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POINTWISE ESTIMATES OF THE SOLUTION TO ONE DIMENSIONAL COMPRESSIBLE NAIVER-STOKES EQUATIONS IN HALF SPACE

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Abstract. In this paper, we study the global existence and pointwise behavior of classical solution to one dimensional isentropic Navier-Stokes equations with mixed type boundary condition in half space. Based on classical energy method for half space problem, the global existence of classical solution is established firstly. Through analyzing the quantitative relationships of Green's function between Cauchy problem and initial boundary value problem, we observe that the leading part of Green's function for the initial boundary value problem is composed of three items: delta function, diffusive heat kernel, and reflected term from the boundary. Then applying Duhamel's principle yields the explicit expression of solution. With the help of accurate estimates for nonlinear wave coupling and the elliptic structure of velocity, the pointwise behavior of the solution is obtained under some appropriate assumptions on the initial data. Our results prove that the solution converges to the equilibrium state at the optimal decay rate $(1+t)^{-\frac{1}{2}}$ in L^∞ norm.

1. **Introduction.** The motion of a viscous, compressible and barotropic fluid is governed by Navier-Stokes (N-S) equations. In Lagrangian coordinate, it is stated as below

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = (\frac{\mu}{v} u_x)_x, \end{cases} \tag{1}$$

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where x is the position, time $t \geq 0$. Let positive constant μ be the viscous coefficient. $v(x,t) > 0$ denotes the specific volume and $u(x,t)$ represents velocity. The pressure p is assumed to be a smooth function of v satisfying

$$p(v) > 0, \quad p'(v) < 0.$$

In order to address more physically relevant situations, the following mixed type boundary for velocity is considered

$$(au_x + bu)\big|_{x=0} = 0, \tag{2}$$

with far field behavior

$$v(x,t) \to v^*, \quad u(x,t) \to 0, \quad \text{as} \quad x \to +\infty, \tag{3}$$

where a, b are constants satisfying $ab \leq 0$ and v^* represents a strictly positive constant. Without loss of generality, we assume $v_* = 1$ in the following of this paper. The initial data are given by

$$(v,u)\big|_{t=0} = (v_0, u_0). \tag{4}$$

There is a lot of literature on the global existence and large time behavior of solutions to compressible fluid models. Let us recall some known results about Cauchy problem to our knowledge. In the case of vacuum absence for initial density, the existence of a global classical solution of (1) has been investigated by Kanel [9]. Later, Kazhikhov made a contribution to the non-isentropic N-S system in [14]. It should be mentioned that the above results focus on large initial data. When small initial data are taken into consideration, Kawashima and Nishida [12] established the global solution to one dimensional non-isentropic N-S system. In relation to the multi-dimensional case, the global classical solution was initially studied by Mastumura and Nishida [23] in a perturbation framework. To reveal the decay property of the solution, one shall study long time behavior. Kawashima [11, 10] obtained the L^2 time decay rate of several general hyperbolic-parabolic systems with applications to related models. The optimal L^2 time decay rate for the three-dimensional full N-S system was investigated by Mastumura and Nishida [22] when the initial data is a small perturbation of constant state in $H^3(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$. They proved

$$\|(\rho - \rho_*, u, \theta - \theta_*)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}}, \tag{5}$$

where ρ_*, θ_* denote positive constants. Later, L^p time decay rate of the solution was studied by Ponce [24]. Their results show that for large time t

$$\|\partial_x^\alpha(\rho - \rho_*, u, \theta - \theta_*)\|_{L^p(\mathbb{R}^n)} \leq Ct^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}, \quad p \geq 2, \tag{6}$$

here $|\alpha| \leq 2$ and $n = 2, 3$. When the external force is taken into account, Li, Matsumura and Zhang [18] proved that for Navier-Stokes-Poisson (N-S-P) system, if the initial data is a small regular perturbation of a constant state in $H^4(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, the perturbed solutions satisfy the following decay rate

$$\|D_x^k(\rho - \rho^*)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}, \quad \|D_x^k(m, \nabla\Psi)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{1}{4}-\frac{k}{2}},$$

where ρ, m represent density and momentum respectively, and $\Psi(x,t)$ denotes electrostatic potential. Later, they continued their work on non-isentropic case, see [28] for details. The methods applied in previous work are based on the spectrum analysis of linearized equations and energy estimates for nonlinear system. As a result, they proved the global existence and established L^2 time decay rate of the solution to compressible fluid models.

It is noted that the time decay rates describe the parabolic properties of the hyperbolic-parabolic system, but it provides less information to the effect of hyperbolic properties. To shed light on revealing the wave propagation related to hyperbolic properties, Zeng [27] has studied L^1 asymptotic behavior for system (1). Their results show that the solution behaves like a heat kernel with diffusion wave. Furthermore, the pointwise behavior of general quasi-linear hyperbolic-parabolic systems of conservation laws was investigated by Liu and Zeng in [20]. To gain a better understanding of the wave propagation for compressible fluids in multi-dimension, Hoff and Zumbrun [5, 6] analyzed the Green's function of an artificial viscosity system related to linearized isentropic N-S equations and obtained the space-time structure of the solution. Later, Liu and Wang [19] directly studied Green's function for isentropic N-S equations. Then applying Duhamel's principle and complex analysis method yields the explicit wave structure of the solution, which exhibits the generalized Huygens' principle in odd dimensions. Afterward, David Li [17] extended their work to the full system on the linear level, where additional new waves are introduced. When the external force is involved, the time-asymptotic shape of the solution for N-S-P system was studied by Wang and Wu in [25], which reveals that the rotating effect of the electric field makes the Huygens' principle observed in [19] for compressible N-S equations invalid here. Later, Wu and Wang [26] generalized their results to the bipolar compressible N-S-P system, which also exhibits a generalized Huygens' principle as the Navier-Stokes equations. This interesting phenomenon is the principal difference from the unipolar Navier-Stokes-Poisson system. We conclude from the above results that the detailed analysis of Green's function to the linear system captures the effect of hyperbolic properties of these compressible flows.

Once the boundary is present, there are also both rich phenomena and significant mathematical challenges, such as slip boundary layer, thermal creep flow, and curvature effects. Let us review some known results about the initial boundary value problem (IBVP) to our knowledge. The global existence and uniqueness of classical solution for one dimensional full N-S system with boundary were established by Kazhikhov and Shelukhin [15] for large initial data. Matsumura and Nishida [21] proved the global existence of full system in three-dimensional half space with small initial data. Later, Kagei and Kobayashi [7, 8] proved the convergence of solutions to the equilibrium state for N-S equations with Dirichlet boundary condition when space dimension $n \geq 2$. In contrast, very few works have been carried out for large time behavior in pointwise sense for IBVP to compressible fluid. The pointwise estimate for p system with damping in half space was studied by Deng [1]. Under some suitable assumptions on the initial data, their results indicate that the solutions satisfy

$$|(\sigma - 1, u)(x,t)| \leq C\Big(\frac{e^{-\frac{x^2}{C(1+t)}}}{\sqrt{1+t}} + e^{-\frac{|x|+t}{C}}\Big), \tag{7}$$

here $\sigma(x,t), u(x,t)$ represent specific volume and velocity. Furthermore, Deng and Wang in [2] studied the Euler equations with damping in 3-D for half space. Then the asymptotic behavior of solutions to the N-S with damping in $\mathbb{R}^n_+$ was investigated around a given constant equilibrium by Du [3]. Afterwards, Du and Wang [4] studied the one-dimensional isentropic N-S equations in Euler coordinate. They showed

$$|\partial_x^\alpha(\rho - 1, m)(x,t)| \leq C(1+t)^{-\alpha/4}\big[(x - c(t+1))^2 + (t+1)\big]^{-1/2},\ |\alpha| \leq 1, \tag{8}$$

where $\rho(x,t), m(x,t)$ denote density and momentum respectively. It then follows from the above inequality that the time decay rate for spatial derivatives is $(1+t)^{-\frac{3}{4}}$ in L^∞ norm. There still leaves room for improvement. Meanwhile, their method cannot be applied to the estimate of high order derivatives. When the equilibrium state is non-constant, Kawashima and Zhu [13] studied the stability of nonlinear waves for the outflow problem of the compressible N-S equations in half line. Recently, Koike [16] obtained the long-time behavior of a mass point moving in viscous fluid based on Green's function and energy method. From the above discussion, it still remains challenges to consider space-time pointwise estimates for N-S equations to the initial boundary value problem in multi-dimension.

This paper is devoted to studying the global existence and large time behavior of solutions for one dimensional compressible N-S system to IBVP in the sense of pointwise. Motivated by the work of Koike in [16], we anticipate improving the L^∞ decay rate of solutions for the derivatives and deducing the estimates of high order derivatives by introducing some new ideas.

In what follows, we make a brief explanation of the main steps of the proof. Based on classical energy method introduced in [21] for the initial boundary value problem, the global existence of a classical solution is studied firstly under some regular assumption on the initial data. After this step, to derive the pointwise estimate of the perturbed solution $U(x,t) = (\varphi, u)^t \triangleq (v-1, u)^t$, we apply Duhamel's principle to yield

$$U(x,t) = \int_0^\infty \mathbb{G}(x,t;y)U_0(y)dy + \int_0^t \int_0^\infty \mathbb{G}(x,t-s;y)F(y,s)dyds. \tag{9}$$

Here $\mathbb{G}(x,t;y)$ denotes the Green's function for the initial boundary value problem of system (1) and F represents the nonlinear term defined by (114). Thus it is crucial to capture the estimate of $\mathbb{G}(x,t;y)$. Fortunately, since the Green's function of Cauchy problem named $G(x,t)$ to system (1) has been studied by Zeng in [27]. The remaining goal is to build up a connection between Green's functions for the Cauchy problem and for the initial boundary value problem. Making use of Laplace transform and Fourier transform, together with boundedness requirement at infinity, we obtain the representation of Green's function

$$\mathbb{G}(x,t;y) = G(x-y,t) - \big(G(x+y,t) + h(x+y,t)\big)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{10}$$

where the function $h(x,t)$ satisfies a ODE that can be solved as (109). We are now in a position to deal with nonlinear system. Assume the solution satisfies the following ansatz

$$\Lambda(t) = \sup_{0\le s\le t}\Big\{\|U\Phi^{-1}\|_{L_x^\infty} + (1+s)\big(\log(2+s)\big)^{-1}\|U_x\|_{L_x^\infty} + (1+s)^{\frac{1}{2}}\|u_{xx}\|_{L_x^\infty}\Big\},$$

where $\Phi(x,t)$ represents the diffusive wave decaying at the L^∞-rate $(1+t)^{-\frac{1}{2}}$ or L^2-rate $(1+t)^{-\frac{1}{4}}$. According to the ansatz and some computational lemmas for wave coupling, the pointwise structure of $U(x,t)$ and $U_x(x,t)$ are investigated directly by using the expression of solution. However, it seems that differentiating (9) with respect to x twice to derive the estimate of second spatial derivative may cause the blow up of solution. To overcome this difficulty, we mainly rely on the elliptic system of velocity. Actually, it is observed that

$$u_{xx} = \frac{1+\varphi}{\mu}u_t + \frac{1+\varphi}{\mu}p(1+\varphi)_x + \frac{u_x\varphi_x}{1+\varphi}. \tag{11}$$

This inspires us to deduce the estimate of u_t firstly. It is easy to see that the solution $U_t = (\varphi_t, u_t)^t$ satisfies the system (195). After a careful observation, we find that the Green's function for this system coincides with $\mathbb{G}(x,t;y)$. Consequently, making use of Duhamel's principle yields

$$U_t(x,t) = \int_0^\infty \mathbb{G}(x,t;y)\partial_t U(y,0)dy + \int_0^t \int_0^\infty \mathbb{G}(x,t-s;y)\partial_s F(y,s)dyds. \quad (12)$$

The estimate of $U_t(x,t)$ is treated in a similar manner as $U(x,t)$ by using (12). It is noteworthy that after we transform the derivative of the nonlinear term to Green's function, the remaining nonlinear terms contain a high order term u_{sy}, seeing (209), which can be dominated by the result of the energy estimate. Thus, the pointwise behavior of $U_t(x,t)$ is derived immediately. Then applying (11) and the estimate of U_x, U_t gives the space-time estimate of u_{xx}. Combing the definition of ansatz and the smallness of δ_0 gives rise to $\Lambda(t) \le C\delta_0$. As a result, we obtain the space-time estimates of solutions. Compared with [4], the L^∞ time-decay rate of the derivative $U_x(x,t)$ is improved from $(1+t)^{-\frac{3}{4}}$ to $(1+t)^{-1}\log(2+t)$, which gives a positive answer to Remark 1.3 in [4]. It should be mentioned that the poinwise behavior of higher order spatial derivatives can also be obtained in this way if we improve higher regularity of the initial data.

The rest of the paper is organized as follows. In section 2, some notations and auxiliary lemmas are introduced for later use. In section 3, we present the main results of this paper. The global existence of classical solutions is investigated in section 4. In section 5, we construct the Green's function for the initial-boundary value problem. The nonlinear pointwise estimates of the solutions are obtained in section 6.

2. **Preliminaries.**

In this section, we will introduce some notations and auxiliary lemmas, which will be used throughout this paper. Write I_n to represent $n \times n$ identity matrix ($n \ge 1$). When $n = 0$, we denote

$$I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (13)$$

In the section 5, we will use the notations as follows

$$A = \begin{pmatrix} 0 & -1 \\ -c^2 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}. \quad (14)$$

The Fourier transform of f is defined as $\hat{f}$ or $\mathfrak{F}[f]$

$$\hat{f}(\xi) = \mathfrak{F}[f] = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx \quad (\xi \in \mathbb{R}).$$

The inverse Fourier transform of f is given by $\mathfrak{F}^{-1}[f]$

$$\mathfrak{F}^{-1}[f](x) = (2\pi)^{-1}\int_{\mathbb{R}} f(\xi)e^{i\xi x}d\xi \quad (x \in \mathbb{R}).$$

The Laplace transform of f is written as $\mathfrak{L}[f]$

$$\mathfrak{L}[f](s) = \int_0^\infty f(t)e^{-st}dt \quad (t \in \mathbb{R}_+).$$

Two functions are introduced as below to state the main results ($i = 1, 2$)

$$\phi_\theta(x,t;\lambda_i) = \big(1+t+|x-\lambda_i(1+t)|\big)^{-\theta/2}, \quad \psi_\theta(x,t;\lambda_i) = \big(1+t+(x-\lambda_i(1+t))^2\big)^{-\theta/2},$$

where $\lambda_1 = c, \lambda_2 = -c$. It should be mentioned that since $x > 0, c > 0$, which implies

$$\phi_\theta(x,t;-c) \le \phi_\theta(x,t;c), \quad \psi_\theta(x,t;-c) \le \psi_\theta(x,t;c). \tag{15}$$

For notational simplicity, we define $\Phi(x,t)$ as

$$\Phi(x,t) = (1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c) + (1+t)^{\frac{3}{8}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c). \tag{16}$$

Through this paper, C denotes a generic positive constant which may vary in different estimates. $f_1 \lesssim f_2$ and $f_1 = \mathcal{O}(1)f_2$ mean that there exists a constant $C > 0$ such that $f_1 \le Cf_2$ for positive functions f_1, f_2. Denote by L^p and $W^{m,p}$ the usual Lebesgue and Sobolev spaces on $\mathbb{R}_+$ and $H^m = W^{m,2}$, with norms $\|\cdot\|_{L^p}, \|\cdot\|_{W^{m,p}}, \|\cdot\|_{H^m}$, respectively.

We next introduce some computational lemmas for wave coupling that will be used in the proof of the main theorem.

Lemma 2.1. *There holds for $k \ge 0$, $C_1 > 0$ and $x \in \mathbb{R}_+$, $r \ge \frac{5}{8}$ such that*

$$\begin{aligned} e^{-\frac{x+t}{C_1}} &\le C(1+t)^{-k}\psi_{\frac{3}{2}}(x,t;c), \\ e^{-\frac{(x-\lambda_i(1+t))^2}{C_1(1+t)}} &\le C(1+t)^{\frac{3}{4}}\psi_{\frac{3}{2}}(x,t;\lambda_i), \\ e^{-\frac{t}{C_1}}(1+x^2)^{-r} &\le C(1+t)^{-k}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c), \\ e^{-\frac{x}{C_1}}(1+t)^{-\frac{3}{2}} &\le C(1+t)^{-\frac{1}{4}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c). \end{aligned}$$

Proof. In order to prove the first inequality, a straightforward calculation gives

$$\begin{aligned} e^{-\frac{x+t}{C_1}} &\le e^{-\frac{t}{2C_1}}e^{-\frac{x+t}{2C_1}} \\ &\le C(1+t)^{-k}(1+t+(x+t+1)^2)^{-\frac{3}{4}}. \end{aligned} \tag{17}$$

Note that if $c \le 1$, then $(x+t+1)^2 \ge (x+c(t+1))^2$. For the case $c > 1$, we have

$$(x+t+1)^2 \ge \frac{1}{c^2}(x+c(t+1))^2, \tag{18}$$

which together with (15) also leads to

$$e^{-\frac{x+t}{C_1}} \le C(1+t)^{-k}\psi_{\frac{3}{2}}(x,t;-c) \le C(1+t)^{-k}\psi_{\frac{3}{2}}(x,t;c). \tag{19}$$

The second inequality is proved by using the fact $e^{-z} \le (1+z)^{-\frac{3}{4}}$ for $z > 0$. Concerning the third inequality, it holds

$$\begin{aligned} e^{-\frac{t}{C_1}}(1+x^2)^{-r} &\le e^{-\frac{t}{C_1}}(1+x)^{-\frac{1}{2}}(1+x^2)^{-\frac{3}{8}} \\ &\le C(1+t)^{-k}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c). \end{aligned} \tag{20}$$

One can similarly deduce the estimate for the last inequality. We hence complete the proof of lemma. □

Lemma 2.2. *The Green's function for system (1) to Cauchy problem satisfies the following estimate*

$$\begin{aligned} \Big|\partial_x^\alpha\Big(G(x,t) - e^{-\frac{c^2}{\mu}t}\delta(x)I_0\Big)\Big| &= \mathcal{O}(1)t^{-\frac{1}{2}-\frac{\alpha}{2}}(e^{-\frac{(x-ct)^2}{2\mu t}} + e^{-\frac{(x+ct)^2}{2\mu t}}) \\ &+ \mathcal{O}(1)(1+t)^{-\frac{1}{2}}t^{-\frac{1}{2}-\frac{\alpha}{2}}(e^{-\frac{(x-ct)^2}{Ct}} + e^{-\frac{(x+ct)^2}{Ct}}) + \mathcal{O}(1)e^{-\frac{|x|+t}{C}}, \end{aligned} \tag{21}$$

where $\delta(x)$ represents the Dirac function, the matrix I_0 is given by (13) and positive constant c denotes the sound wave speed.

Proof. Combing Lemma 5.5 and Lemma 5.6 proved by Liu and Zeng in [20], we can complete the proof of this lemma. See also [27]. □

Lemma 2.3. *Define function*

$$E(x,t;\lambda,D_0)=\int_0^\infty e^{-\omega z}e^{-\frac{(x+z-\lambda t)^2}{D_0 t}}dz. \tag{22}$$

Let $\omega>0,\lambda>0$. Then there exists a positive constant C such that for any given $\epsilon>0$

$$\frac{\partial^k}{\partial x^k}E(x,t;\lambda,D_0)\le \mathcal{O}(1)(t^{-\frac{k}{2}}e^{-\frac{(x-\lambda t)^2}{(D_0+\epsilon)t}}+e^{-\frac{|x|+t}{C}}). \tag{23}$$

Proof. The lemma has been proved by Du and Wang, seeing Lemma 2.1 [4]. □

Lemma 2.4. *For $r\ge\frac{5}{8}$, there exists a positive constant such that*

$$\begin{aligned}K_1&=\int_0^\infty(1+t)^{-1}e^{-\frac{(x-y-\lambda_i(1+t))^2}{2\mu(1+t)}}(1+y^2)^{-r}dy\le C\Phi(x,t),\\ K_2&=\int_0^\infty(1+t)^{-1}e^{-\frac{(x+y-c(1+t))^2}{(2\mu+\epsilon)(1+t)}}(1+y^2)^{-r}dy\le C\Phi(x,t),\end{aligned} \tag{24}$$

where $\lambda_1=c,\lambda_2=-c$ and the definition of $\Phi(x,t)$ is given by (16).

Proof. In order to estimate K_1, three cases are taken into consideration.

(1) $|x-\lambda_i(1+t)|<\sqrt{1+t}$, then we have

$$K_1\le C(1+t)^{-1}\le C(1+t)^{-\frac14}\psi_{\frac32}(x,t;\lambda_i)\le C\Phi(x,t). \tag{25}$$

(2) $\sqrt{1+t}\le|x-\lambda_i(1+t)|\le 1+t$, it holds

$$\begin{aligned}K_1&\le C(1+t)^{-1}\int_{|y|\le\frac{|x-\lambda_i(1+t)|}{2}}e^{-\frac{(x-\lambda_i(1+t))^2}{8\mu(1+t)}}(1+y^2)^{-r}dy\\ &\quad+C(1+t)^{-1}\int_{|y|>\frac{|x-\lambda_i(1+t)|}{2}}e^{-\frac{(x-y-\lambda_i(1+t))^2}{2\mu(1+t)}}(1+y^2)^{-r}dy\\ &\le C(1+t)^{-1}e^{-\frac{(x-\lambda_i(1+t))^2}{8\mu(1+t)}}+C(1+t)^{-\frac12}(1+|x-\lambda_i(1+t)|^2)^{-r}\\ &\le C\Phi(x,t).\end{aligned} \tag{26}$$

(3) $|x-\lambda_i(1+t)|>1+t$, we are able to verify

$$\begin{aligned}K_1&\le C(1+t)^{-1}\int_{|y|\le\frac{|x-\lambda_i(1+t)|}{2}}e^{-\frac{(x-\lambda_i(1+t))^2}{8\mu(1+t)}}(1+y^2)^{-r}dy\\ &\quad+C(1+t)^{-1}\int_{|y|>\frac{|x-\lambda_i(1+t)|}{2}}e^{-\frac{(x-y-\lambda_i(1+t))^2}{2\mu(1+t)}}(1+y^2)^{-r}dy\\ &\le C(1+t)^{-1}e^{-\frac{(x-\lambda_i(1+t))^2}{8\mu(1+t)}}+C(1+t)^{-\frac12}(1+|x-\lambda_i(1+t)|)^{-\frac54}\\ &\le C\Phi(x,t),\end{aligned}$$

where we have used the fact that when $|x-\lambda_i(1+t)|>1+t$, it satisfies

$$(1+|x-\lambda_i(1+t)|)^{-\frac54}\le C\phi_1(x,t;c)\psi_{\frac34}(x,t;c).$$

To conclude, we finally obtain the desired estimate of $(24)_1$. As for K_2, it can be treated in a same manner as K_1. Hence, this completes the proof. □

To gain a better understanding of interactions between different waves, we provide some lemmas as follows. The proof can be followed in a similar argument as Lemma 3.5, 3.6, 3.7, 3.8 in [20] with slight modifications that replacing $x \in \mathbb{R}$ into $x \in \mathbb{R}_+$. Here we omit the details of proof.

Lemma 2.5. *Let $\alpha \geq 0, \beta \geq 0, \bar{\mu} > 0$ and λ be constants. Then for all $x \in \mathbb{R}_+$, $t \geq 0$, we have*

$$\int_0^t \int_{-\infty}^{\infty} (t-s)^{-1}(1+t-s)^{-\frac{\alpha}{2}} e^{-\frac{(x-y-\lambda(t-s))^2}{\bar{\mu}(t-s)}} (1+s)^{-\frac{\beta}{2}} \psi_{\frac{3}{2}}(y,s;\lambda) dyds$$
$$= \begin{cases} \mathcal{O}(1)(1+t)^{-\frac{\gamma}{2}} \psi_{\frac{3}{2}}(x,t;\lambda) \log(2+t), & \text{if } \alpha = 1 \text{ or } \beta = \frac{3}{2} \\ \mathcal{O}(1)(1+t)^{-\frac{\gamma}{2}} \psi_{\frac{3}{2}}(x,t;\lambda), & \text{otherwise,} \end{cases}$$

where $\gamma = \min(\alpha,1) + \min(\beta, \frac{3}{2}) - 1$.

Lemma 2.6. *Let the constants $\alpha \geq 0, \beta \geq 0, \bar{\mu} > 0$ and $\lambda \neq \lambda'$. Then for any fixed $\mathbb{K} > 2|\lambda - \lambda'|$ and all $x \in \mathbb{R}_+, t \geq 0$, we have*

$$\int_0^t \int_{-\infty}^{\infty} (t-s)^{-1}(1+t-s)^{-\frac{\alpha}{2}} e^{-\frac{(x-y-\lambda(t-s))^2}{\bar{\mu}(t-s)}} (1+s)^{-\frac{\beta}{2}} \psi_{\frac{3}{2}}(y,s;\lambda') dyds$$
$$= \mathcal{O}(1)(1+t)^{-\frac{\gamma}{2}} [\psi_{\frac{3}{2}}(x,t;\lambda) + \psi_{\frac{3}{2}}(x,t;\lambda')]$$
$$+ \mathcal{O}(1)|x-\lambda(1+t)|^{-\frac{1}{2}\min(\beta,\frac{5}{2})-\frac{1}{4}} |x-\lambda'(1+t)|^{-\frac{1}{2}\min(\alpha,1)-\frac{1}{2}}$$
$$\cdot \operatorname{char}\left\{ \min(\lambda,\lambda')(1+t) + \mathbb{K}\sqrt{1+t} \leq x \leq \max(\lambda,\lambda')(1+t) - \mathbb{K}\sqrt{1+t} \right\}$$
$$+ \begin{cases} \mathcal{O}(1)(1+t)^{-\frac{\gamma}{2}} \log(1+t) [\psi_{\frac{3}{2}}(x,t;\lambda) + \psi_{\frac{3}{2}}(x,t;\lambda')], & \text{if } \alpha = 1 \\ \mathcal{O}(1)(1+t)^{-\frac{\gamma}{2}} \log(1+t) \psi_{\frac{3}{2}}(x,t;\lambda), & \text{if } \alpha \neq 1 \text{ and } \beta = \frac{3}{2} \\ 0, & \text{otherwise,} \end{cases}$$

where $\gamma = \min(\alpha,1) + \min(\beta,\frac{3}{2}) - 1$ and char is the characteristic function as usual.

Lemma 2.7. *Let $\alpha \geq 0$, $0 \leq \beta \leq 2$, $\bar{\mu} > 0$, and λ be constants. Then for $x \in \mathbb{R}_+$, $t \geq 0$, we have*

$$\int_0^t \int_{-\infty}^{\infty} (t-s)^{-1}(1+t-s)^{-\frac{\alpha}{2}} e^{-\frac{(x-y-\lambda(t-s))^2}{\bar{\mu}(t-s)}} (1+s)^{-\frac{\beta}{2}} \phi_1(y,s;\lambda) \psi_{\frac{3}{4}}(y,s;\lambda) dyds$$
$$= \begin{cases} \mathcal{O}(1)(1+t)^{-\frac{\gamma_1}{2}} \log(2+t) \phi_1(x,t;\lambda) \psi_{\frac{3}{4}}(x,t;\lambda), & \text{if } \alpha = 1, \text{ or } 1 \leq \beta \leq \frac{3}{2} \\ \mathcal{O}(1)(1+t)^{-\frac{\gamma_1}{2}} \phi_1(x,t;\lambda) \psi_{\frac{3}{4}}(x,t;\lambda), & \text{otherwise} \end{cases}$$
$$+ \begin{cases} \mathcal{O}(1)(1+t)^{-\frac{\gamma_2}{2}} \log(2+t) \psi_{\frac{3}{2}}(x,t;\lambda), & \text{if } \alpha = 1, \text{ or } \beta = \frac{3}{2} \\ \mathcal{O}(1)(1+t)^{-\frac{\gamma_2}{2}} \psi_{\frac{3}{2}}(x,t;\lambda), & \text{otherwise,} \end{cases}$$

where $\gamma_1 = \min(\alpha,1) + \frac{1}{2}(\min(\beta,1) + \min(\beta,\frac{3}{2})) - 1$, $\gamma_2 = \min(\alpha,1) + \min(\beta,\frac{3}{2}) - 1$.

Lemma 2.8. *Let the constants $\alpha \geq 0, 0 \leq \beta \leq 2, \bar{\mu} > 0$ and $\lambda \neq \lambda'$. Then for any fixed $\mathbb{K} > 2|\lambda - \lambda'|$ and all $x \in \mathbb{R}_+, t \geq 0$, we have*

$$\int_0^t \int_{-\infty}^{\infty} (t-s)^{-1}(1+t-s)^{-\frac{\alpha}{2}} e^{-\frac{(x-y-\lambda(t-s))^2}{\bar{\mu}(t-s)}} (1+s)^{-\frac{\beta}{2}} \phi_1(y,s;\lambda')\psi_{\frac{3}{4}}(y,s;\lambda')dyds$$
$$= \mathcal{O}(1)(1+t)^{-\frac{\gamma_1}{2}}$$
$$\cdot \begin{cases} \phi_1(x,t;\lambda)\psi_{\frac{3}{4}}(x,t;\lambda) + \phi_1(x,t;\lambda')\psi_{\frac{3}{4}}(x,t;\lambda'), & if\ \alpha \neq 1, \beta \notin [1,\frac{3}{2}] \\ \log(2+t)\phi_1(x,t;\lambda)\psi_{\frac{3}{4}}(x,t;\lambda) + \phi_1(x,t;\lambda')\psi_{\frac{3}{4}}(x,t;\lambda'), & if\ \alpha \neq 1, \beta \in [1,\frac{3}{2}] \\ \log(2+t)[\phi_1(x,t;\lambda)\psi_{\frac{3}{4}}(x,t;\lambda) + \phi_1(x,t;\lambda')\psi_{\frac{3}{4}}(x,t;\lambda')], & if\ \alpha = 1 \end{cases}$$
$$+ \mathcal{O}(1)(1+t)^{-\frac{\gamma_2}{2}} \begin{cases} \psi_{\frac{3}{2}}(x,t;\lambda) + \psi_{\frac{3}{2}}(x,t;\lambda'), & if\ \alpha \neq 1, \beta \neq \frac{3}{2} \\ \log(2+t)\psi_{\frac{3}{2}}(x,t;\lambda) + \psi_{\frac{3}{2}}(x,t;\lambda'), & if\ \alpha \neq 1, \beta = \frac{3}{2} \\ \log(2+t)[\psi_{\frac{3}{2}}(x,t;\lambda) + \psi_{\frac{3}{2}}(x,t;\lambda')], & if\ \alpha = 1 \end{cases}$$
$$+ \mathcal{O}(1)|x-\lambda(1+t)|^{-\frac{1}{2}\min(\beta,2)-\frac{1}{4}+\epsilon}|x-\lambda'(1+t)|^{-\frac{1}{2}\min(\alpha,1)-\frac{1}{2}}$$
$$\cdot \mathrm{char}\big\{\min(\lambda,\lambda')(1+t) + \mathbb{K}\sqrt{1+t} \leq x \leq \max(\lambda,\lambda')(1+t) - \mathbb{K}\sqrt{1+t}\big\},$$

where $\gamma_1 = \min(\alpha,1) + \frac{1}{2}(\min(\beta,1)+\min(\beta,\frac{3}{2})) - 1$, $\gamma_2 = \min(\alpha,1)+\min(\beta,\frac{3}{2}) - 1$ *and* $\epsilon > 0$ *arbitrarily small.*

3. **Main results.** In this paper, we consider the small perturbation of solutions (v,u) near a constant state $(1,0)$. Define a new variable $\varphi = v-1$. Then system (1) is reformulated to

$$\begin{cases} \varphi_t - u_x = 0, \\ u_t + p(1+\varphi)_x = (\dfrac{\mu u_x}{1+\varphi})_x. \end{cases} \tag{27}$$

The boundary condition becomes

$$(au_x + bu)|_{x=0} = 0, \quad \text{and} \quad (\varphi,u) \to (0,0) \quad \text{as} \quad x \to +\infty. \tag{28}$$

The initial data are given by

$$(\varphi,u)|_{t=0} = (\varphi_0,u_0) \triangleq (v_0-1,u_0). \tag{29}$$

To state the main theorem of global existence, we introduce the energy $\mathcal{E}(t)$ and dissipation $\mathcal{D}(t)$ as

$$\begin{aligned} \mathcal{E}(t) &= \|(\varphi,u)\|_{H^4(\mathbb{R}_+)} + \sum_{k=0}^{2}|\partial_t^k\varphi(0,t)|, \\ \mathcal{D}(t) &= \|\varphi_x\|_{H^3(\mathbb{R}_+)} + \|u_x\|_{H^4(\mathbb{R}_+)} + \sum_{k=0}^{2}|\partial_t^k u(0,t)|. \end{aligned} \tag{30}$$

Since the classical solution is taken into consideration in this paper, the following compatible conditions are needed

$$(au_0' + bu_0)|_{x=0} = 0. \tag{31}$$

Theorem 3.1. *(Global existence) Assume the initial data* $(\varphi_0,u_0) \in H^4(\mathbb{R}_+)$ *satisfying compatibility condition* (31). *There exists a small positive constant* δ_0 *such that if*

$$\|(\varphi_0,u_0)\|_{H^4(\mathbb{R}_+)} \leq \delta_0, \tag{32}$$

then the initial boundary value problem (27) $-$ (29) *admits a unique global classical solution when* $ab < 0$ *or* $a = 0$*, or* $b = 0$

$$\begin{aligned}
&\varphi \in C([0,\infty); H^4(\mathbb{R}_+)) \cap C^1([0,\infty); H^3(\mathbb{R}_+)),\\
&u \in C([0,\infty); H^4(\mathbb{R}_+)) \cap C^1([0,\infty); H^2(\mathbb{R}_+)),\\
&\varphi_x \in L^2([0,\infty); H^3(\mathbb{R}_+)), u_x \in L^2([0,\infty); H^4(\mathbb{R}_+)),
\end{aligned} \tag{33}$$

and satisfies for any given time $T > 0$ *that*

$$\sup_{0\le t\le T} \mathcal{E}(t)^2 + \int_0^T \mathcal{D}(t)^2 dt \le C\delta_0^2, \tag{34}$$

where C *is a positive constant independent of time.*

Remark 1. It should be emphasized that the signs of parameters in the boundary condition plays an important role in studying the global existence. When $ab < 0$ or $a = 0$ or $b = 0$, the solution of this problem could be stable under some suitable assumption on the initial data. If $ab > 0$, it should be unstable even in the linear level.

Remark 2. We note that when $a = 0$ or $b = 0$, it reduces to Dirichlet boundary condition or Neumann boundary condition. The existence of these cases can be proved in a same manner as $ab < 0$. For simplicity, we only consider $ab < 0$ in the proof of this theorem.

The linear system of initial boundary value problem (27) $-$ (29) can be written as

$$\begin{cases}
\varphi_t - u_x = 0,\\
u_t - c^2\varphi_x = \mu u_{xx},\\
(au_x + bu)|_{x=0} = 0,\\
(\varphi, u)|_{x\to+\infty} = (0,0),\\
(\varphi, u)|_{t=0} = (\varphi_0, u_0),
\end{cases} \tag{35}$$

here $c = \sqrt{-p'(1)} > 0$. The Green's function $\mathbb{G}(x,t;y)$ of (35) satisfies

$$\begin{cases}
(\partial_t + A\partial_x - B\partial_{xx})\mathbb{G}(x,t;y) = 0, \ x > 0, y > 0, t > 0,\\
\mathbb{G}(x,0;y) = \delta(x-y)I_2,\\
(a\partial_t \quad b)\mathbb{G}(0,t;y) = 0,
\end{cases} \tag{36}$$

where the matrix A, B is defined in (14).

The second theorem captures the explicit poinwise structure of Green's function $\mathbb{G}(x,t;y)$.

Theorem 3.2. *For Green's function* $\mathbb{G}(x,t;y)$*, we have the following estimate for* $0 \le x, y < \infty, t \ge 0$

$$\begin{aligned}
&\left|\partial_x^\alpha\Big(\mathbb{G}(x,t;y) - e^{-\frac{c^2}{\mu}t}(\delta(x-y) - \delta(x+y))I_0\Big)\right|\\
&= \mathcal{O}(1)t^{-\frac{1}{2}-\frac{\alpha}{2}}(e^{-\frac{(x-y-ct)^2}{2\mu t}} + e^{-\frac{(x-y+ct)^2}{2\mu t}} + e^{-\frac{(x+y-ct)^2}{(2\mu+\epsilon)t}})\\
&\quad + \mathcal{O}(1)(e^{-\frac{|x-y|+t}{C}} + e^{-\frac{|x+y|+t}{C}}),
\end{aligned} \tag{37}$$

where $\delta(x)$ *represents the Dirac function, the matrix* I_0 *is given by* (13)*. The positive constant* ϵ *is arbitrarily small.*

Remark 3. The theorem tells us that the leading part of Green's function to IBVP is the heat kernel with convection, which propagates with different characteristic speeds. Compared with the Green's function of Cauchy problem, the reflected term is appeared due to the presence of boundary.

Based on the above theorems, nonlinear pointwise estimates of the solution are obtained as follows.

Theorem 3.3. *Under the assumptions in Theorem 3.1 and define $U_0 = (\varphi_0, u_0)$ satisfying*

$$|U_0(x)| \le C\delta_0(1+x^2)^{-r},\ |U_0'(x)| \le C\delta_0(1+x^2)^{-r},\ \Big|\int_x^\infty U_0(y)dy\Big| \le C\delta_0(1+x^2)^{-r}, \tag{38}$$

where $r \ge \frac{5}{8}$. Then the solution has the following pointwise estimates

$$|(\varphi, u)| \le C\delta_0\Phi(x,t),\ |(\varphi_x, u_x)| \le C\delta_0(1+t)^{-1}\log(2+t),\ |u_{xx}| \le C\delta_0(1+t)^{-\frac{1}{2}},$$

where $\Phi(x,t)$ is given by (16).

Corollary 1. *Applying the assumptions in Theorem 3.3 and the definition of $\Phi(x,t)$, we have the following optimal decay rate of the solution*

$$\|(\varphi, u)(\cdot, t)\|_{L^p(\mathbb{R}_+)} \le C\delta_0(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}, \quad p \in [1, +\infty]. \tag{39}$$

Remark 4. When the non-isentropic N-S equations are taken into consideration, we need to deal with the trouble induced by the liner diffusive wave. For such a complicated problem, we will investigate it in the coming paper.

4. The global existence of classical solution.

First, the local existence of the solution is established as below.

Theorem 4.1. *(Local existence) Assume the initial data $(\varphi_0, u_0) \in H^4(\mathbb{R}_+)$ satisfying compatibility condition (31). Then the initial boundary value problem (27)−(29) admits a unique local classical solution (φ, u) satisfying*

$$\begin{aligned}
&\varphi \in C([0,T]; H^4(\mathbb{R}_+)) \cap C^1([0,T]; H^3(\mathbb{R}_+)),\\
&u \in C([0,T]; H^4(\mathbb{R}_+)) \cap C^1([0,T]; H^2(\mathbb{R}_+)),\\
&\varphi_x \in L^2([0,T]; H^3(\mathbb{R}_+)), u_x \in L^2([0,T]; H^4(\mathbb{R}_+)),
\end{aligned} \tag{40}$$

and satisfies for some given time $T > 0$ that

$$\sup_{0\le t\le T} \mathcal{E}(t)^2 + \int_0^T \mathcal{D}(t)^2 dt \le C\mathcal{E}(0)^2, \tag{41}$$

where C is a positive constant independent of time.

Proof. The construction of local-in-time solution is based on an iteration scheme as in [21]. Here we omit the details. □

To extend the short time classical solution to be a global one, we shall establish the uniform estimates. Hence, it is natural to provide the a-priori assumption for any given $T > 0$

$$\sup_{0\le t\le T} \mathcal{E}(t) \le \delta, \tag{42}$$

here $\delta > 0$ is a sufficiently small constant. We first establish the basic L^2 energy estimate as follows.

Proposition 1. *Assume* (φ, u) *is the classical solution of* (27) *satisfying the assumptions in Theorem* 3.1 *and the a-priori assumption* (42). *Then we have the following estimate for any given* $T > 0$

$$\sup_{0\le t\le T}\left(\|(\varphi,u)\|_{L^2}^2+|\varphi(0,t)|^2\right)+\int_0^T(|u(0,t)|^2+\|u_x\|_{L^2}^2)dt\le C\delta_0^2.$$

Proof. Multiplying $(27)_1$ and $(27)_2$ by $-p(1+\varphi)+p(1)$ and u respectively, then integrating the resulting equation with respect to x in $\mathbb{R}_+$ gives

$$\begin{aligned}&\frac{d}{dt}\Big(\int_{\mathbb{R}_+}\Gamma(\varphi)dx+\frac{1}{2}\|u\|_{L^2}^2\Big)+\big((p(1+\varphi)-p(1))u\big)\Big|_{x=0}^{x=+\infty}\\&+\frac{\mu\omega|u(0,t)|^2}{1+\varphi(0,t)}+\int_{\mathbb{R}_+}\frac{\mu u_x^2}{1+\varphi}dx=0,\end{aligned}\tag{43}$$

where $c=\sqrt{-p'(1)}>0$ and $\omega=-\frac{b}{a}>0$. The function $\Gamma(\varphi)$ is given by

$$\Gamma(\varphi)=-\int_0^{\varphi}(p(1+y)-p(1))dy.$$

A direct computation gives rise to

$$p(1+\varphi)-p(1)=-c^2\varphi+\mathcal{O}(\varphi^2)\ \text{as}\ \varphi\to 0.\tag{44}$$

Utilizing the a-priori assumption (42) and the smallness of the initial data, we capture the following approximation in the next section

$$p(1+\varphi)-p(1)\sim -c^2\varphi.\tag{45}$$

Furthermore, it also holds

$$\Gamma(\varphi)\sim\frac{1}{2}c^2\varphi^2.\tag{46}$$

Therefore, according to the boundary condition, (43) can be written as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(c^2\|\varphi\|_{L^2}^2+\|u\|_{L^2}^2+\frac{c^2}{\omega}|\varphi(0,t)|^2\Big)\\&+\frac{\mu\omega|u(0,t)|^2}{1+\varphi(0,t)}+\int_{\mathbb{R}_+}\frac{\mu u_x^2}{1+\varphi}dx=0.\end{aligned}\tag{47}$$

Integration of (47) in $t\in[0,T]$ gives the desired estimate. This completes the proof. ☐

Next we proceed to establish L^2 energy inequality for the time derivatives of solution.

Proposition 2. *Assume* (φ, u) *is the classical solution of* (27) *satisfying the assumptions in Theorem* 3.1 *and the a-priori assumption* (42). *Then we obtain the estimate for any given* $T > 0$ *satisfying*

$$\begin{aligned}&\sum_{k=1}^{2}\Big(\sup_{0\le t\le T}(\|\partial_t^k(\varphi,u)\|_{L^2}^2+|\partial_t^k\varphi(0,t)|^2)+\int_0^T(|\partial_t^ku(0,t)|^2+\|\partial_t^ku_x\|_{L^2}^2)dt\Big)\\&\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}$$

Proof. We begin with the case of $k = 1$. Differentiate the system (27) with respect to t, then multiply the resulting equations by $-p(1+\varphi)_t$ and u_t respectively. Adding and integrating these equations yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_t\|_{L^2}^2 - \int_{\mathbb{R}_+} p(1+\varphi)_t\varphi_{tt}dx + \int_{\mathbb{R}_+} p(1+\varphi)_t u_{tx}dx \\ &+ \int_{\mathbb{R}_+} p(1+\varphi)_{tx}u_t dx = \int_{\mathbb{R}_+} (\frac{\mu u_x}{1+\varphi})_{tx}u_t dx.\end{aligned} \tag{48}$$

It is easy to verify the following estimate from (45)

$$-\int_{\mathbb{R}_+} p(1+\varphi)_t\varphi_{tt}dx \sim \frac{c^2}{2}\frac{d}{dt}\|\varphi_t\|_{L^2}^2. \tag{49}$$

Making use of integration by parts also gives

$$\int_{\mathbb{R}_+} p(1+\varphi)_t u_{tx}dx + \int_{\mathbb{R}_+} p(1+\varphi)_{tx}u_t dx \sim \frac{c^2}{2\omega}\frac{d}{dt}|\varphi_t(0,t)|^2. \tag{50}$$

Similarly, the last terms in (48) can be addressed as below

$$\begin{aligned}\int_{\mathbb{R}_+} (\frac{\mu u_x}{1+\varphi})_{tx}u_t dx = &-\frac{\mu\omega|u_t(0,t)|^2}{1+\varphi(0,t)} + \frac{\mu u_x(0,t)\varphi_t(0,t)u_t(0,t)}{(1+\varphi(0,t))^2} \\ &- \int_{\mathbb{R}_+} \frac{\mu u_{tx}^2}{1+\varphi}dx + \int_{\mathbb{R}_+} \frac{\mu u_x\varphi_t u_{tx}}{(1+\varphi)^2}dx.\end{aligned} \tag{51}$$

Thus, we conclude from above results that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(c^2\|\varphi_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \frac{c^2}{\omega}|\varphi_t(0,t)|^2) + \frac{\mu\omega|u_t(0,t)|^2}{1+\varphi(0,t)} + \int_{\mathbb{R}_+} \frac{\mu u_{tx}^2}{1+\varphi}dx \\ &= \frac{\mu u_x(0,t)\varphi_t(0,t)u_t(0,t)}{(1+\varphi(0,t))^2} + \int_{\mathbb{R}_+} \frac{\mu u_x\varphi_t u_{tx}}{(1+\varphi)^2}dx.\end{aligned} \tag{52}$$

Via Young's inequality, one has

$$\frac{\mu u_x(0,t)\varphi_t(0,t)u_t(0,t)}{(1+\varphi(0,t))^2} \le C\mathcal{E}(t)\mathcal{D}(t)^2 + \varepsilon\frac{\mu\omega|u_t(0,t)|^2}{1+\varphi(0,t)}. \tag{53}$$

By the a-priori assumption (42) and Proposition 1, we obtain

$$\begin{aligned}\int_{\mathbb{R}_+} \frac{\mu u_x\varphi_t u_{tx}}{(1+\varphi)^2}dx &\le C\|u_x\|_{L^\infty}^2\|\varphi_t\|_{L^2}^2 + \varepsilon\int_{\mathbb{R}_+} \frac{\mu u_{tx}^2}{1+\varphi}dx \\ &\le C\delta\|u_x\|_{H^1}\|u_x\|_{L^2}^2 + \varepsilon\int_{\mathbb{R}_+} \frac{\mu u_{tx}^2}{1+\varphi}dx \\ &\le C\mathcal{E}(t)\mathcal{D}(t)^2 + \varepsilon\int_{\mathbb{R}_+} \frac{\mu u_{tx}^2}{1+\varphi}dx.\end{aligned} \tag{54}$$

It then concludes from above estimates and the smallness of ε to give

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(c^2\|\varphi_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + \frac{c^2}{\omega}|\varphi_t(0,t)|^2) \\ &+ \frac{\mu\omega|u_t(0,t)|^2}{2(1+\varphi(0,t))} + \frac{1}{2}\int_{\mathbb{R}_+} \frac{\mu u_{tx}^2}{1+\varphi}dx \le C\mathcal{E}(t)\mathcal{D}(t)^2.\end{aligned} \tag{55}$$

Integrating time from 0 to T, then applying the a-priori assumption (42) yields

$$\begin{aligned}&\sup_{0\le t\le T}(\|(\varphi_t,u_t)\|_{L^2}^2+|\varphi_t(0,t)|^2)+\int_0^T(|u_t(0,t)|^2+\|u_{tx}\|_{L^2}^2)dt\\&\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}\tag{56}$$

The case of $k=2$ can be treated in a similar argument as $k=1$. Hence, we complete the proof of the proposition. □

To derive the estimate of high order spatial derivatives, we shall obtain the energy inequalities as follows.

Proposition 3. *Assume (φ,u) is the classical solution of (27) satisfying the assumptions in Theorem 3.1 and the a-priori assumption (42). Then we obtain the estimate for any given $T>0$ satisfying*

$$\begin{aligned}&\sum_{k=0}^{1}\Big(\sup_{0\le t\le T}(\|\partial_t^k u_x\|_{L^2}^2+|\partial_t^k u(0,t)|^2)+\int_0^T\|\partial_t^{k+1}u\|_{L^2}^2dt\Big)\\&\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}$$

Proof. Taking the inner product of $(27)_2$ and u_t, then integrating x in $\mathbb{R}_+$

$$\|u_t\|_{L^2}^2+\int_{\mathbb{R}_+}p(1+\varphi)_xu_tdx-\int_{\mathbb{R}_+}(\frac{\mu u_x}{1+\varphi})_xu_tdx=0.\tag{57}$$

In terms of integration by parts, we have

$$\int_{\mathbb{R}_+}p(1+\varphi)_xu_tdx\sim c^2\int_{\mathbb{R}_+}\varphi u_{tx}dx+c^2\varphi(0,t)u_t(0,t).\tag{58}$$

A direct calculation gives

$$c^2\int_{\mathbb{R}_+}\varphi u_{tx}dx=c^2\frac{d}{dt}\int_{\mathbb{R}_+}\varphi u_xdx-c^2\int_{\mathbb{R}_+}u_x^2dx.\tag{59}$$

To estimate the third term in (57), we consider it as follows

$$-\int_{\mathbb{R}_+}(\frac{\mu u_x}{1+\varphi})_xu_tdx=\int_{\mathbb{R}_+}\frac{\mu u_xu_{tx}}{1+\varphi}dx+\frac{\mu u_x(0,t)u_t(0,t)}{1+\varphi(0,t)}.\tag{60}$$

Furthermore, it is obvious to verify

$$\int_{\mathbb{R}_+}\frac{\mu u_xu_{tx}}{1+\varphi}dx=\frac{\mu}{2}\frac{d}{dt}\int_{\mathbb{R}_+}\frac{u_x^2}{1+\varphi}dx+\int_{\mathbb{R}_+}\frac{\mu u_x^2\varphi_t}{2(1+\varphi)^2}dx.\tag{61}$$

Similarly, it also holds

$$\frac{\mu u_x(0,t)u_t(0,t)}{1+\varphi(0,t)}=\frac{\mu\omega}{2}\frac{d}{dt}\Big(\frac{|u(0,t)|^2}{1+\varphi(0,t)}\Big)+\frac{\mu\omega|u(0,t)|^2\varphi_t(0,t)}{2(1+\varphi(0,t))^2}.\tag{62}$$

We obtain the following estimate from above obtained results

$$\begin{aligned}&\frac{\mu}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}_+}\frac{u_x^2}{1+\varphi}dx+\frac{\omega|u(0,t)|^2}{1+\varphi(0,t)}\Big)+\|u_t\|_{L^2}^2\\&=-c^2\frac{d}{dt}\int_{\mathbb{R}_+}\varphi u_x dx+c^2\int_{\mathbb{R}_+}u_x^2dx-c^2\varphi(0,t)u_t(0,t)\\&+\int_{\mathbb{R}_+}\frac{\mu u_x^2\varphi_t}{2(1+\varphi)^2}dx++\frac{\mu\omega|u(0,t)|^2\varphi_t(0,t)}{2(1+\varphi(0,t))^2}.\end{aligned}\tag{63}$$

By the definition of $\mathcal{E}(t),\mathcal{D}(t)$, we deduce

$$\int_{\mathbb{R}_+}\frac{\mu u_x^2\varphi_t}{2(1+\varphi)^2}dx+\frac{\mu\omega|u(0,t)|^2\varphi_t(0,t)}{2(1+\varphi(0,t))^2}\le C\mathcal{E}(t)\mathcal{D}(t)^2.\tag{64}$$

Making use of Proposition 1 and 2 shows

$$\begin{aligned}\sup_{0\le t\le T}\int_{\mathbb{R}_+}\varphi u_x dx&\le C\sup_{0\le t\le T}\|\varphi\|_{L^2}\sup_{0\le t\le T}\|u_x\|_{L^2}\\&\le C\sup_{0\le t\le T}\|\varphi\|_{L^2}\sup_{0\le t\le T}\|\varphi_t\|_{L^2}\\&\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}\tag{65}$$

Via integration by parts, Proposition 1 and 2, we can prove

$$\begin{aligned}\int_0^T\varphi(0,t)u_t(0,t)dt&=\varphi(0,t)u(0,t)\Big|_{t=0}^{t=T}-\int_0^T\varphi_t(0,t)u(0,t)dt\\&\le\sup_{0\le t\le T}|\varphi(0,t)||u(0,t)|+C\delta_0^2+\int_0^T|\varphi_t(0,t)||u(0,t)|dt\\&\le\frac{1}{\omega}\sup_{0\le t\le T}|\varphi(0,t)||\varphi_t(0,t)|+C\delta_0^2+\omega\int_0^T|u(0,t)|^2dt\\&\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}\tag{66}$$

Therefore, integrating (63) with time t over $[0,T]$, then applying (64) $\sim$ (66) yields

$$\sup_{0\le t\le T}(\|u_x\|_{L^2}^2+|u(0,t)|^2)+\int_0^T\|u_t\|_{L^2}^2dt\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\tag{67}$$

It should be noted that the case of $k=1$ can be treated in a similar argument as $k=0$. Hence, we complete the proof. □

We are now ready to establish the L^2 energy estimates of the spatial derivatives.

Proposition 4. *Assume (φ,u) is the classical solution of (27) satisfying the assumptions in Theorem 3.1 and the a-priori assumption (42). Then we obtain the estimate for any given $T>0$ satisfying*

$$\begin{aligned}&\sup_{0\le t\le T}\|(\varphi_x,u_x)\|_{H^3}^2+\int_0^T(\|\varphi_x\|_{H^3}^2+\|u_{xx}\|_{H^3}^2)dt\\&\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned}$$

Proof. $(27)_2$ is reformulated into the following form

$$u_{xx} = \frac{1+\varphi}{\mu}u_t + \frac{1+\varphi}{\mu}p(1+\varphi)_x + \frac{u_x\varphi_x}{1+\varphi}. \tag{68}$$

It then follows from $(27)_1$ and (68) that

$$\varphi_{tx} = \frac{1+\varphi}{\mu}u_t + \frac{1+\varphi}{\mu}p(1+\varphi)_x + \frac{u_x\varphi_x}{1+\varphi}. \tag{69}$$

Multiplying (69) by φ_x, then integrating with respect to x in $\mathbb{R}_+$ gives

$$\frac{1}{2}\frac{d}{dt}\|\varphi_x\|_{L^2}^2 + \int_{\mathbb{R}_+}\frac{c^2(1+\varphi)}{\mu}\varphi_x^2dx = \int_{\mathbb{R}_+}(\frac{1+\varphi}{\mu}u_t + \frac{u_x\varphi_x}{1+\varphi})\varphi_x dx. \tag{70}$$

Thus, by Young's inequality, we deduce

$$\begin{aligned}&\int_{\mathbb{R}_+}(\frac{1+\varphi}{\mu}u_t + \frac{u_x\varphi_x}{1+\varphi})\varphi_x dx\\ &\le \varepsilon\int_{\mathbb{R}_+}\frac{c^2(1+\varphi)}{\mu}\varphi_x^2dx + C\|u_t\|_{L^2}^2 + C\mathcal{E}(t)\mathcal{D}(t)^2.\end{aligned} \tag{71}$$

Because of the smallness of ε and the a-priori assumption (42), one has

$$\frac{1}{2}\frac{d}{dt}\|\varphi_x\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}_+}\frac{c^2(1+\varphi)}{\mu}\varphi_x^2dx \le C\|u_t\|_{L^2}^2 + C\mathcal{E}(t)\mathcal{D}(t)^2. \tag{72}$$

Integrating time t from 0 to T yields

$$\sup_{0\le t\le T}\|\varphi_x\|_{L^2}^2 + \int_0^T\|\varphi_x\|_{L^2}^2dt \le C\delta_0^2 + C\delta\int_0^T\mathcal{D}(t)^2dt, \tag{73}$$

here we have made use of Proposition 3. Since $\varphi_t = u_x$, it is easy to verify from Proposition 2 that

$$\sup_{0\le t\le T}\|u_x\|_{L^2}^2 \le C\delta_0^2 + C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{74}$$

Take L^2 inner product on (68)

$$\begin{aligned}\|u_{xx}\|_{L^2} &\le C(1+\|\varphi\|_{L^\infty})\|u_t\|_{L^2} + C(1+\|\varphi\|_{L^\infty})\|\varphi_x\|_{L^2} + C\|u_x\|_{L^\infty}\|\varphi_x\|_{L^2}\\ &\le C(1+\delta)\|u_t\|_{L^2} + C(1+\delta)\|\varphi_x\|_{L^2}.\end{aligned} \tag{75}$$

By (73) and Proposition 3, we integrate time t in $[0,T]$ to gain

$$\int_0^T\|u_{xx}\|_{L^2}^2dt \le C\delta_0^2 + C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{76}$$

Finally, collecting above inequalities together gives the desired estimate as follows

$$\begin{aligned}&\sup_{0\le t\le T}(\|\varphi_x\|_{L^2}^2 + \|u_x\|_{L^2}^2) + \int_0^T(\|\varphi_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2)dt\\ &\le C\delta_0^2 + C\delta\int_0^T\mathcal{D}(t)^2dt.\end{aligned} \tag{77}$$

The estimate for high order spatial derivatives could be treated in a similar procedure. This completes the proof of the proposition. □

4.1. **The proof of Theorem 3.1.**

Proof. Combining Proposition 1, 2 and 4 together to obtain

$$\sup_{0\le t\le T}\mathcal{E}(t)^2+\int_0^T\mathcal{D}(t)^2dt\le C\delta_0^2+C\delta\int_0^T\mathcal{D}(t)^2dt. \tag{78}$$

Due to the smallness of δ, it is easy to verify

$$\sup_{0\le t\le T}\mathcal{E}(t)^2+\int_0^T\mathcal{D}(t)^2dt\le C\delta_0^2. \tag{79}$$

When the initial data satisfy $C\delta_0^2\le\frac{1}{4}\delta^2$, we could close the a-priori assumption (42). Then, based on the continuous method, the global existence of solution is obtained. The reader can refer to [21] for details. This completes the proof of Theorem 3.1. □

5. **The estimates of Green's function for IBVP.**

To obtain the Green's function of (35), we construct a specific solution for $x\ge 0$ as following

$$\tilde{U}(x,t)=(\tilde{\varphi},\tilde{u})^t=\int_0^\infty G(x-y,t)U_0(y)dy, \tag{80}$$

where $G(x,t)$ denotes the Green's function of Cauchy problem satisfying

$$\begin{cases}(\partial_t+A\partial_x-B\partial_{xx})G(x,t)=0 \quad x\in\mathbb{R},t>0,\\ G(x,0)=\delta(x)I_2.\end{cases} \tag{81}$$

We are able to verify that $\tilde{U}(x,t)$ solves the following system

$$\begin{cases}\tilde{\varphi}_t-\tilde{u}_x=0,x>0,t>0,\\ \tilde{u}_t-c^2\tilde{\varphi}_x=\mu\tilde{u}_{xx},\\ (a\tilde{u}_x+b\tilde{u})|_{x=0}=m(t),\\ (\tilde{\varphi},\tilde{u})|_{x\to+\infty}=(0,0),\\ (\tilde{\varphi},\tilde{u})|_{t=0}=(\varphi_0,u_0),\end{cases} \tag{82}$$

where boundary term $m(t)$ is given by

$$m(t)=(a\partial_t,b)\int_0^\infty G(-y,t)U_0(y)dy. \tag{83}$$

To make the initial data become zero, we define some new variables

$$\bar{\varphi}=\tilde{\varphi}-\varphi,\ \ \bar{u}=\tilde{u}-u, \tag{84}$$

where (φ,u) is the solution of (35). Moreover, we have

$$\begin{cases}\bar{\varphi}_t-\bar{u}_x=0,x>0,t>0,\\ \bar{u}_t-c^2\bar{\varphi}_x=\mu\bar{u}_{xx},\\ (a\bar{u}_x+b\bar{u})|_{x=0}=m(t),\\ (\bar{\varphi},\bar{u})|_{x\to+\infty}=(0,0),\\ (\bar{\varphi},\bar{u})|_{t=0}=(0,0).\end{cases} \tag{85}$$

By a suitable combination, we obtain

$$\begin{cases}\bar{\varphi}_{tt}-c^2\bar{\varphi}_{xx}=\mu\bar{\varphi}_{txx},\\ \bar{u}_{tt}-c^2\bar{u}_{xx}=\mu\bar{u}_{txx}.\end{cases} \tag{86}$$

Making use of Laplace transform in time yields

$$\begin{cases} s^2\mathfrak{L}[\bar{\varphi}] - c^2\mathfrak{L}[\bar{\varphi}]_{xx} = \mu s\mathfrak{L}[\bar{\varphi}]_{xx}, \\ s^2\mathfrak{L}[\bar{u}] - c^2\mathfrak{L}[\bar{u}]_{xx} = \mu s\mathfrak{L}[\bar{u}]_{xx}. \end{cases} \tag{87}$$

Solving above equations and utilizing (85)$_4$ give

$$\mathfrak{L}[\bar{\varphi}] = c_1 e^{-\sigma x}, \quad \mathfrak{L}[\bar{u}] = c_2 e^{-\sigma x}, \tag{88}$$

where c_1, c_2 denote unknowns and $\sigma = \frac{s}{\sqrt{\mu s + c^2}} > 0$. Taking Laplace transform in (85)$_1$ and (85)$_3$ yields

$$\begin{cases} s\mathfrak{L}[\bar{\varphi}] - \mathfrak{L}[\bar{u}]_x = 0, \\ (as\mathfrak{L}[\bar{\varphi}] + b\mathfrak{L}[\bar{u}])|_{x=0} = \mathfrak{L}[m](s). \end{cases} \tag{89}$$

Substituting (88) into (89) gives rise to

$$\begin{cases} c_1 s + c_2\sigma = 0, \\ c_1 as + c_2 b = \mathfrak{L}[m](s). \end{cases} \tag{90}$$

We are able to solve c_1, c_2 as

$$c_1 = \frac{\sigma}{s(a\sigma - b)}\mathfrak{L}[m](s), \quad c_2 = \frac{1}{b - a\sigma}\mathfrak{L}[m](s). \tag{91}$$

To obtain the explicit formula of $\mathfrak{L}[m](s)$, we take Laplace transform in (83)

$$\mathfrak{L}[m](s) = (as, b)\int_0^\infty \mathfrak{L}[G](-y, s)U_0(y)dy. \tag{92}$$

The remaining challenge is to estimate $\mathfrak{L}[G](-y, s)$. Taking Fourier transform in x and Laplace transform in t to (81) provides

$$\begin{pmatrix} s & -i\xi \\ -ic^2\xi & s + \mu\xi^2 \end{pmatrix} \mathfrak{L}\big[\mathfrak{F}[G]\big](\xi, s) = I_2. \tag{93}$$

After tedious computations, one has

$$\mathfrak{L}\big[\mathfrak{F}[G]\big](\xi, s) = \frac{1}{s^2 + (\mu s + c^2)\xi^2}\begin{pmatrix} s + \mu\xi^2 & i\xi \\ ic^2\xi & s \end{pmatrix}. \tag{94}$$

It is easy to observe

$$s^2 + (\mu s + c^2)\xi^2 = (\mu s + c^2)(\sigma^2 + \xi^2). \tag{95}$$

Applying Residue Theorem gives the following results for $(x \neq 0)$

$$\begin{aligned} &\frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ix\xi}}{\sigma^2 + \xi^2}d\xi = \frac{e^{-\sigma|x|}}{2\sigma}, \\ &\frac{1}{2\pi}\int_{\mathbb{R}} \frac{i\xi e^{ix\xi}}{\sigma^2 + \xi^2}d\xi = -\frac{\text{sign}(x)}{2}e^{-\sigma|x|}, \end{aligned} \tag{96}$$

where $\text{sign}(x)$ denotes symbol function. Then we are able to show

$$\mathfrak{L}[G](x, s) = \begin{pmatrix} \frac{\mu\sigma^2}{s^2}\delta(x) + \frac{c^2\sigma^3}{2s^3}e^{-\sigma|x|} & -\frac{\sigma^2}{2s^2}\text{sign}(x)e^{-\sigma|x|} \\ -\frac{c^2\sigma^2}{2s^2}\text{sign}(x)e^{-\sigma|x|} & \frac{\sigma}{2s}e^{-\sigma|x|} \end{pmatrix}. \tag{97}$$

On the one hand, substituting $x = -y(y > 0)$ into (97) leads to

$$\mathfrak{L}[G](-y, s) = \frac{e^{-\sigma y}}{2s^3}\begin{pmatrix} c^2\sigma^3 & \sigma^2 s \\ c^2\sigma^2 s & \sigma s^2 \end{pmatrix}. \tag{98}$$

On the other hand, when $x>0, y>0$, from (97) we have

$$\mathfrak{L}[G](x+y,s)=\frac{e^{-\sigma(x+y)}}{2s^3}\begin{pmatrix} c^2\sigma^3 & -\sigma^2 s\\ -c^2\sigma^2 s & \sigma s^2\end{pmatrix}. \tag{99}$$

By (92) and (98), it is easy to verify

$$\mathfrak{L}[m](s)=\int_0^\infty \frac{(a\sigma+b)}{2s^2}(c^2\sigma^2\varphi_0+\sigma s u_0)e^{-\sigma y}dy. \tag{100}$$

Consequently, one could combine (88), (91) and (99) to obtain

$$\begin{aligned}\mathfrak{L}[\bar\varphi](x,s)&=\frac{a\sigma+b}{a\sigma-b}\int_0^\infty \frac{e^{-\sigma(x+y)}}{2s^3}(c^2\sigma^3\varphi_0+\sigma^2 s u_0)dy,\\ \mathfrak{L}[\bar u](x,s)&=-\frac{a\sigma+b}{a\sigma-b}\int_0^\infty \frac{e^{-\sigma(x+y)}}{2s^3}(c^2\sigma^2 s\varphi_0+\sigma s^2 u_0)dy.\end{aligned} \tag{101}$$

Compared with (99), it is obvious to justify

$$\mathfrak{L}[\bar U](x,s)=\frac{a\sigma+b}{a\sigma-b}\int_0^\infty \mathfrak{L}[G](x+y,s)\begin{pmatrix}1&0\\0&-1\end{pmatrix}U_0(y)dy. \tag{102}$$

It should be emphasized that the Laplace transform of $\tilde U$ is computed as

$$\mathfrak{L}[\tilde U](x,s)=\int_0^\infty \mathfrak{L}[G](x-y,s)U_0(y)dy. \tag{103}$$

Thanks to (102),(103) and (84), which implies

$$\mathfrak{L}[U](x,s)=\int_0^\infty\Big(\mathfrak{L}[G](x-y,s)-\frac{a\sigma+b}{a\sigma-b}\mathfrak{L}[G](x+y,s)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\Big)U_0(y)dy.$$

Thus, the Laplace transform of Green's function for IBVP is expressed as

$$\mathfrak{L}[\mathbb{G}](x,s;y)=\mathfrak{L}[G](x-y,s)-(1+\frac{2b}{a\sigma-b})\mathfrak{L}[G](x+y,s)\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{104}$$

It is easy to see when $a=0$, taking inverse Laplace transform gives rise to

$$\mathbb{G}(x,t;y)=G(x-y,t)+G(x+y,t)\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{105}$$

In a same manner to treat $b=0$, it asserts

$$\mathbb{G}(x,t;y)=G(x-y,t)-G(x+y,t)\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{106}$$

In what follows, we mainly consider the case of $ab<0$. Denote

$$h(x,t)\triangleq\mathfrak{L}^{-1}[\frac{2b}{a\sigma-b}\mathfrak{L}[G](x,s)]. \tag{107}$$

It is noticeable that $h(x,t)$ satisfies the following ordinary equation

$$a\partial_x h(x,t)+bh(x,t)=-2bG(x,t). \tag{108}$$

After a simple calculation, the solution of $h(x,t)$ is expressed as

$$h(x,t)=-2\omega\int_0^\infty e^{-\omega z}G(z+x,t)dz, \tag{109}$$

where $\omega=-\frac{b}{a}>0$. It is natural to take inverse Laplace transform on (104) to get

$$\mathbb{G}(x,t;y)=G(x-y,t)-\big(G(x+y,t)+h(x+y,t)\big)\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{110}$$

With Lemma 2.2 and Lemma 2.3, one has the following lemma for the estimate of $\mathbb{G}(x,t;y)$

$$\begin{aligned}&\Big|\partial_x^\alpha\big(\mathbb{G}(x,t;y)-e^{-\frac{c^2}{\mu}t}(\delta(x-y)-\delta(x+y))I_0\big)\Big|\\&=\mathcal{O}(1)t^{-\frac{1}{2}-\frac{\alpha}{2}}(e^{-\frac{(x-y-ct)^2}{2\mu t}}+e^{-\frac{(x-y+ct)^2}{2\mu t}}+e^{-\frac{(x+y-ct)^2}{(2\mu+\epsilon)t}})\\&+\mathcal{O}(1)(e^{-\frac{|x-y|+t}{C}}+e^{-\frac{|x+y|+t}{C}}).\end{aligned}\tag{111}$$

We hence complete the proof of Theorem 3.2.

6. **Nonlinear pointwise estimates of classical solution.**

Considering the nonlinear system as below

$$\begin{cases}\varphi_t-u_x=0,\\u_t-c^2\varphi_x=\mu u_{xx}+f,\\(\varphi,u)|_{t=0}=(\varphi_0,u_0),\end{cases}\tag{112}$$

where f represents nonlinear term satisfying

$$f=\tilde{f}_x=-\big(p(1+\varphi)-p(1)-p'(1)\varphi+\frac{\mu\varphi u_x}{1+\varphi}\big)_x.\tag{113}$$

Denote

$$U=(\varphi,u)^t,\quad \tilde{F}=(0,\tilde{f})^t,\quad F=(0,f)^t.\tag{114}$$

In the sense of small solution, we can obtain

$$\tilde{f}=\mathcal{O}(1)\big(|\varphi|^2+|\varphi||u_x|\big),\quad f=\mathcal{O}(1)\big(|\varphi||\varphi_x|+|\varphi_x||u_x|+|\varphi||u_{xx}|\big).\tag{115}$$

In order to provide an additional derivative for the initial data, we define

$$W_0(x)\triangleq\int_x^\infty U_0(y)dy,\quad W_0'(x)=-U_0(x).\tag{116}$$

To investigate the poinwise behavior of the solution for (112), we first introduce the following ansatz

$$\Lambda(t)=\sup_{0\le s\le t}\Big\{\|U\Phi^{-1}\|_{L_x^\infty}+(1+s)\big(\log(2+s)\big)^{-1}\|U_x\|_{L_x^\infty}+(1+s)^{\frac{1}{2}}\|u_{xx}\|_{L_x^\infty}\Big\}.\tag{117}$$

From the ansatz, it is easy to verify

$$|U|\le\Lambda(t)\Phi(x,t),\quad |U_x|\le(1+t)^{-1}\log(2+t)\Lambda(t),\quad |u_{xx}|\le(1+t)^{-\frac{1}{2}}\Lambda(t).\tag{118}$$

Indeed, utilizing (115) and (118) yields

$$|\tilde{F}(x,t)|=\mathcal{O}(1)\Lambda(t)^2\big[(1+t)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)+(1+t)^{-\frac{1}{2}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)\big],\tag{119}$$

and

$$|F(x,t)|=\mathcal{O}(1)\Lambda(t)^2\big[(1+t)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)+(1+t)^{-\frac{1}{8}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)\big].\tag{120}$$

Furthermore, by (119) we immediately get

$$|\tilde{F}(0,t)|\le C(1+t)^{-\frac{7}{4}}\Lambda(t)^2.\tag{121}$$

Once the spatial variable is ignored, the time decay rate for $\tilde{F}(x,t),F(x,t)$ can be calculated as

$$|\tilde{F}(x,t)|\le C(1+t)^{-1}\Lambda(t)^2,\quad |F(x,t)|\le C(1+t)^{-1}\Lambda(t)^2.\tag{122}$$

For simplicity, we decompose the Green function $\mathbb{G}(x,t;y)$ into two parts

$$\mathbb{G}(x,t;y) = \mathbb{G}^s(x,t;y) + \mathbb{G}^\ell(x,t;y). \tag{123}$$

Here $\mathbb{G}^s(x,t;y)$ is the short wave part related to singular part in Green's function $\mathbb{G}(x,t;y)$. $\mathbb{G}^\ell(x,t;y)$ represents the long wave part, which dominates the long time behavior. They are given by

$$\mathbb{G}^s(x,t;y) = e^{-\frac{c^2}{\mu}t}(\delta(x-y) - \delta(x+y))I_0, \tag{124}$$

and

$$\mathbb{G}^\ell(x,t;y) \triangleq \mathbb{G}_1^\ell(x,t;y) + \mathbb{G}_2^\ell(x,t;y), \tag{125}$$

where $\mathbb{G}_1^\ell(x,t;y)$ and $\mathbb{G}_2^\ell(x,t;y)$ satisfy the following estimates for any $\alpha > 0$

$$\begin{aligned} |\partial_x^\alpha \mathbb{G}_1^\ell(x,t;y)| &= \mathcal{O}(1)t^{-\frac{1}{2}-\frac{\alpha}{2}}(e^{-\frac{(x-y-ct)^2}{2\mu t}} + e^{-\frac{(x-y+ct)^2}{2\mu t}} + e^{-\frac{(x+y-ct)^2}{(2\mu+\epsilon)t}}), \\ |\partial_x^\alpha \mathbb{G}_2^\ell(x,t;y)| &= \mathcal{O}(1)(e^{-\frac{|x-y|+t}{C}} + e^{-\frac{|x+y|+t}{C}}). \end{aligned} \tag{126}$$

To close the ansatz assumption, we first establish the pointwise estimate of solution $U(x,t)$ and $U_x(x,t)$ as follows.

Proposition 5. *Under the assumptions of Theorem 3.3, there exists a positive constant C such that*

$$\begin{aligned} |U(x,t)| &\le C(\delta_0 + \Lambda(t)^2)\Phi(x,t), \\ |U_x(x,t)| &\le C(1+t)^{-1}\log(2+t)(\delta_0 + \Lambda(t)^2). \end{aligned} \tag{127}$$

Proof. By Duhamel's principle, the solution $U(x,t)$ is expressed as

$$\begin{aligned} U(x,t) &= \int_0^\infty \mathbb{G}(x,t;y)U_0(y)dy + \int_0^t\int_0^\infty \mathbb{G}(x,t-s;y)F(y,s)dyds \\ &\triangleq I + J. \end{aligned} \tag{128}$$

We begin to evaluate the first term by applying (123) $\sim$ (125)

$$\begin{aligned} I &= \int_0^\infty \mathbb{G}_1^\ell(x,t;y)U_0(y)dy + \int_0^\infty \mathbb{G}_2^\ell(x,t;y)U_0(y)dy + \int_0^\infty \mathbb{G}^s(x,t;y)U_0(y)dy \\ &\triangleq I_1 + I_2 + I_3. \end{aligned} \tag{129}$$

Since the case when $0 \le t \le 1$ can be handled easily by using the assumption of Theorem 3.3, we only focus on $t > 1$ in the following. In terms of I_1, changing the initial data U_0 into W_0', then taking integration by parts yields

$$\begin{aligned} |I_1| &= \mathcal{O}(1)t^{-\frac{1}{2}}(\sum_{i=1}^2 e^{-\frac{(x-\lambda_i t)^2}{2\mu t}} + e^{-\frac{(x-ct)^2}{(2\mu+\epsilon)t}})|W_0(0)| \\ &\quad + \mathcal{O}(1)\int_0^\infty t^{-1}(\sum_{i=1}^2 e^{-\frac{(x-y-\lambda_i t)^2}{2\mu t}} + e^{-\frac{(x-y-ct)^2}{(2\mu+\epsilon)t}})|W_0(y)|dy \\ &\triangleq I_{11} + I_{12}. \end{aligned} \tag{130}$$

As a result of (38) and Lemma 2.1, it is easy to confirm

$$\begin{aligned} I_{11} &\le C\delta_0(1+t)^{-\frac{1}{2}}e^{-\frac{(x-c(1+t))^2}{(2\mu+\epsilon)(1+t)}} \\ &\le C\delta_0(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c) \\ &\le C\delta_0\Phi(x,t). \end{aligned} \tag{131}$$

As for I_{12}, using Lemma 2.4 and $t > 1$ gives rise to

$$\begin{aligned} I_{12} &\le C\delta_0 \int_0^{\infty} (1+t)^{-1} (\sum_{i=1}^{2} e^{-\frac{(x-y-\lambda_i(1+t))^2}{2\mu(1+t)}} + e^{-\frac{(x+y-c(1+t))^2}{(2\mu+\epsilon)(1+t)}})(1+y^2)^{-r} dy \\ &\le C\delta_0 \Phi(x,t). \end{aligned} \tag{132}$$

It then follows from (131) and (132) to obtain

$$|I_1| \le C\delta_0 \Phi(x,t). \tag{133}$$

In order to estimate I_2, with the help of Lemma 2.1 and $r \ge \frac{5}{8}$, we find

$$\begin{aligned} |I_2| &\le C\delta_0 \int_0^{\frac{x}{2}} e^{-\frac{|x-y|+t}{C}} (1+y^2)^{-r} dy + C\delta_0 \int_{\frac{x}{2}}^{\infty} e^{-\frac{|x-y|+t}{C}} (1+y^2)^{-r} dy \\ &\le C\delta_0 e^{-\frac{x+t}{2C}} + C\delta_0 e^{-\frac{t}{C}} (1+x^2)^{-r} \\ &\le C\delta_0 \Phi(x,t). \end{aligned} \tag{134}$$

For I_3, we have the following estimate from Lemma 2.1

$$|I_3| \le C\delta_0 e^{-\frac{c^2}{\mu}t} (1+x^2)^{-r} \le C\delta_0 \Phi(x,t). \tag{135}$$

Consequently, we summarize above results together to get

$$|I| \le |I_1| + |I_2| + |I_3| \le C\delta_0 \Phi(x,t). \tag{136}$$

The next goal is to deal with J coming from nonlinear coupling. We rewrite it as

$$\begin{aligned} J &= \int_0^t \int_0^{\infty} \mathbb{G}_1^{\ell}(x,t-s;y) F(y,s) dy ds + \int_0^t \int_0^{\infty} \mathbb{G}_2^{\ell}(x,t-s;y) F(y,s) dy ds \\ &\quad + \int_0^t \int_0^{\infty} \mathbb{G}^{s}(x,t-s;y) F(y,s) dy ds \\ &\triangleq J_1 + J_2 + J_3. \end{aligned} \tag{137}$$

Integration by parts on J_1 with respect to y gives

$$\begin{aligned} |J_1| &= \mathcal{O}(1) \int_0^t \int_0^{\infty} (t-s)^{-1} (\sum_{i=1}^{2} e^{-\frac{(x-y-\lambda_i(t-s))^2}{2\mu(t-s)}} + e^{-\frac{(x+y-c(t-s))^2}{(2\mu+\epsilon)(t-s)}}) |\tilde{F}(y,s)| dy ds \\ &\quad + \mathcal{O}(1) \int_0^t (t-s)^{-\frac{1}{2}} (\sum_{i=1}^{2} e^{-\frac{(x-\lambda_i(t-s))^2}{2\mu(t-s)}} + e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}}) |\tilde{F}(0,s)| ds \\ &\triangleq J_{11} + J_{12}. \end{aligned} \tag{138}$$

It then follows from (119) that

$$\begin{aligned} J_{11} &\lesssim \Lambda(t)^2 \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1} (1+s)^{-\frac{1}{4}} e^{-\frac{(x-y-c(t-s))^2}{2\mu(t-s)}} \psi_{\frac{3}{2}}(y,s;c) dy ds \\ &\quad + \Lambda(t)^2 \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1} (1+s)^{-\frac{1}{4}} e^{-\frac{(x-y+c(t-s))^2}{2\mu(t-s)}} \psi_{\frac{3}{2}}(y,s;c) dy ds \\ &\quad + \Lambda(t)^2 \int_0^t \int_{-\infty}^{\infty} (t-s)^{-1} (1+s)^{-\frac{1}{4}} e^{-\frac{(x-y-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} \psi_{\frac{3}{2}}(y,s;-c) dy ds \end{aligned}$$

$$+\Lambda(t)^2\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{(x-y-c(t-s))^2}{2\mu(t-s)}}\phi_1(y,s;c)\psi_{\frac{3}{4}}(y,s;c)dyds$$
$$+\Lambda(t)^2\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{(x-y+c(t-s))^2}{2\mu(t-s)}}\phi_1(y,s;c)\psi_{\frac{3}{4}}(y,s;c)dyds$$
$$+\Lambda(t)^2\int_0^t\int_{-\infty}^{\infty}(t-s)^{-1}(1+s)^{-\frac{1}{2}}e^{-\frac{(x-y-c(t-s))^2}{(2\mu+\epsilon)(t-s)}}\phi_1(y,s;-c)\psi_{\frac{3}{4}}(y,s;-c)dyds$$
$$\triangleq \mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4+\mathcal{L}_5+\mathcal{L}_6.$$

Let $\alpha=0,\beta=\frac{1}{2},\lambda=c,\bar{\mu}=2\mu$. Then by Lemma 2.5 we have

$$\mathcal{L}_1=\mathcal{O}(1)\Lambda(t)^2(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{139}$$

Set $\alpha=0,\beta=\frac{1}{2},\lambda=-c,\lambda'=c,\bar{\mu}=2\mu$. It can be concluded from Lemma 2.6 to get

$$\begin{aligned}\mathcal{L}_2&=\mathcal{O}(1)\Lambda(t)^2(1+t)^{\frac{1}{4}}[\psi_{\frac{3}{2}}(x,t;-c)+\psi_{\frac{3}{2}}(x,t;c)]\\&\quad+\mathcal{O}(1)\Lambda(t)^2(1+t)^{\frac{3}{8}}[\phi_1(x,t;-c)\psi_{\frac{3}{4}}(x,t;-c)+\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)]\\&=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2,\end{aligned} \tag{140}$$

where we have used that when $-c(1+t)+c\sqrt{1+t}\le x\le c(1+t)-c\sqrt{1+t}$, it holds

$$c\sqrt{1+t}\le|x+c(1+t)|\le 2c(1+t),\quad c\sqrt{1+t}\le|x-c(1+t)|\le 2c(1+t). \tag{141}$$

Obviously, one can deduce

$$\begin{aligned}&|x+c(1+t)|^{-\frac{1}{2}}|x-c(1+t)|^{-\frac{1}{2}}\\&\lesssim(1+t)^{\frac{3}{8}}(\phi_1(x,t;-c)\psi_{\frac{3}{4}}(x,t;-c)+\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)).\end{aligned} \tag{142}$$

With the help of Lemma 2.5, choosing $\alpha=0,\beta=\frac{1}{2},\lambda=-c,\bar{\mu}=2\mu+\epsilon$ gives

$$\mathcal{L}_3=\mathcal{O}(1)\Lambda(t)^2(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;-c)=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{143}$$

Substituting $\alpha=0,\beta=1,\lambda=c$ and $\bar{\mu}=2\mu$ into Lemma 2.7 yields

$$\begin{aligned}\mathcal{L}_4&=\mathcal{O}(1)\Lambda(t)^2[\log(2+t)\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)+\psi_{\frac{3}{2}}(x,t;c)]\\&=\mathcal{O}(1)\Lambda(t)^2[(1+t)^{\frac{3}{8}}\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c)+(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)]\\&=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2.\end{aligned} \tag{144}$$

Via Lemma 2.8, let $\alpha=0,\beta=1,\lambda=c,\lambda'=-c,\bar{\mu}=2\mu$, we are able to show

$$\mathcal{L}_5=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{145}$$

The estimate of $\mathcal{L}_6$ can be derived in a same manner as $\mathcal{L}_4$, which implies

$$\mathcal{L}_6=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{146}$$

In summary, it is easy to obtain the following estimate

$$J_{11}=\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4+\mathcal{L}_5+\mathcal{L}_6=\mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{147}$$

As for the term of J_{12}, making use of (121) gives rise to

$$\begin{aligned} J_{12} \lesssim\, & \Lambda(t)^2 \int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds \\ & + \Lambda(t)^2 \int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{7}{4}} e^{-\frac{(x+c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds \\ & \triangleq X_1 + X_2. \end{aligned} \tag{148}$$

We write X_1 as follows

$$\begin{aligned} X_1 =& \Lambda(t)^2 \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds \\ & + \Lambda(t)^2 \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds \\ \triangleq & \mathcal{N}_1 + \mathcal{N}_2. \end{aligned} \tag{149}$$

In order to estimate $\mathcal{N}_1$, we break the integration interval into two parts:
(1) $|x-c(1+t)| \leq \sqrt{1+t}$

$$\begin{aligned} \mathcal{N}_1 \lesssim\, & \Lambda(t)^2 \int_0^{\frac{t}{2}} t^{-\frac{1}{2}}(1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds \\ \lesssim\, & \Lambda(t)^2 (1+t)^{-\frac{1}{2}} \int_0^{\frac{t}{2}} (1+s)^{-\frac{7}{4}} ds \\ \lesssim\, & \Lambda(t)^2 (1+t)^{\frac{1}{4}} \psi_{\frac{3}{2}}(x,t;c) \\ \lesssim\, & \Phi(x,t)\Lambda(t)^2. \end{aligned} \tag{150}$$

(2) $|x-c(1+t)| > \sqrt{1+t}$, we use the decomposition as below

$$\mathcal{R}_1 = \{0 \leq s \leq \frac{t}{2}; |x-c(1+t)| \geq 2cs\}, \quad \mathcal{R}_2 = \{0 \leq s \leq \frac{t}{2}; |x-c(1+t)| < 2cs\}. \tag{151}$$

It is easy to get

$$\begin{aligned} \mathcal{N}_1 \lesssim\, & \Lambda(t)^2 (1+t)^{-\frac{1}{2}} \int_{\mathcal{R}_1} (1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds \\ & + \Lambda(t)^2 (1+t)^{-\frac{1}{2}} \int_{\mathcal{R}_2} (1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}} ds. \end{aligned} \tag{152}$$

Denote the first term on the right-hand side of (152) as $\mathcal{N}_{11}$. It is computed as

$$\begin{aligned} \mathcal{N}_{11} \lesssim\, & \Lambda(t)^2 (1+t)^{-\frac{1}{2}} \int_0^{\frac{t}{2}} (1+s)^{-\frac{7}{4}} e^{-\frac{(x-c(1+t))^2}{C(1+t)}} ds \\ \lesssim\, & \Lambda(t)^2 (1+t)^{-\frac{1}{2}} e^{-\frac{(x-c(1+t))^2}{C(1+t)}} \\ \lesssim\, & \Lambda(t)^2 (1+t)^{\frac{1}{4}} \psi_{\frac{3}{2}}(x,t;c) \\ \lesssim\, & \Phi(x,t)\Lambda(t)^2. \end{aligned} \tag{153}$$

The second term on the right-hand side of (152) called $\mathcal{N}_{12}$ can be estimated as follows

$$\mathcal{N}_{12} \lesssim \Lambda(t)^2 (1+t)^{-\frac{1}{2}} \psi_{\frac{3}{2}}(x,t;c) \int_0^{\frac{t}{2}} (1+s)^{-\frac{1}{4}} ds$$

$$\begin{aligned}&\lesssim \Lambda(t)^2(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)\\&\lesssim \Phi(x,t)\Lambda(t)^2.\end{aligned} \tag{154}$$

We conclude from (153) and (154) to have

$$\mathcal{N}_1 = \mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{155}$$

In the next part, we plan to deal with $\mathcal{N}_2$. Similarly, it can be decomposed into two cases: (1) $|x-c(1+t)| \le \sqrt{1+t}$:

$$\begin{aligned}\mathcal{N}_2 &\lesssim \Lambda(t)^2\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{7}{4}}e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}}ds\\&\lesssim \Lambda(t)^2(1+t)^{-\frac{7}{4}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}ds\\&\lesssim \Lambda(t)^2(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)\\&\lesssim \Phi(x,t)\Lambda(t)^2.\end{aligned} \tag{156}$$

(2) $|x-c(1+t)| > \sqrt{1+t}$, as mentioned earlier, the integral domain are divided into two parts

$$\mathcal{R}_3 = \left\{\frac{t}{2} \le s \le t; |x-c(1+t)| \ge 2cs\right\}, \quad \mathcal{R}_4 = \left\{\frac{t}{2} \le s \le t; |x-c(1+t)| < 2cs\right\}. \tag{157}$$

To continue the current estimates, using the above decomposition gives

$$\begin{aligned}\mathcal{N}_2 &\lesssim \Lambda(t)^2(1+t)^{-\frac{7}{4}}\int_{\mathcal{R}_3}(t-s)^{-\frac{1}{2}}e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}}ds\\&\quad+\Lambda(t)^2(1+t)^{-\frac{7}{4}}\int_{\mathcal{R}_4}(t-s)^{-\frac{1}{2}}e^{-\frac{(x-c(t-s))^2}{(2\mu+\epsilon)(t-s)}}ds\\&\triangleq \mathcal{N}_{21}+\mathcal{N}_{22}.\end{aligned} \tag{158}$$

With regard to the first term $\mathcal{N}_{21}$, applying the Lemma 2.1 yields

$$\begin{aligned}\mathcal{N}_{21} &\lesssim \Lambda(t)^2(1+t)^{-\frac{7}{4}}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}e^{-\frac{(x-c(1+t))^2}{C(1+t)}}ds\\&\lesssim \Lambda(t)^2(1+t)^{-\frac{5}{4}}e^{-\frac{(x-c(1+t))^2}{C(1+t)}}\\&\lesssim \Lambda(t)^2(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)\\&\lesssim \Phi(x,t)\Lambda(t)^2.\end{aligned} \tag{159}$$

Similar argument also applies to $\mathcal{N}_{22}$, which implies

$$\begin{aligned}\mathcal{N}_{22} &\lesssim \Lambda(t)^2\psi_{\frac{3}{2}}(x,t;c)\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{4}}ds\\&\lesssim \Lambda(t)^2(1+t)^{\frac{1}{4}}\psi_{\frac{3}{2}}(x,t;c)\\&\lesssim \Phi(x,t)\Lambda(t)^2.\end{aligned} \tag{160}$$

It together with (159) and (160) results in

$$\mathcal{N}_2 = \mathcal{O}(1)\Phi(x,t)\Lambda(t)^2, \tag{161}$$

which also leads to

$$X_1 = \mathcal{N}_1+\mathcal{N}_2 = \mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{162}$$

The same result is derived for X_2 in the same way as X_1. Thus, it holds

$$|J_1| = J_{11} + J_{12} = \mathcal{O}(1)\Phi(x,t)\Lambda(t)^2. \tag{163}$$

As for J_2, due to $x, y \geq 0$, we prove the following inequality by (120)

$$\begin{aligned}|J_2| \leq\ & C\Lambda(t)^2 \int_0^t \int_0^\infty e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(y,s;c)dyds \\ & + C\Lambda(t)^2 \int_0^t \int_0^\infty e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{8}}\phi_1(y,s;c)\psi_{\frac{3}{4}}(y,s;c)dyds.\end{aligned} \tag{164}$$

We only consider the first term named $\mathcal{T}_1$ on the right-hand side of (164). It is easy to verify

$$\begin{aligned}\mathcal{T}_1 \lesssim\ & \Lambda(t)^2 \int_0^t \int_0^{\frac{x}{2}} e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(y,s;c)dyds \\ & + \Lambda(t)^2 \int_0^t \int_{\frac{x}{2}}^\infty e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(y,s;c)dyds \\ \triangleq\ & \mathcal{T}_{11} + \mathcal{T}_{12}.\end{aligned} \tag{165}$$

Utilizing Lemma 2.1 gives

$$\begin{aligned}\mathcal{T}_{11} &\lesssim \Lambda(t)^2 \int_0^t e^{-\frac{x}{4C}} e^{-\frac{t-s}{C}}(1+s)^{-1}ds \\ &\lesssim e^{-\frac{x}{4C}}(1+t)^{-1}\Lambda(t)^2 \\ &\lesssim \Phi(x,t)\Lambda(t)^2.\end{aligned} \tag{166}$$

It remains the challenge for $\mathcal{T}_{12}$. We break the integration region into two parts

$$\begin{aligned}\mathcal{T}_{12} \leq\ & \Lambda(t)^2 \int_0^t \int_{\mathcal{R}_5} e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(y,s;c)dyds \\ & + \Lambda(t)^2 \int_0^t \int_{\mathcal{R}_6} e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}\psi_{\frac{3}{2}}(y,s;c)dyds \\ \triangleq\ & \mathcal{H}_1 + \mathcal{H}_2,\end{aligned} \tag{167}$$

where

$$\mathcal{R}_5 = \{y \geq \frac{x}{2}\Big||y-c(1+s)| \geq c(1+s)\}, \quad \mathcal{R}_6 = \{y \geq \frac{x}{2}\Big||y-c(1+s)| < c(1+s)\}.$$

For $y \in \mathcal{R}_5$, it is obvious to get

$$(y-c(1+s))^2 \geq \frac{1}{2}(y-c(1+s))^2 + \frac{1}{2}c^2(1+s)^2 \geq \frac{1}{4}(|y-c(1+s)|+c(1+s))^2 \geq \frac{1}{4}|y|^2.$$

In conclusion, we obtain

$$\begin{aligned}\mathcal{H}_1 \leq\ & C\Lambda(t)^2 \int_0^{\frac{t}{2}} \int_{\mathcal{R}^5} e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}(1+s+y^2)^{-\frac{3}{4}}dyds \\ & + C\Lambda(t)^2 \int_{\frac{t}{2}}^t \int_{\mathcal{R}^5} e^{-\frac{|x-y|+t-s}{C}}(1+s)^{-\frac{1}{4}}(1+s+y^2+(y-c(1+s))^2)^{-\frac{3}{4}}dyds \\ \leq\ & C\Lambda(t)^2 \int_0^{\frac{t}{2}} e^{-\frac{t-s}{C}}(1+s)^{-\frac{1}{4}}(1+x^2)^{-\frac{3}{4}}ds\end{aligned}$$

$$+ C\Lambda(t)^2 \int_{\frac{t}{2}}^{t} e^{-\frac{t-s}{C}} (1+s)^{-\frac{1}{4}} (1+s+x^2+c^2(1+s)^2)^{-\frac{3}{4}} ds$$
$$\le C\Lambda(t)^2 (1+t)^{\frac{1}{4}} \psi_{\frac{3}{2}}(x,t;-c)$$
$$\le C\Phi(x,t)\Lambda(t)^2.$$

Notice that applying the definition of $\mathcal{R}_6$ indicates

$$\frac{x}{2} \le y \le 2c(1+s) \le 2c(1+t) \Rightarrow |x - c(1+t)| \le 3c(1+t). \tag{168}$$

Thus it holds

$$(1+t)^{-\frac{5}{4}} \le C\phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c). \tag{169}$$

In terms of $\mathcal{H}_2$, we are able to prove

$$\begin{aligned} \mathcal{H}_2 &\le C\Lambda(t)^2 \int_0^t e^{-\frac{t-s}{C}} (1+s)^{-1} ds \\ &\le C\Lambda(t)^2 (1+t)^{-1} \\ &\le C\Lambda(t)^2 (1+t)^{\frac{3}{8}} \phi_1(x,t;c)\psi_{\frac{3}{4}}(x,t;c) \\ &\le C\Phi(x,t)\Lambda(t)^2, \end{aligned} \tag{170}$$

which implies

$$\mathcal{T}_1 \le C\Phi(x,t)\Lambda(t)^2.$$

In the same way, we evaluate the second term on the right-hand side of (164) named $\mathcal{T}_2$ as

$$\mathcal{T}_2 \le C\Phi(x,t)\Lambda(t)^2.$$

To this end, we conclude from above results that

$$|J_2| \le C\Phi(x,t)\Lambda(t)^2. \tag{171}$$

After a direct calculation, it is significant to verify

$$I_0 F(y,s) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (0,f)^t = \mathbf{0}, \tag{172}$$

which implies $J_3 = 0$. Combining the estimates of J_1, J_2, J_3 yields

$$|J| \le |J_1| + |J_2| + |J_3| \le C\Phi(x,t)\Lambda(t)^2. \tag{173}$$

It together with (136) also leads to

$$|U(x,t)| \le |I| + |J| \le C(\delta_0 + \Lambda(t)^2)\Phi(x,t). \tag{174}$$

In the following, we intend to deduce the estimate of $U_x(x,t)$. Applying Theorem 3.1 gives

$$|U_x(x,t)| \le C\|U(x,t)\|_{H^4} \le C\delta_0. \tag{175}$$

Thus for $0 \le t \le 2$, we are able to show

$$|U_x(x,t)| \le C\delta_0 \le C\delta_0(1+t)^{-1}\log(2+t). \tag{176}$$

Hence, the case of $t > 2$ is mainly taken into consideration in the following. We solve the solution $U_x(x,t)$ as

$$U_x(x,t) = \int_0^\infty \partial_x \mathbb{G}(x,t;y)U_0(y)dy + \int_0^t \int_0^\infty \partial_x \mathbb{G}(x,t-s;y)F(y,s)dyds. \tag{177}$$

To cope with the first term on the right-hand side of (177), we write it as

$$\int_0^\infty \partial_x \mathbb{G}(x,t;y)U_0(y)dy = \int_0^\infty \partial_x \mathbb{G}^\ell(x,t;y)U_0(y)dy + \int_0^\infty \partial_x \mathbb{G}^s(x,t;y)U_0(y)dy. \tag{178}$$

In terms of (125), (126) and the assumptions in Theorem 3.3, one has

$$\Big|\int_0^\infty \partial_x \mathbb{G}^\ell(x,t;y)U_0(y)dy\Big| \le C(1+t)^{-1}\int_0^\infty |U_0(y)|dy \le C(1+t)^{-1}\delta_0. \tag{179}$$

Applying the expression of $\mathbb{G}^s(x,t;y)$ gives rise to

$$\Big|\int_0^\infty \partial_x \mathbb{G}^s(x,t;y)U_0(y)dy\Big| \le Ce^{-\frac{c^2}{\mu}t}|U_0'(x)| \le C(1+t)^{-1}\delta_0. \tag{180}$$

Therefore, we conclude from above inequalities to gain

$$\Big|\int_0^\infty \partial_x \mathbb{G}(x,t;y)U_0(y)dy\Big| \le C(1+t)^{-1}\delta_0. \tag{181}$$

In order to deal with the second term in (177), it is easy to verify

$$\begin{aligned}
&\int_0^t\int_0^\infty \partial_x \mathbb{G}(x,t-s;y)F(y,s)dyds \\
&= \int_0^t\int_0^\infty \partial_x \mathbb{G}_1^\ell(x,t-s;y)F(y,s)dyds + \int_0^t\int_0^\infty \partial_x \mathbb{G}_2^\ell(x,t-s;y)F(y,s)dyds \\
&+ \int_0^t\int_0^\infty \partial_x \mathbb{G}^s(x,t-s;y)F(y,s)dyds.
\end{aligned} \tag{182}$$

To avoid the singularity of time, we divide the time interval into $[0,t-1]$ and $[t-1,t]$. It holds

$$\begin{aligned}
&\int_0^t\int_0^\infty \partial_x \mathbb{G}_1^\ell(x,t-s;y)F(y,s)dyds \\
&= \int_0^{t-1}\int_0^\infty \partial_x \mathbb{G}_1^\ell(x,t-s;y)F(y,s)dyds + \int_{t-1}^t\int_0^\infty \partial_x \mathbb{G}_1^\ell(x,t-s;y)F(y,s)dyds.
\end{aligned} \tag{183}$$

Denote the first term on the right-hand side of (183) as $\mathcal{Q}_1$. Integration by parts gives

$$\mathcal{Q}_1 = -\int_0^{t-1}\partial_x \mathbb{G}_1^\ell(x,t-s,0)\tilde{F}(0,s)ds - \int_0^{t-1}\int_0^\infty \partial_x^2 \mathbb{G}_1^\ell(x,t-s;y)\tilde{F}(y,s)dyds.$$

Thanks to (121) and (122), it leads us to obtain

$$\begin{aligned}
&\Big|\int_0^{t-1}\partial_x \mathbb{G}_1^\ell(x,t-s,0)\tilde{F}(0,s)ds\Big| \\
&\le C\Lambda(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-\frac{7}{4}}ds \\
&\le C\Lambda(t)^2\int_0^{\frac{t}{2}}(t-s)^{-1}(1+s)^{-\frac{7}{4}}ds + C\Lambda(t)^2\int_{\frac{t}{2}}^{t-1}(t-s)^{-1}(1+s)^{-\frac{7}{4}}ds \\
&\le C(1+t)^{-1}\Lambda(t)^2.
\end{aligned} \tag{184}$$

In the same way, we have

$$\begin{aligned}&\Big|\int_0^{t-1}\int_0^{\infty}\partial_x^2\mathbb{G}_1^{\ell}(x,t-s;y)\tilde{F}(y,s)dyds\Big|\\&\le C\Lambda(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-1}ds\\&\le C\Lambda(t)^2\int_0^{\frac{t}{2}}(t-s)^{-1}(1+s)^{-1}ds+C\Lambda(t)^2\int_{\frac{t}{2}}^{t-1}(t-s)^{-1}(1+s)^{-1}ds\\&\le C(1+t)^{-1}\log(2+t)\Lambda(t)^2.\end{aligned}\tag{185}$$

Combing above results together gives

$$|\mathcal{Q}_1|\le C(1+t)^{-1}\log(2+t)\Lambda(t)^2.\tag{186}$$

We now proceed to estimate the second term on the right-hand side of (183) named $\mathcal{Q}_2$.

$$\begin{aligned}|\mathcal{Q}_2|&\le C\Lambda(t)^2\int_{t-1}^{t}(t-s)^{-\frac{1}{2}}(1+s)^{-1}ds\\&\le C(1+t)^{-1}\Lambda(t)^2,\end{aligned}\tag{187}$$

which together with (186) also leads to

$$\Big|\int_0^t\int_0^{\infty}\partial_x\mathbb{G}_1^{\ell}(x,t-s;y)F(y,s)dyds\Big|\le C(1+t)^{-1}\log(2+t)\Lambda(t)^2.\tag{188}$$

What is left is to estimate the second term in (182). Making use of (125) yields

$$\begin{aligned}&\Big|\int_0^t\int_0^{\infty}\partial_x\mathbb{G}_2^{\ell}(x,t-s;y)F(y,s)dyds\Big|\\&\le C\Lambda(t)^2\int_0^t\int_0^{\infty}(e^{-\frac{|x-y|+t-s}{C}}+e^{-\frac{|x+y|+t-s}{C}})(1+s)^{-1}dyds\\&\le C\Lambda(t)^2\int_0^t e^{-\frac{t-s}{C}}(1+s)^{-1}ds\\&\le C(1+t)^{-1}\Lambda(t)^2.\end{aligned}\tag{189}$$

The third term in (182) can be treated similarly as J_3, which implies

$$\int_0^t\int_0^{\infty}\partial_x\mathbb{G}^s(x,t-s;y)F(y,s)dyds=0.\tag{190}$$

Applying (188), (189) and (190) gives rise to

$$\Big|\int_0^t\int_0^{\infty}\partial_x\mathbb{G}(x,t-s;y)F(y,s)dyds\Big|\le C(1+t)^{-1}\log(2+t)\Lambda(t)^2.\tag{191}$$

Thus, we are able to derive the following estimate for $U_x(x,t)$ via (181) and (191)

$$|U_x(x,t)|\le C(1+t)^{-1}\log(2+t)(\delta_0+\Lambda(t)^2).\tag{192}$$

This completes the proof. □

Proposition 6. *Under the assumptions of Theorem 3.3, there exists a positive constant C such that*

$$|u_{xx}(x,t)|\le C(1+t)^{-\frac{1}{2}}(\delta_0+\delta_0\Lambda(t)+\Lambda(t)^2).\tag{193}$$

Proof. It is natural to verify that making use of Theorem 3.1 gives the estimate for $0 \le t \le 2$

$$|U_t| \le \|U\|_{H^4} \le C\delta_0 \le C(1+t)^{-\frac{1}{2}}\delta_0, \quad |u_{xx}| \le \|U\|_{H^4} \le C\delta_0 \le C(1+t)^{-\frac{1}{2}}\delta_0.$$

Thus, the case of $t > 2$ is mainly taken into account in the next section. Differentiate the system (112) with respect to time t

$$\begin{cases} \varphi_{tt} - u_{tx} = 0, \\ u_{tt} - c^2\varphi_{tx} = \mu u_{txx} + f_t, \\ (\varphi_t, u_t)|_{t=0} = (\varphi_t(x,0), u_t(x,0)), \\ (au_{tx} + bu_t)|_{x=0} = 0. \end{cases} \tag{194}$$

Let us reformulate above system as the operator form

$$\begin{cases} \partial_t U_t + A\partial_x U_t = B\partial_{xx} U_t + F_t, \\ U_t(x, t=0) = U_t(x,0), \\ (a\partial_t \quad b)U_t(0,t;y) = 0. \end{cases} \tag{195}$$

It is easy to verify that the Green's function for system (195) is the same as $\mathbb{G}(x,t;y)$. Thus, the solution is expressed as follows

$$U_t(x,t) = \int_0^\infty \mathbb{G}(x,t;y)\partial_t U(y,0)dy + \int_0^t \int_0^\infty \mathbb{G}(x,t-s;y)\partial_s F(y,s)dyds. \tag{196}$$

In terms of (112), we have

$$\partial_t U(y,0) = (u_0(x)', -p(1+\varphi_0(x))_x + (\frac{\mu u_0(x)'}{1+\varphi_0(x)})_x)^t \triangleq (u_0(x)', g_0(x)')^t, \tag{197}$$

where g_0 is defined as

$$g_0(x) = -p(1+\varphi_0(x)) + p(1) + \frac{\mu u_0(x)'}{1+\varphi_0(x)}.$$

Then via (123), the first term on the right-hand side of (196) is stated as below

$$\int_0^\infty \mathbb{G}(x,t;y)\partial_t U(y,0)dy = \int_0^\infty \mathbb{G}^\ell(x,t;y)\partial_t U(y,0)dy + \int_0^\infty \mathbb{G}^s(x,t;y)\partial_t U(y,0)dy. \tag{198}$$

Integration by parts with x in $\mathbb{R}_+$ gives rise to

$$\begin{aligned} &\Big|\int_0^\infty \mathbb{G}^\ell(x,t;y)\partial_t U(y,0)dy\Big| \\ &\le \Big|\mathbb{G}^\ell(x,t;0)(u_0(0), g_0(0))^t\Big| + \Big|\int_0^\infty \partial_x \mathbb{G}^\ell(x,t;y)(u_0(y), g_0(y))^t dy\Big| \\ &\le C(1+t)^{-\frac{1}{2}}\delta_0. \end{aligned} \tag{199}$$

Applying the definition of $\mathbb{G}^s(x,t;y)$ and assumptions on the initial data yields

$$\Big|\int_0^\infty \mathbb{G}^s(x,t;y)\partial_t U(y,0)dy\Big| \le C(1+t)^{-\frac{1}{2}}\delta_0, \tag{200}$$

which together with (199) also leads to

$$\Big|\int_0^\infty \mathbb{G}(x,t;y)\partial_t U(y,0)dy\Big| \le C(1+t)^{-\frac{1}{2}}\delta_0. \tag{201}$$

We are in a position to deal with the nonlinear term in (196). It holds

$$
\begin{aligned}
&\int_0^t\int_0^\infty \mathbb{G}(x,t-s;y)\partial_s F(y,s)dyds\\
&=\int_0^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds+\int_0^t\int_0^\infty \mathbb{G}^s(x,t-s;y)\partial_s F(y,s)dyds.
\end{aligned}
\tag{202}
$$

In order to avoid the singularity of time, we break the interval of time into $[0,t-1]$ and $[t-1,t]$.

$$
\begin{aligned}
&\int_0^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds\\
&=\int_0^{t-1}\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds+\int_{t-1}^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds.
\end{aligned}
\tag{203}
$$

Integration by parts with regard to time t gives

$$
\begin{aligned}
&\int_0^{t-1}\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds\\
&=\int_0^{t-1}\int_0^\infty \partial_t\mathbb{G}^\ell(x,t-s;y)F(y,s)dyds+\int_0^\infty \mathbb{G}^\ell(x,1;y)F(y,t-1)dy\\
&\quad-\int_0^\infty \mathbb{G}^\ell(x,t;y)F(y,0)dy.
\end{aligned}
\tag{204}
$$

To estimate the first term in (204), making use of integration by parts with respect to x yields

$$
\begin{aligned}
&\Big|\int_0^{t-1}\int_0^\infty \partial_t\mathbb{G}^\ell(x,t-s;y)F(y,s)dyds\Big|\\
&\le\Big|\int_0^{t-1}\int_0^\infty \partial_{tx}\mathbb{G}^\ell(x,t-s;,y)\tilde{F}(y,s)dyds\Big|+\Big|\int_0^{t-1}\partial_t\mathbb{G}^\ell(x,t-s;0)\tilde{F}(0,s)ds\Big|\\
&\le C\Lambda(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-1}ds+C\Lambda(t)^2\int_0^{t-1}(t-s)^{-1}(1+s)^{-\frac{7}{4}}ds\\
&\le C(1+t)^{-\frac{1}{2}}\Lambda(t)^2.
\end{aligned}
\tag{205}
$$

What is left is to cope with the remaining terms in (204). It is easy to have

$$
\Big|\int_0^\infty \mathbb{G}^\ell(x,1;y)F(y,t-1)dy-\int_0^\infty \mathbb{G}^\ell(x,t;y)F(y,0)dy\Big|\le C(1+t)^{-\frac{1}{2}}\Lambda(t)^2. \tag{206}
$$

As a result, it holds

$$
\Big|\int_0^{t-1}\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds\Big|\le C(1+t)^{-\frac{1}{2}}\Lambda(t)^2. \tag{207}
$$

Using integration by parts with respect to x to deduce the second term in (203)

$$
\begin{aligned}
&\Big|\int_{t-1}^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_s F(y,s)dyds\Big|\\
&\le\Big|\int_{t-1}^t\int_0^\infty \partial_x\mathbb{G}^\ell(x,t-s;y)\partial_s\tilde{F}(y,s)dyds\Big|+\Big|\int_{t-1}^t\mathbb{G}^\ell(x,t-s;0)\partial_s\tilde{F}(0,s)ds\Big|
\end{aligned}
$$

$$\begin{aligned}&\le C(\delta_0\Lambda(t)+\Lambda(t)^2)\int_{t-1}^{t}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}}ds\\&\le C(1+t)^{-\frac{1}{2}}(\delta_0\Lambda(t)+\Lambda(t)^2),\end{aligned}\tag{208}$$

where we have used Theorem 3.1 and Proposition 5 to show

$$|\partial_s\tilde{F}(y,s)|\le C(|\varphi||u_y|+|u_y|^2+|\varphi||u_{sy}|)\le C(1+s)^{-\frac{1}{2}}(\delta_0\Lambda(t)+\Lambda(t)^2),\tag{209}$$

and

$$|\partial_s\tilde{F}(0,s)|\le C(1+s)^{-\frac{1}{2}}(\delta_0\Lambda(t)+\Lambda(t)^2).\tag{210}$$

Combing (208) and (207) together also leads to

$$\Big|\int_0^t\int_0^\infty \mathbb{G}^\ell(x,t-s;y)\partial_sF(y,s)dyds\Big|\le C(1+t)^{-\frac{1}{2}}(\delta_0\Lambda(t)+\Lambda(t)^2).\tag{211}$$

Via (124), the second term in (202) is computed as

$$\int_0^t\int_0^\infty \mathbb{G}^s(x,t-s;y)\partial_sF(y,s)dyds=0.\tag{212}$$

As a result of (201), (211) and (212), we have

$$|U_t(x,t)|\le C(1+t)^{-\frac{1}{2}}(\delta_0+\delta_0\Lambda(t)+\Lambda(t)^2).\tag{213}$$

To close the ansatz, we are going to derive the pointwise estimate of u_{xx}. Applying (68) and Proposition 5 yields

$$\begin{aligned}|u_{xx}(x,t)|&\le C(1+|\varphi|)|u_t|+C(1+|\varphi|)|n_x|+C|u_x||\varphi_x|\\&\le C(1+t)^{-\frac{1}{2}}(\delta_0+\delta_0\Lambda(t)+\Lambda(t)^2).\end{aligned}\tag{214}$$

This completes the proof of the proposition. □

6.1. **The proof of Theorem 3.3.**

Proof. Combining (118), Proposition 5 and Proposition 6 together yields

$$\Lambda(t)\le C\delta_0+C\delta_0\Lambda(t)+C\Lambda(t)^2.\tag{215}$$

Since δ_0 is sufficiently small, we deduce that there exists a constant $C>0$ independent of time such that

$$\Lambda(t)\le C\delta_0.\tag{216}$$

Then the pointwise estimates of solutions are stated as

$$\begin{aligned}&|U(x,t)|\le C\delta_0\Phi(x,t),\\&|U_x(x,t)|\le C\delta_0(1+t)^{-1}\log(2+t),\\&|u_{xx}(x,t)|\le C\delta_0(1+t)^{-\frac{1}{2}}.\end{aligned}\tag{217}$$

We hence complete the proof. □

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