# STABILITY OF BACKGROUND PERTURBATION FOR BOLTZMANN EQUATION 

YU-CHU LIN, HAITAO WANG, AND KUNG-CHIEN WU


#### Abstract

Consider the Boltzmann equation in the perturbation regime. Since the macroscopic quantities in the background global Maxwellian are obtained through measurements, there are typically some errors involved. This paper investigates the effect of background variations on the solution for a given initial perturbation. Our findings demonstrate that the solution changes continuously with variations in the background and provide a sharp time decay estimate of the associated errors. The proof relies on refined estimates for the linearized solution operator and a proper decomposition of the nonlinear solution.


## 1. Introduction

1.1. The model. The Boltzmann equation reads

$$
\left\{\begin{array}{l}
\partial_{t} F+\xi \cdot \nabla_{x} F=Q(F, F),  \tag{1}\\
F(0, x, \xi)=F_{0}(x, \xi),
\end{array} \quad(t, x, \xi) \in \mathbb{R}^{+} \times \mathbb{R}^{3} \times \mathbb{R}^{3},\right.
$$

where $F(t, x, \xi)$ is the velocity distribution function for the particles at time $t>0$, position $x=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3}$ and microscopic velocity $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right) \in \mathbb{R}^{3}$. The left-hand side of this equation models the transport of particles and the operator on the right-hand side models the effect of collisions during the transport,

$$
Q(F, G)=\frac{1}{2} \int_{\mathbb{R}^{3} \times S^{2}}\left|\xi-\xi_{*}\right|^{\gamma} B(\vartheta)\left\{F_{*}^{\prime} G^{\prime}+G_{*}^{\prime} F^{\prime}-F_{*} G-F G_{*}\right\} d \xi_{*} d \omega
$$

Here the usual convention, i.e., $F=F(t, x, \xi), F_{*}\left(t, x, \xi_{*}\right), F^{\prime}=F(t, x, \xi)$ and $F_{*}^{\prime}=F\left(t, x, \xi_{*}^{\prime}\right)$, is used; the post-collisional velocities of particles satisfy

$$
\xi^{\prime}=\xi-\left[\left(\xi-\xi_{*}\right) \cdot \omega\right] \omega, \quad \xi_{*}^{\prime}=\xi_{*}+\left[\left(\xi-\xi_{*}\right) \cdot \omega\right] \omega, \omega \in S^{2}
$$

Throughout this paper, we consider the Maxwellian molecules $(\gamma=0)$ and hard potentials $(0<$ $\gamma \leq 1)$; and $B(\vartheta)$ satisfies the Grad cutoff assumption

$$
0<B(\vartheta) \leq C|\cos \vartheta|,
$$

for some constant $C>0$, and $\vartheta$ is defined by

$$
\cos \vartheta=\frac{\left|\left(\xi-\xi_{*}\right) \cdot \omega\right|}{\left|\xi-\xi_{*}\right|} .
$$

The global Maxwellian states $\mathcal{M}_{[\rho, \mu, T]}$, with $[\rho, \mu, T]$ constant,

$$
\mathcal{M}_{[\rho, \mu, T]}=\frac{\rho}{(2 \pi R T)^{3 / 2}} e^{-\frac{|\xi-\mu|^{2}}{2 R T}}
$$

satisfy $Q\left(\mathcal{M}_{[\rho, \mu, T]}, \mathcal{M}_{[\rho, \mu, T]}\right)=0$ and are steady solutions of the Boltzmann equation.
As is well-known, in the perturbative framework, the evolution of the initial perturbation crucially depends on the background Maxwellian. For example, transport coefficients derived from the Boltzmann equation, such as viscosity, heat conductivity, and macroscopic Euler waves, are all determined by the background Maxwellian (see [1, 12, 13, 15, for more information).

[^0]Typically, macroscopic quantities in the global Maxwellian, such as density, velocity, and temperature, are obtained through measurements, which can involve errors. Therefore, it is natural to ask the following questions:
(1) How does the solution change as the background varies for the same initial perturbation?
(2) Can we obtain a sharp estimate for the difference between solutions associated with different background Maxwellians?
In this paper, we aim to answer the aforementioned questions by studying the stability of the solution of (11) under perturbations of the background with respect to macroscopic quantities. Let us now formulate our problem. Let $F^{a}$ and $F^{b}$ be solutions to the Boltzmann equation with the same initial perturbation but for different global Maxwellians. That is,

$$
\left\{\begin{array}{l}
\partial_{t} F^{a}+\xi \cdot \nabla_{x} F^{a}=Q\left(F^{a}, F^{a}\right) \\
F^{a}(0, x, \xi)=F_{i n}^{a}(x, \xi)=\mathcal{M}_{a}(\xi)+F_{0}
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\partial_{t} F^{b}+\xi \cdot \nabla_{x} F^{b}=Q\left(F^{b}, F^{b}\right) \\
F^{b}(0, x, \xi)=F_{i n}^{b}(x, \xi)=\mathcal{M}_{b}(\xi)+F_{0}
\end{array}\right.
$$

respectively, where $\mathcal{M}_{a}(\xi)$ and $\mathcal{M}_{b}(\xi)$ are two global Maxwellians

$$
\mathcal{M}_{a}(\xi)=\frac{\rho_{a}}{\left(2 \pi R T_{a}\right)^{3 / 2}} e^{-\frac{\left|\xi-\mu_{a}\right|^{2}}{2 R T_{a}}}, \quad \mathcal{M}_{b}(\xi)=\frac{\rho_{b}}{\left(2 \pi R T_{b}\right)^{3 / 2}} e^{-\frac{\left|\xi-\mu_{b}\right|^{2}}{2 R T_{b}}}
$$

with $\left|\rho_{a}-\rho_{b}\right|+\left|\mu_{a}-\mu_{b}\right|+\left|T_{a}-T_{b}\right|>0$.
In the perturbation regime, if we let $F^{a}=\mathcal{M}_{a}+\sqrt{\mathcal{M}_{a}} f^{a}$ and $F^{b}=\mathcal{M}_{b}+\sqrt{\mathcal{M}_{b}} f^{b}$, we can obtain that the perturbation functions $f^{\alpha}$, where $\alpha=a, b$, satisfy the following equations:

$$
\left\{\begin{array}{l}
\partial_{t} f^{\alpha}+\xi \cdot \nabla_{x} f^{\alpha}=\mathcal{L}_{\alpha} f^{\alpha}+\Gamma_{\alpha}\left(f^{\alpha}, f^{\alpha}\right)  \tag{2}\\
f^{\alpha}(0, x, \xi)=\varepsilon f_{0}^{\alpha}(x, \xi)
\end{array}\right.
$$

where

$$
\mathcal{L}_{\alpha} h=\frac{2}{\sqrt{\mathcal{M}_{\alpha}}} Q\left(\mathcal{M}_{\alpha}, \sqrt{\mathcal{M}_{\alpha}} h\right), \quad \Gamma_{\alpha}\left(h_{1}, h_{2}\right)=\frac{1}{\sqrt{\mathcal{M}_{\alpha}}} Q\left(\sqrt{\mathcal{M}_{\alpha}} h_{1}, \sqrt{\mathcal{M}_{\alpha}} h_{2}\right)
$$

As the initial perturbations are the same, one has

$$
\begin{equation*}
\sqrt{\mathcal{M}_{a}} f_{0}^{a}=\sqrt{\mathcal{M}_{b}} f_{0}^{b} \tag{3}
\end{equation*}
$$

Moreover, let $G=F^{b}-F^{a}$ and then it satisfies

$$
\left\{\begin{array}{l}
\partial_{t} G+\xi \cdot \nabla_{x} G=Q(G, G)+2 Q\left(F^{a}, G\right)  \tag{4}\\
G(0, x, \xi)=\mathcal{M}_{b}(\xi)-\mathcal{M}_{a}(\xi)
\end{array}\right.
$$

Fixing $\mathcal{M}_{a}$ as a reference background, one can consider $G=\mathcal{M}_{b}-\mathcal{M}_{a}+\sqrt{\mathcal{M}_{a}} g$ and so

$$
\begin{equation*}
g=\frac{\left(F^{b}-\mathcal{M}_{b}\right)-\left(F^{a}-\mathcal{M}_{a}\right)}{\sqrt{\mathcal{M}_{a}}} \tag{5}
\end{equation*}
$$

Plugging this into (4) gives

$$
\left\{\begin{array}{l}
\partial_{t} g+\xi \cdot \nabla_{x} g=\mathcal{L}_{a} g+2 \Gamma_{a}\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)+2 \Gamma_{a}\left(f^{a}, g\right)+\Gamma_{a}(g, g)  \tag{6}\\
g(0, x, \xi)=0
\end{array}\right.
$$

Noting that $\sqrt{\mathcal{M}_{a}} g$ represents the difference between the perturbation solutions, and the objective of this paper is to investigate the quantitative behavior of $g$. Since the Boltzmann equation is
invariant under Galilean transforms, we may assume that one of the Maxwellians has zero macroscopic velocity. Then, without loss of generality and through appropriate normalization, the global Maxwellians $\mathcal{M}_{a}$ and $\mathcal{M}_{b}$ can be assumed as

$$
\begin{equation*}
\mathcal{M}_{a}=\mathcal{M}=\frac{1}{(2 \pi)^{3 / 2}} \exp \left(-\frac{|\xi|^{2}}{2}\right), \mathcal{M}_{b}=\frac{\rho}{(2 \pi \lambda)^{3 / 2}} \exp \left(-\frac{|\xi-\mu|^{2}}{2 \lambda}\right) \tag{7}
\end{equation*}
$$

For simplicity, we hereafter denote $\mathcal{L}_{a}$ and $\Gamma_{a}$ by $\mathcal{L}$ and $\Gamma$.
There are extensive studies on the stability of Boltzmann equation in the literature, with initial data either near the vacuum or near the same global Maxwellian, see [5, 6] and references therein. It is important to note that our problem in equation (6) presents a significantly different setting than the prior research.
1.2. Notations. Let us define some notations used in this paper. We denote $\langle\xi\rangle^{s}=\left(1+|\xi|^{2}\right)^{s / 2}$, $s \in \mathbb{R}$. For the microscopic variable $\xi$, we denote the Lebesgue spaces

$$
|g|_{L_{\xi}^{q}}=\left(\int_{\mathbb{R}^{3}}|g|^{q} d \xi\right)^{1 / q} \text { if } 1 \leq q<\infty, \quad|g|_{L_{\xi}^{\infty}}=\sup _{\xi \in \mathbb{R}^{3}}|g(\xi)|
$$

and the weighted norms can be defined by

$$
|g|_{L_{\xi, \beta}^{q}}=\left(\int_{\mathbb{R}^{3}}\left|\langle\xi\rangle^{\beta} g\right|^{q} d \xi\right)^{1 / q} \text { if } 1 \leq q<\infty, \quad|g|_{L_{\xi, \beta}^{\infty}}=\sup _{\xi \in \mathbb{R}^{3}}\left|\langle\xi\rangle^{\beta} g(\xi)\right|,
$$

and

$$
|g|_{L_{\xi}^{\infty}(m)}=\sup _{\xi \in \mathbb{R}^{3}}\{|g(\xi)| m(\xi)\}
$$

where $\beta \in \mathbb{R}$ and $m$ is a weight function. The $L_{\xi}^{2}$ inner product in $\mathbb{R}^{3}$ will be denoted by $\langle\cdot, \cdot\rangle_{\xi}$, i.e.,

$$
\langle f, g\rangle_{\xi}=\int f(\xi) \overline{g(\xi)} d \xi
$$

For the Boltzmann equation with cut-off potential, the natural norm in $\xi$ is $|\cdot|_{L_{\sigma}^{2}}$, which is defined as

$$
|g|_{L_{\sigma}^{2}}^{2}=\left|\langle\xi\rangle^{\frac{\gamma}{2}} g\right|_{L_{\xi}^{2}}^{2}
$$

For the space variable $x$, we have similar notations, namely,

$$
|g|_{L_{x}^{q}}=\left(\int_{\mathbb{R}^{3}}|g|^{q} d x\right)^{1 / q} \text { if } 1 \leq q<\infty, \quad|g|_{L_{x}^{\infty}}=\sup _{x \in \mathbb{R}^{3}}|g(x)|
$$

Furthermore, we define the high order Sobolev norm: let $s \in \mathbb{N}$ and define

$$
|g|_{H_{\xi}^{s}}=\sum_{|\alpha| \leq s}\left|\partial_{\xi}^{\alpha} g\right|_{L_{\xi}^{2}}, \quad|g|_{H_{x}^{s}}=\sum_{|\alpha| \leq s}\left|\partial_{x}^{\alpha} g\right|_{L_{x}^{2}}
$$

where $\alpha$ is any multi-index with $|\alpha| \leq s$.
Finally, with $\mathcal{X}$ and $\mathcal{Y}$ being norm spaces, we define

$$
\|g\|_{\mathcal{X} \mathcal{Y}}=\left||g|_{\mathcal{Y}}\right|_{\mathcal{X}}
$$

We also denote

$$
\|g\|_{L^{2}}=\|g\|_{L_{\xi}^{2} L_{x}^{2}}=\left(\int_{\mathbb{R}^{3}}|g|_{L_{x}^{2}}^{2} d \xi\right)^{1 / 2}
$$

For simplicity of notations, hereafter, we abbreviate " $\leq C$ " to " $\lesssim$ ", where $C$ is a positive constant depending only on fixed numbers.
1.3. Heuristics and a toy model. Starting from (5), we can express $g$ as

$$
\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}-f^{a}
$$

where $f^{a}$ and $f^{b}$ are the solutions to (2). Let $\mathbb{G}_{\alpha}^{t}$ denote the semi-group generated by the linearized operator $-\xi \cdot \nabla_{x}+\mathcal{L}_{\alpha}$ for $\alpha=a, b$. Since we are considering the perturbative regime, it is reasonable to expect that the behavior of $f^{\alpha}$ is mainly governed by the linearized equation, i.e., $f^{\alpha}(t) \approx \mathbb{G}_{\alpha}^{t} f^{\alpha}(0)$, this is because the decay of the nonlinear part is faster than the linear part, and thus

$$
g \approx \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{t} f^{b}(0)-\mathbb{G}_{a}^{t} f^{a}(0)=\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{t} \frac{\sqrt{\mathcal{M}_{a}}}{\sqrt{\mathcal{M}_{b}}}-\mathbb{G}_{a}^{t}\right) \varepsilon f_{0}^{a}
$$

where the last equality is due to (3).
This observation suggests to us that the behavior of $g$ should be expressed in terms of the difference between the two solution operators for linearized Boltzmann equations. Moreover, for the linearized Boltzmann equation, by the Chapman-Enskog expansion, it is known that the largetime behavior of the solution to the linearized equation is governed by the linear Navier-Stokes equation, where the viscosity and heat conductivity are proportional to the power (precisely, the power is $\frac{2-\gamma}{2}$ ) of the macroscopic temperature (see [2, 7] for more details).

Inspired by the linear Navier-Stokes equation, we consider the possibly simplest toy model, the heat equation. Given two sets of heat equations,

$$
\left\{\begin{array}{l}
\partial_{t} h^{\alpha}+\mu_{\alpha} \cdot \nabla_{x} h^{\alpha}=\lambda^{(2-\gamma) / 2} \Delta h^{\alpha}  \tag{8}\\
\left.h^{\alpha}\right|_{t=0}=h_{0}, \quad \alpha=a, b,
\end{array}\right.
$$

here, positive constants $\mu_{\alpha}$ and $\lambda_{\alpha}$ are used to mimic the macroscopic velocity and temperature, respectively. Using Galilean transformation and suitable scaling, we assume that

$$
\mu_{a}=0, \mu_{b}=\mu, \lambda_{a}=1, \lambda_{b}=\lambda
$$

To analyze the difference between $h^{a}$ and $h^{b}$, we can use an explicit heat kernel representation to yield

$$
h^{b}(t, x)-h^{a}(t, x)=\int_{\mathbb{R}^{3}}\left[\frac{1}{(4 \pi \kappa t)^{3 / 2}} e^{-\frac{|x-\mu t-y|^{2}}{4 \kappa t}}-\frac{1}{(4 \pi t)^{3 / 2}} e^{-\frac{|x-y|^{2}}{4 t}}\right] h_{0}(y) d y, \quad \kappa=\lambda^{(2-\gamma) / 2} .
$$

By calculating the difference between two heat kernels explicitly (see Appendix A), one obtains the sharp estimates for $h^{b}-h^{a}$ :

$$
\left\{\begin{array}{l}
\left|h^{b}-h^{a}\right|_{L_{x}^{\infty}} \leq C(1+t)^{-1}\left|h_{0}\right|_{L_{x}^{1}}\left(|\lambda-1|(1+t)^{-1 / 2}+|\mu|\right)  \tag{9}\\
\left|h^{b}-h^{a}\right|_{L_{x}^{2}} \leq C(1+t)^{-1 / 4}\left|h_{0}\right|_{L_{x}^{1}}\left(|\lambda-1|(1+t)^{-1 / 2}+|\mu|\right)
\end{array}\right.
$$

for $t \geq 1$.
It should be noted that this method heavily relies on the explicit expressions of heat kernels. For Boltzmann equation, while there exist constructions of Green's functions (see [10, 11, 12]), the expressions are not precise enough to obtain sharp estimates for the difference between them.

Alternatively, for the toy model (8), we set $h=h^{a}-h^{b}$ to obtain

$$
\left\{\begin{array}{l}
\partial_{t} h=\Delta h+\mu \cdot \nabla_{x} h^{b}-\left(\lambda^{(2-\gamma) / 2}-1\right) \Delta h^{b}  \tag{10}\\
h(0, x)=0
\end{array}\right.
$$

One could still recover (9) by refining estimates for the heat kernel and $h^{b}$. See Appendix A for details.

Compared with (6), one may view (10) as a simplified analogue of it. Therefore, it is natural to ask whether we can establish similar estimates as (9) for the difference $g$. Our main result provides an affirmative answer to this question.
1.4. Main theorem and idea of proof. In our main theorem, we assume the initial condition $f_{0}^{a}$ satisfies

$$
\left(1+|\xi|^{2}\right)^{\beta / 2} f_{0}^{a} \in L_{\xi}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right), \quad \beta>3 / 2+2 \gamma
$$

in order to ensure the existence of the solution of the Boltzmann equation $f^{a}$ in the space $L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap\right.$ $L_{x}^{2}$ ) and to control the nonlinear part of $g$. Moreover, in view of the assumption (3), when $\left(1+|\xi|^{2}\right)^{\beta / 2} f_{0}^{a} \in L_{\xi}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)$ is assumed, $f_{0}^{b}$ cannot be arbitrary and is determined by

$$
\left(1+|\xi|^{2}\right)^{\beta / 2} \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}=\left(1+|\xi|^{2}\right)^{\beta / 2} f_{0}^{a} \in L_{\xi}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)
$$

where

$$
\begin{aligned}
\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} & =\frac{\rho^{1 / 2}}{\lambda^{3 / 4}} \exp \left\{\frac{|\xi-\mu|^{2}}{4}\left(1-\frac{1}{\lambda}\right)+\frac{2(\xi-\mu) \cdot \mu+|\mu|^{2}}{4}\right\} \\
& =\frac{\rho^{1 / 2}}{\lambda^{3 / 4}} \exp \left\{\frac{\lambda-1}{4 \lambda}\left|\xi+\frac{\mu}{\lambda-1}\right|^{2}+\frac{|\mu|^{2}}{4(\lambda-1)}\right\}
\end{aligned}
$$

hence we need to assume $\lambda>1$ later on. The assumption $\lambda<2$ comes from the estimate of the difference of two Maxwellians in Lemma 14

The main theorem of this paper is stated as follows:
Theorem 1. Let $1<\underline{\lambda}<\lambda<\bar{\lambda}<2,0<\rho<\bar{\rho}, 0<|\mu|<|\bar{\mu}|$. Assume that fa and f $f^{b}$ are solutions to (2) corresponding to $\mathcal{M}_{\alpha}=\mathcal{M}_{a}$ and $\mathcal{M}_{\alpha}=\mathcal{M}_{b}$ defined as (7), with the initial data $\langle\xi\rangle^{\beta} f_{0}^{a} \in\left(L_{x}^{1} \cap L_{x}^{\infty}\right) L_{\xi}^{\infty}, \beta>3 / 2+2 \gamma$, and $\varepsilon>0$ small. Then

$$
\left\|\frac{\left(F^{b}-\mathcal{M}_{b}\right)-\left(F^{a}-\mathcal{M}_{a}\right)}{\sqrt{\mathcal{M}_{a}}}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \leq C \varepsilon(1+t)^{-1}(|\rho-1|+|\mu|+|\lambda-1|)
$$

and

$$
\left\|\frac{\left(F^{b}-\mathcal{M}_{b}\right)-\left(F^{a}-\mathcal{M}_{a}\right)}{\sqrt{\mathcal{M}_{a}}}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \leq C \varepsilon(1+t)^{-1 / 4}(|\rho-1|+|\mu|+|\lambda-1|) .
$$

for some constant $C>0$ dependent of $\bar{\rho}, \underline{\lambda}, \bar{\lambda}$ and $|\bar{\mu}|$, but independent of time $t$.
Some comments on the theorem are as follows:
Remark 2. We say the estimate is sharp in the sense that the decay rate of the estimate is $(1+t)^{-1}$, which is the same as that of the toy model (9). However, for the toy model, the decay rate for speed variation is $(1+t)^{-1}$, while for variation of the diffusion coefficient it is $(1+t)^{-3 / 2}$. In contrast, for the Boltzmann equation, we can only obtain $(1+t)^{-1}$ for variations of all macroscopic quantities. So far, it is unknown whether it is possible to obtain different decay rates for different quantities.
Remark 3. The result is established using the $L_{x}^{\infty}$ framework, and no Sobolev regularity is required. Moreover, the error estimate is given in terms of the $L_{x}^{\infty}$ norm, which appears to be more realistic in terms of measurement.

Remark 4. The theorem requires the strict inequality $1<\underline{\lambda}<\lambda<\bar{\lambda}<2$. However, it can also be proven for the case $\lambda=1$, provided that we assume suitable exponential velocity weight on the initial data.

We will now outline the main idea and strategy to prove our main result. For the equation (6), we use the solution operator $\mathbb{G}^{t}$ for the linearized operator $-\xi \cdot \nabla_{x}+\mathcal{L}$ and Duhamel's principle to represent $g$ as follows:

$$
\begin{align*}
g(t)= & 2 \int_{0}^{t} \mathbb{G}^{t-\tau} \Gamma_{a}\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau  \tag{11}\\
& +2 \int_{0}^{t} \mathbb{G}^{t-\tau} \Gamma_{a}\left(f^{a}, g\right)(\tau) d \tau+\int_{0}^{t} \mathbb{G}^{t-\tau} \Gamma_{a}(g, g)(\tau) d \tau
\end{align*}
$$

Since the first term on the right hand side is expected to dominate the behavior of $g$, our primary objective is to obtain a sharp estimate for it. However, the decay estimates available for the linearized equation (as shown in Theorem (9) and $f^{b}$ (as seen in Theorem 8) can only yield a $(1+t)^{-3 / 4}$ decay rate for the $L_{x}^{\infty}$ norm (as indicated in Remark 15). The challenge of this paper is to improve the decay of this part from $(1+t)^{-3 / 4}$ to $(1+t)^{-1}$. Therefore, we need more precise estimates for both $\mathbb{G}^{t}$ and $f^{b}$. Through spectral analysis, we decompose the operator $\mathbb{G}^{t}$ into three parts:

$$
\mathbb{G}^{t}=\mathbb{G}_{L ; 0}^{t}+\mathbb{G}_{L ; \perp}^{t}+\mathbb{G}_{S}^{t}
$$

where $\mathbb{G}_{L ; 0}^{t}$ is the long wave fluid part, $\mathbb{G}_{L ; \perp}^{t}$ is the long wave non-fluid part, and $\mathbb{G}_{S}^{t}$ is the short wave part. These three parts have different natures. Among them, only $\mathbb{G}_{L ; 0}^{t}$ behaves like the heat kernel in the toy model (8). We exploit its space-time pointwise structure to obtain $L_{x}^{p}-L_{x}^{q}$-type estimates. However, for $\mathbb{G}_{L ; \perp}^{t}$ and $\mathbb{G}_{S}^{t}$, due to insufficient spectral information, we need to work in the $\|\cdot\|_{L_{x}^{2} L_{\xi}^{2}}$ setting, as shown in Proposition 10

Next, motivated by the Liu-Yu's Green's function approach 10 and bootstrap argument, we improve the estimate from $L_{x}^{2} L_{\xi}^{2}$ to $L_{\xi, \beta}^{\infty} L_{x}^{r}$ and obtain a more precise estimate for the semigroup than the classical results [1, 14, [15]. Moreover, we apply the semi-group estimate to the long and short wave parts of the linearized solution to obtain decay rates, which reveal their different structures. These decay estimates for the semi-group $\mathbb{G}^{t}$ (including Propositions 10 and 11. Corollaries 12 and 13) are themselves new.

However, even with the above refined estimates for the semi-group, the slow decay of $f^{b}$ still poses a challenge when applying it to the first term on the right hand side of (11). To address this issue, we decompose $f^{b}$ into linear and nonlinear parts:

$$
f^{b}=\varepsilon \mathbb{G}_{b}^{t} f_{0}^{b}+\int_{0}^{t} \mathbb{G}_{b}^{t-\tau} \Gamma_{b}\left(f^{b}, f^{b}\right) d \tau
$$

Here $\mathbb{G}_{b}^{t}$ is the semi-group generated by $-\xi \cdot \nabla_{x}+\mathcal{L}_{b}$. The nonlinear part decays fast and is not problematic. We decompose linear part further into long wave part $\mathbb{G}_{b, L}^{t} f_{0}^{b}$ and short wave parts $\mathbb{G}_{b, S}^{t} f_{0}^{b}$. We carefully analyze each term in the following integral

$$
\int_{0}^{t}\left(\mathbb{G}_{L ; 0}^{t-\tau}+\mathbb{G}_{L ; \perp}^{t-\tau}+\mathbb{G}_{S}^{t-\tau}\right) \Gamma_{a}\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}}\left(\mathbb{G}_{b ; L}^{\tau} f_{0}^{b}+\mathbb{G}_{b ; S}^{\tau} f_{0}^{b}\right), \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right) d \tau
$$

By doing so, we eventually recover the sharp decay estimate. Notably, only the term

$$
\mathbb{G}_{L ; 0}^{t-\tau}(\cdots) \mathbb{G}_{b ; L}^{\tau}
$$

behaves similarly to the counterpart in the toy model. Once we have obtained the sharp estimate, we propose an appropriate ansatz for $g$ and close the nonlinear problem by an a priori estimate.
1.5. Organization of the paper. The rest of this paper is structured as follows: In Section 2. we begin by introducing some basic properties of the operators $\mathcal{L}$ and $\Gamma$. We then provide a review of previously established results concerning decay estimates of semi-groups and perturbation problems. After that, we develop new refined estimates for the semi-group. Section 3 is dedicated to the proof of the main theorem, while Appendix A contains detailed estimates for the toy model.

## 2. Results for the problem around $\mathcal{M}_{a}$

2.1. Basic estimates for $\mathcal{L}, \Gamma$ and well-posedness results. It is well known that the null space of $\mathcal{L}$ is a five-dimensional vector space with the orthonormal basis $\left\{\chi_{i}\right\}_{i=0}^{4}$, where

$$
\operatorname{Ker}(\mathcal{L})=\left\{\chi_{0}, \chi_{i}, \chi_{4}\right\}=\left\{\mathcal{M}^{1 / 2}, \xi_{i} \mathcal{M}^{1 / 2}, \frac{1}{\sqrt{6}}\left(|\xi|^{2}-3\right) \mathcal{M}^{1 / 2}, i=1,2,3\right\}
$$

Based on this property, we can introduce the macro-micro decomposition: let $\mathrm{P}_{0}$ be the orthogonal projection with respect to the $L_{\xi}^{2}$ inner product onto $\operatorname{Ker}(\mathcal{L})$, and $\mathrm{P}_{1} \equiv \mathrm{Id}-\mathrm{P}_{0}$.

The collision operator $\mathcal{L}$ consists of a multiplicative operator $\nu(\xi)$ and an integral operator $K$ :

$$
\mathcal{L} f=-\nu(\xi) f+K f
$$

where

$$
\nu(\xi)=\int_{\mathbb{R}^{3} \times S^{2}} B(\vartheta)\left|\xi-\xi_{*}\right|^{\gamma} \mathcal{M}\left(\xi_{*}\right) d \xi_{*} d \omega
$$

and

$$
K f=-K_{1} f+K_{2} f
$$

is defined as [3, 4]:

$$
\begin{aligned}
K_{1} f:= & \int_{\mathbb{R}^{3} \times S^{2}} B(\vartheta)\left|\xi-\xi_{*}\right|^{\gamma} \mathcal{M}^{1 / 2}(\xi) \mathcal{M}^{1 / 2}\left(\xi_{*}\right) f\left(\xi_{*}\right) d \xi_{*} d \omega \\
K_{2} f:= & \int_{\mathbb{R}^{3} \times S^{2}} B(\vartheta)\left|\xi-\xi_{*}\right|^{\gamma} \mathcal{M}^{1 / 2}\left(\xi_{*}\right) \mathcal{M}^{1 / 2}\left(\xi^{\prime}\right) f\left(\xi_{*}^{\prime}\right) d \xi_{*} d \omega \\
& +\int_{\mathbb{R}^{3} \times S^{2}} B(\vartheta)\left|\xi-\xi_{*}\right|^{\gamma} \mathcal{M}^{1 / 2}\left(\xi_{*}\right) \mathcal{M}^{1 / 2}\left(\xi_{*}^{\prime}\right) f\left(\xi^{\prime}\right) d \xi_{*} d \omega
\end{aligned}
$$

To begin with, we present a number of basic properties and estimates of the operators $\mathcal{L}, \nu(\xi)$ and $K$, which can be found in [4, 8].
Lemma 5. Let $0 \leq \gamma \leq 1$. For any $g \in L_{\sigma}^{2}$, we have the coercivity of the linearized collision operator $\mathcal{L}$, that is, there exists a positive constant $\nu_{0}$ such that

$$
\langle g, \mathcal{L} g\rangle_{\xi} \leq-\nu_{0}\left|\mathrm{P}_{1} g\right|_{L_{\sigma}^{2}}^{2}
$$

For the multiplicative operator $\nu(\xi)$, there are positive constants $\nu_{0}$ and $\nu_{1}$ such that

$$
\nu_{0}\langle\xi\rangle^{\gamma} \leq \nu(\xi) \leq \nu_{1}\langle\xi\rangle^{\gamma}
$$

For the integral operator $K$,

$$
K f=-K_{1} f+K_{2} f=\int_{\mathbb{R}^{3}}-k_{1}\left(\xi, \xi_{*}\right) f\left(\xi_{*}\right) d \xi_{*}+\int_{\mathbb{R}^{3}} k_{2}\left(\xi, \xi_{*}\right) f\left(\xi_{*}\right) d \xi_{*}
$$

the kernels $k_{1}\left(\xi, \xi_{*}\right)$ and $k_{2}\left(\xi, \xi_{*}\right)$ satisfy

$$
k_{1}\left(\xi, \xi_{*}\right) \lesssim\left|\xi-\xi_{*}\right|^{\gamma} \exp \left\{-\frac{1}{4}\left(|\xi|^{2}+\left|\xi_{*}\right|^{2}\right)\right\}
$$

and

$$
k_{2}\left(\xi, \xi_{*}\right)=a\left(\xi, \xi_{*}, \kappa\right) \exp \left(-\frac{(1-\kappa)}{8}\left[\frac{\left(|\xi|^{2}-\left|\xi_{*}\right|^{2}\right)^{2}}{\left|\xi-\xi_{*}\right|^{2}}+\left|\xi-\xi_{*}\right|^{2}\right]\right)
$$

for any $0<\kappa<1$, together with

$$
a\left(\xi, \xi_{*}, \kappa\right) \leq C_{\kappa}\left|\xi-\xi_{*}\right|^{-1}\left(1+|\xi|+\left|\xi_{*}\right|\right)^{\gamma-1}
$$

Furthermore, from Lemma 5 we have some essential properties for the integral operator $K$.
Lemma 6. Let $0 \leq \gamma \leq 1$ and $\tau \in \mathbb{R}$. Then

$$
\begin{equation*}
|K g|_{L_{\xi, \tau+2-\gamma}^{q}} \lesssim|g|_{L_{\xi, \tau}^{q}}, 1 \leq q \leq \infty \tag{12}
\end{equation*}
$$

and

$$
|K g|_{L_{\xi, \tau-\gamma+3 / 2}^{\infty}}^{\infty} \leq C|g|_{L_{\xi, \tau}^{2}}
$$

Moreover,

$$
\begin{equation*}
\left|\varpi K \varpi^{-1} g\right|_{L_{\xi, \tau+2-\gamma}^{q}} \lesssim|g|_{L_{\xi, \tau}^{q}}, 1 \leq q \leq \infty \tag{13}
\end{equation*}
$$

holds for any weight function of the form

$$
\begin{equation*}
\varpi(\xi)=\exp \left\{\kappa_{0}|\xi|^{2}+\kappa \cdot \xi+\kappa_{4}\right\} \tag{14}
\end{equation*}
$$

with $\kappa_{0}, \kappa, \kappa_{4}$ constant and $\kappa_{0}>0$.

Lemma 7. Let $0 \leq \gamma \leq 1$ and $\tau \geq 0$. Then

$$
\left|\nu^{-1} \varpi \Gamma\left(h_{1}, h_{2}\right)\right|_{L_{\xi, \tau}^{\infty}} \leq\left|\varpi h_{1}\right|_{L_{\xi, \tau}^{\infty}}\left|\varpi h_{2}\right|_{L_{\xi, \tau}^{\infty}},
$$

for any weight function defined as (14).
Theorem 8 (The large time behavior for $0 \leq \gamma \leq 1$, 9 ). Let $\beta>3 / 2+\gamma$ and let $\varpi=\varpi(\xi)$ be any weight function defined as (14). Assume that the initial data $\varepsilon f_{0}^{\alpha}$ satisfies $\varpi f_{0}^{\alpha} \in L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)$ and $\varepsilon>0$ is sufficiently small. Then there is a unique solution $f^{\alpha}$ to (2) in $L_{\xi, \beta}^{\infty}(\varpi)\left(L_{x}^{2} \cap L_{x}^{\infty}\right)$ with

$$
\begin{aligned}
\left\|\varpi f^{\alpha}(t)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} & \leq \varepsilon C_{1}(1+t)^{-\frac{3}{4}}\left\|\varpi f_{0}^{\alpha}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)} \\
\left\|\varpi f^{\alpha}(t)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} & \leq \varepsilon C_{2}(1+t)^{-\frac{3}{2}}\left\|\varpi f_{0}^{\alpha}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}
\end{aligned}
$$

for some positive constants $C_{1}$ and $C_{2}$.
2.2. Refined estimate for the linearized Boltzmann equation. Let $u$ be the solution of the linearized Boltzmann equation

$$
\left\{\begin{array}{l}
\partial_{t} u+\xi \cdot \nabla_{x} u=\mathcal{L} u  \tag{15}\\
u(0, x, \xi)=u_{0}(x, \xi)
\end{array}\right.
$$

and $\mathbb{G}^{t}$ the corresponding solution operator, i.e., $u=\mathbb{G}^{t} u_{0}$. Then we have known that
Theorem 9 (9, 17). Let $u$ be the solution of (15) and $\beta>3 / 2$. Then

$$
\begin{gathered}
\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \leq C(1+t)^{-3 / 4}\left[\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right] \\
\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \leq C\left[\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right]
\end{gathered}
$$

for $u_{0} \in L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap L_{x}^{2}\right)$, and

$$
\begin{aligned}
&\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \leq C(1+t)^{-3 / 2}\left[\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{1}}\right], \\
&\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \leq C(1+t)^{-3 / 4}\left[\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{1}}\right],
\end{aligned}
$$

for $u_{0} \in L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap L_{x}^{1}\right)$. Moreover, if $\mathrm{P}_{0} u_{0}=0$, then we will get extra $(1+t)^{-1 / 2}$ decay rate in each estimate above.

To attain the desired time decay rate, we need to refine these estimates for the linearized Boltzmann equation. According to the semigroup theory, the solution $u$ to (15) can be represented by

$$
u(t, x, \xi)=\mathbb{G}^{t} u_{0}=(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta} e^{(-i \xi \cdot \eta+\mathcal{L}) t} \widehat{u}_{0}(\eta) d \eta
$$

where $\widehat{u}_{0}$ is the Fourier transformation of $u_{0}$ with respect to the space variable $x$. Based on the spectrum analysis of the operator $-i \xi \cdot \eta+\mathcal{L}$, the semigroup $e^{(-i \xi \cdot \eta+\mathcal{L}) t}$ can be decomposed as

$$
\begin{aligned}
e^{(-i \xi \cdot \eta+\mathcal{L}) t} & =\chi_{\delta}(\eta) \sum_{j=0}^{4} e^{\lambda_{j}(\eta) t}\left|e_{j}(\eta)\right\rangle\left\langle e_{j}(\eta)\right|+\chi_{\delta}(\eta) e^{(-i \xi \cdot \eta+\mathcal{L}) t} \Pi_{\eta}^{\perp}+\left(1-\chi_{\delta}(\eta)\right) e^{(-i \xi \cdot \eta+\mathcal{L}) t} \\
& =: \widehat{\mathbb{G}}_{L ; 0}(\eta, t)+\widehat{\mathbb{G}}_{L ; \perp}(\eta, t)+\widehat{\mathbb{G}}_{S}(\eta, t)
\end{aligned}
$$

where $\chi_{\delta}(\eta)$ is a smooth cutoff function with $0 \leq \chi_{\delta} \leq 1, \chi_{\delta}(\eta)=1$ for $|\eta| \leq \frac{\delta}{2}$ and 0 for $|\eta| \geq \delta$, for $\delta>0$ sufficient small. Note that the spectrums of $\widehat{\mathbb{G}}_{L ; \perp}(\eta, t)$ and $\widehat{\mathbb{G}}_{S}(\eta, t)$ are strictly away from imaginary axis (with negative real part). Moreover for $|\eta| \ll 1$, the spectrum $\operatorname{Spec}(\eta)$ of the operator $-i \xi \cdot \eta+\mathcal{L}$ consists of exactly five eigenvalues $\lambda_{j}(\eta)(0 \leq j \leq 4)$ associated with the corresponding eigenfunctions $e_{j}(\eta)([2, ~ 11, ~ 16]): ~$

$$
\begin{aligned}
& \lambda_{j}(\eta)=-i a_{j}|\eta|-A_{j}|\eta|^{2}+O\left(|\eta|^{3}\right) \\
& e_{j}(\eta)=E_{j}+O(|\eta|)
\end{aligned}
$$

here $A_{j}>0,\left\langle e_{j}(-\eta), e_{l}(\eta)\right\rangle_{\xi}=\delta_{j l}, 0 \leq j, l \leq 4$ and

$$
\left\{\begin{array}{l}
a_{0}=\sqrt{\frac{5}{3}}, \quad a_{1}=-\sqrt{\frac{5}{3}}, \quad a_{2}=a_{3}=a_{4}=0 \\
E_{0}=\sqrt{\frac{3}{10}} \chi_{0}+\sqrt{\frac{1}{2}} \omega \cdot \bar{\chi}+\sqrt{\frac{1}{5}} \chi_{4} \\
E_{1}=\sqrt{\frac{3}{10}} \chi_{0}-\sqrt{\frac{1}{2}} \omega \cdot \bar{\chi}+\sqrt{\frac{1}{5}} \chi_{4} \\
E_{2}=-\sqrt{\frac{2}{5}} \chi_{0}+\sqrt{\frac{3}{5}} \chi_{4} \\
E_{3}=\omega_{1} \cdot \bar{\chi} \\
E_{4}=\omega_{2} \cdot \bar{\chi}
\end{array}\right.
$$

where $\bar{\chi}=\left(\chi_{1}, \chi_{2}, \chi_{3}\right), \eta=|\eta| \omega\left(\omega \in S^{2}\right)$ and $\left\{\omega_{1}, \omega_{2}, \omega\right\}$ is an orthonormal basis of $\mathbb{R}^{3}$. And then we define

$$
\begin{aligned}
u_{L ; 0} & =\mathbb{G}_{L ; 0}^{t} u_{0}=(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta} \widehat{\mathbb{G}}_{L ; 0}(\eta, t) \widehat{u}_{0}(\eta) d \eta, \\
u_{L ; \perp} & =\mathbb{G}_{L ; \perp}^{t} u_{0}=(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta} \widehat{\mathbb{G}}_{L ; \perp}(\eta, t) \widehat{u}_{0}(\eta) d \eta, \\
u_{S} & =\mathbb{G}_{S}^{t} u_{0}=(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta} \widehat{\mathbb{G}}_{S}(\eta, t) \widehat{u}_{0}(\eta) d \eta,
\end{aligned}
$$

called the fluid part and nonfluid part of the long wave of $u$, and the short wave of $u$, respectively. In the meanwhile, we define

$$
\begin{gathered}
u_{0 L}:=(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta} \chi_{\delta}(\eta) \widehat{u}_{0}(\eta) d \eta \\
u_{0 S}:=(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta}\left(1-\chi_{\delta}(\eta)\right) \widehat{u}_{0}(\eta) d \eta .
\end{gathered}
$$

One can see that the long wave $u_{L}:=u_{L ; 0}+u_{L ; \perp}$ satisfies the equation

$$
\left\{\begin{array}{l}
\partial_{t} u_{L}+\xi \cdot \nabla_{x} u_{L}=\mathcal{L} u_{L}  \tag{16}\\
u_{L}(0, x, \xi)=u_{0 L}(x, \xi)
\end{array}\right.
$$

and the short wave satisfies

$$
\left\{\begin{array}{l}
\partial_{t} u_{S}+\xi \cdot \nabla_{x} u_{S}=\mathcal{L} u_{S}  \tag{17}\\
u_{S}(0, x, \xi)=u_{0 S}(x, \xi)
\end{array}\right.
$$

According to [8, 12, the wave structure is given by

$$
\left\|\partial_{x}^{\alpha} \mathbb{G}_{L ; 0}^{t}(x, t)\right\| \leq C_{N}(1+t)^{-|\alpha| / 2}\left[\begin{array}{l}
(1+t)^{-2} B_{N}(|x|-\mathbf{c} t, t)+(1+t)^{-3 / 2} B_{N}(|x|, t) \\
+\mathbf{1}_{\{|x| \leq \mathbf{c} t\}}(1+t)^{-3 / 2} B_{3 / 2}(|x|, t)
\end{array}\right]
$$

for all $N \in \mathbb{N}$ if $0 \leq \gamma \leq 1$, where $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ is a multi-index with $|\alpha| \geq 0,\|\cdot\|$ denotes the operator norm from $L_{\xi}^{2}$ to $L_{\xi}^{2}$, the number $\mathbf{c}=\sqrt{\frac{5}{3}}$ is the sound speed, $\mathbf{1}_{D}$ is the characteristic function of the set $D$ and

$$
B_{N}(z, t)=\left(1+\frac{z^{2}}{1+t}\right)^{-N}
$$

We then have the following proposition:
Proposition 10. Let $u$ be a solution to (15) with the initial data $u_{0}$. Then

$$
\begin{gather*}
\left\|\partial_{x}^{\alpha} \mathbb{G}_{L ; 0}^{t} u_{0}\right\|_{L_{x}^{q}\left(L_{\xi}^{2}\right)} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{|\alpha|}{2}}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}, \quad 1 \leq p \leq q \leq \infty,  \tag{18}\\
\left\|\partial_{x}^{\alpha} \mathbb{G}_{L ; \perp}^{t} u_{0}\right\|_{L_{x}^{2} L_{\xi}^{2}} \leq C e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{x}^{2} L_{\xi}^{2}}, \tag{19}
\end{gather*}
$$

$$
\begin{equation*}
\left\|\mathbb{G}_{S}^{t} u_{0}\right\|_{L_{x}^{2} L_{\xi}^{2}} \leq C e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{x}^{2} L_{\xi}^{2}} \tag{20}
\end{equation*}
$$

and moreover, if $\mathrm{P}_{0} u_{0}=0$, then

$$
\left\|\partial_{x}^{\alpha} \mathbb{G}_{L ; 0}^{t} u_{0}\right\|_{L_{x}^{q}\left(L_{\xi}^{2}\right)} \leq C(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}-\frac{|\alpha|}{2}}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}, \quad 1 \leq p \leq q \leq \infty
$$

for some constants $C, c>0$, where $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ is a multi-index with $|\alpha| \geq 0$.
In view of (19), we have

$$
\left\|\mathbb{G}_{L ; \perp}^{t} u_{0}\right\|_{L_{x}^{r} L_{\xi}^{2}} \leq C e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{x}^{2} L_{\xi}^{2}}
$$

for $2 \leq r \leq \infty$. However, we have the $L_{x}^{2}$ estimate for $\mathbb{G}_{S}^{t} u_{0}$ only. In order to obtain the $L_{x}^{\infty}$ estimate, we apply the singular-regular decomposition as those in [8, 10]: We denote the solution operator of the damped transport equation

$$
\left\{\begin{array}{l}
\partial_{t} h+\xi \cdot \nabla_{x} h+\nu(\xi) h=0 \\
h(0, x, \xi)=h_{0}
\end{array}\right.
$$

by $\mathbb{S}^{t}$, i.e., $h(t)=\mathbb{S}^{t} h_{0}$. Then we design the series as

$$
\mathbb{G}^{t} u_{0}=\sum_{j=0}^{m} u^{(j)}+R^{(m)}=W^{(m)}+R^{(m)}
$$

for some $m \in \mathbb{N}$ (precisely, $m \geq 6$ ), where $u^{(j)}$ and $R^{(m)}$ are defined by

$$
u^{(0)}=\mathbb{S}^{t} u_{0}, \quad u^{(j)}=\int_{0}^{t} \mathbb{S}^{t-\tau} K u^{(j-1)}(\tau) d \tau, \quad 1 \leq j \leq m
$$

and

$$
R^{(m)}=\int_{0}^{t} \mathbb{G}^{t-\tau} K u^{(m)}(\tau) d \tau
$$

Combining the singular-regular decomposition with Proposition 10, we are able to get the $L_{x}^{r}$ estimate of $\mathbb{G}^{t} u_{0}$ for $2 \leq r \leq \infty$ :

Proposition 11. Let $u$ be a solution of (15) with the initial data $u_{0}$. Then

$$
\begin{gathered}
\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{x}^{r} L_{\xi}^{2}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right) \\
\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right) \\
\left\|\varpi \mathbb{G}^{t} u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|\varpi u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right),
\end{gathered}
$$ for $\beta \geq 0,2 \leq r \leq \infty$ and $1 \leq p \leq r$. Moreover, if $\mathrm{P}_{0} u_{0}=0$, then we will get extra $(1+t)^{-1 / 2}$ decay rate in the above estimates.

Proof. By Lemma 6

$$
\begin{gather*}
\left\|W^{(m)}\right\|_{L_{\xi}^{q} L_{x}^{p}} \lesssim e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{\xi}^{q} L_{x}^{p}}, 1 \leq p, q \leq \infty  \tag{21}\\
\left\|W^{(m)}\right\|_{L_{x}^{\infty} L_{\xi}^{q}} \lesssim e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{\xi}^{q} L_{x}^{\infty}}, 1 \leq q \leq \infty \tag{22}
\end{gather*}
$$

Utilizing the Mixture Lemma [8, 10] yields

$$
\begin{equation*}
\left\|R^{(m)}\right\|_{H_{x}^{2} L_{\xi}^{2}}=\left\|R^{(m)}\right\|_{L_{\xi}^{2} H_{x}^{2}} \lesssim\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}} \tag{23}
\end{equation*}
$$

Note that $\mathbb{G}^{t} u_{0}=\mathbb{G}_{L ; 0}^{t} u_{0}+\mathbb{G}_{L ; \perp}^{t} u_{0}+\mathbb{G}_{S}^{t} u_{0}=W^{(m)}+R^{(m)}$. In light of Proposition (10, (21) and (23), we find

$$
\left\|\mathbb{G}_{S}^{t} u_{0}-W^{(m)}\right\|_{L_{x}^{\infty} L_{\xi}^{2}},\left\|\mathbb{G}_{S}^{t} u_{0}-W^{(m)}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}
$$

$$
\begin{aligned}
& \lesssim\left\|\mathbb{G}_{S}^{t} u_{0}-W^{(m)}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{3 / 4}\left\|\mathbb{G}_{S}^{t} u_{0}-W^{(m)}\right\|_{L_{\xi}^{2} H_{x}^{2}}^{1 / 4} \\
& \lesssim\left\|\mathbb{G}_{S}^{t} u_{0}-W^{(m)}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{3 / 4}\left\|\left(\mathbb{G}_{L ; 0}^{t} u_{0}+\mathbb{G}_{L ; \perp}^{t} u_{0}\right)-R^{(m)}\right\|_{L_{\xi}^{2} H_{x}^{2}}^{1 / 4} \lesssim e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}},
\end{aligned}
$$

and so

$$
\left\|\mathbb{G}_{S}^{t} u_{0}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \leq\left\|\mathbb{G}_{S}^{t} u_{0}-W^{(m)}\right\|_{L_{x}^{\infty} L_{\xi}^{2}}+\left\|W^{(m)}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \lesssim e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}\right)
$$

due to (22). By the interpolation with (20),

$$
\begin{equation*}
\left\|\mathbb{G}_{S}^{t} u_{0}\right\|_{L_{x}^{r} L_{\xi}^{2}} \lesssim e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right) \tag{24}
\end{equation*}
$$

for $2 \leq r \leq \infty$. Together with (18)-(19), we conclude

$$
\begin{equation*}
\left\|\mathbb{G}^{t} u_{0}\right\|_{L_{x}^{r} L_{\xi}^{2}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right) \tag{25}
\end{equation*}
$$

for $2 \leq r \leq \infty$ and $1 \leq p \leq r$.
Next we will derive the weighted estimate. In view of (15),

$$
u=\mathbb{G}^{t} u_{0}=\mathbb{S}^{t} u_{0}+\int_{0}^{t} \mathbb{S}^{t-\tau} K u(\tau) d \tau
$$

and then

$$
\begin{aligned}
|u|_{L_{x}^{r}} & \leq\left|\mathbb{S}^{t} u_{0}\right|_{L_{x}^{r}}+\int_{0}^{t}\left|\mathbb{S}^{t-\tau} K u(\tau)\right|_{L_{x}^{r}} d \tau \\
& \leq e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{\xi}^{\infty} L_{x}^{r}}+\int_{0}^{t} e^{-\frac{t-\tau}{c}}\|K u(\tau)\|_{L_{\xi}^{\infty} L_{x}^{r}} d \tau \\
& \lesssim e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{\xi}^{\infty} L_{x}^{r}}+\int_{0}^{t} e^{-\frac{t-\tau}{c}}\|u(\tau)\|_{L_{x}^{r} L_{\xi}^{2}} d \tau \\
& \lesssim e^{-\frac{t}{c}}\left\|u_{0}\right\|_{L_{\xi}^{\infty} L_{x}^{r}}+(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right)
\end{aligned}
$$

for $2 \leq r \leq \infty$ and $1 \leq p \leq r$, by using (25). Through the bootstrap argument,

$$
\begin{equation*}
\|u\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right), \tag{26}
\end{equation*}
$$

for $\beta \geq 0$.
Furthermore,

$$
\begin{aligned}
\left|\varpi(\xi)\langle\xi\rangle^{\beta} u\right|_{L_{x}^{r}} & =\left|\varpi(\xi)\langle\xi\rangle^{\beta} \mathbb{S}^{t} u_{0}+\int_{0}^{t} \varpi(\xi)\langle\xi\rangle^{\beta} \mathbb{S}^{t-\tau} K u(\tau) d \tau\right|_{L_{x}^{r}} \\
& \leq \varpi(\xi)\langle\xi\rangle^{\beta}\left|\mathbb{S}^{t} u_{0}\right|_{L_{x}^{r}}+\int_{0}^{t} \varpi(\xi)\langle\xi\rangle^{\beta}\left|\mathbb{S}^{t-\tau} K u(\tau)\right|_{L_{x}^{r}} d \tau \\
& \equiv(I)+(I I) .
\end{aligned}
$$

It readily follows that

$$
(I) \lesssim e^{-\frac{t}{c}}\left\|\varpi u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}
$$

For (II), we split the integrand into two parts $|\xi| \leq \theta$ and $|\xi|>\theta$, where $\theta>0$ will be determined later. That is, for any $\beta \geq 0$,

$$
\begin{aligned}
& \varpi(\xi)\langle\xi\rangle^{\beta}\left|\mathbb{S}^{t-\tau} K u(\tau)\right|_{L_{x}^{r}} \\
\leq & e^{-\nu(\xi)(t-\tau)}\left[\sup _{|\xi| \leq \theta} \varpi(\xi)\langle\xi\rangle^{\beta}|K u(\tau)|_{L_{x}^{r}}+\sup _{|\xi|>\theta} \varpi(\xi)\langle\xi\rangle^{\beta}|K u(\tau)|_{L_{x}^{r}}\right] \\
\lesssim & e^{-\frac{t-\tau}{c}}\left[e^{c_{1}|\theta|^{2}}\|u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+(1+\theta)^{\gamma-2}\|\varpi K u(\tau)\|_{L_{\xi, 2-\gamma+\beta}^{\infty} L_{x}^{r}}\right]
\end{aligned}
$$

$$
\begin{aligned}
& \lesssim e^{-\frac{t-\tau}{c}}\left[e^{c_{1}|\theta|^{2}}\|u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+(1+\theta)^{\gamma-2}\left\|\varpi K \varpi^{-1} \varpi u(\tau)\right\|_{L_{\xi, 2-\gamma+\beta}^{\infty} L_{x}^{r}}\right] \\
& \lesssim e^{-\frac{t-\tau}{c}}\left[e^{c_{1}|\theta|^{2}}\|u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+(1+\theta)^{\gamma-2}\|\varpi u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}\right]
\end{aligned}
$$

by (13) and so

$$
(I I) \lesssim \int_{0}^{t} e^{-\frac{t-\tau}{c}}\left[e^{c_{2}|\theta|^{2}}\|u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+(1+\theta)^{\gamma-2}\|\varpi u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}\right] d \tau
$$

From (26) it follows that

$$
\begin{aligned}
& \|\varpi u(t)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \\
\lesssim & e^{-\frac{t}{c}}\left[\left\|\varpi u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+C(\theta)\left(\left\|u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right)\right] \\
& +C(\theta)(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+(1+\theta)^{\gamma-2} \int_{0}^{t} e^{-\frac{t-\tau}{c}}\|\varpi u(\tau)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} d \tau
\end{aligned}
$$

After $\theta$ is chosen sufficiently large,

$$
\begin{aligned}
& \|\varpi u(t)\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \\
& \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-t / c}\left(\left\|\varpi u_{0}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right)
\end{aligned}
$$

holds for $\beta \geq 0,2 \leq r \leq \infty$ and $1 \leq p \leq r$.
In view of (16), we see $u_{L}=\mathbb{G}^{t} u_{0 L}$. Observe that

$$
\begin{aligned}
\left\|u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} & =\left\|(2 \pi)^{-3} \int_{\mathbb{R}^{3}} e^{i x \cdot \eta} \chi_{\delta}(\eta) \widehat{u}_{0}(\eta) d \eta\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \\
& \leq(2 \pi)^{-3}\left|\int_{|\eta|<\delta}\right| \chi_{\delta}(\eta) \widehat{u}_{0}(\eta)|d \eta|_{L_{\xi, \beta}^{\infty}} \lesssim\left\|u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}
\end{aligned}
$$

and similarly for $\left\|\varpi u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}$. Hence $\left\|\varpi u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim\left\|\varpi u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}$ and $\left\|u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim$ $\left\|u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}$ for $2 \leq r \leq \infty$. It immediately follows from Proposition 11 that
Corollary 12. Let $u$ be a solution of (15) with initial data $u_{0}$. Then $u_{L}=\mathbb{G}^{t} u_{0 L}$ satisfies

$$
\begin{aligned}
&\left\|u_{L}\right\|_{L_{x}^{r} L_{\xi}^{2}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0 L}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left(\left\|u_{0 L}\right\|_{L_{x}^{2} L_{\xi}^{2}}+\left\|u_{0 L}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{2 / r}\left\|u_{0 L}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right), \\
&\left\|u_{L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0 L}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left\|u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
&\left\|\varpi u_{L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} \lesssim(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\left\|u_{0 L}\right\|_{L_{x}^{p} L_{\xi}^{2}}+e^{-\frac{t}{c}}\left\|\varpi u_{0 L}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}
\end{aligned}
$$

for $\beta>3 / 2,2 \leq r \leq \infty$ and $1 \leq p \leq r$. Moreover, if $\mathrm{P}_{0} u_{0}=0$, then we will get extra $(1+t)^{-1 / 2}$ decay rate in the above estimates.

On the other hand, in view of (17),

$$
u_{S}=\mathbb{S}^{t} u_{0 S}+\int_{0}^{t} \mathbb{S}^{t-\tau} K u_{S}(\tau) d \tau
$$

Applying similar argument as those for $u$, together with (24), we get the $L_{x}^{r}$ estimate of the short wave $u_{S}$ for $2 \leq r \leq \infty$ as well.

Corollary 13. Let $u$ be a solution of (15) with initial data $u_{0}$. Then

$$
\begin{aligned}
\left\|u_{S}(t)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} & \lesssim e^{-\frac{t}{c}}\left(\left\|u_{0 S}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}^{1-2 / r}\right) \\
\left\|\varpi u_{S}(t)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}} & \lesssim e^{-\frac{t}{c}}\left(\left\|\varpi u_{0 S}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{r}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{2 / r}\left\|u_{0}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{1-2 / r}\right),
\end{aligned}
$$

for $\beta \geq 0,2 \leq r \leq \infty$ and $1 \leq p \leq r$.
2.3. Estimates for the solution around $\mathcal{M}_{b}$. In this section we mention that the foregoing results stated in Subsections 2.1 and 2.2 are also valid for the solution around $\mathcal{M}_{b}$, up to constants depending on $\rho, \lambda$ and $\mu$. To see this, write $F^{b}(t, x, \xi)=\sigma \widetilde{F}(\widetilde{t}, \widetilde{x}, \widetilde{\xi})$ by the change of variables

$$
\sigma=\rho \lambda^{-3 / 2}, \widetilde{x}=x-\mu t, \tilde{t}=\sqrt{\lambda} t, \widetilde{\xi}=\frac{\xi-\mu}{\sqrt{\lambda}}
$$

and we discover

$$
\left\{\begin{array}{l}
\partial_{\overparen{t}} \widetilde{F}+\widetilde{\xi} \cdot \nabla_{\widetilde{x}} \widetilde{F}=\rho \sqrt{\lambda}^{(\gamma-1)} Q(\widetilde{F}, \widetilde{F})=\widetilde{Q}(\widetilde{F}, \widetilde{F}) \\
\widetilde{F}(0, \widetilde{x}, \widetilde{\xi})=\mathcal{M}_{a}(\widetilde{\xi})+\sqrt{\mathcal{M}_{a}(\widetilde{\xi})} \widetilde{f_{0}}(\widetilde{x}, \widetilde{\xi})
\end{array}\right.
$$

where $\widetilde{f}_{0}(\widetilde{x}, \widetilde{\xi})=\frac{1}{\sqrt{\sigma}} f_{0}^{b}(\widetilde{x}, \mu+\sqrt{\lambda} \widetilde{\xi})$.

## 3. Proof of the Main Theorem

Now we turn to equation (6). In view of (6), $g$ can be expressed by

$$
\begin{equation*}
g=\int_{0}^{t} \mathbb{G}^{t-s}\left\{2 \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)+\left[2 \Gamma\left(f^{a}, g\right)+\Gamma(g, g)\right]\right\} d s \equiv \chi_{1}+\chi_{2} \tag{27}
\end{equation*}
$$

Precisely,

$$
\chi_{1}=2 \int_{0}^{t} \mathbb{G}^{t-s} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}(s), \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right) d s
$$

and

$$
\chi_{2}=\int_{0}^{t} \mathbb{G}^{t-s}\left[2 \Gamma\left(f^{a}, g\right)(s)+\Gamma(g, g)(s)\right] d s
$$

Keep in mind that $\mathrm{P}_{0} \Gamma\left(h_{1}, h_{2}\right)=0$ during the course of the proof, and it will give extra $(1+t)^{-1 / 2}$ time decay rate under the operator $\mathbb{G}^{t}$.

Furthermore, we write $g=g_{1}+g_{2}$, where $g_{1}$ and $g_{2}$ solve the equations

$$
\left\{\begin{array}{l}
\partial_{t} g_{1}+\xi \cdot \nabla_{x} g_{1}+\nu(\xi) g_{1}=2 \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)+\left[2 \Gamma\left(f^{a}, g\right)+\Gamma(g, g)\right] \\
g_{1}(0, x, \xi)=0
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
\partial_{t} g_{2}+\xi \cdot \nabla_{x} g_{2}+\nu(\xi) g_{2}=K g \\
g_{2}(0, x, \xi)=0
\end{array}\right.
$$

respectively. By the Duhamel principle, they can be written in terms of the damped transport operator $\mathbb{S}^{t}$ as

$$
g_{1}(t, x, \xi)=\int_{0}^{t} \mathbb{S}^{t-s}\left[2 \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)+2 \Gamma\left(f^{a}, g\right)+\Gamma(g, g)\right] d s
$$

and

$$
g_{2}(t, x, \xi)=\int_{0}^{t} \mathbb{S}^{t-s} K g(s) d s
$$

In what follows we will estimate $\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}$ and $\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}$. The estimates will be expected to involve the quantity $\left(\mathcal{M}_{b}-\mathcal{M}_{a}\right) / \sqrt{\mathcal{M}_{a}}$. Before proceeding, we show that this quantity can be simply dominated by the parameters $\rho, \mu$ and $\lambda$.
Lemma 14. Let $1<\underline{\lambda}<\lambda<\bar{\lambda}<2,0<\rho<\bar{\rho}$ and $\beta \geq 0$. There exits a constant $C>0$ such that

$$
\left|\frac{\mathcal{M}_{b}-\mathcal{M}_{b}}{\sqrt{\mathcal{M}_{a}}}\langle\xi\rangle^{\beta}\right| \leq C e^{-\frac{2-\bar{\lambda}}{16}|\xi|^{2}}(|\rho-1|+|\lambda-1|+|\mu|)
$$

where $\mathcal{M}_{a}$ and $\mathcal{M}_{b}$ are given as (7).

For the brevity of presentation, we denote the difference of macroscopic quantities by

$$
\begin{equation*}
\mathcal{B}=|\rho-1|+|\lambda-1|+|\mu| . \tag{28}
\end{equation*}
$$

Proof. Define

$$
\mathcal{M}(\theta)=\frac{1+(\rho-1) \theta}{(2 \pi(1+(\lambda-1) \theta))^{3 / 2}} e^{-\frac{|\xi-\theta \mu|^{2}}{2(1+(\lambda-1) \theta)}}
$$

Then by Mean Value Theorem

$$
\begin{aligned}
\mathcal{M}_{b}-\mathcal{M}_{a} & =\mathcal{M}(1)-\mathcal{M}(0) \\
& =\left.\mathcal{M}(\theta)\left[-\frac{3(\lambda-1)}{2(1+\theta(\lambda-1))}+\frac{\mu \cdot(\xi-\theta \mu)}{1+\theta(\lambda-1)}+\frac{(\lambda-1)|\xi-\theta \mu|^{2}}{2(1+\theta(\lambda-1))^{2}}+\frac{\rho-1}{1+\theta(\rho-1)}\right]\right|_{\theta=\theta_{0}}
\end{aligned}
$$

for some $0<\theta_{0}<1$. Then we have

$$
\begin{aligned}
\left|\frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right| & \leq C e^{-\frac{\left|\xi-\theta_{0} \mu\right|^{2}}{2(1+(\lambda-1) \theta)}+\frac{|\xi|^{2}}{4}}\left[|\lambda-1|+|\mu|\left|\xi-\theta_{0} \mu\right|+|\lambda-1|\left|\xi-\theta_{0} \mu\right|^{2}+|\rho-1|\right] \\
& \leq C_{\delta} e^{-\frac{\left|\xi-\theta_{0} \mu\right|^{2}}{2 \lambda(1+\delta)}+\frac{|\xi|^{2}}{4}}[|\rho-1|+|\mu|+|\lambda-1|]
\end{aligned}
$$

where the polynomial $\left|\xi-\theta_{0} \mu\right|$ is absorbed by the exponential function and $\delta$ can be any positive number.

Note that

$$
e^{-\frac{\left|\xi-\theta_{0} \mu\right|^{2}}{2 \lambda(1+\delta)}+\frac{|\xi|^{2}}{4}}=e^{-\frac{2-\lambda(1+\delta)}{4(1+\delta) \lambda}|\xi|^{2}+\frac{\theta_{0} \mu \cdot \xi}{(1+\delta) \lambda}-\frac{\theta_{0}^{2}|\mu|^{2}}{2(1+\delta) \lambda}}
$$

In view of $1<\underline{\lambda}<\lambda<\bar{\lambda}<2$, one can choose $\delta>0$ sufficiently small to ensure the coefficient in front of $|\xi|^{2}$ is negative. It then follows that any polynomial $\langle\xi\rangle^{\beta}$ with $\beta \geq 0$ can be absorbed by the exponential function. This completes the proof.
3.1. $L_{x}^{2}$ Estimate of $g$. Let $T>0$ be any finite number and $\beta>3 / 2+2 \gamma$. Then for $0 \leq t \leq T$,

$$
\begin{align*}
&\langle\xi\rangle^{\beta}\left|g_{1}\right|_{L_{x}^{2}} \\
& \lesssim \int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)\left\|\nu^{-1} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s) d s \\
&+\int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)\left[\left\|\nu^{-1} \Gamma\left(f^{a}, g\right)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s)+\left\|\nu^{-1} \Gamma(g, g)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s)\right] d s  \tag{29}\\
& \lesssim \mathcal{B} \int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s) d s \\
&+\int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)\left[\left\|f^{a}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s)+\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s)\right] d s \\
& \lesssim \varepsilon \mathcal{B} \int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)(1+s)^{-\frac{3}{4}} d s\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}}\left(L_{x}^{\infty} \cap L_{x}^{1}\right) \\
&+\varepsilon \int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)(1+s)^{-\frac{3}{2}-\frac{1}{4}} d s \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
& \quad+\int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)(1+s)^{-\frac{1}{4}-1} d s \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \cdot \sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \\
& \lesssim \varepsilon \mathcal{B}(1+t)^{-3 / 4}+\varepsilon(1+t)^{-7 / 4} \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
&+(1+t)^{-5 / 4} \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}
\end{align*}
$$

by Theorem 8, Lemma 14 and the remarks in Section 2.3. Here $\mathcal{B}$ is the difference of macroscopic quantities defined in (28).

$$
\begin{aligned}
\langle\xi\rangle^{\beta}\left|g_{2}\right|_{L_{x}^{2}} & \lesssim \int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)\left\|\nu(\xi)^{-1} K g\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}(s) d s \\
& \lesssim \int_{0}^{t} e^{-\nu(\xi)(t-s)} \nu(\xi)\|g\|_{L_{\xi, \beta-\gamma} L_{x}^{2}}(s) d s \\
& \lesssim(1+t)^{-1 / 4} \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{2}} .
\end{aligned}
$$

To obtain $\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta-\gamma} L_{x}^{2}}$, we now estimate $\left\|\chi_{1}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{2}}$ and $\left\|\chi_{2}\right\|_{L_{\xi, \beta-\gamma} L_{x}^{2}}$ instead. In view of Theorems 9, 8 and Lemma 14.

$$
\begin{aligned}
& \left\|\chi_{1}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{2}} \\
\lesssim & \int_{0}^{t}(1+t-s)^{-1 / 2}\left\|2\langle\xi\rangle^{\beta-\gamma} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)\right\|_{L_{\xi}^{\infty} L_{x}^{2}}(s) d s \\
\lesssim & \mathcal{B} \int_{0}^{t}(1+t-s)^{-1 / 2}\left\|\langle\xi\rangle^{\beta} \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}\right\|_{L_{\xi}^{\infty} L_{x}^{2}}(s) d s \\
\lesssim & \varepsilon \mathcal{B} \int_{0}^{t}(1+t-s)^{-1 / 2}(1+s)^{-3 / 4} d s \\
\lesssim & \varepsilon \mathcal{B}(1+t)^{-1 / 4},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\|\chi_{2}\right\|_{L_{\xi, \beta-\gamma}^{\infty}}^{L_{x}^{2}} \\
& \lesssim \int_{0}^{t}(1+t-s)^{-5 / 4}\left(\left\|\Gamma\left(f^{a}, g\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{2}}(s)+\left\|\Gamma\left(f^{a}, g\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{1}}(s)\right) d s \\
&+\int_{0}^{t}(1+t-s)^{-5 / 4}\left(\|\Gamma(g, g)\|_{L_{\xi, \beta-\gamma}} L_{x}^{2}(s)+\|\Gamma(g, g)\|_{L_{\xi, \beta-\gamma} L_{x}^{1}}(s)\right) d s \\
& \lesssim \varepsilon \int_{0}^{t}(1+t-s)^{-5 / 4}(1+s)^{-1} d s \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
&+\int_{0}^{t}(1+t-s)^{-5 / 4}(1+s)^{-5 / 4} d s\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right) \\
&+\int_{0}^{t}(1+t-s)^{-5 / 4}(1+s)^{-1 / 2} d s\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& \left\|\chi_{2}\right\|_{L_{\xi, \beta-\gamma}^{\infty}} L_{x}^{2} \\
\lesssim & \varepsilon(1+t)^{-1} \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
& +(1+t)^{-1 / 2}\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}+(1+t)^{-5 / 4}\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2} .
\end{aligned}
$$

Therefore, we obtain

$$
\begin{align*}
(1+t)^{1 / 4}\left\|g_{2}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \lesssim & \varepsilon \mathcal{B}+\varepsilon \sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}  \tag{30}\\
& +\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}+\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2}
\end{align*}
$$

Summing up above estimates, we have the $L_{\xi, \beta}^{\infty} L_{x}^{2}$ estimate of $g$ :

$$
\begin{align*}
(1+t)^{1 / 4}\|g\|_{L_{\xi, \beta} L_{x}^{2}}^{\infty} \lesssim & \varepsilon \mathcal{B}+\varepsilon \sup _{0 \leq s \leq t}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
& +\left(\sup _{0 \leq s \leq t}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}+\left(\sup _{0 \leq s \leq t}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2} \tag{31}
\end{align*}
$$

for $0 \leq t \leq T$.
3.2. $L_{x}^{\infty}$ Estimate on $g$. Applying similar argument as those for the $L_{x}^{2}$ estimate of $g$, we have for $0 \leq t \leq T$ and $\beta>3 / 2+2 \gamma$

$$
\begin{align*}
(1+t)\left\|g_{1}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \lesssim & (1+t)^{-1 / 2} \varepsilon \mathcal{B}+\varepsilon(1+t)^{-3 / 2} \sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \\
& +(1+t)^{-1}\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2} \tag{32}
\end{align*}
$$

and

$$
\langle\xi\rangle^{\beta}\left|g_{2}\right|_{L_{x}^{\infty}} \lesssim(1+t)^{-1} \sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}}
$$

As same as the $L_{x}^{2}$ case, , we will obtain $\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{, \beta-\gamma}} L_{x}^{\infty}$ by estimating $\chi_{1}$ and $\chi_{2}$. For $\chi_{1}$, we have

$$
\left\|\chi_{1}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}} \lesssim \varepsilon(1+t)^{-1} \mathcal{B}
$$

We need some refined estimate for the linearized Boltzmann equation in Section 2.2 to prove this estimate. Since the proof is delicate and lengthy, we postpone the detail to the next subsection (Proposition 16).

For $\chi_{2}$, by Theorems 8 and 9 we have

$$
\begin{aligned}
\left\|\chi_{2}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}} \lesssim & \int_{0}^{t}(1+t-s)^{-5 / 4}\left(\left\|\Gamma\left(f^{a}, g\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}}(s)+\left\|\Gamma\left(f^{a}, g\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{2}}(s)\right) d s \\
& +\int_{0}^{t}(1+t-s)^{-5 / 4}\left(\|\Gamma(g, g)\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}}(s)+\|\Gamma(g, g)\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{2}}(s)\right) d s \\
\lesssim & \varepsilon(1+t)^{-5 / 4}\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right) \\
& +(1+t)^{-5 / 4}\left[\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}+\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2}\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(33)(1+t)\left\|g_{2}\right\|_{L_{\xi, \beta} L_{x}^{\infty}} \lesssim & \varepsilon \mathcal{B}+\varepsilon\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right) \\
& +\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}+\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2}
\end{aligned}
$$

Combining (32) and (33), we conclude the $L_{\xi, \beta}^{\infty} L_{x}^{\infty}$ estimate of $g$ :

$$
\begin{align*}
(1+t)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \lesssim & \varepsilon \mathcal{B}+\varepsilon\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)  \tag{34}\\
& +\left(\sup _{0 \leq s \leq T}(1+s)^{1 / 4}\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}\right)^{2}+\left(\sup _{0 \leq s \leq T}(1+s)\|g\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right)^{2}
\end{align*}
$$

Now let

$$
Q(T)=\sup _{0 \leq s \leq T}\left((1+s)\left\|\langle\xi\rangle^{\beta} g / \mathcal{B}\right\|_{L_{\xi}^{\infty} L_{x}^{\infty}}+(1+s)^{1 / 4}\left\|\langle\xi\rangle^{\beta} g / \mathcal{B}\right\|_{L_{\xi}^{\infty} L_{x}^{2}}\right)
$$

for any finite $T>0$. From (31) and (34) it gives the inequality

$$
Q(T) \leq C_{1} \varepsilon+C_{2} \varepsilon Q(T)+C \mathcal{B} Q^{2}(T)
$$

for any finite $T \geq 0$. Since $g(0, x, \xi)=0$, we get $Q(T) \leq \widetilde{C} \varepsilon$ for some constant $\widetilde{C}>0$ and for all $T \geq 0$ whenever $\varepsilon>0$ is sufficiently small. The proof is completed.
3.3. $L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}$ estimate of $\chi_{1}$. In this section we are devoted to the estimate of $\left\|\chi_{1}\right\|_{L_{\xi, \beta-\gamma} L_{x}^{\infty}}$, $\beta>3 / 2+2 \gamma$.
Remark 15. At the first sight, it readily follows from Theorems 8 and 9 that

$$
\left\|\chi_{1}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}} \lesssim \varepsilon \mathcal{B} \int_{0}^{t}(1+t-s)^{-5 / 4}(1+s)^{-3 / 4} d s \lesssim \varepsilon \mathcal{B}(1+t)^{-3 / 4}
$$

However, this estimate for $\chi_{1}$ is not satisfactory and we can improve it up to the time decay rate $(1+t)^{-1}$ by the refined estimate for the linearized Boltzmann equation.

Proposition 16. Let $\beta>3 / 2+2 \gamma$. Then $\chi_{1}$ defined as (27) satisfies

$$
\left\|\chi_{1}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}} \leq C \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap L_{x}^{1}\right)}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)
$$

for some constant $C>0$ independent of $t$.
To prove this, we represent $f^{b}$ by

$$
f^{b}=\varepsilon \mathbb{G}_{b}^{t} f_{0}^{b}+\int_{0}^{t} \mathbb{G}_{b}^{t-\tau} \Gamma_{b}\left(f^{b}, f^{b}\right)(\tau) d \tau
$$

and then write $\chi_{1}$ as

$$
\begin{aligned}
\chi_{1}= & \int_{0}^{t} \mathbb{G}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau \\
= & \varepsilon \int_{0}^{t} \mathbb{G}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau \\
& +\int_{0}^{t} \mathbb{G}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \int_{0}^{\tau} \mathbb{G}_{b}^{\tau-s} \Gamma_{b}\left(f^{b}, f^{b}\right) d s, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau \\
\equiv & \chi_{11}+\chi_{12}
\end{aligned}
$$

We further split $\chi_{11}$ into two parts

$$
\chi_{111}=\varepsilon \int_{0}^{\frac{t}{2}} \mathbb{G}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau
$$

and

$$
\chi_{112}=\varepsilon \int_{\frac{t}{2}}^{t} \mathbb{G}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau
$$

Among them, $\chi_{112}$ is delicate, so we first deal with this term and use the long-short wave decomposition to split the integrand into six parts

$$
\chi_{112}=\varepsilon \int_{\frac{t}{2}}^{t}\left(\mathbb{G}_{L, 0}^{t-\tau}+\mathbb{G}_{L, \perp}^{t-\tau}+\mathbb{G}_{S}^{t-\tau}\right) \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}}\left(\mathbb{G}_{b, L}^{\tau}+\mathbb{G}_{b, S}^{\tau}\right) f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau
$$

For the purpose of simplification, we denote

$$
\mathcal{T} h:=\Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} h, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)
$$

We will estimate them term by term.
In view of Proposition 10, Corollary 12, and the fact that $\mathrm{P}_{0} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}=0$,

$$
\begin{aligned}
& \left\|\mathbb{G}_{L, 0}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \\
\lesssim & (1+t-\tau)^{-\frac{3}{2 p}-\frac{1}{2}}\left\|\mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{p} L_{\xi}^{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \lesssim(1+t-\tau)^{-\frac{3}{2 p}-\frac{1}{2}}\left\|\Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)\right\|_{L_{x}^{p} L_{\xi}^{2}} \\
& \lesssim(1+t-\tau)^{-\frac{3}{2 p}-\frac{1}{2}} \mathcal{B}\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{p} L_{\xi}^{2}} \\
& \lesssim(1+t-\tau)^{-\frac{3}{2 p}-\frac{1}{2}} \mathcal{B}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{p}} \\
& \lesssim(1+t-\tau)^{-\frac{3}{2 p}-\frac{1}{2}}(1+\tau)^{-\frac{3}{2}\left(\frac{1}{r}-\frac{1}{p}\right)} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{r} L_{\xi}^{2}}\right)
\end{aligned}
$$

for $2 \leq p \leq \infty$ and $1 \leq r \leq p$. Taking $r=1$ and $p>3$ gives

$$
\begin{equation*}
\int_{\frac{t}{2}}^{t}\left\|\mathbb{G}_{L, 0}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} d \tau \lesssim(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right) \tag{35}
\end{equation*}
$$

By the Sobolev inequality,

$$
\begin{aligned}
\left\|\mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} & \lesssim\left\|\nabla_{x}^{2} \mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{1 / 2} \cdot\left\|\nabla_{x} \mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{1 / 2} \\
& \lesssim\left\|\mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \nabla_{x}^{2} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{1 / 2} \cdot\left\|\mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \nabla_{x} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{1 / 2} .
\end{aligned}
$$

In view of Proposition 10 and Corollary 12 ,

$$
\begin{aligned}
\left\|\mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \nabla_{x}^{2} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}} & \lesssim e^{-\frac{t-\tau}{c}}\left\|\mathcal{T} \nabla_{x}^{2} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}} \\
& \lesssim e^{-\frac{t-\tau}{c}}\left\|\Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \nabla_{x}^{2} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)\right\|_{L_{x}^{2} L_{\xi}^{2}} \\
& \lesssim e^{-\frac{t-\tau}{c}} \mathcal{B}\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \nabla_{x}^{2} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}} \\
& \lesssim e^{-\frac{t-\tau}{c} \mathcal{B}}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \nabla_{x}^{2} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} \\
& \lesssim e^{-\frac{t-\tau}{c}}(1+\tau)^{-\frac{3}{4}-\frac{2}{2}} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)
\end{aligned}
$$

Similarly,

$$
\left\|\mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \nabla_{x} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}} \lesssim e^{-\frac{t-\tau}{c}}(1+\tau)^{-\frac{3}{4}-\frac{1}{2}} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)
$$

Therefore,

$$
\left\|\mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \lesssim e^{-\frac{t-\tau}{c}}(1+\tau)^{-\frac{3}{2}} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)
$$

and

$$
\begin{equation*}
\left\|\int_{\frac{t}{2}}^{t} \mathbb{G}_{L, \perp}^{t-\tau} \mathcal{T} \mathbb{G}_{b, L}^{\tau} f_{0}^{b} d \tau\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \lesssim(1+t)^{-\frac{3}{2}} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right) \tag{36}
\end{equation*}
$$

By Proposition 10, Corollary 13 and the fact that $\mathrm{P}_{0} \mathcal{T} \mathbb{G}_{b, S}^{\tau} f_{0}^{b}=0$,

$$
(37) \int_{\frac{t}{2}}^{t}\left\|\left(\mathbb{G}_{L, 0}^{t-\tau}+\mathbb{G}_{L, \perp}^{t-\tau}\right) \mathcal{T} \mathbb{G}_{b, S}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} d \tau \lesssim \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{5}{4}}\left\|\mathcal{T} \mathbb{G}_{b, S}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}} d \tau
$$

$$
\begin{aligned}
& \lesssim \mathcal{B} \int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{5}{4}}\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{2}} d \tau \\
& \lesssim \int_{\frac{t}{2}}^{t} \mathcal{B}(1+t-\tau)^{-\frac{5}{4}}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}} d \tau \\
& \lesssim e^{-\frac{t}{2 c} \mathcal{B}}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 S}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{2}}\right) .
\end{aligned}
$$

In accordance with (24) and Corollary 13 ,

$$
\begin{align*}
& \int_{\frac{t}{2}}^{t}\left\|\mathbb{G}_{S}^{t-\tau} \mathcal{T} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} d \tau  \tag{38}\\
\lesssim & \int_{\frac{t}{2}}^{t} e^{-\frac{t-\tau}{c}}\left[\left\|\mathcal{T} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|\mathcal{T} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}\right] d \tau \\
\lesssim & \int_{\frac{t}{2}}^{t} e^{-\frac{t-\tau}{c} \mathcal{B}}\left(\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{2}}+\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}\right) d \tau \\
\lesssim & \mathcal{B} \int_{\frac{t}{2}}^{t} e^{-\frac{t-\tau}{c}}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{S}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right) d \tau \\
\lesssim & \mathcal{B} \int_{\frac{t}{2}}^{t} e^{-\frac{t-\tau}{c}} e^{-\frac{\tau}{c}} d \tau\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 S}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}+\left\|f_{0}^{b}\right\|_{L_{\xi}^{2}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}\right) \\
\lesssim & e^{-\frac{t}{2 c} \mathcal{B}}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 S}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}+\left\|f_{0}^{b}\right\|_{L_{\xi}^{2}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}\right) .
\end{align*}
$$

By the Sobolev inequality, Proposition 10 and Corollary 12

$$
\begin{aligned}
& \left\|\mathbb{G}_{S}^{t-\tau} \mathcal{T} \mathbb{G}_{L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \\
\lesssim & \left\|\nabla_{x}^{2} \mathbb{G}_{S}^{t-\tau} \mathcal{T} \mathbb{G}_{L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{1 / 2} \cdot\left\|\nabla_{x} \mathbb{G}_{S}^{t-\tau} \mathcal{T} \mathbb{G}_{L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}^{1 / 2} \\
\lesssim & e^{-\frac{t-\tau}{c}}\left\|\mathcal{T} \nabla_{x}^{2} \mathbb{G}_{L}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{1 / 2}\left\|\mathcal{T} \nabla_{x} \mathbb{G}_{L}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{2}}^{1 / 2} \\
\lesssim & e^{-\frac{t-\tau}{c}}(1+\tau)^{\frac{1}{2}\left(-\frac{3}{4}-\frac{2}{2}\right)+\frac{1}{2}\left(-\frac{3}{4}-\frac{1}{2}\right)} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)
\end{aligned}
$$

and so

$$
\begin{equation*}
\int_{\frac{t}{2}}^{t}\left\|\mathbb{G}_{S}^{t-\tau} \mathcal{T} \mathbb{G}_{L}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} d \tau \lesssim \mathcal{B}(1+t)^{-3 / 2}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, B}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right) \tag{39}
\end{equation*}
$$

Gathering (35)-(39), we obtain

$$
\begin{aligned}
& \left\|\chi_{112}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \\
\lesssim & \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}+\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 S}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}+\left\|f_{0}^{b}\right\|_{L_{\xi}^{2}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}\right) \\
\lesssim & \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right),
\end{aligned}
$$

since $\left\|f_{0 L}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}} \lesssim\left\|f_{0}^{b}\right\|_{L_{x}^{1} L \xi_{\xi}^{2}}$ and

$$
\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 S}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}=\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}}\left(f_{0}^{b}-f_{0 L}^{b}\right)\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}} \leq\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0 L}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}
$$

Next we see $\left\|\chi_{111}\right\|_{L_{x}^{\infty} L_{\xi}^{2}}$. In light of Theorem 9 and Proposition 11 and the fact that $\mathrm{P}_{0} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)=$ 0 ,

$$
\begin{aligned}
\left\|\chi_{111}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} & =\varepsilon\left\|\int_{0}^{\frac{t}{2}} \mathbb{G}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right) d \tau\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \\
& \lesssim \varepsilon \mathcal{B} \int_{0}^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}\left(\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}\right\|_{L_{x}^{2} L_{\xi}^{2}}+\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{\xi}^{\tau} f_{0}^{b}\right\|_{L_{\xi}^{2} L_{x}^{\infty}}\right) d \tau \\
& \lesssim \varepsilon \mathcal{B} \int_{0}^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}\right) d \tau \\
& \lesssim \varepsilon \mathcal{B} \int_{0}^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{4}} d \tau\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{2}}+\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty} L_{x}^{\infty}}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right) \\
& \lesssim \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right) .
\end{aligned}
$$

Therefore,
(40) $\left\|\chi_{11}\right\|_{L_{x}^{\infty} L_{\xi}^{2}}=\left\|\chi_{111}+\chi_{112}\right\|_{L_{x}^{\infty} L_{\xi}^{2}} \lesssim \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap L_{x}^{2}\right)}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)$.

Note that $\chi_{11}$ can be expressed by

$$
\chi_{11}=\int_{0}^{t} \mathbb{S}^{t-\tau}\left[K \chi_{11}(\tau)+\varepsilon \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)\right] d \tau
$$

By Proposition 11 and (40), we have

$$
\begin{aligned}
\left\|\chi_{11}\right\|_{L_{x}^{\infty} L_{\xi}^{\infty}} & \lesssim \int_{0}^{t} e^{-\frac{(t-\tau)}{c}}\left(\left\|\chi_{11}(\tau)\right\|_{L_{x}^{\infty} L_{\xi}^{2}}+\varepsilon \mathcal{B}\left\|\nu(\xi) \frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau} f_{0}^{b}\right\|_{L_{x}^{\infty} L_{\xi}^{\infty}}\right) d \tau \\
& \lesssim \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap L_{x x}^{2}\right)}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right)
\end{aligned}
$$

and then

$$
\begin{equation*}
\left\|\chi_{11}\right\|_{L_{x}^{\infty} L_{\xi, \beta-\gamma}^{\infty}} \lesssim \varepsilon(1+t)^{-1} \mathcal{B}\left(\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{\infty} \cap L_{x}^{2}\right)}+\left\|f_{0}^{b}\right\|_{L_{x}^{1} L_{\xi}^{2}}\right) \tag{41}
\end{equation*}
$$

via the bootstrap argument.
Finally, we consider $\left\|\chi_{12}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}}$. Owing to Theorems $\boxtimes$ and 0 we have

$$
\begin{aligned}
& \left\|\chi_{12}\right\|_{L_{\xi, \beta-2 \gamma} L_{x}^{\infty}} \\
\lesssim & \int_{0}^{t}(1+t-\tau)^{-\frac{5}{4}}\left\|\Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \int_{0}^{\tau} \mathbb{G}_{b}^{\tau-s} \Gamma_{b}\left(f^{b}, f^{b}\right) d s, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)\right\|_{L_{\xi, \beta-2 \gamma}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)} d \tau \\
\lesssim & \mathcal{B} \int_{0}^{t}(1+t-\tau)^{-\frac{5}{4}} \int_{0}^{\tau}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau-s} \Gamma_{b}\left(f^{b}, f^{b}\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty}\left(L_{x}^{2} \cap L_{x}^{\infty}\right)} d s d \tau \\
\lesssim & \mathcal{B} \int_{0}^{t}(1+t-\tau)^{-\frac{5}{4}} \int_{0}^{\tau}(1+\tau-s)^{-\frac{5}{4}}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \Gamma_{b}\left(f^{b}, f^{b}\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty}\left(L_{x}^{1} \cap L_{x}^{2} \cap L_{x}^{\infty}\right)} d s d \tau
\end{aligned}
$$

$$
\begin{aligned}
& \lesssim \varepsilon^{2} \mathcal{B} \int_{0}^{t}(1+t-\tau)^{-\frac{5}{4}} \int_{0}^{\tau}(1+\tau-s)^{-\frac{5}{4}}(1+s)^{-\frac{3}{2}} d s d \tau\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}^{2} \\
& \lesssim \varepsilon^{2} \mathcal{B}(1+t)^{-\frac{5}{4}}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}^{2}
\end{aligned}
$$

Note that

$$
\chi_{12}=\int_{0}^{t} \mathbb{S}^{t-\tau} \Gamma\left(\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \int_{0}^{\tau} \mathbb{G}_{b}^{\tau-s} \Gamma_{b}\left(f^{b}, f^{b}\right) d s, \frac{\mathcal{M}_{b}-\mathcal{M}_{a}}{\sqrt{\mathcal{M}_{a}}}\right)(\tau) d \tau+\int_{0}^{t} \mathbb{S}^{t-\tau} K \chi_{12}(\tau) d \tau
$$

Hence in view of (12), Theorem 8 and Proposition 11

$$
\begin{aligned}
& \left|\langle\xi\rangle^{\beta-\gamma} \chi_{12}\right|_{L_{x}^{\infty}} \\
\lesssim & \mathcal{B} \int_{0}^{t} e^{-\nu(\xi)(t-\tau)} \nu(\xi) \int_{0}^{\tau}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} \mathbb{G}_{b}^{\tau-s} \Gamma_{b}\left(f^{b}, f^{b}\right)\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}} d s d \tau \\
& +\int_{0}^{t} e^{-\nu_{0}(t-\tau)}\left\|\chi_{12}\right\|_{L_{\xi, \beta-2 \gamma}^{\infty} L_{x}^{\infty}}(\tau) d \tau \\
\lesssim & \varepsilon^{2} \mathcal{B} \int_{0}^{t} e^{-\nu(\xi)(t-\tau)} \nu(\xi) \int_{0}^{\tau}(1+\tau-s)^{-5 / 4}(1+s)^{-9 / 4} d s d \tau\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}^{2} \\
& +\varepsilon^{2} \mathcal{B} \int_{0}^{t} e^{-\nu_{0}(t-\tau)}(1+\tau)^{-5 / 4} d \tau\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}^{2} \\
\lesssim & \varepsilon^{2} \mathcal{B}(1+t)^{-5 / 4}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}^{2},
\end{aligned}
$$

so that

$$
\begin{equation*}
\left\|\chi_{12}\right\|_{L_{\xi, \beta-\gamma}^{\infty} L_{x}^{\infty}} \lesssim \varepsilon^{2} \mathcal{B}(1+t)^{-5 / 4}\left\|\frac{\sqrt{\mathcal{M}_{b}}}{\sqrt{\mathcal{M}_{a}}} f_{0}^{b}\right\|_{L_{\xi, \beta}^{\infty}\left(L_{x}^{1} \cap L_{x}^{\infty}\right)}^{2} \tag{42}
\end{equation*}
$$

Combining (41) and (42), the proof of Proposition 16 is completed.

## Appendix A. Heat equation

Theorem 17. Let $\mu \in \mathbb{R}^{3}, 1 \leq \lambda \leq 2$. Assume that $h^{a}$ and $h^{b}$ satisfy the heat equations in the whole space $\mathbb{R}^{3}$, i.e.,

$$
\partial_{t} h^{a}=\Delta h^{a}
$$

and

$$
\partial_{t} h^{b}+\mu \cdot \nabla h^{b}=\lambda^{(2-\gamma) / 2} \Delta h^{b}
$$

with initial data $h_{0}^{a}=h_{0}^{b}=h_{0} \in L_{x}^{1}\left(\mathbb{R}^{3}\right)$. Then there exists a constant $C>0$ independent of time such that

$$
\begin{aligned}
& \left|h^{b}-h^{a}\right|_{L_{x}^{\infty}} \leq C(1+t)^{-1}\left|h_{0}\right|_{L_{x}^{1}}\left(|\lambda-1|(1+t)^{-1 / 2}+|\mu|\right) \\
& \left|h^{b}-h^{a}\right|_{L_{x}^{2}} \leq C(1+t)^{-1 / 4}\left|h_{0}\right|_{L_{x}^{1}}\left(|\lambda-1|(1+t)^{-1 / 2}+|\mu|\right)
\end{aligned}
$$

for $t \geq 1$.
To simplify the notation, we set $\kappa=\lambda^{(2-\gamma) / 2}$. We will provide two different methods to prove the theorem. The first method is to study the difference of the two solutions. In view of the exact solution formula associated with the heat kernel, we have

$$
\begin{gathered}
h^{a}(t, x)=\int_{\mathbb{R}^{3}} \frac{1}{(4 \pi t)^{3 / 2}} e^{-\frac{|x-y|^{2}}{4 t}} h_{0}(y) d y \\
h^{b}(t, x)=\int_{\mathbb{R}^{3}} \frac{1}{(4 \pi \kappa t)^{3 / 2}} e^{-\frac{|x-\mu t-y|^{2}}{4 \kappa t}} h_{0}(y) d y
\end{gathered}
$$

so that

$$
h^{a}(t, x)-h^{b}(t, x)=\int_{\mathbb{R}^{3}}\left[\frac{1}{(4 \pi t)^{3 / 2}} e^{-\frac{|x-y|^{2}}{4 t}}-\frac{1}{(4 \pi \kappa t)^{3 / 2}} e^{-\frac{|x-\mu t-y|^{2}}{4 \kappa t}}\right] h_{0}(y) d y
$$

To proceed, we need the following lemma:
Lemma 18. For $\mu \in \mathbb{R}^{3}, 1 \leq \lambda \leq 2, t>0$,

$$
\left|\frac{e^{-\frac{|x|^{2}}{4 t}}}{(4 \pi t)^{3 / 2}}-\frac{e^{-\frac{|x-\mu t|^{2}}{4 \kappa t}}}{(4 \pi \kappa t)^{3 / 2}}\right|_{L_{x}^{p}} \leq C t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}[|\kappa-1|+|\mu| \sqrt{t}], 1 \leq p \leq \infty
$$

for some constant $C>0$ independent of $\lambda, u, p$.
Proof. By mean value theorem

$$
\begin{aligned}
& \frac{e^{-\frac{|x-\mu t|^{2}}{4 \kappa t}}}{(4 \pi \kappa t)^{3 / 2}}-\frac{e^{-\frac{|x|^{2}}{4 t}}}{(4 \pi t)^{3 / 2}}=\left[\frac{e^{-\frac{|x-\theta \mu t|^{2}}{4(1+(\kappa-1) \theta) t}}}{(4 \pi(1+(\kappa-1) \theta) t)^{3 / 2}}\right]_{\theta=0}^{\theta=1} \\
& =\int_{0}^{1} \frac{e^{-\frac{|x-\theta \mu t|^{2}}{4(1+(\kappa-1) \theta) t}}}{(4 \pi(1+(\kappa-1) \theta) t)^{3 / 2}}\left[-\frac{3(\kappa-1)}{2(\theta(\kappa-1)+1)}+\frac{\mu \cdot(x-\theta \mu t)}{2(\theta(\kappa-1)+1)}+\frac{(\kappa-1)|x-\theta \mu t|^{2}}{4 t(\theta(\kappa-1)+1)^{2}}\right] d \theta
\end{aligned}
$$

It then immediately follows that for $1 \leq p \leq \infty$,

$$
\begin{aligned}
& \left|\frac{e^{-\frac{|x-\mu t|^{2}}{\kappa \kappa t}}}{(4 \pi \kappa t)^{3 / 2}}-\frac{e^{-\frac{|x|^{2}}{4 t}}}{(4 \pi t)^{3 / 2}}\right|_{L_{x}^{p}} \\
\leq & \int_{0}^{1}\left|\frac{e^{-\frac{|x-\theta \mu t|^{2}}{4(1+(\kappa-1) \theta) t}}}{(4 \pi(1+(\kappa-1) \theta) t)^{3 / 2}}\left[-\frac{3(\kappa-1)}{2(\theta(\kappa-1)+1)}+\frac{\mu \cdot(x-\theta \mu t)}{2(\theta(\kappa-1)+1)}+\frac{(\kappa-1)|x-\theta \mu t|^{2}}{4 t(\theta(\kappa-1)+1)^{2}}\right]\right|_{L_{x}^{p}} d \theta \\
\lesssim & \int_{0}^{1}\left|\frac{e^{-\frac{|x-\theta \mu t|^{2}}{C t}}}{t^{3 / 2}}[|\kappa-1|+|\mu| \sqrt{t}]\right|_{L_{x}^{p}} d \theta \lesssim t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}[|\kappa-1|+|\mu| \sqrt{t}] .
\end{aligned}
$$

The polynomial $x-\theta \mu t$ is absorbed by exponential function in the second inequality.
From the Young's inequality for convolution, together with Lemma 18, it follows that

$$
\begin{aligned}
\left|h^{b}-h^{a}\right|_{L_{x}^{r}} & \leq\left|\frac{e^{-\frac{|x-\mu t|^{2}}{4 \kappa t}}}{(4 \pi \kappa t)^{3 / 2}}-\frac{e^{-\frac{|x|^{2}}{4 t}}}{(4 \pi t)^{3 / 2}}\right|_{L_{x}^{p}}\left|h_{0}\right|_{L_{x}^{q}} \\
& \leq C t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}[|\kappa-1|+|\mu| \sqrt{t}]\left|h_{0}\right|_{L_{x}^{q}} \\
& \leq C t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}[|\kappa-1|+|\mu| \sqrt{t}]\left|h_{0}\right|_{L_{x}^{q}}
\end{aligned}
$$

where $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}, p, q, r \geq 1$. Therefore, taking $q=1, r=\infty$ and $q=1, r=2$, respectively, gives

$$
\begin{aligned}
& \left|h^{b}-h^{a}\right|_{L_{x}^{\infty}} \leq C\left|h_{0}\right|_{L_{x}^{1}}\left(|\kappa-1| t^{-3 / 2}+|\mu| t^{-1}\right) \\
& \left|h^{b}-h^{a}\right|_{L_{x}^{2}} \leq C\left|h_{0}\right|_{L_{x}^{1}}\left(|\kappa-1| t^{-3 / 4}+|\mu| t^{-1 / 4}\right)
\end{aligned}
$$

Noting that $|\kappa-1| \leq|\lambda-1|$, the proof of Theorem 17 is completed.
Next, we provide an alternative proof for the difference. Let $h=h^{a}-h^{b}$. Then $h$ satisfies the equation

$$
\left\{\begin{array}{l}
\partial_{t} h=\Delta h+\mu \cdot \nabla h^{b}-(\kappa-1) \Delta h^{b} \\
h(0, x)=0
\end{array}\right.
$$

and it is given by

$$
\begin{aligned}
h(t, x)= & \int_{0}^{t} \int_{\mathbb{R}^{3}} \frac{1}{[4 \pi(t-s)]^{3 / 2}} e^{-\frac{|x-y|^{2}}{4(t-s)}}\left[\mu \cdot \nabla h^{b}(s, y)-(\kappa-1) \Delta h^{b}(s, y)\right] d y d s \\
= & \int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{3}} \frac{1}{[4 \pi(t-s)]^{3 / 2}} e^{-\frac{|x-y|^{2}}{4(t-s)}}\left[\mu \cdot \nabla h^{b}(s, y)-(\kappa-1) \Delta h^{b}(s, y)\right] d y d s \\
& +\int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^{3}} \frac{1}{[4 \pi(t-s)]^{3 / 2}} e^{-\frac{|x-y|^{2}}{4(t-s)}}\left[\mu \cdot \nabla h^{b}(s, y)-(\kappa-1) \Delta h^{b}(s, y)\right] d y d s \\
\equiv & h_{1}(t, x)+h_{2}(t, x) .
\end{aligned}
$$

Recall the fact that

$$
\left|\partial_{x}^{\alpha} h^{b}\right|_{L_{x}^{q}} \leq C(1+t)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{|\alpha|}{2}}\left|f_{0}\right|_{L_{x}^{1}}, 1 \leq q \leq \infty
$$

for some constant $C>0$, where $\alpha=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ is a multi-index. Thus, the integration by parts gives

$$
\begin{aligned}
h_{1}(t, x) & =\int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{3}} \frac{1}{[4 \pi(t-s)]^{3 / 2}} e^{-\frac{|x-y|^{2}}{4(t-s)}}\left[\mu \cdot \nabla h^{b}(s, y)-(\kappa-1) \Delta h^{b}(s, y)\right] d y d s \\
& =\int_{0}^{\frac{t}{2}} \int_{\mathbb{R}^{3}} \frac{1}{[4 \pi(t-s)]^{3 / 2}} e^{-\frac{|x-y|^{2}}{4(t-s)}} \frac{-(x-y)}{2(t-s)} \cdot\left[\mu h^{b}(s, y)-(\kappa-1) \nabla h^{b}(s, y)\right] d y d s
\end{aligned}
$$

and for $t \geq 1$,

$$
\begin{aligned}
\left|h_{1}\right|_{L_{x}^{r}} & \leq C\left|f_{0}\right|_{L_{x}^{1}} \int_{0}^{\frac{t}{2}}(t-s)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}\left(|\mu|(1+s)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)}+|\kappa-1|(1+s)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{1}{2}}\right) d s \\
& \lesssim|\mu|(1+t)^{-1+\frac{3}{2 r}}+|\kappa-1|(1+t)^{-\frac{3}{2}+\frac{3}{2 r}}
\end{aligned}
$$

by the Young convolution inequality with $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r}$ and $-\frac{3}{2}\left(1-\frac{1}{q}\right)+\frac{1}{2}>0$. With the same $r$,

$$
\begin{aligned}
\left|h_{2}\right|_{L_{x}^{r}} & \leq \int_{\frac{t}{2}}^{t}\left|\frac{1}{[4 \pi(t-s)]^{3 / 2}} e^{-\frac{|x|^{2}}{4(t-s)}}\right|_{L_{x}^{p}}\left|\mu \cdot \nabla h^{b}-(\kappa-1) \Delta h^{b}\right|_{L_{x}^{q}} d s \\
& \leq C\left|h_{0}\right|_{L_{x}^{1}} \int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}\left(|\mu|(1+s)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-\frac{1}{2}}+|\kappa-1|(1+s)^{-\frac{3}{2}\left(1-\frac{1}{q}\right)-1}\right) d s \\
& \lesssim|\mu|(1+t)^{-1+\frac{3}{2 r}}+|\kappa-1|(1+t)^{-\frac{3}{2}+\frac{3}{2 r}}
\end{aligned}
$$

for $t \geq 1$, by the Young convolution inequality with $\frac{1}{p}+\frac{1}{q}=1+\frac{1}{r},-\frac{3}{2}\left(1-\frac{1}{p}\right)+1>0$. Hence, for $1 \leq r \leq \infty$,

$$
|h|_{L_{x}^{r}} \leq C\left(|\mu|(1+t)^{-1+\frac{3}{2 r}}+|\kappa-1|(1+t)^{-\frac{3}{2}+\frac{3}{2 r}}\right), t \geq 1
$$

where $C>0$ is a constant independent of $\lambda, u$ and $r$. In particular,

$$
\begin{aligned}
& |h|_{L_{x}^{\infty}} \leq C\left(|\mu|(1+t)^{-1}+|\kappa-1|(1+t)^{-\frac{3}{2}}\right) \\
& |h|_{L_{x}^{2}} \leq C\left(|\mu|(1+t)^{-\frac{1}{4}}+|\kappa-1|(1+t)^{-\frac{3}{4}}\right)
\end{aligned}
$$

for $t \geq 1$.
Acknowledgments: This work is partially supported by the National Key R\&D Program of China, Project 2022YFA1000087. Y.-C. Lin is supported by the National Science and Technology Council under the grant NSTC 110-2115-M-006-002-MY2. H.T. Wang is supported by NSFC under Grant No. 12031013 and 12161141004, the Strategic Priority Research Program of Chinese Academy of Sciences under Grant No. XDA25010403. K.-C. Wu is supported by the National Science and Technology Council under the grant NSTC 112-2636-M-006-001 and National Center for Theoretical Sciences.

## References

[1] C. Cercignani, R. Illner, M. Pulvirenti, The mathematical theory of dilute gases, Springer, 2013
[2] R. Ellis, M. Pinsky, The first and second fluid approximation to the linearized Boltzmann equation. J. Math. Pure. Appl. 54 (1975), 125-156.
[3] R. Glassey, The Cauchy problem in kinetic theory, SIAM, Philadelphia, 1996.
[4] H. Grad, Asymptotic theory of the Boltzmann equation, Rarefied Gas Dynamics, J. A. Laurmann, Ed. 1, 26, pp.26-59 Academic Press, New York, 1963.
[5] S.-Y. Ha, $L^{1}$ stability of the Boltzmann equation for the hard sphere model. Arch. Ration. Mech. Anal. 173, 279-296 (2004).
[6] S.-Y. Ha and Q. Xiao, A revisiting of the $L^{2}$-stability theory of the Boltzmann equation near global Maxwellians, J. Stat. Phys. 160 (2015), no. 2, 430-465.
[7] S. Kawashima, A. Matsumura, and T. Nishida, On the Fluid-Dynamical Approximation to the Boltzmann Equation at the Level of the Navier-Stokes Equation, Commun. Math. Phys., 70(1979), 97-124.
[8] Y.-C. Lin, H.T. Wang, K.-C. Wu, Quantitative Pointwise Estimate of the Solution of the Linearized Boltzmann Equation, J. Stat. Phys., 171 (2018), 927-964.
[9] Y.-C. Lin, M.-J. Lyu, H.T. Wang, K.-C. Wu, Space-time behavior of the Boltzmann equation with soft potentials, J. Differential Equations, 322 (2022), 180-236.
[10] T.-P. Liu and S.-H. Yu, The Green function and large time behavior of solutions for the one-dimensional Boltzmann equation, Commun. Pure App. Math., 57 (2004), 1543-1608.
[11] T.-P. Liu, S.-H. Yu, Green's function of Boltzmann equation, 3-D waves, Bull. Inst. Math. Acad. Sin. 1 (2006) 1-78.
[12] T.-P. Liu, S.-H. Yu, Solving Boltzmann equation, Part I : Green's function, Bull. Inst. Math. Acad. Sin. (N.S.), 6 (2011), 151-243.
[13] Y. Sone, Molecular Gas Dynamics. Theory, Techniques, and Applications. Modeling and Simulation in Science, Engineering and Technology. Birkhäuser Inc., Boston (2007)
[14] S. Ukai, T. Yang, The Boltzmann equation in the space $L^{2} \cap L_{\beta}^{\infty}$ : global and time-periodic solutions, Anal. Appl., 4 (2006), no. 3, 263-310.
[15] S. Ukai, T. Yang, Mathematical theory of Boltzmann equation, Lecture Notes Series 8 (2006).
[16] T. Yang, H. Yu, Spectrum analysis of some kinetic equations, Arch. Ration. Mech. Anal., 222 (2016), no. 2, 731-768.
[17] M.Y. Zhong, Optimal time-decay rates of the Boltzmann equation, Sci. China Math., 57 (2014), 807-822.
Yu-Chu Lin, Department of Mathematics, National Cheng Kung University, Tainan, Taiwan
Email address: yuchu@mail.ncku.edu.tw
Haitao Wang, School of Mathematical Sciences, Institute of Natural Sciences, MOE-LSC, IMA-
Shanghai, Shanghai Jiao Tong University, Shanghai, China
Email address: haitallica@sjtu.edu.cn
Kung-Chien Wu, Department of Mathematics, National Cheng Kung University, Tainan, Taiwan and
National Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan
Email address: kungchienwu@gmail.com


[^0]:    Date: August 1, 2023.
    Key words and phrases. Boltzmann equation, Maxwellian states, Stability.
    2020 Mathematics Subject Classification: 35Q20; 82C40.

