The weak Galerkin finite element method for eigenvalue problems

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SJTU, Changchun

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Talk Outline

- WG method for Laplacian eigenvalue problems
  - Basic Theory of weak Galerkin method
  - WG scheme for eigenvalue problem
  - Lower bounds approximation
- Guaranteed lower bound with WG method
  - Guaranteed CR lower bound and GLB with WG
  - Numerical Experiments
- Acceleration of WG method in eigenvalue problems
  - Upper & lower bound
  - Two grid method
  - Shifted-inverse power method
- Conclusion and Ongoing Work
I. WG method for Laplacian eigenvalue problems
Applications of eigenvalue problem

The need of accurate eigenvalues of PDEs arises in:

- **Mathematical aspects**
  - Poincaré constant in the Soblev theory
  - spectral distribution of nonlinear equations...

- **Physical aspects: vibrations and related phenomena.**
  - plasma physics in fusion experiments
  - sympathetic vibration phenomenon...

- **Other fields**
  - petroleum reservoir simulation
  - linear stability of flows in fluid mechanics, ...
Model problem

\[-\Delta u = \lambda u, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial \Omega,\]
\[
\int_{\Omega} u^2 \, d\Omega = 1,
\]

where \(\Omega\) is a polygonal or polyhedral domain in \(\mathbb{R}^d\) \((d = 2, 3)\).
Many efforts have been dedicated to approximating the solution of the elliptic eigenvalue problem. Some numerical methods for solving the elliptic boundary value problem can be applied to the eigenvalue problem, such as

- Finite difference method.
- Finite element method.
- **How about the weak Galerkin method?**
What is the weak Galerkin method?
WG FEMs: Weak Operators + Stabilizer

(DG function)  (WG function)
Consider the second order elliptic problem:

\[-\Delta u = f, \quad \text{in } \Omega\]

\[u = 0, \quad \text{on } \partial \Omega\]

Testing (1) by \(v \in H^1_0(\Omega)\) gives

\[-\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\partial \Omega} \frac{\partial u}{\partial n} v ds = \int_{\Omega} fv dx\]

that is,

\[(\nabla u, \nabla v) = (f, v)\]
Variational form: find \( u \in H^1_0(\Omega) \) such that

\[
(\nabla u, \nabla v) = (f, v), \quad \forall v \in H^1_0(\Omega)
\]

Partition \( \Omega \) into triangles or tetrahedra \( \mathcal{T}_h \), let \( W_h \subset H^1_0(\Omega) \) be a finite dimensional space.

\[
W_h = \{ v \in H^1_0(\Omega); \, v|_T \in P_k(T), \, T \in \mathcal{T}_h \}.
\]

Conforming FEM: find \( u_h \in W_h \) such that

\[
(\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in W_h
\]
Conforming FEM: find $u_h \in W_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h), \quad \forall v_h \in W_h$$

Weak Galerkin FEM Seeking $u_h \in V_h$ satisfying

$$(\nabla_w u_h, \nabla_w v) + s(u_h, v) = (f, v), \quad \forall v_h \in V_h$$

with

$$s(u_h, v_h) = \sum_{T \in T_h} h_T^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T}$$
\( \mathcal{T}_h \): partition of the domain \( \Omega \), shape regular

Figure: Illustration of a shape-regular polygonal element \( T \) and Polytopal Mesh
Weak Finite Element Spaces

construct local discrete elements

\[ W(T, k, j) := \{ v = \{ v_0, v_b \} : v_0 \in P_k(T), v_b \in P_j(\partial T) \} . \]

patch local elements together to get a global space

\[ V_h := \{ v = \{ v_0, v_b \} : \{ v_0, v_b \} |_T \in W(T, k, j), \forall T \in \mathcal{T}_h \} . \]

Weak finite element spaces with homogeneous boundary value:

\[ V^0_h := \{ v = \{ v_0, v_b \} \in V_h, v_b|_{\partial T \cap \partial \Omega} = 0, \forall T \in \mathcal{T}_h \} . \]
The space \([P_\ell(T)]^2\) is used for the computation of \(\nabla_{w,j}\).
Preliminaries

- **WG finite element space**

  \[ V_h = \{(v_0, v_b) : v_0 \in P_k(T), v_b \in P_j(\partial T), v_b = 0 \text{ on } \partial \Omega\} \]

- **Discrete weak gradient** \( \nabla_w v \in [P_\ell(T)]^d \)

  \[
  (\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in [P_\ell(T)]^d
  \]
There are three polynomial spaces involved in WG:

- $P_k(T)$: building block for $\nu_0$ on each element.
- $P_j(\partial T)$: building block for $\nu_b$ on the boundary of each element.
- $P_\ell(T)$: building block for the discrete weak gradient/derivative.

**Issues to consider:**

1. What combination of $k, j, \text{ and } \ell$ shall produce stable finite elements?
There are three polynomial spaces involved in WG:

- $P_k(T)$: building block for $v_0$ on each element.
- $P_j(\partial T)$: building block for $v_b$ on the boundary of each element.
- $P_\ell(T)$: building block for the discrete weak gradient/derivative.

**Issues to consider:**

1. What combination of $k, j,$ and $\ell$ shall produce stable finite elements?

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Wang-Zhai-Z, JSC’2018, 74
Case I: When

\[ j \partial v = Q^b_m(v_0 - v_b) \]

with \( m = \max\{j, \ell\} \), the following hold

- when \( \ell < k - 1 \), the resulting discrete problem is singular.
- when \( \ell \geq k - 1 \) and \( j < \ell \) the convergence order is \( s = \min\{k, j\} \).
- when \( \ell \geq k - 1 \) and \( j \geq \ell \) the convergence order is \( k \).
Case II: When

\[ j\partial v = Q_j^b(v_0 - v_b) \]

the following hold

- when \( \ell < k - 1 \), the resulting discrete problem is singular.
- when \( \ell \geq k - 1, j \geq k - 1 \) and \( j < \ell \) the convergence order is \( s = \min\{k, j\} \).
- when \( \ell \geq k - 1, j \geq k - 1 \) and \( j \geq \ell \) the convergence order is \( k \).
- when \( \ell > k \) the convergence order is \( s = \min\{k, j\} \).
- when \( \ell = k - 1, k \) and \( j \leq k - 2 \) the property is chaos.
Case III: When

\[ j \partial v = v_0 - v_b \]

the following hold

- when \( \ell \geq k - 1 \), the convergence order is \( s = \min\{k, j\} \).
- when \( \ell < k - 1 \), there are two cases:
  - when \( \ell = k - 2, j < k \), the convergence order is \( s = \min\{k, j\} \).
  - otherwise the resulting discrete problem is singular.
The Optimal WG Element for Second Elliptic

The space \([P_{k-1}(T)]^2\) is used for the computation of \(\nabla w_j\)

Figure: A triangular element with acute angles
Standard WG for LEP
Laplacian eigenvalue problem

Model problem

\[-\Delta u = \lambda u, \quad \text{in } \Omega,\]
\[u = 0, \quad \text{on } \partial \Omega,\]
\[\int_{\Omega} u^2 \, d\Omega = 1,\]

where \(\Omega\) is a polygonal or polyhedral domain in \(\mathbb{R}^d (d = 2, 3)\).

Variational form: Find \((\lambda; u) \in (\mathbb{R}; H^1_0(\Omega))\) such that \((u, u) = 1\) and

\[(\nabla u, \nabla v) = \lambda(u, v), \quad \forall v \in H^1_0(\Omega).\]
Preliminaries

- **WG finite element space**

\[ V_h = \{ (v_0, v_b) : v_0 \in P_k(T), v_b \in P_{k-1}(e), v_b = 0 \text{ on } \partial \Omega \} . \]

- **Discrete weak gradient** \( \nabla_w v \in [P_r(T)]^d \) for \( v \in V_h \) on each element \( T \):

\[ (\nabla_w v, q)_T = -(v_0, \nabla \cdot q)_T + \langle v_b, q \cdot n \rangle_{\partial T}, \quad \forall q \in [P_r(T)]^d . \]
Standard WG scheme: Find \((\lambda_h; u_h) \in (\mathbb{R}; V_h)\) such that 
\((u_h, u_h) = 1\) and

\[
(\nabla_w u_h, \nabla_w v_h) + s(u_h, v_h) = \lambda_h(u_0, v_0), \quad \forall v_h \in V_h.
\]

with

\[
s(u_h, v_h) = \sum_{T \in T_h} h_T^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T}.
\]
Suppose the eigenvalue problem has k regularity, and \((\lambda_h; u_h)\) is the j-th eigenpair of the WG scheme. There exists an exact eigenpair \((\lambda; u)\) such that

- \(|\lambda - \lambda_h| \leq Ch^{2k}\),
- \(\|Q_h u - u_h\| \leq Ch^k\),
- \(\|Q_0 u - u_0\| \leq Ch^{k+1}\).
### Table: Errors of Eigenvalues by WG Scheme.

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<td>8.58e-2</td>
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<td>$\lambda_4 - \lambda_{4,h}$</td>
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<td>1.67</td>
<td>1.90</td>
<td>1.97</td>
<td>1.99</td>
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</table>

All the errors of $\lambda - \lambda_h$ are **positive!!!**

What does this mean? Could the $\lambda_h$ obtained by WG be lower bounds?
The eigenvalue is a positive real number.

Due to the Rayleigh quotient and minimum-maximum principle, standard FEMs always provide the upper bounds of eigenvalues.

If we get lower bounds, we have credible intervals for the eigenvalues.

The weak Galerkin finite element method for eigenvalue problems...
Modified WG ensuring lower bounds
Existing work

Mainly two approaches:

- **Post-processing method:**
  
  Lason SINUM’00, Liu and Oishi SINUM’13, Dai, Xu, and Zhou NM’08
  
  - deal with an auxiliary problem
  - can not get the full order error estimate

- **Special nonconforming finite element:**
  
  Lin, Huang et. al. MC’ 08, Carstensen et.al. MC’14, Lin, Xie, and Xu MC’14
  Hu, Huang et. al. NM’ 15, Yang, Lin et. al. ACM’ 12
  
  - hard to construct degree \( \geq 2 \) have been constructed
  - hard to construct in high dimensional cases

Could the **WG** method provide the **asymptotic lower bound** naturally?
Suppose \((\lambda, u)\) is the exact eigenpair, and \((\lambda_h, u_h)\) is the numerical eigenpair. Then for any \(v \in V_h\) we have

\[
\lambda - \lambda_h = \|\nabla u - \nabla_w u\|^2 + s(u_h - v, u_h - v) + 2(\nabla u - \nabla_w v, \nabla_w u_h) - s(v, v).
\]

\[
(\geq Ch^{2k}) \quad (\leq Ch^{2k})
\]

\[
-\lambda_h\|u_0 - v_0\|^2 - \lambda_h(\|u_0\|^2 - \|v_0\|^2) + 2(\nabla u - \nabla_w v, \nabla_w u_h) - s(v, v).
\]

\[
(\leq Ch^{2k+2}) \quad (\leq Ch^{2k+2})
\]

\[
(= 0) \quad (\leq Ch^{2k})
\]
Suppose \((\lambda, u)\) is the exact eigenpair, and \((\lambda_h, u_h)\) is the numerical eigenpair. Then for any \(v \in V_h\) we have

\[
\lambda - \lambda_h = \|\nabla u - \nabla_w u\|^2 + s(u_h - v, u_h - v)
\]

\[
\geq Ch^{2k}
\]

\[
\leq Ch^{2k}
\]

\[
-\lambda_h\|u_0 - v_0\|^2 - \lambda_h(\|u_0\|^2 - \|v_0\|^2)
\]

\[
\leq Ch^{2k+2}
\]

\[
\leq Ch^{2k+2}
\]

\[
+2(\nabla u - \nabla_w v, \nabla_w u_h) - s(v, v).
\]

\[
= 0
\]

\[
\leq Ch^{2k}
\]
Modified WG scheme: Find \((\lambda_h; u_h) \in (\mathbb{R}; V_h)\) such that \((u_0, u_0) = 1\) and

\[
(\nabla_w u_h, \nabla_w v_h) + \tilde{s}(u_h, v_h) = \lambda_h (u_0, v_0), \quad \forall v_h \in V_h,
\]

where

\[
\tilde{s}(v, w) = \sum_{T \in T_h} h_T^{-1+\varepsilon} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial T}
\]

\[
s(v, w) = \sum_{T \in T_h} h_T^{-1} \langle Q_b v_0 - v_b, Q_b w_0 - w_b \rangle_{\partial T}
\]
Lemma

Suppose \((\lambda, u)\) is the exact eigenpair, and \((\lambda_h, u_h)\) is the numerical eigenpair. Then for any \(v \in V_h\) we have

\[
\lambda - \lambda_h = \|\nabla u - \nabla w u\|^2 + \tilde{s}(u_h - v, u_h - v) \\
\quad \geq Ch^{2k} \quad (\leq Ch^{2k - 2\varepsilon}) \\
\quad -\lambda_h\|u_0 - v_0\|^2 - \lambda_h(\|u_0\|^2 - \|v_0\|^2) \\
\quad \leq Ch^{2k+2-2\varepsilon} \quad \quad \leq Ch^{2k+2-2\varepsilon} \\
\quad +2(\nabla u - \nabla w v, \nabla w u_h) - \tilde{s}(v, v) \\
\quad = 0 \quad \quad \quad \quad (\leq Ch^{2k+\varepsilon}) \\
\quad \geq 0 \quad (h \text{ small enough})
\]
Error estimates

**Theorem**

Suppose the eigenvalue problem has \( k \) regularity, and \((\lambda_h; u_h)\) is the \( j \)-th eigenpair of the modified WG scheme. The exact eigenpair is \((\lambda; u)\). Then

\[
0 \leq \lambda - \lambda_h \leq Ch^{2k-2\epsilon}, \\
\| Q_h u - u_h \| \leq Ch^{k-\epsilon}, \\
\| Q_0 u - u_0 \| \leq Ch^{k+1-\epsilon}.
\]

as \( h \) is small enough.

Remark:

The numerical results show that \( \lambda_h \) is still a lower bound of \( \lambda \) even if \( \epsilon = 0 \).
Numerical Experiments
Example

We consider the problem on the square domain $\Omega = (0, 1)^2$. It has the analytic solution

$$\lambda = (m^2 + n^2)\pi^2, \quad u = \sin(m\pi x)\sin(n\pi y),$$

where $m, n$ are arbitrary integers. The first four eigenvalues are $\lambda_1 = 2\pi^2$, $\lambda_2 = 5\pi^2$, $\lambda_3 = 8\pi^2$ and $\lambda_4 = 10\pi^2$, where $\lambda_2$ and $\lambda_4$ have 2 algebraic and geometric multiplicities.
Numerical Experiments

**Table:** Convergence rates for $\varepsilon = 0.1$ and $k = 1$.

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<td>$\lambda_1 - \lambda_{1,h}$</td>
<td>4.69e+0</td>
<td>1.50e+0</td>
<td>4.24e-1</td>
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<td>3.08e-2</td>
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<td>order</td>
<td>1.64</td>
<td>1.82</td>
<td>1.88</td>
<td>1.89</td>
<td>1.90</td>
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<td>2.26e+1</td>
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<td>1.71</td>
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<td>1.79</td>
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The weak Galerkin finite element method for eigenvalue problems
Numerical Experiments

Table: Convergence rates for $\varepsilon = 0.05$ and $k = 1$.

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<td>2.20e+1</td>
<td>8.32e+0</td>
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The weak Galerkin finite element method for eigenvalue problems.
Table: Convergence rates for $\varepsilon = 0$ and $k = 1$.

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<td>1.97</td>
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</table>
Numerical Experiments

Table: Convergence rates for $\varepsilon = 0.1$ and $k = 2$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$1/4$</th>
<th>$1/8$</th>
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<th>$1/64$</th>
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<tbody>
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<td>$\lambda_1 - \lambda_{1,h}$</td>
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Numerical Experiments

**Table:** Convergence rates for $\varepsilon = 0.1$ and $k = 2$.

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<tr>
<td>$|Q_h u_1 - u_{1,h}|$</td>
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Numerical Experiments

Table: Convergence rates for $\varepsilon = 0.1$ and $k = 2$.

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The weak Galerkin finite element method for eigenvalue problems
II. Guaranteed lower bound with WG method
WG scheme: Find \((\lambda_h; u_h) \in (\mathbb{R}; V_h)\) such that \((u_h, u_h) = 1\) and

\[
(\nabla_w u_h, \nabla_w v_h) + s(u_h, v_h) = \lambda_h(u_0, v_0), \quad \forall v_h \in V_h.
\]

with

\[
s(u_h, v_h) = \gamma(h) \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_{\partial T}.
\]
The weak Galerkin finite element method for eigenvalue problems

**Theorem**

Suppose the eigenvalue problem has $k$ regularity, and $(\lambda_h; u_h)$ is the $j$-th eigenpair of the WG scheme. There exists an exact eigenpair $(\lambda; u)$ such that

\[
|\lambda - \lambda_h| \lesssim \gamma(h)^{-2} h^{2k},
\]

\[
\|Q_h u - u_h\| \lesssim \gamma(h)^{-1} h^k,
\]

\[
\|Q_0 u - u_0\| \lesssim \gamma(h)^{-1} h^{k+1}.
\]
Suppose \((\lambda, u)\) is the exact eigenpair, and \((\lambda_h, u_h)\) is the numerical eigenpair. Then for any \(v \in V_h\) we have

\[
\lambda - \lambda_h = \|\nabla u - \nabla_w u\|^2 + s(u_h - v, u_h - v)
\]

\[
(\geq Ch^{2k}) \quad \quad \quad \quad (\leq Ch^{2k})
\]

\[
- \lambda_h \|u_0 - v_0\|^2 - \lambda_h (\|u\|^2 - \|v_0\|^2)
\]

\[
(\leq Ch^{2k+2}) \quad \quad \quad \quad (\leq Ch^{2k+2})
\]

\[
+ 2(\nabla u - \nabla_w v, \nabla_w u_h) - s(v, v).
\]

\[
(= 0) \quad \quad \quad \quad (\leq C\gamma(h)h^{2k})
\]
Question:

- What if $\gamma(h) = C$?
- What is “sufficiently small”?

How to get guaranteed eigenvalue lower bounds (GLB)?
Why do we need guaranteed lower bound?

Crouzeix-Raviart element provides asymptotic lower bounds.

Does Crouzeix-Raviart element always provides lower bounds?
Guaranteed lower bound

\[ \lambda_{CR,1} = 24 \]
\[ \lambda_1 = 2\pi^2 \approx 19.74 \]

\[ \lambda_{CR,1} > \lambda_1 \]
Guaranteed lower bound

\[ \lambda_{CR,1} = 24 \]

\[ \lambda_1 = 2\pi^2 \approx 19.74 \]

\[ \lambda_{CR,1} > \lambda_1 \]
Guaranteed CR lower bound:

\[
\frac{\lambda_{CR,1}}{1 + \kappa^2 \lambda_{CR,1} H^2} \leq \lambda_1,
\]

where \( \kappa = \sqrt{\frac{1}{\pi^2} + \frac{1}{48}} \).

For graded/adaptive meshes, guaranteed CR lower bounds may be inaccurate.
WG scheme: Find $(\lambda_h; u_h) \in (\mathbb{R}; V_h)$ such that $(u_h, u_h) = 1$ and

$$(\nabla_w u_h, \nabla_w v_h) + s(u_h, v_h) = \lambda_h (u_0, v_0), \quad \forall v_h \in V_h.$$ 

with

$$s(u_h, v_h) = \alpha \sum_{T \in \mathcal{T}_h} \sum_{e \subset \partial T} h_T^{-2} |e| |T| \langle Q_b u_0 - u_b, Q_b v_0 - v_b \rangle_e.$$
Poincaré-Friedrichs inequality

**Lemma**

*For any simplex $T$ with diameter $h_T$*

$$
\|f\|_{L^2(T)} \leq \frac{h_T}{\pi} \|\nabla f\|_{L^2(T)}, \quad \forall f \in H^1(T).
$$
Trace identity

**Lemma**

For any function $f \in H^1(T)$ in a simplex $T$ with the side $e$ opposite to the vertex $P_e$ satisfies

$$|T|/|e| \int_{e} f(x) ds = \int_{T} f(x) dx + \frac{1}{d} \int_{T} (x - P_e) \cdot \nabla f dx.$$

Trace inequality

**Lemma**

For any function $f \in H^1(T)$ in a simplex $T$ with the set $\mathcal{E}(T)$ of all sides satisfies

$$|T|h_T^{-2} \sum_{e \in \mathcal{E}(T)} |e|^{-1} \|f\|_{L^2(T)}^2 \leq (2 + (d + 1)/\pi)/\pi \|\nabla f\|_{L^2(T)}^2.$$
A key inequality:

\[ \| \nabla (f - Q_0 f) \|_{L^2(T)} \leq C_{st}(T) \| \nabla f - \mathcal{Q} \nabla f \|_{L^2(T)}. \]

\( Q_0 \) is \( L^2 \) orthogonal projection onto \( P_k(T) \),
\( \mathcal{Q} \) is \( L^2 \) orthogonal projection onto \( [P_{k-1}(T)]^d \).
Denote $\Lambda = C_{st}^2(2 + (d + 1)/\pi)/\pi$, $\delta = \kappa^2 h_{\text{max}}^2$.

**Theorem**

If $\delta \lambda_h + \alpha \Lambda \leq 1$, then $\lambda_h \leq \lambda$.

For the lowest order case $k = 1$:

**Theorem**

If $\alpha \leq \frac{1}{(d + 1)\kappa^2}$ and $h_{\text{max}}^2 \leq \frac{1}{\lambda_h \kappa^2}$, then $\lambda_h \leq \lambda$. 
WG allows for GLB with proper mesh size and stabilisation

The GLB analysis is applicable to dimension $n \geq 2$ cases

For 2D triangulation, WGGLB is an upper bound of CRGLB[Carstensen and Gedicke, Math Comp (2014)]
Consider graded meshes for an L-shaped 2D domain with grading parameter $3/2$. Denote $\lambda_h$ the WGGLB eigenvalues, $\lambda_{CR}$ the CR eigenvalues, and $CRGLB$ the GLB of CR eigenvalues. The numerical results verifies $CRGLB \leq \lambda_h \leq \lambda_{CR}$.

The coarsest mesh.
| $|\mathcal{T}|$ | $h_{\text{max}}$ | CRGLB | $\lambda_h$ | $\lambda_{CR}$ | CRGLB | $\lambda_h$ | $\lambda_{CR}$ |
|---|---|---|---|---|---|---|---|
| 54 | 0.6672 | 6.01 | 6.83 | 8.92 | 8.06 | 9.20 | 14.36 |
| 96 | 0.5156 | 7.08 | 7.82 | 9.19 | 9.95 | 11.20 | 14.70 |
| 150 | 0.4196 | 7.77 | 8.39 | 9.33 | 11.26 | 12.39 | 14.87 |
| 216 | 0.3536 | 8.23 | 8.73 | 9.42 | 12.18 | 13.13 | 14.96 |
| 384 | 0.2689 | 8.77 | 9.10 | 9.51 | 13.29 | 13.97 | 15.06 |
| 600 | 0.2168 | 9.05 | 9.28 | 9.55 | 13.90 | 14.38 | 15.11 |
| 1350 | 0.1461 | 9.36 | 9.47 | 9.59 | 14.58 | 14.82 | 15.15 |
| 2400 | 0.1101 | 9.48 | 9.54 | 9.61 | 14.84 | 14.98 | 15.17 |
| 5400 | 0.0738 | 9.56 | 9.59 | 9.62 | 15.03 | 15.10 | 15.18 |
| 7350 | 0.0633 | 9.58 | 9.60 | 9.63 | 15.07 | 15.12 | 15.19 |
| 9600 | 0.0555 | 9.59 | 9.61 | 9.63 | 15.10 | 15.14 | 15.19 |
| 12150 | 0.0494 | 9.60 | 9.62 | 9.63 | 15.12 | 15.15 | 15.19 |
| 15000 | 0.0445 | 9.61 | 9.62 | 9.63 | 15.13 | 15.16 | 15.19 |

The 1st and 2nd discrete eigenvalues.
III. Acceleration of WG method in eigenvalue problems
Upper & lower bound

Getting both upper and lower bound:

\[ \text{WG} + \text{FEM} \]
\[ \text{(nonlinear)} \quad \text{(nonlinear)} \]
\[ \Downarrow \]

\[ \text{WG} + \text{Postprocessing} \]
\[ \text{(nonlinear)} \quad \text{(interpolation)} \]

See also:
Upper & lower bound

Discontinuous numerical eigenfunction

Continuous numerical eigenfunction

The weak Galerkin finite element method for eigenvalue problems
Step 1. Find $\lambda_h \in \mathbb{R}$, $u_h \in V_h$ such that $b_w(u_h, u_h) = 1$ and 

$$a_w(u_h, v) = \lambda_h b_w(u_h, v), \quad \forall v \in V_h.$$ 

Step 2. Calculate $\tilde{u} = \Pi_h u_h$, where $\Pi_h$ is the interpolation operator.
Step 3. Calculate the Rayleigh quotient 

$$\tilde{\lambda}_h = \frac{a(\tilde{u}, \tilde{u})}{b(\tilde{u}, \tilde{u})}.$$
Lemma

For any $v_h \in V_h$ in the WG finite element space, we have the following estimate

$$\|v_h - \Pi_h v_h\| \leq C h \|v\|_V,$$

where $\Pi_h$ is the interpolation operator.

Lemma

Suppose $(\lambda, u)$ satisfies

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V.$$

Then for any $w \in V$, $b(w, w) \neq 0$ we have

$$\frac{a(w, w)}{b(w, w)} - \lambda = \frac{a(w - u, w - u)}{b(w, w)} - \lambda \frac{b(w - u, w - u)}{b(w, w)}.$$
Table: Numerical results for the Lshape domain with $k = 2$.

<table>
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<tr>
<th>$h$</th>
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<tbody>
<tr>
<td>$\lambda_{1,h}$</td>
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<td>9.55615</td>
<td>9.61581</td>
<td>9.63083</td>
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<td>10.07505</td>
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<td>29.52156</td>
<td>29.52148</td>
</tr>
</tbody>
</table>
Existing literatures:

......
Two grid & two space method

Idea:

large scale $\Rightarrow$ small scale + large scale
nonlinear nonlinear linear

Two grid: fine mesh + coarse mesh + fine mesh
$h$ $H = \sqrt{h}$ $h$

Two space: high order + low order + high order
Two grid method

Step 1: Find \((\lambda_H, u_H) \in \mathbb{R} \times V_H\), such that

\[
a_w(u_H, v_H) = \lambda_H b_w(u_H, v_H), \quad \forall v_H \in V_H.
\]

Step 2: Find \(\tilde{u}_h \in V_h\) such that

\[
a_w(\tilde{u}_h, v_h) = \lambda_H b_w(u_H, v_h), \quad \forall v_h \in V_h.
\]

Step 3:

\[
\tilde{\lambda}_h = \frac{a_w(\tilde{u}_h, \tilde{u}_h)}{b_w(\tilde{u}_h, \tilde{u}_h)}.
\]

Lower bounds still holds!
**Table**: The errors for the eigenvalue approximation $\tilde{\lambda}_h$ with $k = 1, \gamma(h) = h^{0.1}$.

<table>
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<th></th>
<th>$H$</th>
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<td>3.9019</td>
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<td></td>
</tr>
<tr>
<td>$\lambda_3 - \tilde{\lambda}_{3,h}$</td>
<td>1.7347e0</td>
<td>1.1339e-1</td>
<td>7.9261e-3</td>
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<tr>
<td>order</td>
<td>3.9353</td>
<td>3.8386</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda_4 - \tilde{\lambda}_{4,h}$</td>
<td>3.9686e0</td>
<td>3.0206e-1</td>
<td>2.0167e-2</td>
<td></td>
</tr>
<tr>
<td>order</td>
<td>3.7157</td>
<td>3.9048</td>
<td></td>
<td></td>
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<tr>
<td>$\lambda_5 - \tilde{\lambda}_{5,h}$</td>
<td>5.7315e0</td>
<td>4.4163e-1</td>
<td>3.1474e-2</td>
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</tr>
<tr>
<td>order</td>
<td>3.6980</td>
<td>3.8106</td>
<td></td>
<td></td>
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<tr>
<td>$\lambda_6 - \tilde{\lambda}_{6,h}$</td>
<td>5.1859e0</td>
<td>4.5160e-1</td>
<td>2.9977e-2</td>
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<tr>
<td>order</td>
<td>3.5215</td>
<td>3.9131</td>
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</tbody>
</table>
Existing literatures:
......
Shifted-inverse power method

Idea:

<table>
<thead>
<tr>
<th>fine mesh</th>
<th>coarse mesh</th>
<th>fine mesh</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$H = \frac{\sqrt[4]{h}}{h}$</td>
<td>$h$</td>
</tr>
<tr>
<td>nonlinear</td>
<td>nonlinear</td>
<td>linear</td>
</tr>
</tbody>
</table>

Two grid: $h = 1/256 \Rightarrow H = \sqrt{h} = 1/16$

Shifted-inverse power: $h = 1/256 \Rightarrow H = \sqrt[4]{h} = 1/4$
Step 1: Find \((\lambda_H, u_H) \in \mathbb{R} \times V_H\), such that
\[
a_w(u_H, v_H) = \lambda_H b_w(u_H, v_H), \quad \forall v_H \in V_H.
\]
Step 2: Find \(\tilde{u}_h \in V_h\) such that
\[
a_w(\tilde{u}_h, v_h) - \lambda_H b_w(\tilde{u}_h, v_h) = b_w(u_H, v_h), \quad \forall v_h \in V_h. \quad (1)
\]
Step 3:
\[
\tilde{\lambda}_h = \frac{a_w(\tilde{u}_h, \tilde{u}_h)}{b_w(\tilde{u}_h, \tilde{u}_h)}.
\]
Lower bounds still holds!
Table: The errors for the eigenvalue approximations $\lambda - \tilde{\lambda}_h$.

<table>
<thead>
<tr>
<th>$H$</th>
<th>$1/8$</th>
<th>$1/16$</th>
<th>$1/32$</th>
<th>$1/64$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>1/512</td>
<td>1/512</td>
<td>1/512</td>
<td>1/512</td>
</tr>
<tr>
<td>$\lambda_1 - \tilde{\lambda}_{1,h}$</td>
<td>5.9031e-4</td>
<td>5.9045e-4</td>
<td>5.9045e-4</td>
<td>5.9045e-4</td>
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<tr>
<td>$\lambda_2 - \tilde{\lambda}_{2,h}$</td>
<td>3.7769e-3</td>
<td>3.8294e-3</td>
<td>3.8295e-3</td>
<td>3.8295e-3</td>
</tr>
<tr>
<td>$\lambda_3 - \tilde{\lambda}_{3,h}$</td>
<td>3.7805e-3</td>
<td>3.8294e-3</td>
<td>3.8295e-3</td>
<td>3.8295e-3</td>
</tr>
<tr>
<td>$\lambda_4 - \tilde{\lambda}_{4,h}$</td>
<td>9.0348e-3</td>
<td>9.4457e-3</td>
<td>9.4464e-3</td>
<td>9.4464e-3</td>
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<tr>
<td>$\lambda_5 - \tilde{\lambda}_{5,h}$</td>
<td>1.2305e-2</td>
<td>1.5744e-2</td>
<td>1.5750e-2</td>
<td>1.5750e-2</td>
</tr>
<tr>
<td>$\lambda_6 - \tilde{\lambda}_{6,h}$</td>
<td>1.2293e-2</td>
<td>1.5744e-2</td>
<td>1.5750e-2</td>
<td>1.5750e-2</td>
</tr>
</tbody>
</table>
Conclusion and Ongoing Work
A high order asymptotic lower bound has been obtained.
Lower bounds of different types of eigenvalue problem in both 2D and 3D cases has been obtained.
The method is efficient for polytopal meshes.
Guaranteed lower bound for $k=1$ order polynomials.
Some acceleration techniques are applied to WG method.
Ongoing work

- Guaranteed lower bound for high order polynomials.
- More applications: Spectral of atom ...
Thank You for Your Attention...