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POINTWISE WAVE BEHAVIOR OF THE NON-ISENTROPIC NAVIER-STOKES EQUATIONS IN HALF SPACE*

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Abstract. In this paper, we aim to study the global well-posedness and pointwise behavior of the classical solution to one-dimensional non-isentropic compressible Navier-Stokes equations in half space. Based on H^s energy method, we first establish the global existence and uniqueness. To derive the accurate pointwise estimate of the solution, Green's function for the initial boundary value problem is investigated. It is shown that Green's function can be expressed in terms of a fundamental solution to the Cauchy problem. Then applying Duhamel's principle and nonlinear analysis yields the space-time estimate of the solution under some suitable assumptions on the initial data, which exhibits the rich wave structure. As a corollary, we prove that the solution converges to the equilibrium state at an algebraic time decay rate $(1+t)^{-1/2}$ in L^∞ norm with respect to the spatial variable.

Keywords. Navier-Stokes equations; pointwise estimate; Green's function; half space; non-isentropic.

AMS subject classifications. 35B40; 35M13; 76N10; 76N30.

1. Introduction

It's known that one-dimensional full compressible Navier-Stokes (CNS) equations describe the motion of a viscous, compressible, heat-conductive, and Newtonian polytropic fluid. In Lagrangian coordinates, it is written as follows

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \big(\frac{\tilde{\mu}}{v} u_x\big)_x, \\ (e + \frac{1}{2}u^2)_t + (pu)_x = \big(\frac{\kappa}{v}\theta_x + \frac{\tilde{\mu}}{v} u u_x\big)_x, \end{cases} \tag{1.1}$$

where $v(x,t) > 0$, $u(x,t)$, $p(x,t)$, $e(x,t)$ and $\theta(x,t)$ denote the specific volume, the velocity, the pressure, the internal energy and the absolute temperature respectively. Assume the viscosity $\tilde{\mu}$ and the coefficient of heat conductivity κ are positive constants. For simplicity, the monatomic gas model is considered in this paper, which means

$$p = \frac{R\theta}{v}, \quad e = c_v \theta, \tag{1.2}$$

where R is a positive constant and $c_v = \frac{R}{\gamma - 1} > 0$.

Most of the interesting phenomena in fluid dynamics are in connection with the presence of a physical boundary, such as slip boundary layer, thermal creep flow, and curvature effects. To this end, we are devoted to the study of the global existence and pointwise wave behavior of one-dimensional compressible Navier-Stokes equations in a half-line. According to the result of numerical simulation and physical experiment,

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the no-slip boundary condition for velocity and the adiabatic boundary condition for temperature are commonly used, which are described as

$$(u,\theta_x)|_{x=0}=(0,0), \quad (v,u,\theta)|_{x\to+\infty}=(v_*,0,\theta_*), \tag{1.3}$$

where v_* and θ_* denote the positive constants. Define $E=e+\frac{1}{2}u^2$. Then, the system (1.1) is reformulated as

$$\begin{cases} v_t-u_x=0, \\ u_t+\dfrac{R}{c_v}(\dfrac{E}{v}-\dfrac{u^2}{2v})_x=(\dfrac{\tilde{\mu}}{v}u_x)_x, \\ E_t+\dfrac{R}{c_v}(\dfrac{uE}{v}-\dfrac{u^3}{2v})_x=\dfrac{1}{c_v}(\dfrac{\kappa}{v}E_x+\dfrac{\tilde{\mu}c_v-\kappa}{v}uu_x)_x. \end{cases} \tag{1.4}$$

It is obvious that the boundary condition (1.3) becomes

$$(u,E_x)|_{x=0}=(0,0), \quad (v,u,E)|_{x\to+\infty}=(v_*,0,E_*), \tag{1.5}$$

with $E_*=c_v\theta_*$. We supply (1.4) with the following initial data

$$(v,u,E)|_{t=0}=(v_0,u_0,E_0). \tag{1.6}$$

When the Cauchy problem is taken into consideration, there is much important progress on the global existence and large-time behavior of classical solutions to the compressible fluid models. The global classical solution of isentropic compressible Navier-Stokes equations has been studied by Kanel [9] with positive initial density. Later, Kazhikhov made an important contribution to the non-isentropic case in [14]. It should be noted that the above references focus on big initial data. When it comes to small initial data, the global smooth solution of compressible Navier-Stokes equations in 3D was initiated by Mastumura and Nishida [21] and in one-dimensional space was proved by Kawashima and Nishida [12]. The optimal L^2 time decay rate for the three-dimensional full CNS system was investigated by Mastumura and Nishida [22] and the $L^p(p\geq 2)$ time decay rate was established by Ponce [24] under the assumption that the initial data is a small perturbation of constant state in $H^3(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$. Kawashima [10, 11] obtained the L^2 time decay rates of several general hyperbolic-parabolic systems with applications to related models.

In order to reveal the wave propagation related to hyperbolic properties of compressible flow, Zeng [25] initially studied the large-time behavior in L^1 norm of the compressible, isentropic, viscous 1-D flow. Their results show that the solution behaves like a heat kernel with convection, which propagates with the sound speed in opposite directions. Later, the pointwise estimates of general quasilinear hyperbolic-parabolic systems of conservation laws were established by Liu and Zeng [20]. To describe the wave propagation for compressible fluids in multi-dimensions, Hoff and Zumbrun [5, 6] constructed the Green's function of an artificial viscosity system related to linearized isentropic CNS system and investigated the space-time structure of the solution. Liu and Wang [19] analyzed the Green's function for isentropic compressible Navier-Stokes equations and derived the pointwise convergence of the solution to diffusive waves with the optimal time decay rate, which reveals the important phenomenon of the weaker Huygens' principle due to the existence of the stronger dispersion effects in odd-dimensional space. Afterwards, David Li [17] generalized their work to the non-isentropic CNS system on the linear level, where additional new waves are introduced to show the Huygens' principle.

Once the boundary is involved, there are many important mathematical results. The global existence and uniqueness of classical solution for 1D full CNS system with boundary were investigated by Kazhikhov and Shelukhin [15]. Matsumura and Nishida [23] proved the global existence of this system in three-dimensional half space with small initial data. Later, Kagei and Kobayashi [7, 8] obtained the convergence rate of the solution to the equilibrium state based on the delicate analysis of semigroup and energy method for isentropic CNS system with Dirichlet boundary condition to the momentum. On the other hand, there are a few results carried out on large-time behavior in the pointwise sense for the IBVP to compressible models. The pointwise estimate for p system with damping in a half-line was investigated by Deng [2]. It is shown that the solution decays exponentially with respect to space and time under some suitable assumptions on the initial data. This was generalized later to high dimensional space in Deng and Wang [1], and in Du [3]. It should be emphasized that the methods applied in previous works are not applicable to the compressible Navier-Stokes equations due to the existence of the viscosity term. To this end, Du-Wang [4] and Li-Tang-Wang [18] were devoted to the study of the space-time pointwise estimate of one-dimensional isentropic CNS system in half space, where the rich wave structure of the solution is observed. When the perturbation state is non-constant, Kawashima and Zhu [13] analyzed the stability of nonlinear waves for the outflow problem of the compressible CNS system in half space. Recently, Koike [16] studied the large-time behavior of the motion of a point mass moving in isentropic compressible fluid based on Green's function and energy method.

Nevertheless, it remains challenging to study the pointwise wave behavior of the solution perturbed around a given constant state to 1D non-isentropic CNS system (1.4) with boundary. The first challenge is to estimate the second-order spatial derivatives of the solution, which can not be obtained by directly differentiating the integral representation of the solution twice due to the existence of boundary. To resolve this difficulty, we employ a similar idea applied in the proof of global existence. Once the estimates of the solution and the first derivatives in space are established, we follow in a similar manner to derive the decay rates of the first derivatives in time. Then utilizing the elliptic structure of the velocity and temperature yields the decay rates of the second spatial derivatives to close the assumption on the ansatz. The second difficulty lies in that the highest order terms coming from the nonlinear part can not be bounded by the ansatz since the compressible Navier-Stokes system is quasilinear. To settle the problem, we make use of the results on the global existence, which provides that the highest order terms are bounded by the initial data. As a price, the time decay rates of derivatives become slow. Moreover, the biggest challenge is induced by the degeneration of the characteristic velocity for Green's function. Compared with the previous work [18] on the isentropic CNS equations, the boundary terms have slower time decay rates so that we can not close the assumption of the ansatz when taking integration by parts to obtain the nonlinear stability. Fortunately, we observe the special structure of Green's function for IBVP, in terms of the no-slip boundary condition for velocity and the adiabatic boundary condition for temperature. Indeed, our results show that the Green's function $\mathbb{G}(x,t;y)$ of the initial boundary value problem can be expressed as below

$$\mathbb{G}(x,t;y) = G(x-y,t) + G(x+y,t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.7}$$

where $G(x,t)$ denotes the Green's function of Cauchy problem that has been well studied in [20] and [25]. After we transform the derivative into $\mathbb{G}(x,t;y)$ to obtain the nonlinear estimates, it is observed that the most troublesome boundary term vanishes due to the fine structure of Green's function.

This paper is organized as follows. In Section 2, some notations and auxiliary lemmas are introduced for later use. In Section 3, we present the main theorems of this paper. The global existence of the classical solution is proved in Section 4. The estimate of Green's function to the initial boundary value problem is investigated in Section 5. In Section 6, we obtain the nonlinear pointwise estimates of the solution.

2. Preliminaries

In this section, we firstly introduce some notations, which will be used throughout this paper. Let

$$\Pi_0=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad \Pi_1=\begin{pmatrix}0&-\frac{\sigma}{\mu}&0\\-\frac{\sigma}{\mu}&0&0\\0&0&0\end{pmatrix}. \tag{2.1}$$

In Section 5, we will use the notations as follows

$$\mathbb{A}=\begin{pmatrix}0&-\sigma&0\\-\sigma&0&\eta\\0&\eta&0\end{pmatrix},\qquad \mathbb{B}=\begin{pmatrix}0&0&0\\0&\mu&0\\0&0&\nu\end{pmatrix}, \tag{2.2}$$

where the positive constants σ,ν,μ,η are defined by (3.2).

The Fourier transform of f is denoted by $\hat{f}$

$$\hat{f}(\xi)=\int_{\mathbb{R}}f(x)e^{-ix\xi}dx\quad(\xi\in\mathbb{R}).$$

The inverse Fourier transform of f is given by $\mathfrak{F}^{-1}[f]$

$$\mathfrak{F}^{-1}[f](x)=(2\pi)^{-1}\int_{\mathbb{R}}\hat{f}(\xi)e^{i\xi x}d\xi\quad(x\in\mathbb{R}).$$

The Laplace transform of f is written as $\mathcal{L}[f]$

$$\mathcal{L}[f](s)=\int_0^\infty f(t)e^{-st}dt\quad(t\in\mathbb{R}_+).$$

Two functions are introduced to state the main results $(i=1,2,3)$

$$\psi_\alpha(x,t;l_i)=\big(1+t+(x-l_i(1+t))^2\big)^{-\alpha/2},\quad \phi_\alpha(x,t;l_i)=\big(1+t+|x-l_i(1+t)|\big)^{-\alpha/2}, \tag{2.3}$$

where $l_1=-c,\ l_2=0,\ l_3=c$ and the constant $c>0$ is defined in (2.12).

The diffusive wave function $F(x,t)$ is given by

$$F(x,t)=\sum_{i=2}^{3}(1+t)^{\frac{3}{8}}\phi_1(x,t;l_i)\psi_{\frac{3}{4}}(x,t;l_i). \tag{2.4}$$

Through this paper, C denotes a generic positive constant that may vary in different estimates. $A\lesssim B$ and $A=\mathcal{O}(1)B$ mean that there exists a uniform positive constant C such that $A\le CB$.

Next, we introduce some lemmas that will be used in the proof of the main theorems.

LEMMA 2.1. *It holds for $k \geq 0$, $C_1 > 0$ and $x \in \mathbb{R}_+$ that*

$$
\begin{aligned}
e^{-\frac{x+t}{C_1}} &\leq C(1+t)^{-k}\psi_{\frac{3}{2}}(x,t;c),\\
e^{-\frac{(x-l_i(1+t))^2}{C_1(1+t)}} &\leq C(1+t)^{\frac{3}{4}}\psi_{\frac{3}{2}}(x,t;l_i),\\
e^{-\frac{t}{C_1}}(1+x^2)^{-\frac{5}{8}} &\leq C(1+t)^{-k}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c),\\
e^{-\frac{x}{C_1}}(1+t)^{-\frac{3}{2}} &\leq C(1+t)^{-\frac{1}{4}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c).
\end{aligned}
$$

Proof. In order to prove the first inequality, after a direct calculation, we have

$$
\begin{aligned}
e^{-\frac{x+t}{C_1}} &\leq e^{-\frac{t}{2C_1}}e^{-\frac{x+t}{2C_1}}\\
&\leq C(1+t)^{-k}(1+t+(x+t+1)^2)^{-\frac{3}{4}}.
\end{aligned} \tag{2.5}
$$

Note that if $c \leq 1$, then $(x+t+1)^2 \geq (x+c(t+1))^2$. For the case $c > 1$, it holds

$$
(x+t+1)^2 \geq \frac{1}{c^2}(x+c(t+1))^2. \tag{2.6}
$$

Then it is easy to verify

$$
e^{-\frac{x+t}{C_1}} \leq C(1+t)^{-k}\psi_{\frac{3}{2}}(x,t;c). \tag{2.7}
$$

The second inequality is proved by using the fact $e^{-z} \leq (1+z)^{-\frac{3}{4}}$ for $z > 0$. Concerning the third inequality, we have

$$
\begin{aligned}
e^{-\frac{t}{C_1}}(1+x^2)^{-\frac{5}{8}} &\leq e^{-\frac{t}{C_1}}(1+x)^{-\frac{1}{2}}(1+x^2)^{-\frac{3}{8}}\\
&\leq C(1+t)^{-k}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c).
\end{aligned} \tag{2.8}
$$

The last inequality can be treated in a similar argument as above. This completes the proof. ∎

Let $G(x,t)$ be Green's function of the Cauchy problem (3.11) satisfying

$$
\begin{cases}
\partial_t G(x,t) + \mathbb{A}\partial_x G(x,t) = \mathbb{B}\partial_x^2 G(x,t), & x \in \mathbb{R}, t > 0,\\
G(x,0) = \delta(x) I_3,
\end{cases} \tag{2.9}
$$

where I_3 represents a 3×3 identity matrix and the constant matrices $\mathbb{A}$ and $\mathbb{B}$ are given by (2.2).

LEMMA 2.2. *The Green's function $G(x,t)$ of the Cauchy problem (3.11) satisfies the following estimate for $x \in \mathbb{R}, t > 0$*

$$
G(x,t) = G^*(x,t) + \mathcal{O}(1)(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}}\sum_{i=1}^{3} e^{-\frac{(x-l_i t)^2}{Ct}} + e^{-\frac{\sigma^2}{\mu}t}\delta(x)\Pi_0, \tag{2.10}
$$

where the function $G^*(x,t)$ is defined by

$$
G^*(x,t) = \sum_{i=1}^{3} \frac{1}{\sqrt{4\pi\mu_i t}} e^{-\frac{(x-l_i t)^2}{4\mu_i t}} P_i. \tag{2.11}
$$

The matrix Π_0 *is given by* (2.1) *and the projections* $P_i (i=1,2,3)$ *satisfy*

$$P_1 = \frac{1}{2c^2}\begin{pmatrix} \sigma^2 & \sigma c & -\sigma\eta \\ \sigma c & c^2 & -c\eta \\ -\sigma\eta & -c\eta & \eta^2 \end{pmatrix}, \quad P_2 = \frac{1}{c^2}\begin{pmatrix} \eta^2 & 0 & \sigma\eta \\ 0 & 0 & 0 \\ \sigma\eta & 0 & \sigma^2 \end{pmatrix}, \quad P_3 = \frac{1}{2c^2}\begin{pmatrix} \sigma^2 & -\sigma c & -\sigma\eta \\ -\sigma c & c^2 & c\eta \\ -\sigma\eta & c\eta & \eta^2 \end{pmatrix}.$$

The constants l_i *and* μ_i *satisfy*

$$\begin{aligned} &l_1 = -c, \quad l_2 = 0, \quad l_3 = c, \quad c = \sqrt{\sigma^2+\eta^2}, \\ &\mu_1 = \mu_3 = \frac{\mu}{2} + \frac{\nu\eta^2}{2c^2}, \quad \mu_2 = \frac{\nu\sigma^2}{c^2}. \end{aligned} \tag{2.12}$$

Proof. Since matrix $\mathbb{A}$ is symmetric, applying Theorem 6.2 [20] completes the proof. ∎

Lemma 2.3. *The Green's function* $G(x,t)$ *of the Cauchy problem* (3.11) *has the property for* $x\in\mathbb{R}, t>0$

$$\left|\partial_x^\alpha G(x,t) - \partial_x^\alpha G^*(x,t) - e^{-\frac{\sigma^2}{\mu}t}\sum_{j=0}^{\alpha}\delta^{(\alpha-j)}(x)\Pi_j(t)\right| \le C(1+t)^{-\frac12}t^{-\frac{\alpha+1}{2}}\sum_{i=1}^{3}e^{-\frac{(x-l_it)^2}{Ct}},$$

where $G^*(x,t)$ *is given by* (2.11), $\delta^{(k)}$ *is the k-th derivative of the Dirac delta function and* $\Pi_j = \Pi_j(t)$ *is a* 3×3 *polynomial matrix. Especially, the matrices* Π_0, Π_1 *are given by* (2.1).

Proof. Thanks to Theorem 6.15 [20] and Lemma 2.2, combining them together completes the proof. ∎

To gain a better understanding of interactions between different waves, we provide some lemmas as below for $x\in\mathbb{R}$, which can be found in Lemmas 3.7, 3.8 in [20].

Lemma 2.4. *Let* $\alpha\ge 0$, $0\le\beta\le 2$, $\bar\mu>0$, *and* λ *be constants. Then for* $x\in\mathbb{R}$, $t\ge 0$, *we have*

$$\begin{aligned} &\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}(1+t-s)^{-\frac{\alpha}{2}}e^{-\frac{(x-y-\lambda(t-s))^2}{\bar\mu(t-s)}}(1+s)^{-\frac{\beta}{2}}\phi_1(y,s;\lambda)\psi_{\frac34}(y,s;\lambda)dyds \\ =&\begin{cases} \mathcal{O}(1)(1+t)^{-\frac{\gamma_1}{2}}\log(2+t)\phi_1(x,t;\lambda)\psi_{\frac34}(x,t;\lambda), & \text{if } \alpha=1, \text{ or } 1\le\beta\le\frac32, \\ \mathcal{O}(1)(1+t)^{-\frac{\gamma_1}{2}}\phi_1(x,t;\lambda)\psi_{\frac34}(x,t;\lambda), & \text{otherwise}, \end{cases} \\ &+\begin{cases} \mathcal{O}(1)(1+t)^{-\frac{\gamma_2}{2}}\log(2+t)\psi_{\frac32}(x,t;\lambda), & \text{if } \alpha=1, \text{ or } \beta=\frac32, \\ \mathcal{O}(1)(1+t)^{-\frac{\gamma_2}{2}}\psi_{\frac32}(x,t;\lambda), & \text{otherwise}, \end{cases} \end{aligned}$$

where $\gamma_1 = \min(\alpha,1)+\frac12(\min(\beta,1)+\min(\beta,\frac32))-1$, $\gamma_2 = \min(\alpha,1)+\min(\beta,\frac32)-1$.

Lemma 2.5. *Let the constants* $\alpha\ge 0, 0\le\beta\le 2, \bar\mu>0$ *and* $\lambda\ne\lambda'$. *Then for any fixed* $K>2|\lambda-\lambda'|$ *and all* $x\in\mathbb{R}, t\ge 0$, *we have*

$$\begin{aligned} &\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}(1+t-s)^{-\frac{\alpha}{2}}e^{-\frac{(x-y-\lambda(t-s))^2}{\bar\mu(t-s)}}(1+s)^{-\frac{\beta}{2}}\phi_1(y,s;\lambda')\psi_{\frac34}(y,s;\lambda')dyds \\ =&\mathcal{O}(1)(1+t)^{-\frac{\gamma_1}{2}} \\ &\cdot\begin{cases} \phi_1(x,t;\lambda)\psi_{\frac34}(x,t;\lambda)+\phi_1(x,t;\lambda')\psi_{\frac34}(x,t;\lambda'), & \text{if } \alpha\ne1, \beta\notin[1,\frac32], \\ \log(2+t)\phi_1(x,t;\lambda)\psi_{\frac34}(x,t;\lambda)+\phi_1(x,t;\lambda')\psi_{\frac34}(x,t;\lambda'), & \text{if } \alpha\ne1, \beta\in[1,\frac32], \\ \log(2+t)[\phi_1(x,t;\lambda)\psi_{\frac34}(x,t;\lambda)+\phi_1(x,t;\lambda')\psi_{\frac34}(x,t;\lambda')], & \text{if } \alpha=1, \end{cases} \end{aligned}$$

$$+\mathcal{O}(1)(1+t)^{-\frac{\gamma_2}{2}}\begin{cases}\psi_{\frac{3}{2}}(x,t;\lambda)+\psi_{\frac{3}{2}}(x,t;\lambda'), & if\ \alpha\neq 1,\beta\neq\frac{3}{2},\\ \log(2+t)\psi_{\frac{3}{2}}(x,t;\lambda)+\psi_{\frac{3}{2}}(x,t;\lambda'), & if\ \alpha\neq 1,\beta=\frac{3}{2},\\ \log(2+t)[\psi_{\frac{3}{2}}(x,t;\lambda)+\psi_{\frac{3}{2}}(x,t;\lambda')], & if\ \alpha=1,\end{cases}$$
$$+\mathcal{O}(1)|x-\lambda(1+t)|^{-\frac{1}{2}\min(\beta,2)-\frac{1}{4}+\epsilon}|x-\lambda'(1+t)|^{-\frac{1}{2}\min(\alpha,1)-\frac{1}{2}}$$
$$\cdot\mathrm{char}\{\min(\lambda,\lambda')(1+t)+K\sqrt{1+t}\leq x\leq\max(\lambda,\lambda')(1+t)-K\sqrt{1+t}\},$$

where $\gamma_1=\min(\alpha,1)+\frac{1}{2}(\min(\beta,1)+\min(\beta,\frac{3}{2}))-1,\gamma_2=\min(\alpha,1)+\min(\beta,\frac{3}{2})-1,$ *and* $\epsilon>0$ *can be arbitrarily small.*

3. Main results

In this paper, we consider a small perturbation of the solution near a constant state $(v_*,0,E_*)$. Denote

$$n=\frac{v-v_*}{v_*},\quad w=\frac{u}{c_1},\quad \phi=\frac{E-E_*}{c_2E_*},\tag{3.1}$$

and

$$c_1=\sqrt{\frac{RE_*}{c_v}},\quad c_2=\sqrt{\frac{R}{c_v}},\quad \sigma=\frac{c_1}{v_*},\quad \nu=\frac{\kappa}{c_vv_*},\quad \mu=\frac{\tilde{\mu}}{v_*},\quad \eta=\frac{R\sqrt{E_*}}{c_vv_*}.\tag{3.2}$$

Then, the system (1.4) is reformulated as

$$\begin{cases}n_t-\sigma w_x=0,\\ w_t+\sigma\Big(\frac{1+c_2\phi}{1+n}\Big)_x=\Big(\frac{\mu w_x}{1+n}\Big)_x+\Big(\frac{\sigma c_2^2w^2}{2(1+n)}\Big)_x,\\ \phi_t+\eta\Big(\frac{w(1+c_2\phi)}{1+n}\Big)_x=\nu\Big(\frac{\phi_x}{1+n}\Big)_x+\Big(\frac{c_2(\mu-\nu)ww_x}{1+n}\Big)_x+\Big(\frac{\eta c_2^2w^3}{2(1+n)}\Big)_x.\end{cases}\tag{3.3}$$

We equip (3.3) with the following boundary condition

$$(w,\phi_x)|_{x=0}=(0,0),\quad (n,w,\phi)|_{x\to+\infty}=(0,0,0).\tag{3.4}$$

The initial data is given by

$$(n,w,\phi)|_{t=0}=(n_0,w_0,\phi_0).\tag{3.5}$$

To state the theorem of global existence, we introduce the definitions of energy $\mathcal{E}(t)$ and dissipation $\mathcal{D}(t)$

$$\begin{aligned}\mathcal{E}(t)&=\|n\|_{H^4(\mathbb{R}_+)}+\|w\|_{H^4(\mathbb{R}_+)}+\|\phi\|_{H^4(\mathbb{R}_+)},\\ \mathcal{D}(t)&=\|n_x\|_{H^3(\mathbb{R}_+)}+\|w_x\|_{H^4(\mathbb{R}_+)}+\|\phi_x\|_{H^4(\mathbb{R}_+)}.\end{aligned}\tag{3.6}$$

Since the classical solution is taken into consideration in the present paper, the following compatible conditions are needed

$$\begin{cases}w_0(0)=\phi_0'(0)=0,\\ \Big\{\sigma\Big(\frac{1+c_2\phi_0}{1+n_0}\Big)_x-\Big(\frac{\mu w_{0x}}{1+n_0}\Big)_x-\Big(\frac{\sigma c_2^2w_0^2}{2(1+n_0)}\Big)_x\Big\}\Big|_{x=0}=0,\\ \Big\{\eta\Big(\frac{w(1+c_2\phi_0)}{1+n_0}\Big)_x-\nu\Big(\frac{\phi_{0x}}{1+n_0}\Big)_x-\Big(\frac{c_2(\mu-\nu)w_0w_{0x}}{1+n_0}\Big)_x-\Big(\frac{\eta c_2^2w_0^3}{2(1+n_0)}\Big)_x\Big\}_x\Big|_{x=0}=0.\end{cases}\tag{3.7}$$

THEOREM 3.1 (Global existence). *Assume the initial data $(n_0,w_0,\phi_0)\in H^4(\mathbb{R}_+)$ satisfying the compatibility conditions (3.7). There exists a small positive constant such that if*

$$\|(n_0,w_0,\phi_0)\|_{H^4(\mathbb{R}_+)}\le \varepsilon_0, \tag{3.8}$$

then the initial boundary value problem (3.3)-(3.5) admits a unique classical solution (n,w,ϕ) satisfying

$$\begin{aligned}
&n\in C([0,\infty);H^4(\mathbb{R}_+))\cap C^1([0,\infty);H^3(\mathbb{R}_+)),\\
&w\in C([0,\infty);H^4(\mathbb{R}_+))\cap C^1([0,\infty);H^2(\mathbb{R}_+)),\\
&\phi\in C([0,\infty);H^4(\mathbb{R}_+))\cap C^1([0,\infty);H^2(\mathbb{R}_+)),\\
&n_x\in L^2([0,\infty);H^3(\mathbb{R}_+)), w_x,\phi_x\in L^2([0,\infty);H^4(\mathbb{R}_+)).
\end{aligned} \tag{3.9}$$

Furthermore, it holds that for any given time $T>0$

$$\sup_{0\le t\le T}\mathcal{E}(t)^2+\int_0^T \mathcal{D}(t)^2dt\le C\varepsilon_0^2, \tag{3.10}$$

where C is a positive constant independent of time.

We consider the linearized system of (3.3)

$$\begin{cases}
n_t-\sigma w_x=0,\\
w_t-\sigma n_x+\eta\phi_x=\mu w_{xx},\\
\phi_t+\eta w_x=\nu\phi_{xx},
\end{cases} \tag{3.11}$$

with the initial data

$$(n,w,\phi)|_{t=0}=(n_0,w_0,\phi_0),$$

which satisfies the following boundary condition

$$(w,\phi_x)|_{x=0}=(0,0),\quad (n,w,\phi)|_{x\to+\infty}=(0,0,0). \tag{3.12}$$

The Green's function of the linear system (3.11)-(3.12) satisfies

$$\begin{cases}
\partial_t\mathbb{G}(x,t;y)+\mathbb{A}\partial_x\mathbb{G}(x,t;y)=\mathbb{B}\partial_x^2\mathbb{G}(x,t;y),\ x>0,y>0,t>0,\\
\mathbb{G}(x,0;y)=\delta(x-y)I_3,\ x>0,y>0,\\
(0,1,0)\mathbb{G}(0,t;y)=(0,0,0),\\
\{(0,0,1)\partial_x\mathbb{G}(x,t;y)\}|_{x=0}=(0,0,0),
\end{cases} \tag{3.13}$$

where $\mathbb{A},\mathbb{B}$ are defined in (2.2) and I_3 represents a 3×3 identity matrix. The second theorem is stated about the estimate of $\mathbb{G}(x,t;y)$.

THEOREM 3.2. *The Green's function $\mathbb{G}(x,t;y)$ of the linear system (3.11)-(3.12) has the following estimate for $x,y\in\mathbb{R}_+,t\ge 0$*

$$\begin{aligned}
&\left|\partial_x^\alpha\mathbb{G}(x,t;y)-e^{-\frac{\sigma^2}{\mu}t}\sum_{j=0}^{\alpha}\delta^{(\alpha-j)}(x-y)\Pi_j(t)\right|\\
=&O(1)t^{-\frac{1}{2}-\frac{\alpha}{2}}\sum_{i=1}^{3}\big(e^{-\frac{(x-y-l_it)^2}{Ct}}+e^{-\frac{(x+y-l_it)^2}{Ct}}\big),
\end{aligned} \tag{3.14}$$

where $\delta^{(k)}$ is the k-th derivative of the Dirac delta function and $\Pi_j = \Pi_j(t)$ is a 3×3 polynomial matrix. Especially, Π_0, Π_1 is given by (2.1) and the constants l_i is introduced in (2.12).

Based on the above theorems, nonlinear pointwise estimates of the solution are obtained.

THEOREM 3.3. *Under the assumptions in Theorem 3.1 and define $U_0 = (n_0, w_0, \phi_0)^t$ satisfying*

$$|\partial_x^\alpha U_0(x)| \le C\varepsilon_0 (1+x^2)^{-\frac{5}{8}}, \quad \Big|\int_x^\infty U_0(y)dy\Big| \le C\varepsilon_0(1+x^2)^{-\frac{5}{8}}, \tag{3.15}$$

for $\alpha = 0,1$. Then the solution obtained in Theorem 3.1 obeys the following pointwise estimate

$$|(n,w,\phi)(x,t)| \le C\varepsilon_0 F(x,t), \tag{3.16}$$

where $F(x,t)$ represents the diffusive wave given by (2.4). Furthermore, the decay rates of spatial derivatives of the solution satisfy

$$|(n_x,w_x,\phi_x)(x,t)| \le C\varepsilon_0(1+t)^{-1}\log(2+t), \qquad |(w_{xx},\phi_{xx})(x,t)| \le C\varepsilon_0(1+t)^{-\frac{1}{2}}. \tag{3.17}$$

COROLLARY 3.1. *Applying the assumptions in Theorem 3.3 and the definition of $F(x,t)$, we have the following L^p time decay rate of the solution*

$$\|(n,w,\phi)(x,t)\|_{L^p(\mathbb{R}_+)} \le C\varepsilon_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad p\in(1,+\infty]. \tag{3.18}$$

In what follows, we describe a brief explanation of the main steps of the proof. The global existence is established firstly based on the classical H^s energy method applied in [23]. To capture the pointwise behavior of the solution, the delicate structure of Green's function is needed to be investigated in the first place. Then applying Duhamel's principle yields the accurate expression of the solution. By defining a suitable ansatz, we anticipate obtaining nonlinear stability. Applying the results of energy estimates and the Green's function together close the estimates of the highest order derivatives. With the strong wave interaction, the nonlinear analysis exhibits a rich wave structure. As a result, the perturbed solution behaves like a diffusive wave, which propagates with different wave speeds. Meanwhile, we also derive the algebraic time decay rate of the solution in L^∞ norm about the spatial variable.

4. The Global existence of classical solutions

The local existence of solutions are established as follows.

THEOREM 4.1 (Local existence). *Assume the initial data $(n_0,w_0,\phi_0)\in H^4(\mathbb{R}_+)$ satisfying compatibility conditions (3.7). Then, the initial boundary value problem (3.3)-(3.5) admits a unique local classical solution (n,w,ϕ) satisfying the following estimates for some $T_*>0$*

$$\begin{aligned}
&n\in C([0,T_*];H^4(\mathbb{R}_+))\cap C^1([0,T_*];H^3(\mathbb{R}_+)),\\
&w\in C([0,T_*];H^4(\mathbb{R}_+))\cap C^1([0,T_*];H^2(\mathbb{R}_+)),\\
&\phi\in C([0,T_*];H^4(\mathbb{R}_+))\cap C^1([0,T_*];H^2(\mathbb{R}_+)),\\
&n_x\in L^2([0,T_*];H^3(\mathbb{R}_+)), w_x,\phi_x\in L^2([0,T_*];H^4(\mathbb{R}_+)).
\end{aligned} \tag{4.1}$$

Furthermore, it holds for some given time $T_*>0$

$$\sup_{0\le t\le T_*}\mathcal{E}(t)^2+\int_0^{T_*}\mathcal{D}(t)^2dt\le C\mathcal{E}(0)^2, \tag{4.2}$$

where C *is a positive constant independent of time.*

Proof. The construction of local-in-time solutions is based on an iteration scheme in [23]. Here we omit the details. □

To extend the short-time classical solution to be a global one, it is essential to establish the uniform estimates. Hence we provide the a-priori assumption that for any given time $T>0$, it holds

$$\sup_{0\le t\le T}\mathcal{E}(t)\le\delta, \tag{4.3}$$

where δ is a suitably small positive constant. We first give the basic L^2 energy estimate of the solution.

Proposition 4.1. *Assume* (n,w,ϕ) *is the classical solution of the initial boundary value problem* (3.3)-(3.5) *satisfying the assumptions in Theorem* 3.1 *and the a-priori assumption* (4.3). *Then, we obtain the following estimate for any given time* $T>0$

$$\sup_{0\le t\le T}\left(\|n\|_{L^2}^2+\|w\|_{L^2}^2+\|\phi\|_{L^2}^2\right)+\int_0^T(\|w_x\|_{L^2}^2+\|\phi_x\|_{L^2}^2)dt\le C\varepsilon_0^2.$$

Proof. To derive the L^2 energy estimate, we employ the following equivalent form of system (1.1)

$$\begin{cases} v_t-u_x=0,\\ u_t+p_x=(\dfrac{\tilde\mu}{v}u_x)_x,\\ c_v\theta_t+pu_x=(\dfrac{\kappa}{v}\theta_x)_x+\dfrac{\tilde\mu u_x^2}{v}. \end{cases} \tag{4.4}$$

Using (3.1) provides

$$v=(1+n)v_*,\qquad u=c_1w.$$

Moreover, we define

$$\theta=(1+c_2\bar\theta)\theta_*.$$

Then system (4.4) is reformulated to

$$\begin{cases} n_t-\sigma w_x=0,\\ w_t+\sigma(\dfrac{1+c_2\bar\theta}{1+n})_x=(\dfrac{\mu w_x}{1+n})_x,\\ \bar\theta_t+\eta(\dfrac{1+c_2\bar\theta}{1+n})w_x=\nu(\dfrac{\bar\theta_x}{1+n})_x+\dfrac{\mu c_2w_x^2}{1+n}, \end{cases} \tag{4.5}$$

where the positive constants c_2,σ,ν,μ,η are given by (3.2). Define the total energy $X(t)$ as

$$X(t)=\int_{\mathbb{R}_+}\left[R(n-\log(1+n))+\frac12Rw^2+c_v(c_2\bar\theta-\log(1+c_2\bar\theta))\right]dx. \tag{4.6}$$

Applying (4.3) gives rise to

$$X(t)\sim\frac{1}{2}R\int_{\mathbb{R}_+}(n^2+w^2+\bar{\theta}^2)dx. \tag{4.7}$$

Taking integration by parts yields

$$\frac{d}{dt}X(t)+\int_{\mathbb{R}_+}\frac{\mu Rw_x^2}{(1+n)(1+c_2\bar{\theta})}dx+\int_{\mathbb{R}_+}\frac{\nu R\bar{\theta}_x^2}{(1+n)(1+c_2\bar{\theta})^2}dx=0. \tag{4.8}$$

Then we integrate time from 0 to t and take supremum with respect to $t\in[0,T]$ to prove

$$\sup_{0\le t\le T}(\|n\|_{L^2}^2+\|w\|_{L^2}^2+\|\bar{\theta}\|_{L^2}^2)+\int_0^T(\|w_x\|_{L^2}^2+\|\bar{\theta}_x\|_{L^2}^2)dt\le C\varepsilon_0^2. \tag{4.9}$$

Due to $\phi=\bar{\theta}+\frac{1}{2}c_2w^2$, we immediately obtain

$$\sup_{0\le t\le T}(\|n\|_{L^2}^2+\|w\|_{L^2}^2+\|\bar{\theta}\|_{L^2}^2)+\int_0^T(\|w_x\|_{L^2}^2+\|\phi_x\|_{L^2}^2)dt\le C\varepsilon_0^2. \tag{4.10}$$

This completes the proof. □

To capture the estimates of high order derivatives, one rewrites (3.3) as below

$$\begin{cases} n_t-\sigma w_x=0,\\ w_t-\sigma n_x+\eta\phi_x=\mu w_{xx}+N_1,\\ \phi_t+\eta w_x=\nu\phi_{xx}+N_2, \end{cases} \tag{4.11}$$

which satisfies the following boundary condition

$$(w,\phi_x)|_{x=0}=(0,0),\quad (n,w,\phi)|_{x\to+\infty}=(0,0,0). \tag{4.12}$$

The initial data is given by

$$(n,w,\phi)|_{t=0}=(n_0,w_0,\phi_0). \tag{4.13}$$

$N_1,\tilde{N}_1,N_2,\tilde{N}_2$ represent nonlinear terms satisfying

$$\begin{aligned} N_1&=\partial_x\tilde{N}_1=\Big(\frac{\eta\phi n-\sigma n^2-\mu nw_x}{1+n}+\frac{\sigma c_2^2w^2}{2(1+n)}\Big)_x,\\ N_2&=\partial_x\tilde{N}_2=\Big(\frac{\eta nw-c_2\eta w\phi-\nu n\phi_x}{1+n}+\frac{c_2(\mu-\nu)ww_x}{1+n}+\frac{\eta c_2^2w^3}{2(1+n)}\Big)_x. \end{aligned} \tag{4.14}$$

For later use, set

$$\tilde{N}=(0,\tilde{N}_1,\tilde{N}_2)^t,\qquad N=(0,N_1,N_2)^t. \tag{4.15}$$

Let $U=(n,w,\phi)^t$ be the solution of nonlinear system (4.11). Then we rewrite the nonlinear system (4.11) into a simple form

$$\begin{cases} \partial_tU(x,t)+\mathbb{A}\partial_xU(x,t)=\mathbb{B}\partial_x^2U(x,t)+N(x,t),\\ U(x,0)=U_0, \end{cases} \tag{4.16}$$

where $U_0=(n_0,w_0,\phi_0)^t$.

In what follows, we are prepared to deduce the energy estimates of the time derivatives of the solution.

Proposition 4.2. *Assume (n,w,ϕ) is the classical solution of the initial boundary value problem (3.3)-(3.5) satisfying the assumptions in Theorem 3.1 and the a-priori assumption (4.3). Then, we obtain the following estimate for any given time $T>0$*

$$\sum_{k=1}^{2}\Big(\sup_{0\le t\le T}(\|\partial_t^k n\|_{L^2}^2+\|\partial_t^k w\|_{L^2}^2+\|\partial_t^k \phi\|_{L^2}^2)+\int_0^T(\|\partial_t^k w_x\|_{L^2}^2+\|\partial_t^k \phi_x\|_{L^2}^2)dt\Big)$$
$$\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.$$

Proof. Differentiate the system (4.11) with regard to t, then multiply the resulting equations by n_t,w_t and ϕ_t respectively. Adding and integrating these equations yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|n_t\|_{L^2}^2+\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)+\mu\|w_{tx}\|_{L^2}^2+\nu\|\phi_{tx}\|_{L^2}^2\\&=\int_{\mathbb{R}_+}\partial_tN_1w_tdx+\int_{\mathbb{R}_+}\partial_tN_2\phi_tdx\\&=-\int_{\mathbb{R}_+}\partial_t\tilde{N}_1w_{tx}dx-\int_{\mathbb{R}_+}\partial_t\tilde{N}_2\phi_{tx}dx,\end{aligned}\tag{4.17}$$

where the last step has used integration by parts. Applying the a-priori assumption (4.3) leads to

$$\begin{aligned}&\|\partial_t\tilde{N}_1\|_{L^2}\\&\le C\|ww_t+n_t\phi+\phi n_t+nn_t+n_tw_x+nw_{tx}\|_{L^2}+C\|(w^2+n\phi+n^2+nw_x)n_t\|_{L^2}\\&\le C(\|w\|_{L^\infty}\|w_t\|_{L^2}+\|\phi\|_{L^\infty}\|n_t\|_{L^2}+\|n\|_{L^\infty}\|n_t\|_{L^2}+\|n_t\|_{L^2}\|w_x\|_{L^\infty}\\&\quad+\|n\|_{L^\infty}\|w_{tx}\|_{L^2})+C(\|w\|_{L^\infty}^2+\|n\|_{L^\infty}\|\phi\|_{L^\infty}+\|n\|_{L^\infty}^2+\|n\|_{L^\infty}\|w_x\|_{L^\infty})\|n_t\|_{L^2}\\&\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C\|n\|_{L^\infty}\|w_{tx}\|_{L^2}\\&\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C\delta\|w_{tx}\|_{L^2}.\end{aligned}\tag{4.18}$$

It also holds

$$\|\partial_t\tilde{N}_2\|_{L^2}\le C\sqrt{\mathcal{E}(t)}\mathcal{D}(t)+C\delta\|\phi_{tx}\|_{L^2}.\tag{4.19}$$

Then we immediately compute

$$\begin{aligned}&-\int_{\mathbb{R}+}\partial_t\tilde{N}_1w_{tx}dx-\int_{\mathbb{R}+}\partial_t\tilde{N}_2\phi_{tx}dx\\&\le\|\partial_t\tilde{N}_1\|_{L^2}\|w_{tx}\|_{L^2}+\|\partial_t\tilde{N}_2\|_{L^2}\|\phi_{tx}\|_{L^2}\\&\le\varepsilon(\|w_{tx}\|_{L^2}^2+\|\phi_{tx}\|_{L^2}^2)+C_\varepsilon(\|\partial_t\tilde{N}_1\|_{L^2}^2+\|\partial_t\tilde{N}_2\|_{L^2}^2)\\&\le(\varepsilon+CC_\varepsilon\delta^2)\|w_{tx}\|_{L^2}^2+(\varepsilon+CC_\varepsilon\delta^2)\|\phi_{tx}\|_{L^2}^2+C\mathcal{E}(t)\mathcal{D}(t)^2,\end{aligned}\tag{4.20}$$

which, together with (4.17), also leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|n_t\|_{L^2}^2+\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)+\mu\|w_{tx}\|_{L^2}^2+\nu\|\phi_{tx}\|_{L^2}^2\\&\le(\varepsilon+CC_\varepsilon\delta^2)\|w_{tx}\|_{L^2}^2+(\varepsilon+CC_\varepsilon\delta^2)\|\phi_{tx}\|_{L^2}^2+C\mathcal{E}(t)\mathcal{D}(t)^2.\end{aligned}\tag{4.21}$$

Integrating (4.21) with respect to time from 0 to t, then taking supremum in $t\in[0,T]$ and making use of the smallness of ε,δ give rise to

$$\begin{aligned}&\sup_{0\le t\le T}(\|n_t\|_{L^2}^2+\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)+\int_0^T(\|w_{tx}\|_{L^2}^2+\|\phi_{tx}\|_{L^2}^2)dt\\ \le& C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}\tag{4.22}$$

Since the case of $k=2$ can be treated in a similar argument as $k=1$, we omit the details. This completes the proof. □

PROPOSITION 4.3. *Assume (n,w,ϕ) is the classical solution of the initial boundary value problem (3.3)-(3.5) satisfying the assumptions in Theorem 3.1 and the a-priori assumption (4.3). Then, we obtain the following estimate for any given time $T>0$*

$$\begin{aligned}&\sum_{k=0}^{1}\Big(\sup_{0\le t\le T}(\|\partial_t^k w_x\|_{L^2}^2+\|\partial_t^k\phi_x\|_{L^2}^2)+\int_0^T(\|\partial_t^{k+1}w\|_{L^2}^2+\|\partial_t^{k+1}\phi\|_{L^2}^2)dt\Big)\\ \le& C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}$$

Proof. Multiplying $(4.11)_2,(4.11)_3$ by w_t,ϕ_t respectively, then integrating the resulting equations with respect to x in $\mathbb{R}_+$ reaches the following equality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\mu\|w_x\|_{L^2}^2+\nu\|\phi_x\|_{L^2}^2)+\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2\\ =&\sigma\int_{\mathbb{R}_+}n_xw_tdx-\eta\int_{\mathbb{R}_+}(\phi_xw_t+w_x\phi_t)dx+\int_{\mathbb{R}_+}(N_1w_t+N_2\phi_t)dx.\end{aligned}\tag{4.23}$$

Using integration by parts, the first term is computed as

$$\sigma\int_{\mathbb{R}_+}n_xw_tdx=-\sigma\int_{\mathbb{R}_+}nw_{tx}dx=-\int_{\mathbb{R}_+}nn_{tt}dx=-\frac{d}{dt}\int_{\mathbb{R}_+}nn_tdx+\sigma^2\|w_x\|_{L^2}^2.\tag{4.24}$$

With the help of Young's inequality, it provides

$$\begin{aligned}&\eta\int_{\mathbb{R}_+}(\phi_xw_t+w_x\phi_t)dx\\ \le&\eta\|\phi_x\|_{L^2}\|w_t\|_{L^2}+\eta\|w_x\|_{L^2}\|\phi_t\|_{L^2}\\ \le&\varepsilon(\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)+C_\varepsilon(\|\phi_x\|_{L^2}^2+\|w_x\|_{L^2}^2).\end{aligned}\tag{4.25}$$

In order to deal with the nonlinear terms, by the definition of N_1,N_2, we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+}(N_1w_t+N_2\phi_t)dx\\ \le&\|N_1\|_{L^2}\|w_t\|_{L^2}+\|N_2\|_{L^2}\|\phi_t\|_{L^2}\\ \le&\varepsilon(\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)+C\mathcal{E}(t)\mathcal{D}(t)^2.\end{aligned}\tag{4.26}$$

Since ε is sufficiently small, it is easy to confirm

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\mu\|w_x\|_{L^2}^2+\nu\|\phi_x\|_{L^2}^2)+\frac{1}{2}(\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)\\ \le&-\frac{d}{dt}\int_{\mathbb{R}_+}nn_tdx+\sigma^2\|w_x\|_{L^2}^2+C_\varepsilon(\|\phi_x\|_{L^2}^2+\|w_x\|_{L^2}^2)+C\mathcal{E}(t)\mathcal{D}(t)^2.\end{aligned}\tag{4.27}$$

Making use of Proposition 4.2 gives

$$\sup_{0\le t\le T}\int_{\mathbb{R}_+}|nn_t|dx\le\sup_{0\le t\le T}\|n\|_{L^2}\sup_{0\le t\le T}\|n_t\|_{L^2}\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{4.28}$$

Integrating (4.27) with respect to time in $[0,t]$ and taking supremum in $t\in[0,T]$, then applying (4.3), Proposition 4.1 and Proposition 4.2 yields

$$\begin{aligned}&\frac{1}{2}\sup_{0\le t\le T}(\mu\|w_x\|_{L^2}^2+\nu\|\phi_x\|_{L^2}^2)+\frac{1}{2}\int_0^T(\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)dt\\ \le&\sup_{0\le t\le T}\int_{\mathbb{R}_+}|nn_t|dx+C\varepsilon_0^2+\sigma^2\int_0^T\|w_x\|_{L^2}^2dt\\ &\quad+C_\varepsilon\int_0^T(\|\phi_x\|_{L^2}^2+\|w_x\|_{L^2}^2)dt+C\delta\int_0^T\mathcal{D}(t)^2dt\\ \le& C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}$$

As a result, we deduce

$$\begin{aligned}&\sup_{0\le t\le T}(\|w_x\|_{L^2}^2+\|\phi_x\|_{L^2}^2)+\int_0^T(\|w_t\|_{L^2}^2+\|\phi_t\|_{L^2}^2)dt\\ \le& C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned} \tag{4.29}$$

The case of $k=1$ can be treated in a similar way as $k=0$. Hence, we complete the proof. ∎

Based on the above propositions, we shall establish L^2 energy estimates of spatial derivatives.

PROPOSITION 4.4. *Assume (n,w,ϕ) is the classical solution of the initial boundary value problem (3.3)-(3.5) satisfying the assumptions in Theorem 3.1 and the a-priori assumption (4.3). Then, we obtain the following estimate for any given time $T>0$*

$$\begin{aligned}&\sup_{0\le t\le T}(\|n_x\|_{H^3}^2+\|w_x\|_{H^3}^2+\|\phi_x\|_{H^3}^2)+\int_0^T(\|n_x\|_{H^3}^2+\|w_{xx}\|_{H^3}^2+\|\phi_{xx}\|_{H^3}^2)dt\\ \le& C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}$$

Proof. Write $(4.11)_2,(4.11)_3$ into the following form

$$\begin{aligned}w_{xx}&=\frac{1}{\mu}(w_t-\sigma n_x+\eta\phi_x-N_1),\\ \phi_{xx}&=\frac{1}{\nu}(\phi_t+\eta w_x-N_2).\end{aligned} \tag{4.30}$$

It is easy to verify

$$n_{tx}=\frac{\sigma}{\mu}(w_t-\sigma n_x+\eta\phi_x-N_1). \tag{4.31}$$

Multiplying (4.31) by n_x, then integrating with respect to x in $\mathbb{R}_+$ yields

$$\frac{1}{2}\frac{d}{dt}\|n_x\|_{L^2}^2+\frac{\sigma^2}{\mu}\|n_x\|_{L^2}^2=\frac{\sigma}{\mu}\int_{\mathbb{R}_+}(w_t+\eta\phi_x-N_1)n_x dx. \tag{4.32}$$

By Young's inequality, we derive

$$\begin{aligned}&\frac{\sigma}{\mu}\int_{\mathbb{R}_+}(w_t+\eta\phi_x-N_1)n_x dx\\ \le& C\|w_t\|_{L^2}\|n_x\|_{L^2}+C\|\phi_x\|_{L^2}\|n_x\|_{L^2}+C\mathcal{E}(t)\mathcal{D}(t)^2\\ \le&\varepsilon\|n_x\|_{L^2}^2+C_\varepsilon(\|w_t\|_{L^2}^2+\|\phi_x\|_{L^2}^2)+C\mathcal{E}(t)\mathcal{D}(t)^2.\end{aligned} \tag{4.33}$$

Integrating (4.32) with respect to time from 0 to t and taking supremum in $t\in[0,T]$, then making use of Proposition 4.1 and Proposition 4.3 gives rise to

$$\sup_{0\le t\le T}\|n_x\|_{L^2}^2+\int_0^T\|n_x\|_{L^2}^2dt\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{4.34}$$

Utilizing Proposition 4.3 proves

$$\sup_{0\le t\le T}(\|w_x\|_{L^2}^2+\|\phi_x\|_{L^2}^2)\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{4.35}$$

Take L^2 inner product on $(4.30)_1$ to show

$$\|w_{xx}\|_{L^2}\le\frac{1}{\mu}(\|w_t\|_{L^2}+\sigma\|n_x\|_{L^2}+\eta\|\phi_x\|_{L^2}+\|N_1\|_{L^2}). \tag{4.36}$$

In the same way, we have

$$\|\phi_{xx}\|_{L^2}\le\frac{1}{\nu}(\|\phi_t\|_{L^2}+\eta\|w_x\|_{L^2}+\|N_2\|_{L^2}). \tag{4.37}$$

By (4.34), (4.35) and Proposition 4.3, we obtain

$$\int_0^T\|w_{xx}\|_{L^2}^2dt\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt, \tag{4.38}$$

and

$$\int_0^T\|\phi_{xx}\|_{L^2}^2dt\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{4.39}$$

It then deduces from the above results that

$$\begin{aligned}&\sup_{0\le t\le T}(\|n_x\|_{L^2}^2+\|w_x\|_{L^2}^2+\|\phi_x\|_{L^2}^2)+\int_0^T(\|n_x\|_{L^2}^2+\|w_{xx}\|_{L^2}^2+\|\phi_{xx}\|_{L^2}^2)dt\\ \le& C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned} \tag{4.40}$$

As for high order derivatives, we apply a similar argument as above. This completes the proof. ∎

4.1. The proof of Theorem 3.1.

Proof. Making use of Propositions 4.1, 4.2 and 4.4 together yields

$$\sup_{0\le t\le T}\mathcal{E}(t)^2+\int_0^T\mathcal{D}(t)^2dt\le C\varepsilon_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{4.41}$$

Since δ is sufficiently small, we obtain

$$\sup_{0\le t\le T}\mathcal{E}(t)^2+\int_0^T\mathcal{D}(t)^2dt\le C\varepsilon_0^2. \tag{4.42}$$

Let the initial data satisfy $C\varepsilon_0^2\le\frac{1}{2}\delta^2$, which closes the a-priori assumption (4.3). Based on the continuity argument, the global existence of the solution is established. The reader can refer to [23] for details. This completes the proof of Theorem 3.1. □

5. The estimate of Green's function for IBVP

To obtain the Green's function for IBVP (3.11)-(3.12), we construct a specific solution for $x\in\mathbb{R}_+,t\ge0$

$$\tilde{U}(x,t)=(\tilde{n},\tilde{w},\tilde{\phi})^t=\int_0^\infty G(x-y,t)U_0(y)dy, \tag{5.1}$$

where $G(x,t)$ is Green's function of Cauchy problem satisfying

$$\begin{cases}\partial_tG(x,t)+\mathbb{A}\partial_xG(x,t)=\mathbb{B}\partial_x^2G(x,t), & x\in\mathbb{R},t>0,\\ G(x,0)=\delta(x)I_3.\end{cases} \tag{5.2}$$

It is obvious to verify that $\tilde{U}(x,t)$ solves the following problem

$$\begin{cases}\tilde{n}_t-\sigma\tilde{w}_x=0,x>0,t>0,\\ \tilde{w}_t-\sigma\tilde{n}_x+\eta\tilde{\phi}_x=\mu\tilde{w}_{xx},\\ \tilde{\phi}_t+\eta\tilde{w}_x=\nu\tilde{\phi}_{xx},\\ (\tilde{n},\tilde{w},\tilde{\phi})|_{t=0}=(n_0,w_0,\phi_0),\end{cases} \tag{5.3}$$

with the following boundary condition

$$(\tilde{w},\tilde{\phi}_x)|_{x=0}=(m(t),j(t)),\qquad(\tilde{n},\tilde{w},\tilde{\phi})|_{x\to+\infty}=(0,0,0), \tag{5.4}$$

where $m(t),j(t)$ are given by

$$\begin{aligned}m(t)&=(0,1,0)\int_0^\infty G(-y,t)U_0(y)dy,\\ j(t)&=\Big\{(0,0,1)\int_0^\infty\partial_xG(x-y,t)U_0(y)dy\Big\}\Big|_{x=0}.\end{aligned} \tag{5.5}$$

Denote

$$\bar{U}=\tilde{U}-U=(\bar{n},\bar{w},\bar{\phi})^t=(\tilde{n}-n,\tilde{w}-w,\tilde{\phi}-\phi)^t, \tag{5.6}$$

where $U=(n,w,\phi)^t$ is the solution of linear system (3.11). After a direct calculation, a new system is derived as follows

$$\begin{cases} \bar{n}_t-\sigma\bar{w}_x=0, x>0, t>0,\\ \bar{w}_t-\sigma\bar{n}_x+\eta\bar{\phi}_x=\mu\bar{w}_{xx},\\ \bar{\phi}_t+\eta\bar{w}_x=\nu\bar{\phi}_{xx},\\ (\bar{n},\bar{w},\bar{\phi})|_{t=0}=(0,0,0). \end{cases} \tag{5.7}$$

Meanwhile, the boundary condition becomes

$$(\bar{w},\bar{\phi}_x)|_{x=0}=(m(t),j(t)), \qquad (\bar{n},\bar{w},\bar{\phi})|_{x\to+\infty}=(0,0,0). \tag{5.8}$$

Through an appropriate combination, we derive the equation of $\bar{w}$

$$\bar{w}_{ttt}-(\mu+\nu)\bar{w}_{ttxx}-(\sigma^2+\eta^2)\bar{w}_{txx}+\mu\nu\bar{w}_{txxxx}+\nu\sigma^2\bar{w}_{xxxx}=0. \tag{5.9}$$

Then, taking Laplace transform in time gives

$$\nu(\sigma^2+\mu s)\mathcal{L}[\bar{w}]_{xxxx}-(as^2+bs)\mathcal{L}[\bar{w}]_{xx}+s^3\mathcal{L}[\bar{w}]=0, \tag{5.10}$$

where $a=\mu+\nu$, $b=\sigma^2+\eta^2$. Solve this ordinary differential equation to give

$$\mathcal{L}[\bar{w}]=c_1e^{-\lambda_1 x}+\bar{c}_1e^{\lambda_1 x}+c_2e^{-\lambda_2 x}+\bar{c}_2e^{\lambda_2 x}. \tag{5.11}$$

The unknowns $c_i,\bar{c}_i (i=1,2)$ are determined by boundary data. Let λ_1,λ_2 satisfy

$$\lambda_1=\sqrt{\frac{as^2+bs-\sqrt{(as^2+bs)^2-4\nu(\sigma^2+\mu s)s^3}}{2\nu(\sigma^2+\mu s)}},$$
$$\lambda_2=\sqrt{\frac{as^2+bs+\sqrt{(as^2+bs)^2-4\nu(\sigma^2+\mu s)s^3}}{2\nu(\sigma^2+\mu s)}}.$$

We take the branch such that $\mathrm{Re}\lambda_i>0$ when $\mathrm{Re}\ s>0$. It should be mentioned that the solution of $\mathcal{L}[\bar{\phi}]$ can also be obtained identically. Hence, we deduce that it can be stated as

$$\mathcal{L}[\bar{\phi}]=d_1e^{-\lambda_1 x}+\bar{d}_1e^{\lambda_1 x}+d_2e^{-\lambda_2 x}+\bar{d}_2e^{\lambda_2 x}. \tag{5.12}$$

Note that the first equation of (5.7) tells us

$$\mathcal{L}[\bar{n}]=\frac{\sigma}{s}\mathcal{L}[\bar{w}]_x. \tag{5.13}$$

In virtue of (5.8), one obtains

$$\bar{c}_1=\bar{c}_2=0, \quad \bar{d}_1=\bar{d}_2=0.$$

Making use of (5.8), (5.11) and (5.12) gives

$$\begin{cases} c_1+c_2=\mathcal{L}[m],\\ \lambda_1d_1+\lambda_2d_2=-\mathcal{L}[j]. \end{cases} \tag{5.14}$$

Taking Laplace transform on system $(5.7)_2$ and $(5.7)_3$, then combining (5.14) yields

$$\begin{cases} (\mu s+\sigma^2)(c_1\lambda_1^2+c_2\lambda_2^2)=s^2\mathcal{L}[m]+\eta s\mathcal{L}[j], \\ \eta(c_1\lambda_1+c_2\lambda_2)+(\nu\lambda_1^2-s)d_1+(\nu\lambda_2^2-s)d_2=0. \end{cases} \tag{5.15}$$

By a direct calculation of (5.14) and (5.15), we obtain

$$\begin{cases} c_1=\dfrac{1}{\lambda_1^2-\lambda_2^2}\Big[(\dfrac{s^2}{\sigma^2+\mu s}-\lambda_2^2)\mathcal{L}[m]+\dfrac{\eta s}{\sigma^2+\mu s}\mathcal{L}[j]\Big], \\ c_2=\dfrac{1}{\lambda_1^2-\lambda_2^2}\Big[(\lambda_1^2-\dfrac{s^2}{\sigma^2+\mu s})\mathcal{L}[m]-\dfrac{\eta s}{\sigma^2+\mu s}\mathcal{L}[j]\Big], \end{cases} \tag{5.16}$$

and

$$\begin{cases} d_1=-\dfrac{1}{(\lambda_1^2-\lambda_2^2)(s+\nu\lambda_1\lambda_2)}[\dfrac{\eta\lambda_2 s^2}{(\sigma^2+\mu s)}+\eta\lambda_1\lambda_2^2]\mathcal{L}[m] \\ \quad -\dfrac{1}{(\lambda_1-\lambda_2)(s+\nu\lambda_1\lambda_2)}[s-\nu\lambda_2^2+\dfrac{\lambda_2 s\eta^2}{(\sigma^2+\mu s)(\lambda_1+\lambda_2)}]\mathcal{L}[j], \\ d_2=\dfrac{1}{(\lambda_1^2-\lambda_2^2)(s+\nu\lambda_1\lambda_2)}[\dfrac{\eta\lambda_1 s^2}{\sigma^2+\mu s}+\eta\lambda_1^2\lambda_2]\mathcal{L}[m] \\ \quad +\dfrac{1}{(\lambda_1-\lambda_2)(s+\nu\lambda_1\lambda_2)}[s-\nu\lambda_1^2+\dfrac{\lambda_1 s\eta^2}{(\sigma^2+\mu s)(\lambda_1+\lambda_2)}]\mathcal{L}[j]. \end{cases} \tag{5.17}$$

Finally, the remaining task is to calculate $\mathcal{L}[G](-y,s)$. Taking Fourier and Laplace transforms in (5.2) provides

$$\begin{pmatrix} s & -i\sigma\xi & 0 \\ -i\sigma\xi & s+\mu\xi^2 & i\eta\xi \\ 0 & i\eta\xi & s+\nu\xi^2 \end{pmatrix}\mathcal{L}[\hat{G}](\xi,s)=I_3. \tag{5.18}$$

After taking inverse of the matrix, we get

$$\mathcal{L}[\hat{G}](\xi,s)=\frac{1}{\Delta}\begin{pmatrix} s^2+(as+\eta^2)\xi^2+\mu\nu\xi^4 & i\sigma\xi(s+\nu\xi^2) & \sigma\eta\xi^2 \\ i\sigma\xi(s+\nu\xi^2) & s(s+\nu\xi^2) & -i\eta\xi s \\ \sigma\eta\xi^2 & -i\eta\xi s & s^2+(\sigma^2+\mu s)\xi^2 \end{pmatrix},$$

where

$$\Delta=\nu(\sigma^2+\mu s)\xi^4+(as^2+bs)\xi^2+s^3. \tag{5.19}$$

It is easy to verify

$$\Delta=\nu(\sigma^2+\mu s)(\xi^2+\lambda_1^2)(\xi^2+\lambda_2^2).$$

In terms of Residue Theorem, the following results hold for $x\neq 0$

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{e^{ix\xi}}{(\xi^2+\lambda_1^2)(\xi^2+\lambda_2^2)}d\xi=\frac{\lambda_1e^{-\lambda_2|x|}-\lambda_2e^{-\lambda_1|x|}}{2\lambda_1\lambda_2(\lambda_1^2-\lambda_2^2)},\\
&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{i\xi e^{ix\xi}}{(\xi^2+\lambda_1^2)(\xi^2+\lambda_2^2)}d\xi=\frac{\operatorname{sign}(x)(e^{-\lambda_1|x|}-e^{-\lambda_2|x|})}{2(\lambda_1^2-\lambda_2^2)},\\
&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\xi^2 e^{ix\xi}}{(\xi^2+\lambda_1^2)(\xi^2+\lambda_2^2)}d\xi=\frac{\lambda_1e^{-\lambda_1|x|}-\lambda_2e^{-\lambda_2|x|}}{2(\lambda_1^2-\lambda_2^2)},\\
&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{i\xi^3 e^{ix\xi}}{(\xi^2+\lambda_1^2)(\xi^2+\lambda_2^2)}d\xi=\frac{\operatorname{sign}(x)(\lambda_2^2e^{-\lambda_2|x|}-\lambda_1^2e^{-\lambda_1|x|})}{2(\lambda_1^2-\lambda_2^2)},\\
&\frac{1}{2\pi}\int_{\mathbb{R}}\frac{\xi^4 e^{ix\xi}}{(\xi^2+\lambda_1^2)(\xi^2+\lambda_2^2)}d\xi=\frac{\lambda_2^3e^{-\lambda_2|x|}-\lambda_1^3e^{-\lambda_1|x|}}{2(\lambda_1^2-\lambda_2^2)},
\end{aligned}$$

where sign(x) is a symbol function.
Thus, we obtain the Laplace transform of the Green's function of the Cauchy problem

$$\begin{aligned}
&\mathcal{L}[G](x,s)\\
&=\frac{e^{-\lambda_1|x|}}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\begin{pmatrix}\frac{-s^2}{\lambda_1}+(as+\eta^2)\lambda_1-\mu\nu\lambda_1^3 & \operatorname{sign}(x)(-\sigma s+\sigma\nu\lambda_1^2) & \sigma\eta\lambda_1\\ \operatorname{sign}(x)(-\sigma s+\sigma\nu\lambda_1^2) & \frac{-s^2}{\lambda_1}+\nu s\lambda_1 & \operatorname{sign}(x)\eta s\\ \sigma\eta\lambda_1 & \operatorname{sign}(x)\eta s & \frac{-s^2}{\lambda_1}+(\sigma^2+\mu s)\lambda_1\end{pmatrix}\\
&\quad+\frac{e^{-\lambda_2|x|}}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\begin{pmatrix}\frac{s^2}{\lambda_2}-(as+\eta^2)\lambda_2+\mu\nu\lambda_2^3 & \operatorname{sign}(x)(\sigma s-\sigma\nu\lambda_2^2) & -\sigma\eta\lambda_2\\ \operatorname{sign}(x)(\sigma s-\sigma\nu\lambda_2^2) & \frac{s^2}{\lambda_2}-\nu s\lambda_2 & -\operatorname{sign}(x)\eta s\\ -\sigma\eta\lambda_2 & -\operatorname{sign}(x)\eta s & \frac{s^2}{\lambda_2}-(\sigma^2+\mu s)\lambda_2\end{pmatrix}.
\end{aligned}\tag{5.20}$$

Then, if $y>0$, we could obtain

$$\begin{aligned}
&\mathcal{L}[G](-y,s)\\
&=\frac{e^{-\lambda_1 y}}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\begin{pmatrix}\frac{-s^2}{\lambda_1}+(as+\eta^2)\lambda_1-\mu\nu\lambda_1^3 & -\sigma s+\sigma\nu\lambda_1^2 & \sigma\eta\lambda_1\\ -\sigma s+\sigma\nu\lambda_1^2 & \frac{-s^2}{\lambda_1}+\nu s\lambda_1 & \eta s\\ \sigma\eta\lambda_1 & \eta s & \frac{-s^2}{\lambda_1}+(\sigma^2+\mu s)\lambda_1\end{pmatrix}\\
&\quad+\frac{e^{-\lambda_2 y}}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\begin{pmatrix}\frac{s^2}{\lambda_2}-(as+\eta^2)\lambda_2+\mu\nu\lambda_2^3 & \sigma s-\sigma\nu\lambda_2^2 & -\sigma\eta\lambda_2\\ \sigma s-\sigma\nu\lambda_2^2 & \frac{s^2}{\lambda_2}-\nu s\lambda_2 & -\eta s\\ -\sigma\eta\lambda_2 & -\eta s & \frac{s^2}{\lambda_2}-(\sigma^2+\mu s)\lambda_2\end{pmatrix}.
\end{aligned}\tag{5.21}$$

We are in a position to capture the accurate expressions of $\mathcal{L}[m],\mathcal{L}[j]$.

$$\begin{aligned}
\mathcal{L}[m]=&\int_0^\infty(0,1,0)\mathcal{L}[G](-y,s)U_0(y)dy\\
=&\frac{1}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\Big\{\int_0^\infty e^{-\lambda_1 y}((\sigma\nu\lambda_1^2-\sigma s)n_0+(\nu s\lambda_1-\frac{s^2}{\lambda_1})w_0+\eta s\phi_0)dy\\
&+\int_0^\infty e^{-\lambda_2 y}((\sigma s-\sigma\nu\lambda_2^2)n_0+(\frac{s^2}{\lambda_2}-\nu s\lambda_2)w_0-\eta s\phi_0)dy\Big\},
\end{aligned}\tag{5.22}$$

and

$$\mathcal{L}[j]=\int_0^\infty\Big\{(0,0,\partial_x)\mathcal{L}[G](x-y,s)U_0(y)dy\Big\}\Big|_{x=0}$$

$$=\frac{1}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\Big\{\int_0^\infty \lambda_1 e^{-\lambda_1 y}(\sigma\eta\lambda_1 n_0+\eta s w_0+(-\frac{s^2}{\lambda_1}+(\sigma^2+\mu s)\lambda_1)\phi_0)dy$$
$$+\int_0^\infty \lambda_2 e^{-\lambda_2 y}(-\sigma\eta\lambda_2 n_0-\eta s w_0+(\frac{s^2}{\lambda_2}-(\sigma^2+\mu s)\lambda_2)\phi_0)dy\Big\}. \tag{5.23}$$

After a tedious computation, we are able to prove

$$\mathcal{L}[\bar{w}](x,s)$$
$$=\frac{1}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\Big\{\int_0^\infty e^{-\lambda_1(x+y)}\big((-\sigma s+\sigma\nu\lambda_1^2)n_0+(-\frac{s^2}{\lambda_1}+\nu s\lambda_1)w_0+\eta s\phi_0\big)dy$$
$$+\int_0^\infty e^{-\lambda_2(x+y)}\big((\sigma s-\sigma\nu\lambda_2^2)n_0+(\frac{s^2}{\lambda_2}-\nu s\lambda_2)w_0-\eta s\phi_0\big)dy\Big\}. \tag{5.24}$$

In the same way, $\mathcal{L}[\bar{n}]$ is calculated as

$$\mathcal{L}[\bar{n}](x,s)$$
$$=\frac{1}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\Big\{\int_0^\infty e^{-\lambda_1(x+y)}\big((\frac{s^2}{\lambda_1}-(as+\eta^2)\lambda_1+\mu\nu\lambda_1^3)n_0$$
$$+(\sigma s-\sigma\nu\lambda_1^2)w_0-\sigma\eta\lambda_1\phi_0\big)dy+\int_0^\infty e^{-\lambda_2(x+y)}\big((-\frac{s^2}{\lambda_2}+(as+\eta^2)\lambda_2-\mu\nu\lambda_2^3)n_0$$
$$+(-\sigma s+\sigma\nu\lambda_2^2)w_0+\sigma\eta\lambda_2\phi_0\big)dy\Big\}. \tag{5.25}$$

In a same manner for $\mathcal{L}[\bar{\phi}]$, one has

$$\mathcal{L}[\bar{\phi}](x,s)=\frac{1}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\Big\{\int_0^\infty e^{-\lambda_1(x+y)}\big(-\sigma\eta\lambda_1 n_0-\eta s w_0$$
$$+(\frac{s^2}{\lambda_1}-(\sigma^2+\mu s)\lambda_1)\phi_0\big)dy+\int_0^\infty e^{-\lambda_2(x+y)}\big(\sigma\eta\lambda_2 n_0+\eta s w_0$$
$$+(-\frac{s^2}{\lambda_2}+(\sigma^2+\mu s)\lambda_2)\phi_0\big)dy\Big\}. \tag{5.26}$$

Meanwhile, by (5.20), we also obtain

$$\mathcal{L}[G](x+y,s)$$
$$=\frac{e^{-\lambda_1(x+y)}}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\begin{pmatrix} \frac{-s^2}{\lambda_1}+(as+\eta^2)\lambda_1-\mu\nu\lambda_1^3 & -\sigma s+\sigma\nu\lambda_1^2 & \sigma\eta\lambda_1 \\ -\sigma s+\sigma\nu\lambda_1^2 & \frac{-s^2}{\lambda_1}+\nu s\lambda_1 & \eta s \\ \sigma\eta\lambda_1 & \eta s & \frac{-s^2}{\lambda_1}+(\sigma^2+\mu s)\lambda_1 \end{pmatrix}$$
$$+\frac{e^{-\lambda_2(x+y)}}{2\nu(\sigma^2+\mu s)(\lambda_1^2-\lambda_2^2)}\begin{pmatrix} \frac{s^2}{\lambda_2}-(as+\eta^2)\lambda_2+\mu\nu\lambda_2^3 & \sigma s-\sigma\nu\lambda_2^2 & -\sigma\eta\lambda_2 \\ \sigma s-\sigma\nu\lambda_2^2 & \frac{s^2}{\lambda_2}-\nu s\lambda_2 & -\eta s \\ -\sigma\eta\lambda_2 & -\eta s & \frac{s^2}{\lambda_2}-(\sigma^2+\mu s)\lambda_2 \end{pmatrix}. \tag{5.27}$$

According to (5.24)-(5.25) and (5.27), it is significant to observe

$$\mathcal{L}[\bar{n}](x,s)=(-1,0,0)\int_0^\infty \mathcal{L}[G](x+y,s)U_0(y)dy,$$
$$\mathcal{L}[\bar{w}](x,s)=(0,1,0)\int_0^\infty \mathcal{L}[G](x+y,s)U_0(y)dy, \tag{5.28}$$
$$\mathcal{L}[\bar{\phi}](x,s)=(0,0,-1)\int_0^\infty \mathcal{L}[G](x+y,s)U_0(y)dy.$$

As a result, it holds

$$\mathcal{L}[\bar{U}] = \int_0^\infty \mathcal{L}[G](x+y,s) \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} U_0(y)dy. \tag{5.29}$$

Therefore, we derive the solution based on (5.1) and (5.6)

$$\mathcal{L}[U](x,s) = \int_0^\infty \Big(\mathcal{L}[G](x-y,s) + \mathcal{L}[G](x+y,s) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\Big) U_0(y)dy. \tag{5.30}$$

Then, the Laplace transform of $\mathbb{G}(x,t;y)$ has the following form

$$\mathcal{L}[\mathbb{G}](x,s;y) = \mathcal{L}[G](x-y,s) + \mathcal{L}[G](x+y,s) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.31}$$

Taking Laplace inverse transform, we ultimately get

$$\mathbb{G}(x,t;y) = G(x-y,t) + G(x+y,t) \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.32}$$

According to Lemma 2.3 and the formula of $G^*(x,t)$, the leading part of $\mathbb{G}(x,t;y)$ has the following estimate

$$\begin{aligned} &\Big|\partial_x^\alpha \mathbb{G}(x,t;y) - e^{-\frac{\sigma^2}{\mu}t} \sum_{j=0}^{\alpha} \delta^{(\alpha-j)}(x-y)\Pi_j(t)\Big| \\ &= O(1) t^{-\frac{1}{2}-\frac{\alpha}{2}} \sum_{i=1}^{3} \big(e^{-\frac{(x-y-l_i t)^2}{Ct}} + e^{-\frac{(x+y-l_i t)^2}{Ct}}\big). \end{aligned} \tag{5.33}$$

We hence complete the proof of Theorem 3.2.

6. Nonlinear pointwise estimates

In this section, we are devoted to the study of nonlinear stability. We first define

$$W_0(x) = \int_x^\infty U_0(y)dy. \tag{6.1}$$

To investigate the poinwise behavior of the solution, we introduce the ansatz as follows

$$M(t) = \sup_{0\le s\le t} \Big\{ \|UF^{-1}\|_{L_x^\infty} + (1+s)\big(\log(2+s)\big)^{-1} \|U_x\|_{L_x^\infty} + (1+s)^{\frac{1}{2}} \|(w_{xx},\phi_{xx})\|_{L_x^\infty} \Big\}. \tag{6.2}$$

The wave function $F(x,t)$ is defined as

$$F(x,t) = \sum_{i=2}^{3} (1+t)^{\frac{3}{8}} \phi_1(x,t;l_i) \psi_{\frac{3}{4}}(x,t;l_i), \tag{6.3}$$

where $\psi_\alpha(x,t;l_i), \phi_\alpha(x,t;l_i)$ are given by (2.3) and the constants l_i satisfy $l_2 = 0,\ l_3 = c$.

By the definition of $M(t)$, it is easy to see

$$\begin{aligned}
&|U(x,t)| \leq M(t)F(x,t),\\
&|U_x(x,t)| \leq (1+t)^{-1}\log(2+t)M(t),\\
&|w_{xx}(x,t)| \leq (1+t)^{-\frac{1}{2}}M(t),\\
&|\phi_{xx}(x,t)| \leq (1+t)^{-\frac{1}{2}}M(t).
\end{aligned} \tag{6.4}$$

Applying (4.14) and (6.4) yields

$$|\tilde{N}(x,t)| \leq CM(t)^2\sum_{i=2}^{3}(1+t)^{-\frac{1}{2}}\phi_1(x,t;l_i)\psi_{\frac{3}{4}}(x,t;l_i). \tag{6.5}$$

According to the boundary condition (4.12), we immediately obtain

$$\tilde{N}(0,t) = (0,\tilde{N}_1,0)^t. \tag{6.6}$$

Once the spatial variable is ignored, the time decay rates for $\tilde{N}(x,t)$ and $N(x,t)$ are computed as follows

$$|\tilde{N}(x,t)| \leq C(1+t)^{-1}M(t)^2, \quad |N(x,t)| \leq C(1+t)^{-1}M(t)^2. \tag{6.7}$$

For later use, we decompose the Green's function $\mathbb{G}(x,t;y)$ into two parts

$$\mathbb{G}(x,t;y) \triangleq \mathbb{G}^s(x,t;y) + \mathbb{G}^\ell(x,t;y), \tag{6.8}$$

where $\mathbb{G}^s(x,t;y)$ is the short wave part corresponding to the singular part of $\mathbb{G}(x,t;y)$ satisfying

$$\partial_x^\alpha \mathbb{G}^s(x,t;y) = e^{-\frac{\sigma^2}{\mu}t}\sum_{j=0}^{\alpha}\delta^{(\alpha-j)}(x-y)\Pi_j(t). \tag{6.9}$$

$\mathbb{G}^\ell(x,t;y)$ represents the long wave part, which dominates the large-time behavior. It holds for any integers $k_1,k_2 \geq 0$ that

$$\left|\partial_x^{k_1}\partial_y^{k_2}\mathbb{G}^\ell(x,t;y)\right| = \mathcal{O}(1)t^{-\frac{1}{2}-\frac{k_1+k_2}{2}}\sum_{i=1}^{3}\left(e^{-\frac{(x-y-l_it)^2}{Ct}} + e^{-\frac{(x+y-l_it)^2}{Ct}}\right). \tag{6.10}$$

We are prepared to develop the pointwise estimate of the solution $U(x,t)$.

PROPOSITION 6.1. *Under the assumptions of Theorem 3.3, there exists a positive constant C such that*

$$|U(x,t)| \leq C(\varepsilon_0 + M(t)^2)F(x,t), \tag{6.11}$$

$$|U_x(x,t)| \leq C(1+t)^{-1}\log(2+t)(\varepsilon_0 + M(t)^2), \tag{6.12}$$

where the wave function $F(x,t)$ is defined in (2.4).

Proof. By Duhamel's principle, the solution $U(x,t)$ of nonlinear system (4.16) is written as

$$U(x,t) = \int_0^\infty \mathbb{G}(x,t;y)U_0(y)dy + \int_0^t\int_0^\infty \mathbb{G}(x,t-s;y)N(y,s)dyds. \tag{6.13}$$

We begin it by applying (6.8)-(6.10)

$$\int_0^\infty \mathbb{G}(x,t;y)U_0(y)dy=\int_0^\infty \mathbb{G}^\ell(x,t;y)U_0(y)dy+\int_0^\infty \mathbb{G}^s(x,t;y)U_0(y)dy \\ \triangleq I_1+I_2. \tag{6.14}$$

Since the case when $t\le 1$ can be handled easily by using the assumption of Theorem 3.3, we assume $t>1$ in the proof of (6.11). To estimate I_1, we change the initial data U_0 into W_0', then taking integration by parts yields

$$\begin{aligned}|I_1|=&\mathcal{O}(1)t^{-\frac12}\sum_{i=1}^3 e^{-\frac{(x-l_it)^2}{Ct}}|W_0(0)|\\ &+\mathcal{O}(1)\int_0^\infty t^{-1}\sum_{i=1}^3(e^{-\frac{(x-y-l_it)^2}{Ct}}+e^{-\frac{(x+y-l_it)^2}{Ct}})|W_0(y)|dy\\ \triangleq& I_{11}+I_{12}.\end{aligned} \tag{6.15}$$

Making use of (3.15) and Lemma 2.3 yields

$$\begin{aligned}I_{11}&\le C\varepsilon_0(1+t)^{-\frac12}(e^{-\frac{(x-c(1+t))^2}{C(1+t)}}+e^{-\frac{x^2}{C(1+t)}}+e^{-\frac{(x+c(1+t))^2}{C(1+t)}})\\ &\le C\varepsilon_0(1+t)^{\frac14}(\psi_{\frac32}(x,t;c)+\psi_{\frac32}(x,t;0)+\psi_{\frac32}(x,t;-c))\\ &\le C\varepsilon_0F(x,t).\end{aligned} \tag{6.16}$$

It is easy to verify

$$I_{12}\le C\varepsilon_0\int_0^\infty (1+t)^{-1}\sum_{i=1}^3(e^{-\frac{(x-y-l_i(1+t))^2}{C(1+t)}}+e^{-\frac{(x+y-l_i(1+t))^2}{C(1+t)}})(1+y^2)^{-\frac58}dy. \tag{6.17}$$

For simplicity, we define

$$\Lambda_1=\int_0^\infty (1+t)^{-1}e^{-\frac{(x-y-l_i(1+t))^2}{C(1+t)}}(1+y^2)^{-\frac58}dy, \tag{6.18}$$

$$\Lambda_2=\int_0^\infty (1+t)^{-1}e^{-\frac{(x+y-l_i(1+t))^2}{C(1+t)}}(1+y^2)^{-\frac58}dy. \tag{6.19}$$

In order to estimate Λ_1, it is essential to divide the integration domain into three cases

(1) $|x-l_i(1+t)|<\sqrt{1+t}$, it holds

$$\begin{aligned}\Lambda_1&\le C(1+t)^{-1}\le C(1+t)^{\frac14}\psi_{\frac32}(x,t;l_i)\\ &\le C(1+t)^{\frac38}\phi_1(x,t;l_i)\psi_{\frac34}(x,t;l_i)\le CF(x,t).\end{aligned} \tag{6.20}$$

(2) $\sqrt{1+t}\le|x-l_i(1+t)|\le 1+t$, we have

$$\begin{aligned}\Lambda_1\le& C(1+t)^{-1}\int_{|y|\le\frac{|x-l_i(1+t)|}{2}}e^{-\frac{(x-l_i(1+t))^2}{4C(1+t)}}(1+y^2)^{-\frac58}dy\\ &+C(1+t)^{-1}\int_{|y|>\frac{|x-l_i(1+t)|}{2}}e^{-\frac{(x-y-l_i(1+t))^2}{C(1+t)}}(1+y^2)^{-\frac58}dy\\ \le& C(1+t)^{-1}e^{-\frac{(x-l_i(1+t))^2}{4C(1+t)}}+C(1+t)^{-\frac12}(1+|x-l_i(1+t)|^2)^{-\frac58}\\ \le& CF(x,t).\end{aligned} \tag{6.21}$$

(3) $|x-l_i(1+t)|>1+t$, one has

$$\begin{aligned}\Lambda_1 \le& C(1+t)^{-1}\int_{|y|\le\frac{|x-l_i(1+t)|}{2}} e^{-\frac{(x-l_i(1+t))^2}{4C(1+t)}}(1+y^2)^{-\frac{5}{8}}dy\\&+C(1+t)^{-1}\int_{|y|>\frac{|x-l_i(1+t)|}{2}} e^{-\frac{(x-y-l_i(1+t))^2}{C(1+t)}}(1+y^2)^{-\frac{5}{8}}dy\\ \le& C(1+t)^{-1}e^{-\frac{(x-l_i(1+t))^2}{4C(1+t)}}+C(1+t)^{-\frac{1}{2}}(1+|x-l_i(1+t)|)^{-\frac{5}{4}}\\ \le& CF(x,t),\end{aligned}\tag{6.22}$$

where we use the fact that when $|x-l_i(1+t)|>1+t$, it holds

$$(1+|x-l_i(1+t)|)^{-\frac{5}{4}}\le C\phi_1(x,t;l_i)\psi_{\frac{3}{4}}(x,t;l_i).$$

The estimates of Λ_1 and Λ_2 are similar. Then, we immediately obtain

$$I_{12}\le C\varepsilon_0 F(x,t).\tag{6.23}$$

It then follows from (6.16) and (6.23) that

$$|I_1|=I_{11}+I_{12}\le C\varepsilon_0 F(x,t).\tag{6.24}$$

For I_2, we immediately deduce the following estimate from Lemma 2.1

$$|I_2|\le C\varepsilon_0 e^{-\frac{\sigma^2}{\mu}t}(1+x^2)^{-\frac{5}{8}}\le C\varepsilon_0 F(x,t).\tag{6.25}$$

Consequently, we summarize above results together

$$\Big|\int_0^\infty \mathbb{G}(x,t;y)U_0(y)dy\Big|\le|I_1|+|I_2|\le C\varepsilon_0 F(x,t).\tag{6.26}$$

The next goal is to deal with the nonlinear term. Applying integration by parts yields

$$\begin{aligned}&\int_0^t\int_0^\infty \mathbb{G}(x,t-s;y)N(y,s)dyds\\=&-\int_0^t \mathbb{G}(x,t-s;0)\tilde{N}(0,s)ds-\int_0^t\int_0^\infty \partial_y\mathbb{G}(x,t-s;y)\tilde{N}(y,s)dyds\\ \triangleq& J_1+J_2.\end{aligned}\tag{6.27}$$

By (5.32) and (6.6), it is crucial to observe

$$J_1=-\int_0^t (G(x,t-s)+G(x,t-s)\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix})(0,\tilde{N}_1(0,s),0)^t ds=(0,0,0)^t.\tag{6.28}$$

In view of (6.8), we rewrite J_2 as

$$\begin{aligned}J_2=&-\int_0^t\int_0^\infty \partial_y\mathbb{G}^\ell(x,t-s;y)\tilde{N}(y,s)dyds-\int_0^t\int_0^\infty \partial_y\mathbb{G}^s(x,t-s;y)\tilde{N}(y,s)dyds\\ \triangleq& J_{21}+J_{22}.\end{aligned}\tag{6.29}$$

By the definition of $\mathbb{G}^s$, in terms of integration by parts, it is easy to verify

$$\begin{aligned}
J_{22} &= -\int_0^t\int_0^\infty \partial_y \mathbb{G}^s(x,t-s;y)\tilde{N}(y,s)dyds \\
&= \int_0^t \mathbb{G}^s(x,t-s;0)\tilde{N}(0,s)ds + \int_0^t\int_0^\infty \mathbb{G}^s(x,t-s;y)N(y,s)dyds \\
&= \int_0^t e^{-\frac{\sigma^2}{\mu}(t-s)}\delta(x)\Pi_0\tilde{N}(0,s)ds + \int_0^t\int_0^\infty e^{-\frac{\sigma^2}{\mu}(t-s)}\delta(x-y)\Pi_0 N(y,s)dyds \\
&= (0,0,0)^t,
\end{aligned} \tag{6.30}$$

where we have used the fact that

$$\Pi_0\tilde{N}(0,s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(0,\tilde{N}_1(0,s),\tilde{N}_2(0,s))^t = (0,0,0)^t, \tag{6.31}$$

and

$$\Pi_0 N(y,s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}(0,N_1(y,s),N_2(y,s))^t = (0,0,0)^t. \tag{6.32}$$

With the help of (6.5), we ensure

$$\begin{aligned}
|J_{21}| &\le C\int_0^t\int_0^\infty (t-s)^{-1}\sum_{i=1}^3\left(e^{-\frac{(x-y-l_i(t-s))^2}{C(t-s)}} + e^{-\frac{(x+y-l_i(t-s))^2}{C(t-s)}}\right)|\tilde{N}(y,s)|dyds \\
&\le CM(t)^2\int_0^t\int_{-\infty}^\infty (t-s)^{-1}\sum_{i=1}^3\left(e^{-\frac{(x-y-l_i(t-s))^2}{C(t-s)}} + e^{-\frac{(x+y-l_i(t-s))^2}{C(t-s)}}\right) \\
&\quad \times\sum_{i=2}^3 (1+s)^{-\frac{1}{2}}\phi_1(y,s;l_i)\psi_{\frac{3}{4}}(y,s;l_i)dyds.
\end{aligned} \tag{6.33}$$

Without loss of generality, we only consider the following two terms.

Substituting $\alpha=0,\beta=1,\lambda=c$ and $\bar{\mu}=C$ into Lemma 2.4 yields

$$\begin{aligned}
&\int_0^t\int_{-\infty}^\infty (t-s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{(x-y-c(t-s))^2}{C(t-s)}}\phi_1(y,s;c)\psi_{\frac{3}{4}}(y,s;c)dyds \\
&= \mathcal{O}(1)[\log(2+t)\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c) + \psi_{\frac{3}{2}}(x,t;c)] \\
&= \mathcal{O}(1)[(1+t)^{\frac{3}{8}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c) + (1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)] \\
&= \mathcal{O}(1)(1+t)^{\frac{3}{8}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c) \\
&= \mathcal{O}(1)F(x,t).
\end{aligned} \tag{6.34}$$

Let $\alpha=0,\beta=1,\lambda=c,\lambda'=-c,\bar{\mu}=C$. Via Lemma 2.5, we are able to show

$$\begin{aligned}
&\int_0^t\int_{-\infty}^\infty (t-s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{(x-y+c(t-s))^2}{C(t-s)}}\phi_1(y,s;c)\psi_{\frac{3}{4}}(y,s;c)dyds \\
&= O(1)[\log(2+t)\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c) + \phi_1(x,t;-c)\psi_{\frac{3}{4}}(x,t;-c)] \\
&\quad + O(1)[\psi_{\frac{3}{2}}(x,t;c) + \psi_{\frac{3}{2}}(x,t;-c)] \\
&\quad + O(1)|x-c(1+t)|^{-\frac{3}{4}+\epsilon}|x+c(1+t)|^{-\frac{1}{2}} \\
&\qquad \cdot\text{char}\left\{-c(1+t)+K\sqrt{1+t} \le x \le c(1+t)-K\sqrt{1+t}\right\},
\end{aligned}$$

where $\epsilon>0$ can be arbitrarily small and $K>2c$. Choosing $\epsilon=\frac{1}{4}$, we have

$$\begin{aligned}&\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{(x-y+c(t-s))^2}{C(t-s)}}\phi_1(y,s;c)\psi_{\frac{3}{4}}(y,s;c)dyds\\&=O(1)(1+t)^{\frac{3}{8}}[\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)+\phi_1(x,t;-c)\psi_{\frac{3}{4}}(x,t;-c)]\\&\quad+O(1)|x-c(1+t)|^{-\frac{1}{2}}|x+c(1+t)|^{-\frac{1}{2}}\cdot\text{char}\Big\{-c(1+t)+c\sqrt{1+t}\le x\le c(1+t)-c\sqrt{1+t}\Big\}\\&=\mathcal{O}(1)F(x,t),\end{aligned}$$

where we use the fact that when $-c(1+t)+c\sqrt{1+t}\le x\le c(1+t)-c\sqrt{1+t}$, it holds

$$\begin{aligned}c\sqrt{1+t}\le|x-c(1+t)|\le 2c(1+t),\\c\sqrt{1+t}\le|x+c(1+t)|\le 2c(1+t).\end{aligned}\tag{6.35}$$

As a result, one can deduce

$$\begin{aligned}&|x-c(1+t)|^{-\frac{1}{2}}|x+c(1+t)|^{-\frac{1}{2}}\\&\lesssim(1+t)^{\frac{3}{8}}[\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)+\phi_1(x,t;-c)\psi_{\frac{3}{4}}(x,t;-c)].\end{aligned}\tag{6.36}$$

It should be pointed out that the remainders in (6.33) can be treated in a similar manner. We hence obtain

$$|J_{21}|\le CM(t)^2F(x,t).\tag{6.37}$$

As a consequence, it provides

$$|J_2|\le|J_{21}|+|J_{22}|\le CM(t)^2F(x,t),\tag{6.38}$$

which, together with (6.28), also leads to

$$\Big|\int_0^t\int_0^{\infty}\mathbb{G}(x,t-s;y)N(y,s)dyds\Big|\le|J_1|+|J_2|\le CM(t)^2F(x,t).\tag{6.39}$$

Therefore we derive the pointwise estimates of the solution $U(x,t)$ satisfying

$$\begin{aligned}|U(x,t)|&\le\Big|\int_0^{\infty}\mathbb{G}(x,t;y)U_0(y)dy\Big|+\Big|\int_0^t\int_0^{\infty}\mathbb{G}(x,t-s;y)N(y,s)dyds\Big|\\&\le C(\varepsilon_0+M(t)^2)F(x,t).\end{aligned}\tag{6.40}$$

In what follows, we plan to derive the estimate of the spatial derivative $U_x(x,t)$. Theorem 3.1 provides

$$|U_x(x,t)|\le C\|U(x,t)\|_{H^4}\le C\varepsilon_0.\tag{6.41}$$

Thus, for $0\le t\le 2$, we easily deduce

$$|U_x(x,t)|\le C\varepsilon_0\le C\varepsilon_0(1+t)^{-1}\log(2+t).\tag{6.42}$$

In the following, we will focus on $t>2$. First, $U_x(x,t)$ can be written as

$$U_x(x,t)=\int_0^{\infty}\partial_x\mathbb{G}(x,t;y)U_0(y)dy+\int_0^t\int_0^{\infty}\partial_x\mathbb{G}(x,t-s;y)N(y,s)dyds.\tag{6.43}$$

According to (6.8), the first term on the right-hand side of (6.43) is decomposed into two parts

$$\int_0^\infty \partial_x \mathbb{G}(x,t;y)U_0(y)dy = \int_0^\infty \partial_x \mathbb{G}^\ell(x,t;y)U_0(y)dy + \int_0^\infty \partial_x \mathbb{G}^s(x,t;y)U_0(y)dy. \quad (6.44)$$

By the definition (6.10) and the assumption in Theorem 3.3, we get

$$\Big|\int_0^\infty \partial_x \mathbb{G}^\ell(x,t;y)U_0(y)dy\Big| \le C(1+t)^{-1}\int_0^\infty |U_0(y)|dy \le C\varepsilon_0(1+t)^{-1}. \quad (6.45)$$

In terms of integration by parts and (6.9), we obtain

$$\begin{aligned}\Big|\int_0^\infty \partial_x \mathbb{G}^s(x,t;y)U_0(y)dy\Big| &\le \Big|\int_0^\infty e^{-\frac{\sigma^2}{\mu}t}(\delta^{(1)}(x-y)\Pi_0 + \delta(x-y)\Pi_1)U_0(y)dy\Big| \\ &\le Ce^{-\frac{\sigma^2}{\mu}t}(|U_0'(x)|+|U_0(x)|) \\ &\le C\varepsilon_0(1+t)^{-1}. \end{aligned} \quad (6.46)$$

Therefore, we deduce

$$\Big|\int_0^\infty \partial_x \mathbb{G}(x,t;y)U_0(y)dy\Big| \le C\varepsilon_0(1+t)^{-1}. \quad (6.47)$$

Similarly, it is easy to verify

$$\begin{aligned}&\int_0^t\int_0^\infty \partial_x \mathbb{G}(x,t-s;y)N(y,s)dyds \\ &= \int_0^t\int_0^\infty \partial_x \mathbb{G}^\ell(x,t-s;y)N(y,s)dyds + \int_0^t\int_0^\infty \partial_x \mathbb{G}^s(x,t-s;y)N(y,s)dyds. \end{aligned} \quad (6.48)$$

To avoid the singularity of time, it is natural to divide the domain of time into $[0,t-1]$ and $[t-1,t]$. We write the first term in (6.48) as below

$$\begin{aligned}&\int_0^t\int_0^\infty \partial_x \mathbb{G}^\ell(x,t-s;y)N(y,s)dyds \\ &= \int_0^{t-1}\int_0^\infty \partial_x \mathbb{G}^\ell(x,t-s;y)N(y,s)dyds + \int_{t-1}^t\int_0^\infty \partial_x \mathbb{G}^\ell(x,t-s;y)N(y,s)dyds. \end{aligned} \quad (6.49)$$

Integrating by parts, we observe

$$\begin{aligned}&\int_0^{t-1}\int_0^\infty \partial_x \mathbb{G}^\ell(x,t-s;y)N(y,s)dyds \\ &= -\int_0^{t-1}\int_0^\infty \partial_x\partial_y \mathbb{G}^\ell(x,t-s;y)\tilde{N}(y,s)dyds - \int_0^{t-1}\partial_x \mathbb{G}^\ell(x,t-s;0)\tilde{N}(0,s)ds. \end{aligned} \quad (6.50)$$

Thanks to (6.7) and (6.10), which leads us to obtain

$$\begin{aligned}&\Big|\int_0^{t-1}\int_0^\infty \partial_x\partial_y \mathbb{G}^\ell(x,t-s;y)\tilde{N}(y,s)dyds\Big| \\ &\le CM(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-1}ds \end{aligned}$$

$$\begin{aligned}
&\le CM(t)^2\int_0^{\frac{t}{2}}(t-s)^{-1}(1+s)^{-1}ds+CM(t)^2\int_{\frac{t}{2}}^{t-1}(t-s)^{-1}(1+s)^{-1}ds\\
&\le C(1+t)^{-1}\log(2+t)M(t)^2.
\end{aligned}\tag{6.51}$$

In the same way, we have

$$\begin{aligned}
&\Big|\int_0^{t-1}\partial_x\mathbb{G}^\ell(x,t-s;0)\tilde{N}(0,s)ds\Big|\\
&\le CM(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-1}ds\\
&\le C(1+t)^{-1}\log(2+t)M(t)^2,
\end{aligned}\tag{6.52}$$

which, together with (6.51) and (6.50), also guides to

$$\Big|\int_0^{t-1}\int_0^\infty\partial_x\mathbb{G}^\ell(x,t-s;y)N(y,s)dyds\Big|\le C(1+t)^{-1}\log(2+t)M(t)^2.\tag{6.53}$$

Now we are in a position to estimate the second term in (6.49). That is

$$\begin{aligned}
&\Big|\int_{t-1}^{t}\int_0^\infty\partial_x\mathbb{G}^\ell(x,t-s;y)N(y,s)dyds\Big|\\
&\le CM(t)^2\int_{t-1}^{t}(t-s)^{-\frac{1}{2}}(1+s)^{-1}ds\\
&\le C(1+t)^{-1}M(t)^2.
\end{aligned}\tag{6.54}$$

Thus, combining (6.53) and (6.54) together gives

$$\Big|\int_0^{t}\int_0^\infty\partial_x\mathbb{G}^\ell(x,t-s;y)N(y,s)dyds\Big|\le C(1+t)^{-1}\log(2+t)M(t)^2.\tag{6.55}$$

For the second term in (6.48), we have

$$\begin{aligned}
&\int_0^{t}\int_0^\infty\partial_x\mathbb{G}^s(x,t-s;y)N(y,s)dyds\\
&=\int_0^{t}\int_0^\infty e^{-\frac{\sigma^2}{\mu}(t-s)}(\delta^{(1)}(x-y)\Pi_0+\delta(x-y)\Pi_1)N(y,s)dyds\\
&=\int_0^{t}\int_0^\infty e^{-\frac{\sigma^2}{\mu}(t-s)}\delta(x-y)\Pi_1N(y,s)dyds\\
&=\int_0^{t}e^{-\frac{\sigma^2}{\mu}(t-s)}\Pi_1N(x,s)ds,
\end{aligned}\tag{6.56}$$

where we have used (6.32).

Since Π_1 is a constant matrix given by (2.1), we have

$$\begin{aligned}
&\Big|\int_0^{t}e^{-\frac{\sigma^2}{\mu}(t-s)}\Pi_1N(x,s)ds\Big|\\
&\le C\int_0^{t}e^{-\frac{\sigma^2}{\mu}(t-s)}|N(x,s)|dx
\end{aligned}$$

$$\le CM(t)^2\int_0^t e^{-\frac{\sigma^2}{\mu}(t-s)}(1+s)^{-1}ds$$
$$\le C(1+t)^{-1}M(t)^2. \tag{6.57}$$

Hence, we have

$$\Big|\int_0^t\int_0^\infty \partial_x\mathbb{G}^s(x,t-s;y)N(y,s)dyds\Big|\le C(1+t)^{-1}\log(2+t)M(t)^2.$$

Combining above results together provides

$$\Big|\int_0^t\int_0^\infty \partial_x\mathbb{G}(x,t-s;y)N(y,s)dyds\Big|\le C(1+t)^{-1}\log(2+t)M(t)^2. \tag{6.58}$$

To this end, we finally derive the estimate of $U_x(x,t)$ from (6.47) and (6.58)

$$|U_x(x,t)|\le C(1+t)^{-1}\log(2+t)(\varepsilon_0+M(t)^2). \tag{6.59}$$

This completes the proof. □

To close the ansatz, it remains to get the estimates of w_{xx} and ϕ_{xx}. Thus, we will complete it in the next proposition.

Proposition 6.2. *Under the assumptions of Theorem 3.3, there exists a positive constant C such that*

$$|w_{xx}(x,t)|\le C(1+t)^{-\frac12}(\varepsilon_0+\varepsilon_0M(t)+M(t)^2), \tag{6.60}$$

$$|\phi_{xx}(x,t)|\le C(1+t)^{-\frac12}(\varepsilon_0+\varepsilon_0M(t)+M(t)^2). \tag{6.61}$$

Proof. By Theorem 3.1, we easily check that for $0\le t\le 2$

$$\begin{aligned}&|U_t|\le C\|U\|_{H^4}\le C\varepsilon_0\le C(1+t)^{-\frac12}\varepsilon_0,\\&|w_{xx}|\le C\|U\|_{H^4}\le C\varepsilon_0\le C(1+t)^{-\frac12}\varepsilon_0,\\&|\phi_{xx}|\le C\|U\|_{H^4}\le C\varepsilon_0\le C(1+t)^{-\frac12}\varepsilon_0.\end{aligned} \tag{6.62}$$

Thus in the next section, we will assume time $t>2$. Differentiate the system (3.3) with respect to time

$$\begin{cases}n_{tt}-\sigma w_{tx}=0,\\ w_{tt}-\sigma n_{tx}+\eta\phi_{tx}=\mu w_{txx}+\partial_tN_1,\\ \phi_{tt}+\eta w_{tx}=\nu\phi_{txx}+\partial_tN_2,\\ (n_t,w_t,\phi_t)|_{t=0}=(n_t(x,0),w_t(x,0),\phi_t(x,0)),\end{cases} \tag{6.63}$$

with following boundary condition

$$(w_t,\phi_{tx})|_{x=0}=(0,0),\quad (n_t,w_t,\phi_t)|_{x\to+\infty}=(0,0,0). \tag{6.64}$$

For simplicity, we write it into the operator form

$$\begin{cases}\partial_tU_t(x,t)+\mathbb{A}\partial_xU_t(x,t)=\mathbb{B}\partial_x^2U_t(x,t)+N_t(x,t),\\ U_t(x,t=0)=U_t(x,0).\end{cases} \tag{6.65}$$

Since the Green's function for (6.65) is the same as $\mathbb{G}(x,t;y)$, with the help of Duhamel's principle, we obtain

$$\partial_t U(x,t) = \int_0^\infty \mathbb{G}(x,t;y)\partial_s U(y,0)dy + \int_0^t \int_0^\infty \mathbb{G}(x,t-s;y)\partial_s N(y,s)dyds. \tag{6.66}$$

Via the compatibility condition (3.7), one has

$$n_t(x,0) = \sigma w_x(x,0), \quad w_t(x,0) = g_x, \quad \phi_t(x,0) = h_x, \tag{6.67}$$

where the functions $g(x)$ and $h(x)$ are introduced below

$$\begin{aligned} g(x) &= -\frac{\sigma(1+c_2)\phi_0}{1+n_0} + \frac{\mu w_{0x}}{1+n_0} + \frac{\sigma c_2^2 w_0^2}{2(1+n_0)}, \\ h(x) &= -\frac{\eta w_0(1+c_2\phi_0)}{1+n_0} + \frac{\nu\phi_{0x}}{1+n_0} + \frac{c_2(\mu-\nu)w_0 w_{0x}}{1+n_0} + \frac{\eta c_2^2 w_0^3}{2(1+n_0)}. \end{aligned} \tag{6.68}$$

Using (6.8) yields

$$\begin{aligned} &\int_0^\infty \mathbb{G}(x,t;y)\partial_s U(y,0)dy \\ &= \int_0^\infty \mathbb{G}^\ell(x,t;y)\partial_s U(y,0)dy + \int_0^\infty \mathbb{G}^s(x,t;y)\partial_s U(y,0)dy. \end{aligned} \tag{6.69}$$

Applying integration by parts with respect to x and (6.68), we get

$$\begin{aligned} &\left|\int_0^\infty \mathbb{G}^\ell(x,t;y)\partial_s U(y,0)dy\right| \\ &\le \left|\mathbb{G}^\ell(x,t;0)(\sigma w_0(0),g(0),h(0))^t\right| + \left|\int_0^\infty \partial_y \mathbb{G}^\ell(x,t;y)(\sigma w_0(y),g(y),h(y))^t dy\right| \\ &\le C\varepsilon_0(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{6.70}$$

According to the definition of $\mathbb{G}^s(x,t;y)$ and assumptions on initial data, we are able to show

$$\left|\int_0^\infty \mathbb{G}^s(x,t;y)\partial_s U(y,0)dy\right| \le C\varepsilon_0(1+t)^{-\frac{1}{2}}. \tag{6.71}$$

Combining the above estimates together shows

$$\left|\int_0^\infty \mathbb{G}(x,t;y)\partial_s U(y,0)dy\right| \le C\varepsilon_0(1+t)^{-\frac{1}{2}}. \tag{6.72}$$

In order to estimate the nonlinear term in (6.66), we write it as

$$\begin{aligned} &\int_0^t \int_0^\infty \mathbb{G}(x,t-s;y)\partial_s N(y,s)dyds \\ &= \int_0^t \int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds + \int_0^t \int_0^\infty \mathbb{G}^s(x,t-s;y)\partial_s N(y,s)dyds. \end{aligned} \tag{6.73}$$

We divide the time domain into $[0,t-1]$ and $[t-1,t]$ to avoid the time singularity.

$$\int_0^t \int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds$$

$$
\begin{aligned}
&= \int_0^{t-1}\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds \\
&\quad + \int_{t-1}^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds. \qquad (6.74)
\end{aligned}
$$

Taking integration by parts with regard to time, one has

$$
\begin{aligned}
&\int_0^{t-1}\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds \\
=&\int_0^{t-1}\int_0^\infty \partial_t\mathbb{G}^\ell(x,t-s;y)N(y,s)dyds \\
&+\int_0^\infty \mathbb{G}^\ell(x,1;y)N(y,t-1)dy-\int_0^\infty \mathbb{G}^\ell(x,t;y)N(y,0)dy. \qquad (6.75)
\end{aligned}
$$

Integrating by parts with respect to x, the first term in (6.75) can be handled as below

$$
\begin{aligned}
&\Big|\int_0^{t-1}\int_0^\infty \partial_t\mathbb{G}^\ell(x,t-s;y)N(y,s)dyds\Big| \\
\le&\Big|\int_0^{t-1}\int_0^\infty \partial_{ty}\mathbb{G}^\ell(x,t-s;y)\tilde N(y,s)dyds\Big|+\Big|\int_0^{t-1}\partial_t\mathbb{G}^\ell(x,t-s;0)\tilde N(0,s)ds\Big| \\
\le&CM(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-1}ds+CM(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-1}ds \\
\le&C(1+t)^{-\frac12}M(t)^2. \qquad (6.76)
\end{aligned}
$$

The remaining terms in (6.75) can also be treated in a similar method

$$
\Big|\int_0^\infty \mathbb{G}^\ell(x,1;y)N(y,t-1)dy-\int_0^\infty \mathbb{G}^\ell(x,t;y)N(y,0)dy\Big|\le C(1+t)^{-\frac12}M(t)^2.
$$

To this end, we ultimately deduce

$$
\Big|\int_0^{t-1}\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds\Big|\le C(1+t)^{-\frac12}M(t)^2. \qquad (6.77)
$$

By the definition of $N(x,t)$, applying integration by parts with respect to x gives rise to

$$
\begin{aligned}
&\Big|\int_{t-1}^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds\Big| \\
\le&\Big|\int_{t-1}^t\int_0^\infty \partial_y\mathbb{G}^\ell(x,t-s;y)\partial_s\tilde N(y,s)dyds\Big|+\Big|\int_{t-1}^t \mathbb{G}^\ell(x,t-s;0)\partial_s\tilde N(0,s)ds\Big| \\
\le&C(\varepsilon_0M(t)+M(t)^2)\int_{t-1}^t(t-s)^{-\frac12}(1+s)^{-\frac12}ds \\
\le&C(1+t)^{-\frac12}(\varepsilon_0M(t)+M(t)^2), \qquad (6.78)
\end{aligned}
$$

where we use the fact that

$$
|\partial_s\tilde N(y,s)|\le C(1+s)^{-\frac12}(\varepsilon_0M(t)+M(t)^2),\quad |\partial_s\tilde N(0,s)|\le C(1+s)^{-\frac12}(\varepsilon_0M(t)+M(t)^2). \qquad (6.79)
$$

Combining it and (6.77) also leads to

$$\Big|\int_0^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s N(y,s)dyds\Big| \le C(1+t)^{-\frac{1}{2}}(\varepsilon_0 M(t)+M(t)^2). \tag{6.80}$$

Via (6.9), the second term in (6.73) is calculated as

$$\int_0^t\int_0^\infty \mathbb{G}^s(x,t-s;y)\partial_s N(y,s)dyds=(0,0,0)^t. \tag{6.81}$$

As a result of (6.80) and (6.81), one has

$$|U_t(x,t)|\le C(1+t)^{-\frac{1}{2}}(\varepsilon_0+\varepsilon_0 M(t)+M(t)^2). \tag{6.82}$$

Now we are in a position to deal with w_{xx} and ϕ_{xx}. Making use of (4.30) and Proposition 4.1 yields

$$\begin{aligned}|w_{xx}(x,t)|&\le \frac{1}{\mu}(|w_t|+\sigma|n_x|+\eta|\phi_x|+|N_1|)\\ &\le C(1+t)^{-\frac{1}{2}}(\varepsilon_0+\varepsilon_0 M(t)+M(t)^2),\end{aligned} \tag{6.83}$$

and

$$\begin{aligned}|\phi_{xx}(x,t)|&\le \frac{1}{\nu}(|\phi_t|+\eta|w_x|+|N_2|)\\ &\le C(1+t)^{-\frac{1}{2}}(\varepsilon_0+\varepsilon_0 M(t)+M(t)^2).\end{aligned} \tag{6.84}$$

This completes the proof. □

6.1. The proof of Theorem 3.3.

Proof. Combining (6.4), Proposition 6.1 and Proposition 6.2 together proves

$$M(t)\le C\varepsilon_0+C\varepsilon_0 M(t)+CM(t)^2. \tag{6.85}$$

Since ε_0 is sufficiently small, we deduce that there exists a constant $C>0$ independent of time such that

$$M(t)\le C\varepsilon_0. \tag{6.86}$$

Therefore the pointwise estimates of the solution are written as

$$\begin{aligned}&|U(x,t)|\le C\varepsilon_0 F(x,t), \quad |U_x(x,t)|\le C\varepsilon_0(1+t)^{-1}\log(2+t),\\ &|w_{xx}(x,t)|\le C\varepsilon_0(1+t)^{-\frac{1}{2}}, \quad |\phi_{xx}(x,t)|\le C\varepsilon_0(1+t)^{-\frac{1}{2}}.\end{aligned} \tag{6.87}$$

This completes the proof of Theorem 3.3. □

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