Derivation of the Vlasov equation for short range interactions

Peter Pickl, joint work with Manuela Feistl

Mathematisches Institut
LMU München

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The microscopic system

- N interacting particles (for example stars), Newtonian dynamics
- Trajectory in phase space:
  $X = (Q, P) = (q_1, q_2, \dots q_N, p_1, p_2, \dots p_N) \in \mathbb{R}^{6N}$
- q_j: position of particle j
  p_j momentum (=speed) of particle j
- Newtonian dynamics: $\dot{Q} = P$
  $\dot{P} = F(Q)$
  Force on j^{th} particle: $(F)_j = N^{-1+4\beta} \sum_{k \neq j} f(N^{\beta}(q_j - q_k))$
- Macroscopic: law of motion for particle density

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Vlasov equation

Law of large numbers: Replace force by its expectation value.

- $\rho(t, q(t), p(t)) = \text{const}$
- $\frac{d}{dt}\rho(t, q(t), p(t)) = 0$
- $\frac{\partial}{\partial t}\rho + \nabla_q \rho \cdot \dot{q} + \nabla_p \rho \cdot \dot{p} = 0$
- Continuity equation: $\frac{\partial}{\partial t}\rho + \nabla_q \rho \cdot p + \nabla_p \rho \cdot \overline{f} = 0$
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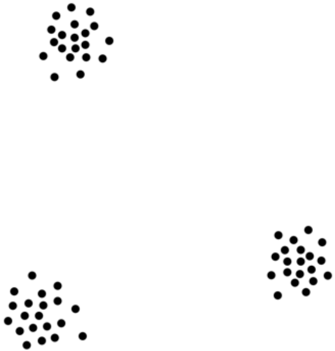
Dynamics of clusters

ϵ^{-1} cluster, ϵN particles each



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- Goal: compare trajectory X_t with density ρ_t.
- Idea: bring ρ_t to level of trajectories
- Vlasov equation: $\frac{\partial}{\partial t}\rho + \nabla_q \rho \cdot p + \nabla_p \rho \cdot (\rho \star_q f) = 0$
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Our results

- Niklas Boers, P.P. (2015):
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- Dustin Lazarovici, P.P. (2015):
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- Expectation value easier to control than $\mathbb{P}$: $\mathbb{P}(A) = \mathbb{E}(\chi_A)$
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- Idea: Use the second order of the equation to improve β

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