Quantum Transport is cheaper

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Joint works with E. Caglioti and F. Golse

NOWADAYS for most of us

OPTIMAL TRANSPORT = STAY HOME !

"Humor is the politeness of despair"
Boris Vian or Oscar Wilde or ...

Quantum analogue of Wasserstein distances :

quantization of Wasserstein

How to metrize the space of quantum states ?

other than in Lebesgue, Schatten norms

Quantum optimal transport is cheaper :

thanks to quantum paradigm

Towards general quantum optimal transport :

structure of optimal couplings

Bipartite matching

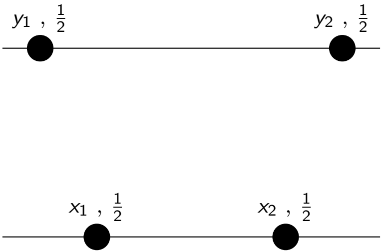
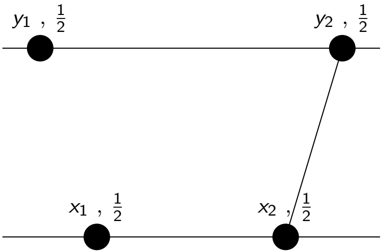


FIGURE: equal masses

Bipartite matching



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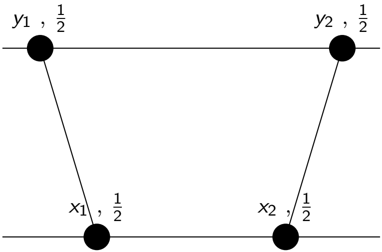

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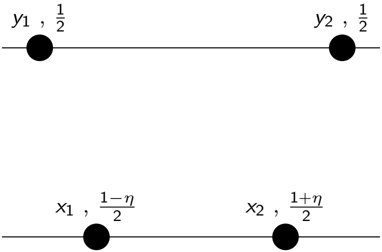
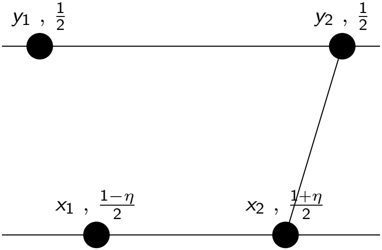


FIGURE: different masses : $\eta > 0$

Bipartite matching



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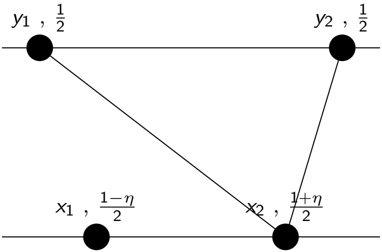

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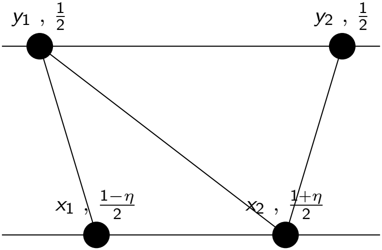

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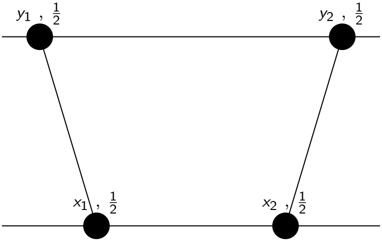

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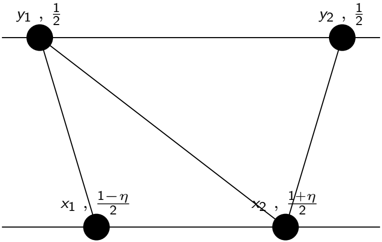


FIGURE: different masses

Wasserstein distances

$\pi \in \mathcal{P}(\mathbb{R}^{2n})$ coupling of $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$: $\int \pi(x,y)dy = \mu(x), \int \pi(x,y)dx = \nu(y)$

2-Wasserstein : $$W_2^2(\mu,\nu) = \inf_{\substack{\pi \text{ coupling} \\ \mu \text{ and } \nu}} \int (x-y)^2 \pi(dx,dy)$$

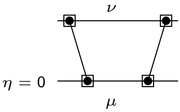
It's a distance !

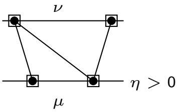
$$W_2(\delta_{q_1}, \delta_{q_2}) = |q_1 - q_2|$$

Bipartite matching

$$\mu = \frac{1-\eta}{2}\delta_{x_1} + \frac{1+\eta}{2}\delta_{x_2}$$

$$\nu = \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}$$





Quantum Wasserstein

$$
\begin{array}{rcl}
\mu,\nu\in\mathcal{P}(\mathbb{R}^{2d}) & \longrightarrow & \text{density matrices } (R,S\geq 0, \operatorname{tr} R,S=1) \text{ on } L^2(\mathbb{R}^d \\
\pi & \longrightarrow & \Pi > 0. \operatorname{tr}\Pi = 1 \text{ on } L^2(\mathbb{R}^{2d}) \\
 & & \text{coupling } R \text{ and } S \text{ i.e. } \operatorname{tr}_2\Pi = R, \operatorname{tr}_1\Pi = S \\
c=(x-y)^2 & \longrightarrow & C=(x-y)^2+(-i\hbar\nabla_x-(-i\hbar\nabla_y))^2-2d\hbar \\
 & & = -\hbar^2\Delta_{x-y}+(x-y)^2-2d\hbar, \ \inf\sigma(C)=0. \\
\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} c\pi & \longrightarrow & \operatorname{tr}(C\Pi)
\end{array}
$$

Quantum Wassertein between R, S density matrices

$$
\mathrm{MK}_\hbar(R,S) = \inf_{\substack{\Pi \\ \text{coupling} \\ R \text{ and } S}} \operatorname{tr}(C\Pi)
$$

It is not a distance ! (e.g. $\mathrm{MK}_\hbar(R,R)$ is not $=0$ for all R) but

$$
W_2(\tilde{W}_\hbar[R],\tilde{W}_\hbar[S]) - 4d\hbar \ \leq \ \mathrm{MK}_\hbar(R,S) \ \leq \ W_2(\sigma^{\text{Töplitz}}[R],\sigma^{\text{Töplitz}}[S])
$$

An exercise : how to metrize the space of quantum states

quantum information : how to distinguish two states ?

$$\psi_q(x) = (\pi\hbar)^{-d/4} e^{-\frac{(x-q)^2}{2\hbar}}$$

Question : which "distance" between ψ_{q_1} and ψ_{q_2} ?

$$\left\|\frac{\psi_{q_1}-\psi_{q_2}}{\sqrt{2}}\right\|_{L^2(\mathbb{R}^d)} = \sqrt{1-e^{-\frac{|q_1-q_2|^2}{4\hbar}}} \underset{\hbar \text{ small}}{\sim} (1-\delta^{\text{Kroneker}}_{q_1,q_2})$$

$$\underset{\hbar\to 0}{\to} \text{ discrete (trivial) topology on space}$$

$$\left\|\frac{\psi-e^{i\theta}\psi}{\sqrt{2}}\right\|_{L^2(\mathbb{R}^d)} = |1-\cos\theta|\ \|\psi\|_{L^2(\mathbb{R}^d)} \text{ : non-physical phase dependence}$$

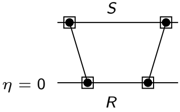
$$\left\|\frac{\psi_{q_1}-\psi_{q_2}}{\sqrt{2}}\right\|_{L^2(\mathbb{R}^d)} = \|\ |\psi_{q_1}\rangle\langle\psi_{q_1}| - |\psi_{q_2}\rangle\langle\psi_{q_2}|\ \|_{HS}$$

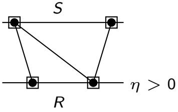
Quantum Wassertein : $MK_\hbar(|\psi_{q_1}\rangle\langle\psi_{q_1}|, |\psi_{q_2}\rangle\langle\psi_{q_2}|) = |q_1-q_2|$

Bipartite matching

$$R = \frac{1-\eta}{2}|\psi_{x_1}\rangle\langle\psi_{x_1}| + \frac{1+\eta}{2}|\psi_{x_2}\rangle\langle\psi_{x_2}|$$

$$S = \frac{1}{2}|\psi_{y_1}\rangle\langle\psi_{y_1}| + \frac{1}{2}|\psi_{y_2}\rangle\langle\psi_{y_2}|$$





Quantum optimal transport is cheaper

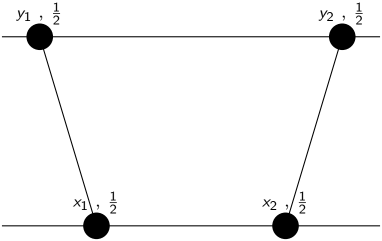


FIGURE: equal masses

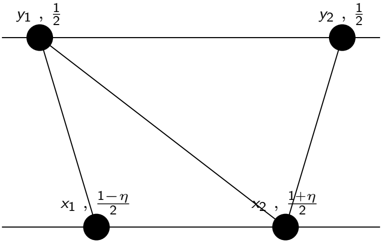


FIGURE: different masses

C : $W_2(\frac{1-\eta}{2}\delta_{x_1} + \frac{1+\eta}{2}\delta_{x_2}, \frac{1}{2}\delta_{y_1} + \frac{1}{2}\delta_{y_2}) = C_{classical}$

Q : $\mathrm{MK}_\hbar(\frac{1-\eta}{2}|\psi_{x_1}\rangle\langle\psi_{x_1}| + \frac{1+\eta}{2}|\psi_{x_2}\rangle\langle\psi_{x_2}|, \frac{1}{2}|\psi_{y_1}\rangle\langle\psi_{y_1}| + \frac{1}{2}|\psi_{y_2}\rangle\langle\psi_{y_2}|) = C_{quantum}$

$\eta = 0$ (equal masses) $C_{quantum} = C_{classical}$

$\eta > 0$ (non equal masses) $C_{quantum} < C_{classical}$

due to quantum terms without classical counterparts

$\eta = 0$ (equal masses)

$$\Pi^{optimal}_{quantum} \leftrightarrow \pi^{optimal}_{classical}$$

$$\begin{aligned} \Pi^{optimal}_{quantum} &= \tfrac{1}{2}(|\psi_{x_1}\rangle\langle\psi_{x_1}| \otimes |\psi_{y_1}\rangle\langle\psi_{y_1}| + |\psi_{x_2}\rangle\langle\psi_{x_2}| \otimes |\psi_{y_2}\rangle\langle\psi_{y_2}|) \\ &= \mathrm{OP}^{T}_{\hbar}[\pi^{optimal}_{classical}] \end{aligned}$$

$$\Longrightarrow C_{quantum} = C_{classical}$$

$\eta > 0$ (non equal masses)

$$\begin{aligned} \Pi^{optimal}_{quantum} &\neq \mathrm{OP}^{T}_{\hbar}[\pi^{optimal}_{classical}] \\ \exists\ \Pi_{quantum} &= \cdots + |\psi_{x_1}\rangle\langle\psi_{x_2}| \otimes |\psi_{y_1}\rangle\langle\psi_{y_1}| + \ldots \\ \mathrm{Tr}\ C\Pi_{quantum} &< C_{classical} \end{aligned}$$

$$\Longrightarrow C_{quantum} < C_{classical}$$

$$\{\text{quantum couplings}\} \gg \{\text{classical couplings}\}$$

Towards general quantum transport

Monge problem : $\min_{T:\mathbb{R}^n \to \mathcal{B}^n} \int C(x, T(x))\mu(dx)$, $\mu \in \mathcal{P}(\mathbb{R}^n)$ density, $C(x,y) =$ cost

e.g. $C(x, T(x)) = |x - T(x)|^2$,

not always well defined $\to$ modern "optimal transport theory"

$\leftrightarrow$ Wasserstein $= \inf_{\pi} \int (x-y)^2 \pi(dx, dy)$

Knott-Smith-Brenier $\to \exists f$ *convex* / $(y - \nabla f(x))\pi_{optimal} = 0$

$(\Rightarrow \int (x-y)^2 \pi_{optimal}(dx, dy) = \int C(x, T(x))\mu(dx),\ T(x) = \nabla f(x))$

Quantum ?

under strong hypothesis (for the moment !)

$$(y - \nabla f(x))\pi_{optimal} = 0 \quad \longrightarrow \quad \Pi_{optimal}^{\frac{1}{2}}(I \otimes y - A \otimes I)\Pi_{optimal}^{\frac{1}{2}} = 0$$

$$\longrightarrow \quad \Pi_{optimal}^{\frac{1}{2}}(I \otimes (-i\hbar\nabla_y) - B \otimes I)\Pi_{optimal}^{\frac{1}{2}} = 0$$

$$T = \nabla f \quad \longleftrightarrow \quad A = \frac{1}{i\hbar}[-i\hbar\nabla, F],\ B = -\frac{1}{i\hbar}[y, F]$$

$$(\nabla_q f = \{p, f\},\ \nabla_p f = -\{q, f\})$$