the hypothesis of those theorems for small enough balls around X_0. Therefore

THEOREM 5. *If $X_0 \in \partial_{\mathrm{red}}(\Omega^+)$ and $L = \Delta$, $\partial\Omega^+$ is a $C^{1,\alpha}$ surface in a neighbourhood of X_0.*

REFERENCES

[A-C]   H.W. ALT - L.A. CAFFARELLI, *Existence and Regularity for a minimal problem with a free boundary*, J. Reine Angew. Math **325** (1981), 105-144.

[A-C-F] H.W. ALT - L.A. CAFFARELLI - A. FRIEDMAN, *Variational problems with two phases and their free boundaries*, T.A.M.S. **282** No. 2 (1984), 431-461.

[C,I]   L.A. CAFFARELLI, *A Harnack inequality approach to the regularity of free boundaries. Part* I: *Lipschitz free boundaries are* $C^{1,\alpha}$, Revista Matematica Iberoamericana, to appear.

[C,II]   L.A. CAFFARELLI, *A Harnack inequality approach to the regularity of free boundaries. Part* II: *Flat free boundaries are Lipschitz*, to appear.

[L-S-W] W. LITTMAN - G. STAMPACCHIA - H. WEINBERGER, *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. di Pisa (3) **17** (1963), 43-77.

[G]   E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Monographs in Mathematics, 1984.

[G-T]   GILBARG - TRUDINGER, *Elliptic P.D.E. of Second order*, 2nd Ed., Springer, New York, 1983.

Department of Mathematics
University of Chicago
Chicago, IL 60637