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Space-time behavior of the solution to the Boltzmann equation with soft potentials [☆]

Yu-Chu Lin [a], Ming-Jiea Lyu [a], Haitao Wang [c], Kung-Chien Wu [a,b,*]

[a] *Department of Mathematics, National Cheng Kung University, Tainan, Taiwan*

[b] *National Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan*

[c] *School of Mathematical Sciences, Institute of Natural Sciences, MOE-LSE and CMA-Shanghai, Shanghai Jiao Tong University, Shanghai, China*

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Abstract

In this paper, we get the quantitative space-time behavior of the full Boltzmann equation with soft potentials ($-2 < \gamma < 0$) in the close to equilibrium setting, under some velocity decay assumption, but without any Sobolev regularity assumption on the initial data. We find that both the large time and spatial behaviors depend on the velocity decay of the initial data and the exponent γ. The key step in our strategy is to obtain the L^∞ bound of a suitable weighted full Boltzmann equation directly, rather than using Green's function and Duhamel's principle to construct the pointwise structure of the solution as in [25]. This provides a new thinking in the related study.

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Keywords: Boltzmann equation; Maxwellian states; Pointwise estimate

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* Corresponding author at: Department of Mathematics, National Cheng Kung University, Tainan, Taiwan.
E-mail addresses: yuchu@mail.ncku.edu.tw (Y.-C. Lin), mingjiealyu@gmail.com (M.-J. Lyu), haitallica@sjtu.edu.cn (H. Wang), kcwu@mail.ncku.edu.tw (K.-C. Wu).

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1. Introduction

1.1. The models

Consider the following Boltzmann equation:

$$\begin{cases} \partial_t F + \xi \cdot \nabla_x F = Q(F, F), \\ F(0, x, \xi) = F_0(x, \xi), \end{cases} \tag{1.1}$$

where $F(t, x, \xi)$ is the distribution function of the particles at time $t > 0$, position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and microscopic velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. The left-hand side of this equation models the transport of particles and the operator on the right-hand side models the effect of collisions on the transport with

$$Q(F, G) = \frac{1}{2} \int_{\mathbb{R}^3 \times S^2} |\xi - \xi_*|^\gamma B(\vartheta) \left\{ F'_* G' + G'_* F' - F_* G - G_* F \right\} d\xi_* d\omega.$$

Here the usual conventions, i.e., $F = F(t, x, \xi)$, $F_* = F(t, x, \xi_*)$, $F' = F(t, x, \xi')$ and $F'_* = F(t, x, \xi'_*)$, are used.

In this paper, we consider the soft potentials ($-2 < \gamma < 0$); and $B(\vartheta)$ satisfies the Grad's angular cutoff assumption

$$0 < B(\vartheta) \le C \left| \cos \vartheta \right|,$$

for some constant $C > 0$. Moreover, the post-collisional velocities satisfy

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi + [(\xi - \xi_*) \cdot \omega]\omega,$$

and ϑ is defined by

$$\cos \vartheta = \frac{|(\xi - \xi_*) \cdot \omega|}{|\xi - \xi_*|}.$$

It is well known that the global Maxwellians are steady-state solutions to the Boltzmann equation (1.1). Therefore, it is natural to consider the Boltzmann equation (1.1) around a global Maxwellian

$$\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2} \right),$$

with the standard perturbation $f(t, x, \xi)$ to $\mathcal{M}$ as

$$F = \mathcal{M} + \mathcal{M}^{1/2} f, \quad F_0 = \mathcal{M} + \eta \mathcal{M}^{1/2} f_0,$$

where $\eta > 0$ is sufficiently small. After substituting F and F_0 into (1.1), the equation for the perturbation f is

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f), \\ \\ f(0, x, \xi) = \eta f_0(x, \xi) = \dfrac{F_0 - \mathcal{M}}{\sqrt{\mathcal{M}}}, \end{cases} \tag{1.2}$$

where $L = -\nu(\xi) + K$ is the linearized collision operator defined as

$$Lf = \mathcal{M}^{-1/2}\left[Q(\mathcal{M}, \mathcal{M}^{1/2} f) + Q(\mathcal{M}^{1/2} f, \mathcal{M})\right],$$

and Γ is the nonlinear operator defined as

$$\Gamma(f, f) = \mathcal{M}^{-1/2} Q(\mathcal{M}^{1/2} f, \mathcal{M}^{1/2} f).$$

It is well-known that the null space of L is a five-dimensional vector space with the orthonormal basis $\{\chi_i\}_{i=0}^4$, where

$$Ker(L) = \{\chi_0, \chi_i, \chi_4\} = \left\{\mathcal{M}^{1/2},\ \xi_i \mathcal{M}^{1/2},\ \frac{1}{\sqrt{6}}(|\xi|^2 - 3)\mathcal{M}^{1/2}\right\}, \quad i = 1, 2, 3.$$

Based on this property, we can introduce the macro-micro decomposition: let P_0 be the orthogonal projection with respect to the L^2_ξ inner product onto $\mathrm{Ker}(L)$, and $\mathrm{P}_1 \equiv \mathrm{Id} - \mathrm{P}_0$.

1.2. Notation

Before the presentation of the main theorem, let us define some notations used in this paper. We denote $\langle \xi \rangle^s = (1 + |\xi|^2)^{s/2}$ and $\langle \xi \rangle_D^s = (D^2 + |\xi|^2)^{s/2}$, where $D > 0$, $s \in \mathbb{R}$. For the microscopic variable ξ, we denote

$$|g|_{L^q_\xi} = \Big(\int_{\mathbb{R}^3} |g|^q d\xi\Big)^{1/q} \text{ if } 1 \le q < \infty, \qquad |g|_{L^\infty_\xi} = \sup_{\xi \in \mathbb{R}^3} |g(\xi)|,$$

and the weighted norms can be defined by

$$|g|_{L^q_{\xi,\beta}} = \Big(\int_{\mathbb{R}^3} \left|\langle \xi \rangle^\beta g\right|^q d\xi\Big)^{1/q} \text{ if } 1 \le q < \infty, \qquad |g|_{L^\infty_{\xi,\beta}} = \sup_{\xi \in \mathbb{R}^3} \left|\langle \xi \rangle^\beta g(\xi)\right|,$$

and

$$|g|_{L^\infty_\xi(m)} = \sup_{\xi \in \mathbb{R}^3} \{|g(\xi)| m(\xi)\},$$

where $\beta \in \mathbb{R}$ and m is a weight function. The L^2_ξ inner product in $\mathbb{R}^3$ will be denoted by $\langle \cdot, \cdot \rangle_\xi$, i.e.,

$$\langle f, g \rangle_\xi = \int f(\xi) \overline{g(\xi)} d\xi.$$

For the Boltzmann equation, the natural norm in ξ is $|\cdot|_{L^2_\sigma}$, which is defined as

$$|g|^2_{L^2_\sigma} = \left|\langle\xi\rangle^{\frac{\gamma}{2}} g\right|^2_{L^2_\xi}.$$

For the space variable x, we have similar notations, namely,

$$|g|_{L^q_x} = \left(\int_{\mathbb{R}^3} |g|^q dx\right)^{1/q} \quad \text{if } 1 \le q < \infty, \qquad |g|_{L^\infty_x} = \sup_{x\in\mathbb{R}^3} |g(x)|.$$

Furthermore, we define the high order Sobolev norm: let $s \in \mathbb{N}$ and define

$$|g|_{H^s_\xi} = \sum_{|\alpha|\le s} \left|\partial^\alpha_\xi g\right|_{L^2_\xi}, \qquad |g|_{H^s_x} = \sum_{|\alpha|\le s} \left|\partial^\alpha_x g\right|_{L^2_x},$$

where α is any multi-index with $|\alpha| \le s$.

Finally, with $\mathcal{X}$ and $\mathcal{Y}$ being normed spaces, we define

$$\|g\|_{\mathcal{X}\mathcal{Y}} = \left||g|_{\mathcal{Y}}\right|_{\mathcal{X}},$$

and for simplicity, we denote

$$\|g\|_{L^2} = \|g\|_{L^2_\xi L^2_x} = \left(\int_{\mathbb{R}^3} |g|^2_{L^2_x} d\xi\right)^{1/2}.$$

The domain decomposition plays an important role in our analysis, so we introduce a cut-off function $\chi : \mathbb{R} \to \mathbb{R}$, which is a smooth non-increasing function, $\chi(s) = 1$ for $s \le 1$, $\chi(s) = 0$ for $s \ge 2$ and $0 \le \chi \le 1$. Moreover, we define $\chi_R(s) = \chi(s/R)$ for positive R.

For simplicity of notations, hereafter, we abbreviate "$\le C$" to "$\lesssim$", where C is a positive constant depending only on fixed numbers.

1.3. Review of previous works and main result

In the literature, there are a lot of works concerning the large time behavior of the solution for various models of the Boltzmann equation, such as the hard sphere, hard potentials and soft potentials.

In the literature, there are several energy methods for the study of the Boltzmann equations near Maxwellian in the whole space. The direct energy method through the micro-macro decomposition was initiated by Liu-Yu [24] and developed by Liu-Yang-Yu [26] and Guo [13] independently in two different ways. In between there is another energy method introduced by Kawashima [18], which is based on constructing compensating function for the thirteen moments of Boltzmann equation. Under some suitable Sobolev regularity assumptions on the initial condition, combining energy estimate with the spectrum method [10,11,31] or compensating function method [5,18,32], one can get the time decay rate. For more details, the reader is referred to the reference therein. In addition, people are aware that the large time behavior is governed by the long wave part in terms of the Fourier variables of the linearized equation, no matter for the hard sphere, hard potential or soft potential.

For Boltzmann equation in a bounded domain, an important $L^2 - L^\infty$ theory was developed in [14] to obtain the global existence and the exponential decay rate of the solution around a global Maxwellian for hard potentials associated with appropriate boundary conditions. See also [23] for its extension to soft potential in a bounded domain, where a sub-exponential decay rate is obtained. One is also referred to [7,15,19] for the recent advancements of this theory.

On the other hand, it is noted that the inter-molecular potential can influence the spatially asymptotic behavior for the stationary linearized Boltzmann half space problem (i.e., the Milne problem). Indeed, [2] obtained exponential decay for the hard sphere case, [1,8] obtained arbitrary polynomial decay for the hard potential upon assuming corresponding velocity weights on boundary data, and [9] obtained sub-exponential decay for the hard potential upon assuming Gaussian weight. Thus, it would be interesting to investigate the space-time behaviors of the solutions for different potentials. To this end, the pointwise approach has been initiated by [25,27,28] for the full nonlinear hard sphere case, and then generalized by [20–22] to hard and soft potential cases on the linear level.

However, the nonlinear problems for hard potential and soft potential have not been settled. In this paper, the spatially asymptotic behavior and uniform time decay for fully nonlinear Boltzmann equation with soft potential are established. The similar result for hard potential is also stated without proof, which is actually easier. It is worth mentioning that our results do not require any Sobolev regularity of the initial data. The main results are stated as follows.

Theorem 1 *(The large time behavior for $-2<\gamma<0$). Let $-2<\gamma<0$, $0<p_1\le 2$, $p_2>3/2$, $\hat{\varepsilon}\ge 0$ sufficiently small, and $j>0$ sufficiently large. Assume that the initial data ηf_0 satisfies $f_{w_30}=w_3 f_0\in L^\infty_{\xi,p_2+3j}(L^1_x\cap L^\infty_x)$ where $w_3=e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}$ ($\hat{\varepsilon}\ge 0$), and $\eta>0$ is sufficiently small. Then there is a unique solution f to (1.2) in $L^\infty_{\xi,p_2+2j}(e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}})L^2_x\cap L^\infty_{\xi,p_2+2j}(e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}})L^\infty_x$ with*

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2}L^2_x}\le \eta C_1(1+t)^{-\frac{3}{4}}\left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j}L^1_x}+\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j}L^\infty_x}\right), \tag{1.3}$$

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2}L^\infty_x}\le \eta C_2(1+t)^{-\frac{3}{2}}\left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+3j}L^1_x}+\|w_3 f_0\|_{L^\infty_{\xi,p_2+3j}L^\infty_x}\right), \tag{1.4}$$

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2+2j}L^2_x}\le \eta \bar{C}_1\left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j}L^1_x}+\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j}L^\infty_x}\right), \tag{1.5}$$

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2+2j}L^\infty_x}\le \eta \bar{C}_2\left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j}L^1_x}+\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j}L^\infty_x}\right), \tag{1.6}$$

for some positive constants C_1, C_2, $\bar{C}_1$, $\bar{C}_2$ depending on γ, $\hat{\varepsilon}$, p_1, p_2, and j.

We here mention that whenever $\hat{\varepsilon}=0$, $f_{w_3}=f$ is the solution to the equation (1.2).

Theorem 2 *(The spatially asymptotic behavior for $-2<\gamma<0$). Let $-2<\gamma<0$ and let f be a solution to the Boltzmann equation (1.2) with initial data ηf_0, where f_0 is compactly supported in the x-variable for all ξ:*

$$f_0(x,\xi)\equiv 0 \textit{ for } |x|\ge 1,\ \xi\in\mathbb{R}^3,$$

and $\eta > 0$ is sufficiently small.

(i) Let $0 < \varsigma \ll 1$. Suppose that $|f_0|_{L^\infty_x} \in L^\infty_{\xi, p+\beta+3j}$ for some $p \geq 1$, $\beta > 3/2$, and $j > 0$ large enough. Then:

If $-1 < \gamma < 0$, there exists $M > 0$ such that for $\langle x \rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^2(\langle x \rangle + Mt)^{\frac{-p}{1-\gamma}} \left\| \langle \xi \rangle^{p+\beta+3j} f_0 \right\|_{L^\infty_\xi L^\infty_x}.$$

If $\gamma = -1$, there exists $M > 0$ such that for $\langle x \rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^{2+\varsigma}(\langle x \rangle + Mt)^{\frac{-p}{1-\gamma}} \left\| \langle \xi \rangle^{p+\beta+3j} f_0 \right\|_{L^\infty_\xi L^\infty_x}.$$

If $-2 < \gamma < -1$, there exists $M > 0$ such that for $\langle x \rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^{7+\frac{5}{\gamma}}(\langle x \rangle + Mt)^{\frac{-p}{1-\gamma}} \left\| \langle \xi \rangle^{p+\beta+3j} f_0 \right\|_{L^\infty_\xi L^\infty_x}.$$

(ii) Let $0 < \varsigma \ll 1$. Suppose that $|f_0|_{L^\infty_x} \in L^\infty_\xi(e^{\hat{\varepsilon}\langle \xi \rangle^p} \langle \xi \rangle^{p+\beta+3j})$ for some $0 < p \leq 2$, $\beta > 3/2$, $\hat{\varepsilon} > 0$ sufficiently small, and $j > 0$ large enough. Then:

If $-1 < \gamma < 0$, there exist $M > 0$ and $0 < \varepsilon < \hat{\varepsilon}$ such that for $\langle x \rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^2 e^{-\varepsilon(\langle x \rangle + Mt)^{\frac{p}{p+1-\gamma}}} \|e^{\hat{\varepsilon}\langle \xi \rangle^p} \langle \xi \rangle^{p+\beta+3j} f_0\|_{L^\infty_\xi L^\infty_x}.$$

If $\gamma = -1$, there exist $M > 0$ and $0 < \varepsilon < \hat{\varepsilon}$ such that for $\langle x \rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^{2+\varsigma} e^{-\varepsilon(\langle x \rangle + Mt)^{\frac{p}{p+1-\gamma}}} \|e^{\hat{\varepsilon}\langle \xi \rangle^p} \langle \xi \rangle^{p+\beta+3j} f_0\|_{L^\infty_\xi L^\infty_x}.$$

If $-2 < \gamma < -1$, there exist $M > 0$ and $0 < \varepsilon < \hat{\varepsilon}$ such that for $\langle x \rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^{7+\frac{5}{\gamma}} e^{-\varepsilon(\langle x \rangle + Mt)^{\frac{p}{p+1-\gamma}}} \|e^{\hat{\varepsilon}\langle \xi \rangle^p} \langle \xi \rangle^{p+\beta+3j} f_0\|_{L^\infty_\xi L^\infty_x}.$$

In fact, we have also established the corresponding results for the full nonlinear Boltzmann equation with hard potential cases (i.e., $0 \leq \gamma < 1$). The proof in that case is almost the same as in the soft potential one and most of the parallel lemmas can be obtained more easily. To avoid a lengthy discussion, we focus on the soft potential case in this paper and just state the results for the hard potential as below.

Theorem 3 *(The large time behavior for $0 \leq \gamma < 1$). Let $0 \leq \gamma < 1$, $0 < p_1 \leq 2$, $p_2 > 3/2$, and let $\hat{\varepsilon} \geq 0$ be sufficiently small. Assume that the initial f_0 satisfies $f_{w_3 0} = w_3 f_0 \in L^\infty_{\xi, p_2+\gamma}(L^1_x \cap L^\infty_x)$ where $w_3 = e^{\hat{\varepsilon}\langle \xi \rangle^{p_1}}$, and $\eta > 0$ is sufficiently small. Then there exists a unique solution f to (1.2) in $L^\infty_{\xi, p_2+\gamma}(e^{\hat{\varepsilon}\langle \xi \rangle^{p_1}}) L^2_x \cap L^\infty_{\xi, p_2+\gamma}(e^{\hat{\varepsilon}\langle \xi \rangle^{p_1}}) L^\infty_x$ with*

$$\|f_{w_3}\|_{L^\infty_{\xi,p_2+\gamma}L^\infty_x} \lesssim \eta(1+t)^{-3/2}\left(\|f_{w_30}\|_{L^\infty_{\xi,p_2+\gamma}L^\infty_x} + \|f_{w_30}\|_{L^\infty_{\xi,p_2}L^1_x}\right), \tag{1.7}$$

and

$$\|f_{w_3}\|_{L^\infty_{\xi,p_2+\gamma}L^2_x} \lesssim \eta(1+t)^{-3/4}\left(\|f_{w_30}\|_{L^\infty_{\xi,p_2+\gamma}L^\infty_x} + \|f_{w_30}\|_{L^\infty_{\xi,p_2}L^1_x}\right). \tag{1.8}$$

We here mention again that $f_{w_3} = f$ is the solution to the equation (1.2) whenever $\hat{\varepsilon} = 0$.

Theorem 4 *(The spatially asymptotic behavior for $0 \le \gamma < 1$). Let $0 \le \gamma < 1$ and let f be a solution to the Boltzmann equation (1.2) with initial data ηf_0, where f_0 is compactly supported in the x-variable for all ξ:*

$$f_0(x,\xi) \equiv 0 \text{ for } |x| \ge 1,\ \xi \in \mathbb{R}^3,$$

and $\eta > 0$ is sufficiently small.

(i) Suppose that $|f_0|_{L^\infty_x} \in L^\infty_{\xi,p+\beta+\gamma/2}$ for some $p \ge 1$ and $\beta > 3/2$. Then there exists $M > 0$ such that for $\langle x\rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^{1/2}(\langle x\rangle + Mt)^{\frac{-p}{1-\gamma}}\|f_0\|_{L^\infty_{\xi,p+\beta+\gamma/2}L^\infty_x}.$$

(ii) Suppose that $|f_0|_{L^\infty_x} \in L^\infty_\xi(e^{\hat{\varepsilon}\langle\xi\rangle^p}\langle\xi\rangle^{p+\beta+\gamma/2})$ for some $0 < p \le 2$, $\beta > 3/2$, $\hat{\varepsilon} > 0$ sufficiently small. Then there exist $M > 0$ and $0 < \varepsilon < \hat{\varepsilon}$ such that for $\langle x\rangle > 2Mt$,

$$|f(t,x)|_{L^\infty_{\xi,\beta}} \lesssim \eta(1+t)^{1/2}e^{-\varepsilon(\langle x\rangle+Mt)^{\frac{p}{p+1-\gamma}}}\|e^{\hat{\varepsilon}\langle\xi\rangle^p}\langle\xi\rangle^{p+\beta+\gamma/2}f_0\|_{L^\infty_\xi L^\infty_x}.$$

1.4. Method of proof and plan of the paper

In order to study the spatially asymptotic behavior of the solution f to the full nonlinear Boltzmann equation (1.2), the following weight functions will be taken into account (which are motivated by the linear results [21,22]):

Weight function w_1. Let $\delta > 0$ be sufficiently small, D, $M \ge 1$ sufficiently large and $p \ge 1$. Define w_1 as

$$w_1(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{p}{1-\gamma}}\left(1-\chi\left(\frac{\delta(\langle x\rangle - Mt)}{\langle\xi\rangle_D^{1-\gamma}}\right)\right) + 3\langle\xi\rangle_D^p\,\chi\left(\frac{\delta(\langle x\rangle - Mt)}{\langle\xi\rangle_D^{1-\gamma}}\right). \tag{1.9}$$

Weight function w_2. Let ϵ, $\delta > 0$ be sufficiently small, $M > 0$ sufficiently large and $0 < p \le 2$. Define w_2 as

$$w_2(t,x,\xi) = e^{\epsilon\rho(t,x,\xi)} \tag{1.10}$$

with

$$\rho(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{p}{p+1-\gamma}}\left(1-\chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right)\right) + 3\langle\xi\rangle^{p}\chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right).$$

Weight function w_3**.** Let $\hat{\varepsilon} \geq 0$ be sufficiently small and $0 < p_1 \leq 2$. Define w_3 as

$$w_3(\xi) = e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}. \tag{1.11}$$

Here we mention that the coefficients 5 and 3 can be replaced by other combinations of positive constants a and b with $a \geq b > 0$, meeting the desired requirement $\partial_t w_i \leq 0$ $(i = 1, 2)$. Now, let $f_{w_i} = w_i f$, $i = 1, 2$. Then f_{w_i} $(i = 1, 2)$ satisfies the equation

$$\begin{cases} \partial_t f_{w_i} + \xi \cdot \nabla_x f_{w_i} - \dfrac{(\partial_t w_i + \xi \cdot \nabla_x w_i)}{w_i} f_{w_i} = L_{w_i} f_{w_i} + \Gamma_{w_i}(f_{w_i}, f), \\ f_{w_i}(0,x,\xi) = \eta w_i(0,x,\xi) f_0(x,\xi) \equiv \eta f_{w_i 0}(x,\xi). \end{cases} \tag{1.12}$$

Here $L_{w_i} f_{w_i} = \left(w_i L w_i^{-1}\right) f_{w_i} = \left(-\nu(\xi) + K_{w_i}\right) f_{w_i}$, $\Gamma_{w_i}(f_{w_i}, f) = w_i \Gamma(w_i^{-1} f_{w_i}, f)$.

Therefore, in order to get the spatially asymptotic behavior of the solution f to (1.2), the key step of our strategy is to obtain the L^∞ bound for the solution u to the weighted linearized Boltzmann equation with a source term as below:

$$\begin{cases} \partial_t u + \xi \cdot \nabla_x u - \dfrac{(\partial_t w_i + \xi \cdot \nabla_x w_i)}{w_i} u = L_{w_i} u + \Gamma_{w_i}(g_i, h_i), \\ u(0,x,\xi) = \eta w_i(0,x,\xi) f_0(x,\xi) \equiv \eta f_{w_i 0}(x,\xi), \end{cases} \tag{1.13}$$

where g_i and h_i are prescribed, $i = 1, 2$. With the sharp estimate of f, a priori estimate of f_{w_i}, and substituting $g_i = f_{w_i}$, $h_i = f$, we can obtain the L^∞ bound of f_{w_i}.

Note that for $\langle x\rangle > 2Mt$, we have

$$\langle x\rangle - Mt > \frac{\langle x\rangle}{3} + \frac{Mt}{3},$$

therefore one has

$$w_1(t,x,\xi) \gtrsim \left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{p}{1-\gamma}} \gtrsim \left[\langle x\rangle + Mt\right]^{\frac{p}{1-\gamma}}$$

and

$$\rho(t,x,\xi) \gtrsim \left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{p}{p+1-\gamma}} \gtrsim \left[\langle x\rangle + Mt\right]^{\frac{p}{p+1-\gamma}}.$$

According to the L^∞ bound of f_{w_i}, it provides the spatial asymptotic behavior of the solution in Theorem 2 and Theorem 4.

The procedure relies on large time decay of the solution f to nonlinear problem for initial data living in ξ-weighted space. Using compensating function methods and the wave-remainder decomposition, we first obtain the large time behavior of the linearized equation in normed spaces $L^2_\xi L^2_x$ and $L^2_\xi L^\infty_x$. By applying Ukai's bootstrap argument to the integral equation, we improve the estimates to the weighted spaces $L^\infty_\xi\left(e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\langle\xi\rangle^{p_2}\right)L^2_x$, $L^\infty_\xi\left(e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\langle\xi\rangle^{p_2}\right)L^\infty_x$, etc. Furthermore, given a source term $\Gamma(h_1,h_2)$ with prescribed time decay (see (5.15)), we establish the large time behavior for inhomogeneous equation, through Duhamel principle in terms of Green's function and damped transport operator, together with refined estimates for $\Gamma(h_1,h_2)$. The estimate for the nonlinear term Γ is more exquisite in the soft potential case ($-2<\gamma<0$). In particular, in Lemma 15, the extra decay (-1) in (2.39) is important in studying the linearized equation with a source term $\Gamma(h_1,h_2)$. With the help of an extra interpolation inequality (Lemma 29), it enables us to get the time decay of $\Gamma(h_1,h_2)$ from h_1 and h_2 through these refined estimates for Γ. The large time behavior of the nonlinear problem (1.2) then follows from an iteration scheme. Due to the interpolation argument, we only get the large time in the L^∞_x at the rate of $(1+t)^{-\frac{3}{4}}$ at first glance, then we recover the rate of $(1+t)^{-\frac{3}{2}}$ by a bootstrap process (see Section 5).

Next we turn to the L^∞ bound of the solution u to the equation (1.13). We combine the wave-remainder decomposition, the energy estimate, and the regularization estimates to conclude the proof. In the sequel, we explain the idea in more details. The wave-remainder decomposition is based on a Picard-type iteration, which is manipulated to construct the increasingly regular particle-like waves. The pointwise estimate for the wave part is obtained from the property of the time-dependent damped transport operator (defined in (3.9) and (3.11). It is noted damped transport equation in weighted equation is not an autonomous differential equation, so one needs to consider the evolution operator rather than simple semi-group. The energy estimate is used to analyze the remainder term. In the course of this procedure, the regularization estimate (see Lemma 24) plays a crucial role, which allows us to show the remainder becomes regular, and in turn do the higher order energy estimate. Also thanks to the regularization estimate, we obtain the pointwise estimate without regularity assumption on the initial data. Finally, we bootstrap the remainder part from L^2_ξ to $L^\infty_{\xi,\beta}$ ($\beta>3/2$) so that the velocity norms of the remainder part and the wave part become consistent.

Here we would like to remark three points in the proof: (1) due to the weaker damping term (i.e., $-2<\gamma<0$), one needs to trade off velocity decay for time decay either to get the decay of f or to control the growth of u, so the delicate velocity-weight-gaining properties of K_{w_i}, Γ, Γ_{w_i} (see Lemmas 11, 15 and 17) are fully used in the estimates; (2) although the bootstrap from L^2_ξ to L^∞_ξ is frequently used in the proof, it is not obvious the integral operator K owns this property if $-2<r\le-3/2$. Thanks to Riesz-Thorin interpolation theorem, K has $L^4_{\xi,1-\gamma}$-L^2_ξ estimate in the case $-2<\gamma\le-3/2$ (see Lemma 8). Associated with $L^\infty_{\xi,7/4-\gamma}$-$L^4_\xi$ estimate, K eventually has $L^\infty_{\xi,7/4-\gamma}$-$L^4_\xi$-$L^2_\xi$ estimate in the case $-2<\gamma\le-3/2$; (3) In the proof of Lemma 24, it reveals that the mixture of the two operators $\mathbb{S}^t_{w_i}$ and K_{w_i} can transport the regularity in the microscopic velocity ξ induced by K_{w_i} to the regularity in the space x. It is worth mentioning that K_{w_i} is an integral operator from L^2_ξ to H^1_ξ only when $\gamma>-2$ (see Lemma 11), this is the reason why we restrict ourselves to the case $\gamma>-2$. The removal of this restriction is left to the future.

Lastly, we want to compare the method in this paper with those in [25], which studied the nonlinear Boltzmann equation with hard sphere, and gave the only space-time pointwise structure result of the nonlinear solution so far. There, it is crucial that the estimates of linear problem can be obtained in the same weighted space as the initial data, which allows for the nonlinear iteration, then the authors achieve the estimate of nonlinear problem. However, for hard potential, as well as soft potential, this methodology does not work since one needs extra weights for maintaining the space-time structure even for the linear equation (see [20–22]). As a comparison, to obtain the spatially asymptotic behavior of the nonlinear equation, we circumvent the difficulty of nonlinear iteration due to mismatch of velocity weight on the linear level, and directly study the $L^\infty_{\xi,\beta}L^\infty_x$ estimate of the solution f_{w_i} to the weighted full Boltzmann equation (1.12). This is a new idea in the related studies. In addition, it should be mentioned that although the spatially asymptotic behavior and time decay of the nonlinear solution are achieved by this new method, we still do not have the space-time pointwise description of the solution as precise as the result in [25]. For soft potential, this is understandable since there is no detailed spectral information of the linearized operator, which contains the fluid behavior of the solution, and thus the pointwise structure inside the finite Mach region. For hard potential, one can indeed obtain the pointwise structure for linearized equation, but cannot close the nonlinear iteration due to the loss of velocity weight in linear estimate. Therefore, it is still challenging to investigate the space-time pointwise structure of the nonlinear Boltzmann equation with potentials other than hard sphere.

The rest of this paper is organized as follows: We first present some basic properties concerning the operators L, Γ and the corresponding weighted operators L_{w_i} $(i = 1, 2, 3)$ and Γ_{w_i} $(i = 1, 2, 3)$ in Section 2. After that, we study the weighted linearized Boltzmann equation with a source term in Section 3. With these preparations and the large time behavior (Theorem 1), we demonstrate the spatially asymptotic behavior (Theorem 2) in Section 4, and postpone the proof of Theorem 1 until Section 5.

2. Preliminaries

As mentioned in the Introduction section, we will study the weighted equation (1.12) first. Before proceeding, some basic properties concerning the operators L, Γ and the corresponding weighted operators L_{w_i} $(i = 1, 2, 3)$ and Γ_{w_i} $(i = 1, 2, 3)$, need to be studied. The linearized collision operator L, which was analyzed extensively by Grad [16], consists of a multiplicative operator $\nu(\xi)$ and an integral operator K:

$$Lf = -\nu(\xi)f + Kf, \tag{2.1}$$

where

$$\nu(\xi) = \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}(\xi_*)d\xi_*d\omega,$$

and

$$Kf = -K_1 f + K_2 f \tag{2.2}$$

is defined as [16]:

$$\begin{aligned} K_1 f &= \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi)\mathcal{M}^{1/2}(\xi_*)f(\xi_*)d\xi_*d\omega, \\ K_2 f &= \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi_*)\mathcal{M}^{1/2}(\xi')f(\xi'_*)d\xi_*d\omega \\ &\quad + \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi_*)\mathcal{M}^{1/2}(\xi'_*)f(\xi')d\xi_*d\omega. \end{aligned}$$

To begin with, we present a number of properties and estimates of the operators L, $\nu(\xi)$ and K, which can be found in [3,6,8,16,22,30].

Lemma 5. *Let $-2 < \gamma < 0$. For any $g \in L^2_\sigma$, we have the coercivity of the linearized collision operator L, that is, there exists a positive constant ν_0 such that*

$$\langle g, Lg \rangle_\xi \le -\nu_0 \left| \mathrm{P}_1 g \right|^2_{L^2_\sigma} . \tag{2.3}$$

For the multiplicative operator $\nu(\xi)$, there are positive constants ν_0 and ν_1 such that

$$\nu_0 \langle \xi \rangle^\gamma \le \nu(\xi) \le \nu_1 \langle \xi \rangle^\gamma , \tag{2.4}$$

and for each multi-index α,

$$\left| \partial^\alpha_\xi \nu(\xi) \right| \lesssim \langle \xi \rangle^{\gamma - |\alpha|} . \tag{2.5}$$

For the integral operator K,

$$Kf = -K_1 f + K_2 f = \int\limits_{\mathbb{R}^3} -k_1(\xi, \xi_*) f(\xi_*) d\xi_* + \int\limits_{\mathbb{R}^3} k_2(\xi, \xi_*) f(\xi_*) d\xi_*,$$

the kernels $k_1(\xi, \xi_*)$ and $k_2(\xi, \xi_*)$ satisfy

$$k_1(\xi, \xi_*) \lesssim |\xi - \xi_*|^\gamma \exp\left\{ -\frac{1}{4} \left(|\xi|^2 + |\xi_*|^2 \right) \right\},$$

and

$$k_2(\xi, \xi_*) = a\left(\xi, \xi_*, \kappa\right) \exp\left(-\frac{(1-\kappa)}{8} \left[\frac{\left(|\xi|^2 - |\xi_*|^2 \right)^2}{|\xi - \xi_*|^2} + |\xi - \xi_*|^2 \right] \right),$$

for any $0 < \kappa < 1$, together with

$$a(\xi, \xi_*, \kappa) \le \begin{cases} C_\kappa |\xi - \xi_*|^{-1} (1 + |\xi| + |\xi_*|)^{\gamma - 1}, & \text{if } -1 < \gamma < 0, \\ C_\kappa |\xi - \xi_*|^{-1} \left| \ln |\xi - \xi_*| \right| (1 + |\xi| + |\xi_*|)^{\gamma - 1}, & \text{if } \gamma = -1, \\ C_\kappa |\xi - \xi_*|^{\gamma} (1 + |\xi| + |\xi_*|)^{\gamma - 1}, & \text{if } -2 < \gamma < -1, \end{cases}$$

and their derivatives as well have similar estimates, i.e.,

$$|\nabla_\xi k_1(\xi, \xi_*)|,\ |\nabla_{\xi_*} k_1(\xi, \xi_*)| \lesssim |\xi - \xi_*|^{\gamma - 1} \exp\left\{ -\frac{1}{4} \left(|\xi|^2 + |\xi_*|^2 \right) \right\},$$

$$|\nabla_\xi k_2(\xi, \xi_*)|,\ |\nabla_{\xi_*} k_2(\xi, \xi_*)| \lesssim |\nabla_\xi a\left(\xi, \xi_*, \kappa\right)| \exp\left(-\frac{(1-\kappa)}{8} \left[\frac{\left(|\xi|^2 - |\xi_*|^2 \right)^2}{|\xi - \xi_*|^2} + |\xi - \xi_*|^2 \right] \right),$$

with

$$|\nabla_\xi a(\xi,\xi_*,\kappa)| \le \begin{cases} C_\kappa \frac{|\xi|}{|\xi-\xi_*|^2}(1+|\xi|+|\xi_*|)^{\gamma-1}, & \text{if } -1<\gamma<0, \\ C_\kappa \frac{|\xi|}{|\xi-\xi_*|^2}\,|\ln|\xi-\xi_*||\,(1+|\xi|+|\xi_*|)^{\gamma-1}, & \text{if } \gamma=-1, \\ C_\kappa \frac{|\xi|}{|\xi-\xi_*|^{1-\gamma}}(1+|\xi|+|\xi_*|)^{\gamma-1}, & \text{if } -2<\gamma<-1. \end{cases}$$

Immediately from Lemma 5, we have the following lemma.

Lemma 6. *Let* $-2<\gamma<0$ *and* $\tau\in\mathbb{R}$. *Then*

$$\int_{\mathbb{R}^3} |k(\xi,\xi_*)|\,\langle\xi_*\rangle^\tau\, d\xi_* \lesssim \langle\xi\rangle^{\tau+\gamma-2}, \quad \int_{\mathbb{R}^3} |k(\xi,\xi_*)|\,\langle\xi\rangle^\tau\, d\xi \lesssim \langle\xi_*\rangle^{\tau+\gamma-2}, \tag{2.6}$$

$$\int_{\mathbb{R}^3} \left|\nabla_\xi k(\xi,\xi_*)\right|\langle\xi_*\rangle^\tau\, d\xi_* \lesssim \langle\xi\rangle^{\tau+\gamma-1}, \quad \int_{\mathbb{R}^3} \left|\nabla_\xi k(\xi,\xi_*)\right|\langle\xi\rangle^\tau\, d\xi \lesssim \langle\xi_*\rangle^{\tau+\gamma-1}. \tag{2.7}$$

Consequently, we have

$$|Kg|_{H^1_\xi} \lesssim |g|_{L^2_{\xi,\gamma-1}}, \quad \left|K\nabla_\xi g\right|_{L^2_\xi} \lesssim |g|_{L^2_{\xi,\gamma-1}}, \tag{2.8}$$

and

$$|Kg|_{L^q_{\xi,\tau+2-\gamma}} \lesssim |g|_{L^q_{\xi,\tau}},\ 1\le q\le\infty. \tag{2.9}$$

Lemma 7. *Let* $\tau\in\mathbb{R}$. *Then if* $-3/2<\gamma<0$,

$$\int_{\mathbb{R}^3} (1+|\xi_*|)^\tau k^2(\xi,\xi_*)\, d\xi_* \lesssim \langle\xi\rangle^{\tau+2\gamma-3}, \tag{2.10}$$

and if $-2<\gamma\le-3/2$,

$$\int_{\mathbb{R}^3} (1+|\xi_*|)^\tau k^q(\xi,\xi_*)\, d\xi_* \lesssim \langle\xi\rangle^{\tau+q(\gamma-1)-1} \tag{2.11}$$

provided $1\le q<\frac{3}{-\gamma}$. *Consequently, if* $-3/2<\gamma<0$,

$$|Kg|_{L^\infty_{\xi,\tau-\gamma+3/2}} \le C|g|_{L^2_{\xi,\tau}}, \tag{2.12}$$

and if $-2<\gamma\le-3/2$,

$$|Kg|_{L^\infty_{\xi,\tau+2-\gamma-\frac{1}{q}}} \le C|g|_{L^q_{\xi,\tau}} \tag{2.13}$$

provided $q>\frac{3}{3+\gamma}$.

Taking $q = 3$ in (2.13) and $q = 1$ in (2.9), respectively, we have,

$$|Kg|_{L^\infty_{\xi,1-\gamma}} \le C|g|_{L^3_\xi},$$

and

$$|Kg|_{L^1_{\xi,1-\gamma}} \le C|g|_{L^1_\xi}.$$

Applying the Riesz-Thorin interpolation theorem to the linear operator $\langle\xi\rangle^{1-\gamma} K$, we obtain the following estimate, which is useful in the proofs of Theorems 1 and 18 whenever $-2 < \gamma \le -3/2$.

Lemma 8. *For $-2 < \gamma \le -3/2$,*

$$|Kg|_{L^4_{\xi,1-\gamma}} \le C|g|_{L^2_\xi}. \tag{2.14}$$

To proceed, we need the estimates associated with the weight function w_i, $i = 1, 2, 3$. By straightforward computation, w_1 and w_2 have the derivative estimates as below.

Lemma 9. *Let $-2 < \gamma < 0$. For the weight function w_1, we have*

$$\left|w_1^{-1}\partial_t w_1\right| \lesssim \delta M \langle\xi\rangle_D^{\gamma-1}, \quad \left|w_1^{-1}\nabla_x w_1\right| \lesssim \delta \langle\xi\rangle_D^{\gamma-1}, \quad \left|w_1^{-1}\nabla_\xi w_1\right| \lesssim \langle\xi\rangle_D^{-2} |\xi|, \tag{2.15}$$

$$\left|w_1^{-1}\nabla_x(\partial_t w_1)\right| \lesssim \delta^2 M \langle\xi\rangle_D^{2\gamma-2}, \quad \left|w_1^{-1}\nabla_x(\xi\cdot\nabla_x w_1)\right| \lesssim \delta^2 \langle\xi\rangle_D^{2\gamma-2} |\xi|, \tag{2.16}$$

$$\left|w_1^{-1}\nabla_\xi(\partial_t w_1)\right| \lesssim \delta M \langle\xi\rangle_D^{\gamma-3} |\xi|, \quad \left|w_1^{-1}\nabla_\xi(\xi\cdot\nabla_x w_1)\right| \lesssim \delta \langle\xi\rangle_D^{\gamma-1}. \tag{2.17}$$

For the weight function w_2, its exponent $\rho(t,x,\xi)$ satisfies

$$|\partial_t\rho| \lesssim \delta M \langle\xi\rangle^{\gamma-1}, \quad |\nabla_x\rho| \lesssim \delta\langle\xi\rangle^{\gamma-1}, \quad \left|\nabla_\xi\rho\right| \lesssim \langle\xi\rangle^{p-2}|\xi|,$$

$$|\nabla_x(\partial_t\rho)| \lesssim \delta^2 M \langle\xi\rangle^{-p+2\gamma-2}, \quad |\nabla_x(\xi\cdot\nabla_x\rho)| \lesssim \delta^2\langle\xi\rangle^{-p+2\gamma-2}|\xi|,$$

$$\left|\nabla_\xi(\partial_t\rho)\right| \lesssim \delta M\langle\xi\rangle^{\gamma-3}|\xi|, \quad \left|\nabla_\xi(\xi\cdot\nabla_x\rho)\right| \lesssim \delta\langle\xi\rangle^{\gamma-1},$$

where $0 < p \le 2$.

Moreover, we have

Lemma 10. *For $-2 < \gamma < 0$,*

$$\left|\frac{w_1(t,x,\xi)}{w_1(t,x,\xi_*)} - 1\right| \le CD^{-\{p\wedge 2\}}\left[1 + \left||\xi|^2 - |\xi_*|^2\right|\right]^{\frac{p}{2}}, \; p \ge 1, \tag{2.18}$$

$$\left|\frac{w_2(t,x,\xi)}{w_2(t,x,\xi_*)}-1\right| \le \epsilon c_p \left||\xi|^2-|\xi_*|^2\right|^{\frac{p}{2}} \exp\left(\epsilon c_p \left||\xi|^2-|\xi_*|^2\right|^{\frac{p}{2}}\right), 0<p\le 2, \tag{2.19}$$

$$\left|\frac{w_3(t,x,\xi)}{w_3(t,x,\xi_*)}-1\right| \le \hat{\varepsilon} \left||\xi|^2-|\xi_*|^2\right|^{\frac{p_1}{2}} \exp\left(\hat{\varepsilon} \left||\xi|^2-|\xi_*|^2\right|^{\frac{p_1}{2}}\right), 0<p_1\le 2. \tag{2.20}$$

Here the constants $c_p > 0$ and $C > 0$ are dependent only upon p and γ.

Proof. Let $s = \langle \xi \rangle_D$ and $s_1 = \langle \xi_* \rangle_D$. Then

$$\left|\frac{\partial w_1}{\partial s}(t,x,s)\right| \le C_1 s^{p-1},$$

and thus

$$\begin{aligned}|w_1(t,x,s)-w_1(t,x,s_1)| &= \left|(s-s_1)\int_0^1 \partial_s w_1(t,x,\theta s+(1-\theta)s_1)\,d\theta\right| \\ &\le C_1\left|(s-s_1)\int_0^1 (\theta s+(1-\theta)s_1)^{p-1}\,d\theta\right| \\ &\le C_1'\left|\left(s^p-s_1^p\right)\right|.\end{aligned}$$

Also, since $w_1(t,x,\xi_*) \gtrsim \langle\xi_*\rangle_D^p$ and $D \ge 1$, we can deduce that for $1 \le p < 2$,

$$\begin{aligned}\left|\frac{w_1(t,x,\xi)}{w_1(t,x,\xi_*)}-1\right| &= \left|\frac{w_1(t,x,\xi)-w_1(t,x,\xi_*)}{w_1(t,x,\xi_*)}\right| \\ &\lesssim \frac{\left|\langle\xi\rangle_D^p-\langle\xi_*\rangle_D^p\right|}{\langle\xi_*\rangle_D^p} \lesssim \frac{1}{D^p}\left||\xi|^2-|\xi_*|^2\right|^{p/2},\end{aligned}$$

and for $p \ge 2$,

$$\begin{aligned}\left|\frac{w_1(t,x,\xi)}{w_1(t,x,\xi_*)}-1\right| &\lesssim \frac{\left|\langle\xi\rangle_D^p-\langle\xi_*\rangle_D^p\right|}{\langle\xi_*\rangle_D^p} \\ &\lesssim \frac{\left||\xi|^2-|\xi_*|^2\right|}{\langle\xi_*\rangle_D^p}\int_0^1 \left(\theta\langle\xi\rangle_D^2+(1-\theta)\langle\xi_*\rangle_D^2\right)^{\frac{p}{2}-1} d\theta \\ &\lesssim \frac{\left||\xi|^2-|\xi_*|^2\right|}{D^2}\int_0^1 \frac{\left(D^2+|\xi_*|^2+\theta\left(|\xi|^2-|\xi_*|^2\right)\right)^{\frac{p}{2}-1}}{\left(D^2+|\xi_*|^2\right)^{\frac{p}{2}-1}} d\theta \\ &\lesssim \frac{1}{D^2}\left||\xi|^2-|\xi_*|^2\right|\left(1+\left||\xi|^2-|\xi_*|^2\right|\right)^{\frac{p}{2}-1}.\end{aligned}$$

Combining the above two estimates, we can conclude (2.18).

Similar to w_1, we have

$$|\rho(t,x,\xi)-\rho(t,x,\xi_*)| \le c_p \left|\langle\xi\rangle^p - \langle\xi_*\rangle^p\right| \le c_p \left||\xi|^2-|\xi_*|^2\right|^{p/2}$$

for $0<p\le 2$, so that

$$\begin{aligned}\left|\frac{w_2(t,x,\xi)}{w_2(t,x,\xi_*)}-1\right| &= \left|e^{\epsilon(\rho(t,x,\xi)-\rho(t,x,\xi_*))}-1\right| \\ &\le \epsilon\,|\rho(t,x,\xi)-\rho(t,x,\xi_*)|\,e^{\epsilon|\rho(t,x,\xi)-\rho(t,x,\xi_*)|} \\ &\le \epsilon c_p \left||\xi|^2-|\xi_*|^2\right|^{\frac{p}{2}} \exp\left(\epsilon c_p\left||\xi|^2-|\xi_*|^2\right|^{\frac{p}{2}}\right),\end{aligned}$$

as desired. As for w_3, the proof is straightforward, that is,

$$\begin{aligned}\left|\frac{w_3(t,x,\xi)}{w_3(t,x,\xi_*)}-1\right| &= \left|e^{\hat\varepsilon\left(\langle\xi\rangle^{p_1}-\langle\xi_*\rangle^{p_1}\right)}-1\right| \\ &\le \hat\varepsilon\left|\langle\xi\rangle^{p_1}-\langle\xi_*\rangle^{p_1}\right|\exp\left(\hat\varepsilon\left|\langle\xi\rangle^{p_1}-\langle\xi_*\rangle^{p_1}\right|\right) \\ &\le \hat\varepsilon\left||\xi|^2-|\xi_*|^2\right|^{\frac{p_1}{2}}\exp\left(\hat\varepsilon\left||\xi|^2-|\xi_*|^2\right|^{\frac{p_1}{2}}\right)\end{aligned}$$

for $0<p_1\le 2$. The proof of this lemma is completed. □

With the help of the estimates on the weight functions, we obtain some useful estimates regarding the integral operator K in the weighted spaces. For simplicity of notations, we define $K_{w_i}=w_i(t,x,\xi)\,K w_i^{-1}(t,x,\xi_*)$, $i=1,2,3$.

Lemma 11. *Let $\tau\in\mathbb{R}$. For $-2<\gamma<0$ and $i=1,2,3$, we have*

$$\int_{\mathbb{R}^3}\left|w_i(t,x,\xi)k(\xi,\xi_*)w_i^{-1}(t,x,\xi_*)\right|\langle\xi_*\rangle^\tau\,d\xi_* \lesssim \langle\xi\rangle^{\tau+\gamma-2}, \tag{2.21}$$

$$\int_{\mathbb{R}^3}\left|w_i(t,x,\xi)k(\xi,\xi_*)w_i^{-1}(t,x,\xi_*)\right|\langle\xi\rangle^\tau\,d\xi \lesssim \langle\xi_*\rangle^{\tau+\gamma-2}, \tag{2.22}$$

$$\int_{\mathbb{R}^3}\left|w_i(t,x,\xi)\left(\nabla_\xi k(\xi,\xi_*)\right)w_i^{-1}(t,x,\xi_*)\right|\langle\xi_*\rangle^\tau\,d\xi_* \lesssim \langle\xi\rangle^{\tau+\gamma-1}, \tag{2.23}$$

$$\int_{\mathbb{R}^3}\left|w_i(t,x,\xi)\left(\nabla_\xi k(\xi,\xi_*)\right)w_i^{-1}(t,x,\xi_*)\right|\langle\xi\rangle^\tau\,d\xi \lesssim \langle\xi_*\rangle^{\tau+\gamma-1}, \tag{2.24}$$

uniformly in t and x; consequently, we have

$$\left|K_{w_i}q(t,x,\xi)\right|_{L^\infty_{\xi,\tau+2-\gamma}} \lesssim |q(t,x,\cdot)|_{L^\infty_{\xi,\tau}}, \tag{2.25}$$

$$\left|\langle\xi\rangle^{\tau} K_{w_i} q(t,x,\xi)\right|_{L^2_\xi} \lesssim \left|\langle\xi\rangle^{\tau-2+\gamma} q(t,x,\cdot)\right|_{L^2_\xi}, \tag{2.26}$$

$$\left|\langle\xi\rangle^{\tau} \nabla_\xi K_{w_i} q(t,x,\xi)\right|_{L^2_\xi} \lesssim \left|\langle\xi\rangle^{\tau-1+\gamma} q(t,x,\cdot)\right|_{L^2_\xi}. \tag{2.27}$$

Furthermore, if $-3/2 < \gamma < 0$,

$$\left|K_{w_i} q(t,x,\xi)\right|_{L^\infty_{\xi,\tau}} \lesssim \left|\langle\xi\rangle^{\tau-3/2+\gamma} q(t,x,\cdot)\right|_{L^2_\xi}; \tag{2.28}$$

if $-2 < \gamma \le -3/2$,

$$|K_{w_i} q(t,x,\xi)|_{L^\infty_{\xi,\tau}} \lesssim |\langle\xi\rangle^{\tau-2+\gamma+1/s} q(t,x,\cdot)|_{L^s_\xi}, \tag{2.29}$$

provided $s > \frac{3}{3+\gamma}$, and

$$|K_{w_i} q(t,x,\xi)|_{L^4_{\xi,1-\gamma}} \lesssim |q(t,x,\cdot)|_{L^2_\xi}. \tag{2.30}$$

Proof. Since $k = -k_1 + k_2$ and the estimate for k_1 can be obtained easily, we just prove (2.21) for k_2 whenever the weight function is w_1 and then a similar argument can be applied to the estimates (2.21)-(2.24) for the weight functions w_i, $i = 1, 2, 3$. Now, rewrite

$$\begin{aligned}
& w_1(t,x,\xi)k_2(\xi,\xi_*)w_1^{-1}(t,x,\xi_*) - k_2(\xi,\xi_*) \\
&= \left\{ a\left(\xi,\xi_*,\frac{1}{2}\right) \exp\left(-\frac{1}{32}\left[\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2} + |\xi-\xi_*|^2\right]\right)\right\} \\
&\quad \times \left\{ \exp\left(-\frac{1}{32}\left[\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2} + |\xi-\xi_*|^2\right]\right) \times \left(\frac{w_1(t,x)}{w_{1*}(t,x)} - 1\right)\right\} \\
&\equiv q_2(\xi,\xi_*)s(D,\xi,\xi_*).
\end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2} + |\xi-\xi_*|^2 \ge 2\left||\xi|^2-|\xi_*|^2\right|.$$

In view of (2.18), we obtain

$$\sup_{\xi,\xi_*} |s(D,\xi,\xi_*)| \to 0 \text{ as } D \to \infty,$$

which implies that $\sup_{\xi,\xi_*} |s(D,\xi,\xi_*)| < 1$ for all $D \ge 1$ sufficiently large. Moreover, in view of Lemma 6, we know

$$\int_{\mathbb{R}^3} |k_2(\xi,\xi_*)| \langle\xi_*\rangle^{\tau} d\xi_* \lesssim \langle\xi\rangle^{\tau+\gamma-2},$$

$$\int_{\mathbb{R}^3} |q_2(\xi,\xi_*)| \langle\xi_*\rangle^\tau d\xi_* \lesssim \langle\xi\rangle^{\tau+\gamma-2}.$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left| w_1(t,x,\xi) k_2(\xi,\xi_*) w_1^{-1}(t,x,\xi_*)\right| \langle\xi_*\rangle^\tau d\xi_* \\
&\lesssim \int_{\mathbb{R}^3} |k_2(\xi,\xi_*)| \langle\xi_*\rangle^\tau d\xi_* + \int_{\mathbb{R}^3} |q_2(\xi,\xi_*) s(D,\xi,\xi_*)| \langle\xi_*\rangle^\tau d\xi_* \\
&\lesssim \langle\xi\rangle^{\tau+\gamma-2}.
\end{aligned}$$

Combining the above estimates (2.21)-(2.24) together with (2.15), we can deduce (2.25)-(2.27).

Mimicking the proof of (2.21), together with (2.12)-(2.14), we obtain (2.28)-(2.30). The proof of this lemma is completed. □

Remark 12. Similar to the proof of Lemma 10, together with the conservation of energy, we have

$$\left|\frac{w_1(t,x,\xi)}{w_1(t,x,\xi_*')} - 1\right| \lesssim D^{-\{2\wedge p\}} \left[1 + \left||\xi'|^2 - |\xi_*|^2\right|\right]^{\frac{p}{2}},\ p \geq 1, \tag{2.31}$$

$$\left|\frac{w_1(t,x,\xi)}{w_1(t,x,\xi')} - 1\right| \lesssim D^{-\{2\wedge p\}} \left[1 + \left||\xi_*'|^2 - |\xi_*|^2\right|\right]^{\frac{p}{2}},\ p \geq 1, \tag{2.32}$$

$$\left|\frac{w_2(t,x,\xi)}{w_2(t,x,\xi_*')} - 1\right| \lesssim \epsilon c_p \left||\xi'|^2 - |\xi_*|^2\right|^{\frac{p}{2}} \exp\left(\epsilon c_p \left||\xi'|^2 - |\xi_*|^2\right|^{\frac{p}{2}}\right),\ 0 < p \leq 2, \tag{2.33}$$

$$\left|\frac{w_2(t,x,\xi)}{w_2(t,x,\xi')} - 1\right| \lesssim \epsilon c_p \left||\xi_*'|^2 - |\xi_*|^2\right|^{\frac{p}{2}} \exp\left(\epsilon c_p \left||\xi_*'|^2 - |\xi_*|^2\right|^{\frac{p}{2}}\right),\ 0 < p \leq 2. \tag{2.34}$$

Here the constant $c_p > 0$ is the same as in Lemma 10. On the other hand, for the weight function w_3, we have

$$\begin{aligned}
\left|\frac{w_3(t,x,\xi)}{w_3(t,x,\xi_*)} - 1\right| &= \left|e^{\hat{\varepsilon}\left(\langle\xi\rangle^{p_1} - \langle\xi_*\rangle^{p_1}\right)} - 1\right| \leq \exp\left(\hat{\varepsilon}\left|\langle\xi\rangle^{p_1} - \langle\xi_*\rangle^{p_1}\right|\right) \\
&= \exp\left(\hat{\varepsilon}\left|\left(1+|\xi|^2\right)^{\frac{p_1}{2}} - \left(1+|\xi_*|^2\right)^{\frac{p_1}{2}}\right|\right) \\
&\leq \exp\left(\hat{\varepsilon}\left||\xi|^2 - |\xi_*|^2\right|^{\frac{p_1}{2}}\right),
\end{aligned} \tag{2.35}$$

since $0 < p_1/2 \leq 1$. By the conservation of energy,

$$\left|\frac{w_3(t,x,\xi)}{w_3(t,x,\xi_*')} - 1\right| \leq \exp\left(\hat{\varepsilon}\left||\xi'|^2 - |\xi_*|^2\right|^{\frac{p_1}{2}}\right),\ 0 < p_1 \leq 2, \tag{2.36}$$

$$\left|\frac{w_3(t,x,\xi)}{w_3(t,x,\xi')}-1\right| \le \exp\left(\hat{\varepsilon}\left|\left|\xi_*'\right|^2-|\xi_*|^2\right|^{\frac{p_1}{2}}\right), 0<p_1\le 2. \tag{2.37}$$

Furthermore, we consider the linear operator $\mathcal{L}_{w_i}$, $i=1,2$, defined as

$$\mathcal{L}_{w_i}h=-\xi\cdot\nabla_x h+(\partial_t w_i+\xi\cdot\nabla_x w_i)w_i^{-1}h+L_{w_i}h.$$

By straightforward computation, we obtain the energy estimate for the linear part as below.

Lemma 13 *(Weighted energy estimate for the linear part). Let $-2<\gamma<0$. If $\delta>0$ is sufficiently small, and D, $M\ge 1$ are sufficiently large with δM sufficiently small, then*

$$\begin{aligned}
&\sum_{j=0}^{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla_x^j h\nabla_x^j\mathcal{L}_{w_1}h\,dx\,d\xi\\
&\le-\left(\nu_0-C_1D^{-2}-C_2\delta-C_3\delta M\right)\sum_{j=0}^{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle\xi\rangle^{\gamma}\left(\mathrm{P}_1\nabla_x^j h\right)^2dx\,d\xi\\
&\quad-\left(C_4\delta M-C_2\delta-C_1D^{-2}\right)\sum_{j=0}^{2}\int_{H_+^D}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{-1}\left|\mathrm{P}_0\nabla_x^j h\right|^2dx\,d\xi\\
&\quad+\left(C_1D^{-2}+C_2\delta+C_5\delta M\right)\sum_{j=0}^{2}\int_{H_0^D}\left|\mathrm{P}_0\nabla_x^j h\right|^2dx\,d\xi+C_1D^{-2}\sum_{j=0}^{2}\int_{H_-^D}\left|\mathrm{P}_0\nabla_x^j h\right|^2dx\,d\xi,
\end{aligned}$$

where

$$\begin{aligned}
H_+^D&=\{(x,\xi):\delta\left(\langle x\rangle-Mt\right)>2\langle\xi\rangle_D^{1-\gamma}\},\\
H_0^D&=\{(x,\xi):\langle\xi\rangle_D^{1-\gamma}\le\delta\left(\langle x\rangle-Mt\right)\le 2\langle\xi\rangle_D^{1-\gamma}\},\\
H_-^D&=\{(x,\xi):\delta\left(\langle x\rangle-Mt\right)<\langle\xi\rangle_D^{1-\gamma}\}.
\end{aligned}$$

If $\epsilon>0$, $\delta>0$ are sufficiently small and M is sufficiently large such that δM is large but $\delta M\ll\epsilon^{-1}$, then

$$\begin{aligned}
&\sum_{j=0}^{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\nabla_x^j h\nabla_x^j\mathcal{L}_{w_2}h\,dx\,d\xi\\
&\le-\left(\nu_0-\epsilon\delta C_2-\epsilon\delta MC_3\right)\sum_{j=0}^{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle\xi\rangle^{\gamma}\left(\mathrm{P}_1\nabla_x^j h\right)^2dx\,d\xi\\
&\quad-\epsilon\left(\delta MC_4-\delta C_2-C_1\right)\sum_{j=0}^{2}\int_{H_+^1}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0\nabla_x^j h\right|^2dx\,d\xi
\end{aligned}$$

$$+\epsilon\left(\delta C_2+\delta M C_5+C_1\right)\sum_{j=0}^{2}\int_{H_0^1}\left|\mathrm{P}_0\nabla_x^j h\right|^2 dx d\xi+\epsilon C_1\sum_{j=0}^{2}\int_{H_-^1}\left|\mathrm{P}_0\nabla_x^j h\right|^2 dx d\xi,$$

where

$$H_+^1=\{(x,\xi):\delta\left(\langle x\rangle-Mt\right)>2\langle\xi\rangle^{p+1-\gamma}\},$$
$$H_0^1=\{(x,\xi):\langle\xi\rangle^{p+1-\gamma}\leq\delta\left(\langle x\rangle-Mt\right)\leq 2\langle\xi\rangle^{p+1-\gamma}\},$$
$$H_-^1=\{(x,\xi):\delta\left(\langle x\rangle-Mt\right)<\langle\xi\rangle^{p+1-\gamma}\}.$$

The rest of this section is devoted to estimates for the nonlinear operators Γ and Γ_{w_i}. Before going on, we point out an essential lemma, which is proved by Guo [12, Lemma 2] and is used frequently in the following discussion. In addition, we split Γ into two parts Γ_{gain} and Γ_{loss} as below:

$$\begin{aligned}\Gamma(g,h)&\equiv\Gamma_{gain}(g,h)-\Gamma_{loss}(g,h)\\&=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2}B(\vartheta)|\xi-\xi_*|^\gamma\mathcal{M}_*^{1/2}\left[g_*'h'+g'h_*'\right]d\xi_*d\omega\\&\quad-\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2}B(\vartheta)|\xi-\xi_*|^\gamma\mathcal{M}_*^{1/2}\left[g_*h+gh_*\right]d\xi_*d\omega.\end{aligned}$$

Lemma 14. *[12, Lemma 2] Let $\varsigma>-3$, $l(\xi)\in C^\infty\left(\mathbb{R}^3\right)$ and $g(\xi)\in C^\infty\left(\mathbb{R}^3\backslash\{0\}\right)$. Assume that for any multi-index α, there is $C_\alpha>0$ such that*

$$\left|\partial^\alpha g(\xi)\right|\leq C_\alpha|\xi|^{\varsigma-|\alpha|},$$
$$\left|\partial^\alpha l(\xi)\right|\leq C_\alpha e^{-\tau_\alpha|\xi|^2},$$

with some $\tau_\alpha>0$. Then there is $C_\alpha^*>0$ such that

$$\left|\partial^\alpha(g*l)(\xi)\right|\leq C_\alpha^*\langle\xi\rangle^{\varsigma-|\alpha|}.$$

Lemma 15. *Let $-2<\gamma<0$, $\hat{\varepsilon}\geq 0$, $0<p_1\leq 2$ and $\lambda\geq 0$. Then*

$$|\Gamma_{loss}(g,h)|_{L_\xi^\infty\left(\langle\xi\rangle^\lambda e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}\lesssim|g|_{L_\xi^\infty}|h|_{L_\xi^\infty\left(\langle\xi\rangle^{\lambda+\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}+|h|_{L_\xi^\infty}|g|_{L_\xi^\infty\left(\langle\xi\rangle^{\lambda+\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)},\tag{2.38}$$

$$|\Gamma_{gain}(g,h)|_{L_\xi^\infty\left(\langle\xi\rangle^\lambda e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}\lesssim|g|_{L_\xi^\infty\left(\langle\xi\rangle^{\lambda+\gamma-1}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}|h|_{L_\xi^\infty\left(\langle\xi\rangle^{\lambda+\gamma-1}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}.\tag{2.39}$$

In particular,

$$\left|\nu^{-1}\Gamma(g,h)\right|_{L_{\xi,\lambda}^\infty}\lesssim|g|_{L_{\xi,\lambda}^\infty}|h|_{L_{\xi,\lambda}^\infty}.\tag{2.40}$$

Proof. It readily follows from Lemma 14 that

$$
\begin{aligned}
&\left|\langle\xi\rangle^{\lambda} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\Gamma_{loss}(g,h)\right| \\
&\lesssim \frac{1}{2}\langle\xi\rangle^{\lambda} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} \int\limits_{\mathbb{R}^3\times\mathbb{S}^2} B(\vartheta)|\xi-\xi_*|^{\gamma}\mathcal{M}_*^{1/2}\left[|g_*||h|+|g||h_*|\right]d\xi_* d\omega \\
&\lesssim |g|_{L^\infty_\xi}|h|_{L^\infty_\xi\left(\langle\xi\rangle^{\lambda+\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)} + |h|_{L^\infty_\xi}|g|_{L^\infty_\xi\left(\langle\xi\rangle^{\lambda+\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)},
\end{aligned}
$$

so that (2.38) holds. Since the conservation of energy implies that $\langle\xi\rangle \lesssim \langle\xi'\rangle\langle\xi'_*\rangle$ and $\langle\xi\rangle^{p_1} \le \langle\xi'\rangle^{p_1}+\langle\xi'_*\rangle^{p_1}$, we can obtain (2.39) by following the argument as in [4, Proposition 5.1]. Finally, (2.40) follows by taking $\hat{\varepsilon}=0$ and replacing λ by $\lambda-\gamma$ simultaneously in (2.38) and (2.39). The proof of this lemma is completed. □

Lemma 16.

$$
\left|\langle f,\Gamma(g,h)\rangle_\xi\right| \lesssim |f|_{L^2_\sigma}\left(|g|_{L^2_\sigma}|h|_{L^\infty_\xi}+|g|_{L^\infty_\xi}|h|_{L^2_\sigma}\right), \tag{2.41}
$$

$$
\left|\nu^{-1}\Gamma(g,h)\right|_{L^2_\xi} \lesssim |g|_{L^\infty_\xi}|h|_{L^2_\xi}+|g|_{L^2_\xi}|h|_{L^\infty_\xi}. \tag{2.42}
$$

Proof. The idea of the proof comes from [30, Lemma 3] and we give the complete proof in the Appendix section. □

Lemma 17. *Let* $\lambda \ge 0$. *Then*

$$
\left|\nu^{-1}\Gamma_{w_1}(g,h)\right|_{L^\infty_{\xi,\lambda}} \lesssim |g|_{L^\infty_{\xi,\lambda}}|\langle\xi\rangle^p h|_{L^\infty_{\xi,\lambda}}, \tag{2.43}
$$

$$
\left|\nu^{-1}\Gamma_{w_1}(g,h)\right|_{L^2_\xi} \lesssim |g|_{L^\infty_\xi}\left|\langle\xi\rangle^p h\right|_{L^2_\xi}+|g|_{L^2_\xi}\left|\langle\xi\rangle^p h\right|_{L^\infty_\xi}, \tag{2.44}
$$

$$
\left|\langle f,\Gamma_{w_1}(g,h)\rangle_\xi\right| \lesssim |f|_{L^2_\sigma}\left(|g|_{L^2_\sigma}\left|\langle\xi\rangle^p h\right|_{L^\infty_\xi}+|g|_{L^\infty_\xi}\left|\langle\xi\rangle^p h\right|_{L^2_\sigma}\right), \tag{2.45}
$$

where $p \ge 1$;

$$
\left|\nu^{-1}\Gamma_{w_2}(g,h)\right|_{L^\infty_{\xi,\lambda}} \lesssim |g|_{L^\infty_{\xi,\lambda}}|\langle\xi\rangle^p e^{\epsilon c_p\langle\xi\rangle^p}h|_{L^\infty_{\xi,\lambda}}, \tag{2.46}
$$

$$
\left|\nu^{-1}\Gamma_{w_2}(g,h)\right|_{L^2_\xi} \lesssim |g|_{L^\infty_\xi}\left|\langle\xi\rangle^p e^{\epsilon c_p\langle\xi\rangle^p}h\right|_{L^2_\xi}+|g|_{L^2_\xi}\left|\langle\xi\rangle^p e^{\epsilon c_p\langle\xi\rangle^p}h\right|_{L^\infty_\xi}, \tag{2.47}
$$

$$
\left|\langle f,\Gamma_{w_2}(g,h)\rangle_\xi\right| \lesssim |f|_{L^2_\sigma}\left(|g|_{L^2_\sigma}\left|\langle\xi\rangle^p e^{\epsilon c_p\langle\xi\rangle^p}h\right|_{L^\infty_\xi}+|g|_{L^\infty_\xi}\left|\langle\xi\rangle^p e^{\epsilon c_p\langle\xi\rangle^p}h\right|_{L^2_\sigma}\right), \tag{2.48}
$$

where $0<p\le 2$ *and the constant* $c_p>0$ *is the same as in Lemma 10;*

$$\left|\nu^{-1}\Gamma_{w_3}(g,h)\right|_{L^\infty_{\xi,\lambda}} \lesssim |g|_{L^\infty_{\xi,\lambda}} \left|e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}h\right|_{L^\infty_{\xi,\lambda}}, \tag{2.49}$$

$$\left|\nu^{-1}\Gamma_{w_3}(g,h)\right|_{L^2_\xi} \lesssim |g|_{L^\infty_\xi} \left|e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}h\right|_{L^2_\xi} + |g|_{L^2_\xi}\left|e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}h\right|_{L^\infty_\xi}, \tag{2.50}$$

where $0 < p_1 \le 2$.

Proof. Direct calculation shows that for $i = 1, 2, 3$,

$$\begin{aligned}&\Gamma_{w_i}(g,h) - \Gamma(g,h)\\ &= \int_{\mathbb{R}^3}\int_{S^2} B(\theta)|\xi-\xi_*|^\gamma \sqrt{\mathcal{M}_*}\left[g'h'_*\left(\frac{w_i}{w'_i}-1\right) + h'g'_*\left(\frac{w_i}{w'_{i*}}-1\right) - g_*h\left(\frac{w_i}{w_{i*}}-1\right)\right]d\omega d\xi_*.\end{aligned}$$

On the other hand, the conservation of energy implies that $\langle\xi\rangle^\beta \lesssim \langle\xi'_*\rangle^\beta \langle\xi'\rangle^\beta$ for $\beta \ge 0$. Using these facts together with (2.40)–(2.42), Lemma 10 and Remark 12, we get the desired estimates. □

3. Weighted linearized Boltzmann equation with a source term

In this section, we are concerned with the following inhomogeneous problem:

$$\begin{cases}\partial_t u + \xi\cdot\nabla_x u - [\partial_t w_i(t,x,\xi) + \xi\cdot\nabla_x w_i(t,x,\xi)]\, w_i^{-1}u = L_{w_i}u + \Gamma_{w_i}(g_i,h_i),\\ u(0,x,\xi) = \eta f_{w_i 0},\end{cases} \tag{3.1}$$

for $i = 1, 2$, where g_i and h_i are prescribed. The proofs are almost the same, so that we focus on the case in which the weight function is w_1 and just state the result for the weight function w_2 (see Theorem 26). Now, let $p \ge 1$. We are concerned with the following inhomogeneous equation:

$$\begin{cases}\partial_t u + \xi\cdot\nabla_x u + \tilde{\nu}u = K_{w_1}u + \Gamma_{w_1}(g_1,h_1),\\ u(0,x,\xi) = \eta f_{w_1 0}.\end{cases} \tag{3.2}$$

After choosing $\delta > 0$ and δM small enough, we have

$$\tilde{\nu}(t,x,\xi) = \nu(\xi) - [\partial_t w_1(t,x,\xi) + \xi\cdot\nabla_x w_1(t,x,\xi)]\, w_1^{-1} \ge \frac{\nu(\xi)}{2},$$

due to (2.15). Let $T > 0$ and $\beta > 3/2$. Assume that $f_{w_1 0} \in L^\infty_{\xi,\beta}L^2_x \cap L^\infty_{\xi,\beta}L^\infty_x$. Also assume the source term $\Gamma_{w_1}(g_1,h_1)$ satisfies

$$C^\infty_{g_1,T} = \sup_{0\le t\le T}(1+t)^{-A}\|g_1\|_{L^\infty_{\xi,\beta}L^\infty_x} < \infty, \quad C^2_{g_1,T} = \sup_{0\le t\le T}\|g_1\|_{L^\infty_{\xi,\beta}L^2_x} < \infty, \tag{3.3}$$

for some constant $A \ge 1/2$, and

$$C^{\infty}_{h_1,T} = \sup_{0\le t\le T} (1+t)^{\frac{3}{2}} \left\| \langle\xi\rangle^{p} h_1 \right\|_{L^\infty_{\xi,\beta}L^\infty_x} < \infty. \tag{3.4}$$

Here we mention that throughout this section we abbreviate "$\le C$" to "$\lesssim$" in which all the constants C are independent of T.

Theorem 18. *Let $\beta > 3/2$ and $0 < \varsigma \ll 1$. Assume that $f_{w_1 0} \in L^\infty_{\xi,\beta}L^2_x \cap L^\infty_{\xi,\beta}L^\infty_x$ and that g_1, h_1 satisfy (3.3) and (3.4), respectively. Then the solution u to the equation (3.2) satisfies*

$$\begin{aligned} &\|u\|_{L^\infty_{\xi,\beta}L^\infty_x} \\ &\lesssim \eta \left\|f_{w_1 0}\right\|_{L^\infty_{\xi,\beta}L^\infty_x} + (1+t)^{-3/2+A+\varsigma} C^\infty_{g_1,T} C^\infty_{h_1,T} \\ &\quad + \left[(1+\delta M)\left(\eta \left\|f_{w_1 0}\right\|_{L^\infty_{\xi,\beta}L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T}\right)\right] \cdot \begin{cases} (1+t)^2, & \text{if } -1<\gamma<0, \\ (1+t)^{2+\varsigma}, & \text{if } \gamma=-1, \\ (1+t)^{7+\frac{5}{\gamma}}, & \text{if } -2<\gamma<-1, \end{cases} \end{aligned} \tag{3.5}$$

$0 \le t \le T$.

To prove this theorem, we design a Picard-type iteration, treating $K_{w_1}u$ as source term. Specifically, the zeroth order approximation $u^{(0)}$ is defined as

$$\begin{cases} \partial_t u^{(0)} + \xi\cdot\nabla_x u^{(0)} + \tilde{\nu} u^{(0)} = \Gamma_{w_1}(g_1, h_1), \\ u^{(0)}(0,x,\xi) = \eta f_{w_1 0}, \end{cases}$$

and the difference $u - u^{(0)}$ satisfies

$$\begin{cases} \partial_t (u-u^{(0)}) + \xi\cdot\nabla_x (u-u^{(0)}) + \tilde{\nu}(u-u^{(0)}) = K_{w_1}(u-u^{(0)}) + K_{w_1}u^{(0)}, \\ (u-u^{(0)})(0,x,\xi) = 0. \end{cases}$$

We can define the i^{th} order approximation $u^{(i)}$, $i\ge 1$, inductively as

$$\begin{cases} \partial_t u^{(i)} + \xi\cdot\nabla_x u^{(i)} + \tilde{\nu} u^{(i)} = K_{w_1}u^{(i-1)}, \\ u^{(i)}(0,x,\xi) = 0. \end{cases}$$

Now, the wave part and the remainder part can be defined as follows:

$$W^{(m)}_{w_1} = \sum_{i=0}^{m} u^{(i)}, \quad \mathcal{R}^{(m)}_{w_1} = u - W^{(m)}, \tag{3.6}$$

$\mathcal{R}^{(m)}_{w_1}$ solving the equation

$$\begin{cases} \partial_t \mathcal{R}_{w_1}^{(m)} + \xi \cdot \nabla_x \mathcal{R}_{w_1}^{(m)} + \tilde{\nu} \mathcal{R}_{w_1}^{(m)} = K_{w_1} \mathcal{R}_{w_1}^{(m)} + K_{w_1} u^{(m)}, \\ \\ \mathcal{R}_{w_1}^{(m)}(0, x, \xi) = 0. \end{cases} \tag{3.7}$$

In the sequel, we shall estimate the wave part and remainder part in order.

3.1. Estimates on the wave part

Let us consider the time-related damped transport equation

$$\begin{cases} \partial_t h + \xi \cdot \nabla_x h + \tilde{\nu} h = 0, \\ \\ h(0, x, \xi) = h_0(x, \xi), \end{cases} \tag{3.8}$$

and denote the solution operator of the time-related damped transport equation (3.8) by $\mathbb{S}_{w_1}(t)$, namely,

$$\mathbb{S}_{w_1}(t) h_0(x, \xi) = h_0(x - t\xi, \xi) \exp\left(- \int_0^t \tilde{\nu}(r, x - (t - r)\xi, \xi) dr \right). \tag{3.9}$$

Next, consider the inhomogeneous problem

$$\begin{cases} \partial_t h + \xi \cdot \nabla_x h + \tilde{\nu}(t, x, \xi) h = q(t, x, \xi), \\ \\ h(0, x, \xi) = 0, \end{cases} \tag{3.10}$$

and then we have

$$h(t, x, \xi) = \int_0^t q(s, x - (t - s)\xi, \xi) \exp\left(- \int_s^t \tilde{\nu}(r, x - (t - r)\xi, \xi) dr \right) ds.$$

Furthermore, we define the operator $\mathbb{S}_{w_1}(t; s)$ as

$$\mathbb{S}_{w_1}(t; s) q(s, x, \xi) \equiv q(s, x - (t - s)\xi, \xi) \exp\left(- \int_s^t \tilde{\nu}(r, x - (t - r)\xi, \xi) dr \right), \tag{3.11}$$

for $0 \le s \le t$, so that the solution h to (3.10) can be represented by

$$h(t, x, \xi) = \int_0^t \mathbb{S}_{w_1}(t; s) q(s, x, \xi) ds.$$

Under this notation, we as well have

$$\mathbb{S}_{w_1}(t;0) f_0(x,\xi) = \mathbb{S}_{w_1}(t) f_0(x,\xi).$$

Thereupon, each item of the wave part $W_{w_1}^{(m)} = \sum_{i=0}^{m} u^{(i)}$ can be expressed as

$$u^{(0)} = \eta \mathbb{S}_{w_1}(t) f_{w_1 0}(x,\xi) + \int_0^t \mathbb{S}_{w_1}(t;s)\Gamma_{w_1}(g_1,h_1)(s,x,\xi)ds,$$

$$u^{(i)} = \int_0^t \mathbb{S}_{w_1}(t;s)\left[K_{w_1} u^{(i-1)}\right](s,x,\xi)ds, \, i \geq 1,$$

in terms of the operator $\mathbb{S}_{w_1}(t;s)$.

Through Lemma 19, it is easy to see some properties of the operators $\mathbb{S}_{w_1}(t)$ and $\mathbb{S}_{w_1}(t;s)$ $(0 \leq s \leq t)$.

Lemma 19. *[3, Lemma 12.1]*

$$\sup_{\xi} e^{-t(1+|\xi|)^{-\alpha}} (1+|\xi|)^{-\lambda} \leq C(1+t)^{-\lambda/\alpha},$$

for $t \geq 0$, $\alpha > 0$, $\lambda > 0$.

Lemma 20. *Let* $\tau \geq 0$ *and* $\lambda \geq 0$. *Then*

$$\left\| \mathbb{S}_{w_1}(t) h_0(x,\xi) \right\|_{L^\infty_{\xi,\lambda} L^\infty_x} \lesssim (1+s)^{\frac{\tau}{\gamma}} \left\| h_0 \right\|_{L^\infty_{\xi,\lambda+\tau} L^\infty_x}, \tag{3.12}$$

$$\left\| \mathbb{S}_{w_1}(t;s) q(s,x,\xi) \right\|_{L^\infty_{\xi,\lambda} L^\infty_x} \lesssim (1+t-s)^{\frac{\tau}{\gamma}} \left\| q(s,\cdot,\cdot) \right\|_{L^\infty_{\xi,\lambda+\tau} L^\infty_x}, \tag{3.13}$$

$$\left\| \mathbb{S}_{w_1}(t;s) q(s,x,\xi) \right\|_{L^\infty_{\xi,\lambda} L^2_x} \lesssim (1+t-s)^{\frac{\tau}{\gamma}} \left\| q(s,\cdot,\cdot) \right\|_{L^\infty_{\xi,\lambda+\tau} L^2_x}, \tag{3.14}$$

$$\left\| \mathbb{S}_{w_1}(s) h_0(x,\xi) \right\|_{L^2} \lesssim (1+t)^{\frac{\tau}{\gamma}} \left\| \langle \xi \rangle^{\tau} h_0 \right\|_{L^2}, \tag{3.15}$$

$$\left\| \mathbb{S}_{w_1}(t;s) q(s,x,\xi) \right\|_{L^2} \lesssim (1+t-s)^{\frac{\tau}{\gamma}} \left\| \langle \xi \rangle^{\tau} q(s,\cdot,\cdot) \right\|_{L^2}, \tag{3.16}$$

for $0 \leq s \leq t \leq T$.

Now we are ready to prove the $L^\infty_{\xi,\beta} L^\infty_x$ estimate and L^2 estimate for the wave part $W_{w_1}^{(m)} = \sum_{i=0}^{m} u^{(i)}$.

Lemma 21 *($L^\infty_{\xi,\beta} L^\infty_x$ estimate of $u^{(i)}$). Let* $\beta > 3/2$, $0 < \varsigma \ll 1$. *Assume that* $f_{w_1 0} \in L^\infty_{\xi,\beta} L^\infty_x$ *and that* g_1 *and* h_1 *satisfy* (3.3) *and* (3.4). *Then for* $i \in \mathbb{N} \cup \{0\}$,

$$\left\| u^{(i)} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} \lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} + (1+t)^{-\frac{3}{2}+A+\varsigma} C^\infty_{g_1,T} C^\infty_{h_1,T}, \tag{3.17}$$

$0 \leq t \leq T$.

Proof. In view of (2.43), (3.12), (3.13), together with the assumptions of (3.3) and (3.4), we have

$$\begin{aligned}
&\left|\langle\xi\rangle^{\beta} u^{(0)}(t,x,\xi)\right| \\
&\leq \eta \left|\langle\xi\rangle^{\beta} \mathbb{S}_{w_1}(t) f_{w_1 0}(x,\xi)\right| + \int_0^t \left|\langle\xi\rangle^{\beta} \mathbb{S}_{w_1}(t;s)\Gamma_{w_1}(g_1,h_1)(s,x,\xi)\right| ds \qquad (3.18)\\
&\lesssim \eta \left\| f_{w_1 0}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} + \int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}} \left\|\nu^{-1}(\xi)\Gamma_{w_1}(g_1,h_1)(s,\cdot,\cdot)\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} ds\\
&\lesssim \eta \left\| f_{w_1 0}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} + \int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}} (1+s)^{-\frac{3}{2}+A} C^{\infty}_{g_1,T} C^{\infty}_{h_1,T} ds\\
&\lesssim \eta \left\| f_{w_1 0}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} + \left[(1+t)^{-\frac{3}{2}+A} \ln(1+t)\right] C^{\infty}_{g_1,T} C^{\infty}_{h_1,T}\\
&\lesssim \eta \left\| f_{w_1 0}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} + (1+t)^{-\frac{3}{2}+A+\varsigma} C^{\infty}_{g_1,T} C^{\infty}_{h_1,T}.
\end{aligned}$$

This completes the estimate for $u^{(0)}$.

For $u^{(1)}$, it follows from (2.25), (3.13) and (3.18) that

$$\begin{aligned}
\left|\langle\xi\rangle^{\beta} u^{(1)}(t,x,\xi)\right| &\leq \int_0^t \left|\mathbb{S}_{w_1}(t;s)\,\langle\xi\rangle^{\beta} K_{w_1} u^{(0)}(s,x,\xi)\right| ds\\
&\lesssim \int_0^t (1+t-s)^{\frac{2-\gamma}{\gamma}} \left\|\langle\xi\rangle^{-\gamma+2} K_{w_1} u^{(0)}(s,\cdot)\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} ds\\
&\lesssim \int_0^t (1+t-s)^{\frac{2-\gamma}{\gamma}} \left\| u^{(0)}(s,\cdot)\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} ds\\
&\lesssim \eta \left\| f_{w_1 0}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} + (1+t)^{-\frac{3}{2}+A+\varsigma} C^{\infty}_{g_1,T} C^{\infty}_{h_1,T}.
\end{aligned}$$

Since for $i \geq 2$,

$$\left|\langle\xi\rangle^{\beta} u^{(i)}(t,x,\xi)\right| = \left|\int_0^t \mathbb{S}_{w_1}(t;s)\,\langle\xi\rangle^{\beta} K_{w_1} u^{(i-1)}(s,x,\xi) ds\right|,$$

we can prove

$$\left\| u^{(i)}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} \lesssim \eta \left\| f_{w_1 0}\right\|_{L^{\infty}_{\xi,\beta}L^{\infty}_x} + (1+t)^{-\frac{3}{2}+A+\varsigma} C^{\infty}_{g_1,T} C^{\infty}_{h_1,T},$$

by induction on $i \geq 1$. □

Lemma 22 *(L^2 estimate of $u^{(i)}$, $i \geq 0$). Let $\beta > 3/2$. Assume that $f_{w_1 0} \in L^\infty_{\xi,\beta} L^2_x \cap L^\infty_{\xi,\beta} L^\infty_x$ and that g_1 and h_1 satisfy (3.3) and (3.4). Then for $i \in \mathbb{N} \cup \{0\}$,*

$$\left\| u^{(i)} \right\|_{L^2} \lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + (1+t)^{-1} C^2_{g_1,T} C^\infty_{h_1,T},$$

$0 \leq t \leq T$.

Proof. In view of (2.44) and the assumptions of (3.3) and (3.4),

$$\begin{aligned} \left\| \langle \xi \rangle^{-\gamma} \Gamma_{w_1}(g_1, h_1)(s, \cdot) \right\|_{L^2} &\lesssim \|g_1\|_{L^\infty_{\xi,\beta} L^2_x} \left\| \langle \xi \rangle^{p} h_1 \right\|_{L^\infty_{\xi,\beta} L^\infty_x} \\ &\lesssim (1+s)^{-\frac{3}{2}} C^2_{g_1,T} C^\infty_{h_1,T}. \end{aligned} \tag{3.19}$$

Therefore, using (2.26), (3.15), and (3.16) gives

$$\begin{aligned} \left\| u^{(0)} \right\|_{L^2} &= \left\| \eta \mathbb{S}_{w_1}(t) f_{w_1 0}(x,\xi) + \int_0^t \mathbb{S}_{w_1}(t;s) \Gamma_{w_1}(g_1,h_1)(s,x,\xi) ds \right\|_{L^2} \\ &\lesssim \eta \left\| f_{w_1 0} \right\|_{L^2} + \left(\int_0^t (1+t-s)^{-1} (1+s)^{-\frac{3}{2}} ds \right) C^2_{g_1,T} C^\infty_{h_1,T} \\ &\lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + (1+t)^{-1} C^2_{g_1,T} C^\infty_{h_1,T}. \end{aligned} \tag{3.20}$$

Note that

$$u^{(1)}(t,x,\xi) = \int_0^t \mathbb{S}_{w_1}(t;s) \left[K_{w_1} u^{(0)} \right](s,x,\xi) ds.$$

Using (2.26), (3.16) and (3.20) gives

$$\begin{aligned} \left\| u^{(1)} \right\|_{L^2} &\lesssim \int_0^t (1+t-s)^{\frac{2-\gamma}{\gamma}} \left\| u^{(0)}(s,\cdot,\cdot) \right\|_{L^2} ds \\ &\lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + \left(\int_0^t (1+t-s)^{\frac{2-\gamma}{\gamma}} (1+s)^{-1} ds \right) C^2_{g_1,T} C^\infty_{h_1,T} \\ &\lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + (1+t)^{-1} C^2_{g_1,T} C^\infty_{h_1,T}. \end{aligned}$$

Similarly, for $i \geq 2$,

$$u^{(i)}(t,x,\xi) = \int_0^t \mathbb{S}_{w_1}(t;s) \left[K_{w_1} u^{(i-1)} \right](s,x,\xi) ds,$$

and thus we can prove

$$\left\| u^{(i)} \right\|_{L^2} \lesssim \eta \, \| f_{w_1 0} \|_{L^\infty_{\xi,\beta} L^2_x} + (1+t)^{-1} C^2_{g_1,T} C^\infty_{h_1,T},$$

inductively for all $i \in \mathbb{N}$, by using (2.26), (3.16). □

3.2. Regularization estimate

In the previous subsection, we have carried out the $L^\infty_{\xi,\beta} L^\infty_x$ ($\beta > 3/2$) and L^2 estimates for the wave part $W^{(m)}_{w_1}$. To obtain the pointwise estimate on $\mathcal{R}^{(m)}_{w_1}$, we still need the H^2_x regularization estimate for $\mathcal{R}^{(m)}_{w_1}$. In light of (3.7), we turn to the H^2_x regularization estimate for $u^{(m)}$ in advance. To proceed, we introduce a differential operator:

$$\mathcal{D}_t = t \nabla_x + \nabla_\xi .$$

This operator $\mathcal{D}_t$ is important since it commutes with the free transport operator, i.e.,

$$[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0,$$

where $[A, B] = AB - BA$ is the commutator.

We remark that the crucial operator $\mathcal{D}_t$ was firstly introduced in the paper by Gualdani, Mischler and Mouhot [17], and Wu [33] applied it to reprove the Mixture Lemma used in [20,25,27,28].

The following lemma will be used to prove the regularization estimate:

Lemma 23. *For any $\tau \in \mathbb{R}$,*

$$\left\| \mathbb{S}_{w_1}(t;s) \left[\nabla_x, K_{w_1} \right] q(s,x,\xi) \right\|_{L^2} \lesssim (1+t-s)^{\frac{2-\gamma}{\gamma}} \| q(s,\cdot,\cdot) \|_{L^2} , \tag{3.21}$$

$$\left\| \mathcal{D}_{t-s} \mathbb{S}_{w_1}(t;s) q(s,x,\xi) \right\|_{L^2} \lesssim \left((1+t-s)^{\frac{\tau+1}{\gamma}} + \delta M (1+t-s)^{\frac{\tau+\gamma}{\gamma}} \right) \left\| \langle \xi \rangle^\tau q(s,\cdot,\cdot) \right\|_{H^1_\xi L^2_x} . \tag{3.22}$$

Consequently,

$$\left\| \mathcal{D}_{t-s} \mathbb{S}_{w_1}(t;s) K_{w_1} q(s,x,\xi) \right\|_{L^2} \lesssim \left((1+t-s)^{\frac{2-\gamma}{\gamma}} + \delta M (1+t-s)^{\frac{1}{\gamma}} \right) \| q(s,\cdot,\cdot) \|_{L^2} . \tag{3.23}$$

Proof. The estimate of (3.21) immediately follows from (2.15), (2.26), (3.15), and the estimate (3.23) is a consequence of (2.26), (2.27), and (3.22) by picking $\tau = -\gamma + 1$. Thus, it remains to prove (3.22). By the definition of $\mathbb{S}_{w_1}(t;s)$,

$$\begin{aligned} & \mathcal{D}_{t-s} \mathbb{S}_{w_1}(t;s) q(s,x,\xi) \\ & = \mathcal{D}_{t-s} \left(q(s, x-(t-s)\xi, \xi) \right) \exp \left(- \int_s^t \tilde{\nu}(r, x-(t-r)\xi, \xi) dr \right) \end{aligned}$$

$$+\left[q(s,x-(t-s)\xi,\xi)\left(\mathcal{D}_{t-s}\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\right)\right.$$
$$\left.\cdot\exp\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\right].$$

Since

$$\begin{aligned}&\nabla_\xi\left(q(s,x-(t-s)\xi,\xi)\right)\\&=(s-t)\nabla_x q(s,x-(t-s)\xi,\xi)+\nabla_\xi q(s,x-(t-s)\xi,\xi),\end{aligned}$$

we have

$$\mathcal{D}_{t-s}\left(q(s,x-(t-s)\xi,\xi)\right)=\nabla_\xi q(s,x-(t-s)\xi,\xi),$$

which implies that

$$\begin{aligned}&\left\|\mathcal{D}_{t-s}\left(q(s,x-(t-s)\xi,\xi)\right)\exp\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\right\|_{L^2}\\&\lesssim(1+t-s)^{\frac{\tau}{\gamma}}\left\|\langle\xi\rangle^\tau\nabla_\xi q(s,\cdot,\cdot)\right\|_{L^2}.\end{aligned}$$

In view of (2.15),

$$|\mathcal{D}_{t-s}\tilde{\nu}(r,x-(t-r)\xi,\xi)|\lesssim(t-s)\,\delta M\langle\xi\rangle^\gamma+(t-r)\delta M\langle\xi\rangle^\gamma+\langle\xi\rangle^{\gamma-1},$$

so that

$$\begin{aligned}&\left|\exp\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\left(\mathcal{D}_{t-s}\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\right)\right|\\&\lesssim\left((t-s)\langle\xi\rangle^{\gamma-1}+\delta M(t-s)^2\langle\xi\rangle^\gamma\right)\exp\left(-\frac{\nu(\xi)}{2}(t-s)\right).\end{aligned}$$

It implies that

$$\begin{aligned}&\left\|q(s,x-(t-s)\xi,\xi)\left(\mathcal{D}_{t-s}\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\right)\right.\\&\left.\quad\cdot\exp\left(-\int_s^t \tilde{\nu}(r,x-(t-r)\xi,\xi)dr\right)\right\|_{L^2}\\&\lesssim\left((1+t-s)^{\frac{\tau+1-\gamma}{\gamma}+1}+\delta M(1+t-s)^{\frac{\tau-\gamma}{\gamma}+2}\right)\left\|\langle\xi\rangle^\tau q(s,\cdot,\cdot)\right\|_{L^2}.\end{aligned}$$

Therefore, we can conclude

$$\begin{aligned}&\left\|\mathcal{D}_{t-s}\mathbb{S}_{w_1}(t;s)q(s,x,\xi)\right\|_{L^2}\\&\lesssim\left((1+t-s)^{\frac{\tau+1}{\gamma}}+\delta M(1+t-s)^{\frac{\tau+\gamma}{\gamma}}\right)\left\|\langle\xi\rangle^{\tau}q(s,\cdot,\cdot)\right\|_{H^1_\xi L^2_x}.\end{aligned}$$

The proof of this lemma is completed. □

Now, we are ready to get the H^2_x regularization estimate. In fact, we find that it is enough to get the H^2_x regularization estimate by taking $m=6$.

Lemma 24 *(H^2_x regularization estimate on $u^{(6)}$). Let ς be any positive number with $0<\varsigma\ll 1$. Then there exists a constant $C_{\varsigma,\gamma,p}>0$ such that*

$$\begin{aligned}&\left\|u^{(6)}(t,x,\xi)\right\|_{L^2_\xi H^2_x}\\&\le C_{\varsigma,\gamma,p}\cdot\begin{cases}(1+\delta M)\left[\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+C^2_{g_1,T}C^\infty_{h_1,T}\right], & \text{if } -1<\gamma<0,\\(1+\delta M)\left[\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+C^2_{g_1,T}C^\infty_{h_1,T}\right](1+t)^{\varsigma}, & \text{if } \gamma=-1,\\(1+\delta M)\left[\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+C^2_{g_1,T}C^\infty_{h_1,T}\right](1+t)^{5+\frac{5}{\gamma}}, & \text{if } -2<\gamma<-1,\end{cases}\end{aligned}$$

$0\le t\le T$.

Proof. In view of Lemma 22,

$$\left\|u^{(6)}(t,x,\xi)\right\|_{L^2}\lesssim\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+(1+t)^{-1}C^2_{g_1,T}C^\infty_{h_1,T}.$$

Next, we prove the estimate for the first x-derivative of $u^{(6)}$. Note that

$$\begin{aligned}&\nabla_x u^{(6)}(t,x,\xi)\\&=\nabla_x\int_0^t\int_0^{s_1}\int_0^{s_2}\mathbb{M}_1\mathbb{M}_2\left(\frac{s_1-s_2}{s_1-s_3}\mathbb{M}_3\right)u^{(3)}(s_3,\cdot,\cdot)ds_3ds_2ds_1\\&\quad+\nabla_x\int_0^t\int_0^{s_1}\int_0^{s_2}\mathbb{M}_1\mathbb{M}_2\left(\frac{s_2-s_3}{s_1-s_3}\mathbb{M}_3\right)u^{(3)}(s_3,\cdot,\cdot)ds_3ds_2ds_1\\&=\int_0^t\int_0^{s_1}\int_0^{s_2}\frac{1}{s_1-s_3}\mathbb{M}_1\left(\mathcal{D}_{s_1-s_2}-\nabla_\xi\right)\mathbb{M}_2\mathbb{M}_3u^{(3)}(s_3,\cdot,\cdot)\,ds_3ds_2ds_1\end{aligned}$$

$$+\int_0^t\int_0^{s_1}\int_0^{s_2}\frac{1}{s_1-s_3}\mathbb{M}_1\mathbb{M}_2\left(\mathcal{D}_{s_2-s_3}-\nabla_\xi\right)\mathbb{M}_3u^{(3)}(s_3,\cdot,\cdot)\,ds_3ds_2ds_1$$

$$+\int_0^t T(s_1,x,\xi,t)\,ds_1,$$

where $\mathbb{M}_i=\mathbb{S}_{w_1}(s_{i-1};s_i)[K_{w_1}]_{s_i}$, $[K_{w_1}]_{s_i}=w_1(s_i,x,\xi)\,Kw_1^{-1}(s_i,x,\xi_*)$, $s_0\equiv t$, and

$$\int_0^t T(s_1,x,\xi,t)\,ds_1$$

$$=\int_0^t\int_0^{s_1}\int_0^{s_2}\left[\nabla_x,\mathbb{S}_{w_1}(t;s_1)\right][K_{w_1}]_{s_1}\mathbb{M}_2\mathbb{M}_3u^{(3)}(s_3,\cdot,\cdot)\,ds_3ds_2ds_1$$

$$+\int_0^t\int_0^{s_1}\int_0^{s_2}\mathbb{S}_{w_1}(t;s_1)\left[\nabla_x,[K_{w_1}]_{s_1}\right]\mathbb{M}_2\mathbb{M}_3u^{(3)}(s_3,\cdot,\cdot)\,ds_3ds_2ds_1$$

$$+\int_0^t\int_0^{s_1}\int_0^{s_2}\frac{s_2-s_3}{s_1-s_3}\mathbb{M}_1\left[\nabla_x,\mathbb{S}_{w_1}(s_1;s_2)\right][K_{w_1}]_{s_2}\mathbb{M}_3u^{(3)}(s_3,\cdot,\cdot)\,ds_3ds_2ds_1$$

$$+\int_0^t\int_0^{s_1}\int_0^{s_2}\frac{s_2-s_3}{s_1-s_3}\mathbb{M}_1\mathbb{S}_{w_1}(s_1;s_2)\left[\nabla_x,[K_{w_1}]_{s_2}\right]\mathbb{M}_3u^{(3)}(s_3,\cdot,\cdot)\,ds_3ds_2ds_1.$$

Note that

$$\left\|\left[\nabla_x,\mathbb{S}_{w_1}(s_{i-1};s_i)\right][K_{w_1}]_{s_i}q(s_i,x,\xi)\right\|_{L^2}\lesssim(1+s_{i-1}-s_i)^{\frac{2-\gamma}{\gamma}}\|q(s_i,\cdot,\cdot)\|_{L^2}$$

and

$$\left\|\mathbb{S}_{w_1}(s_{i-1};s_i)\left[\nabla_x,[K_{w_1}]_{s_i}\right]q(s_i,x,\xi)\right\|_{L^2}\lesssim(1+s_{i-1}-s_i)^{\frac{2-\gamma}{\gamma}}\|q(s_i,\cdot,\cdot)\|_{L^2},$$

for $i=1,2$. By (2.26), (2.27), (3.16), (3.23) and Lemma 22, we obtain

$$\left\|\nabla_xu^{(6)}(t,x,\xi)\right\|_{L^2}$$

$$\lesssim(1+\delta M)\int_0^t\int_0^{s_1}\int_0^{s_2}\left\{(1+t-s_1)^{\frac{2-\gamma}{\gamma}}(1+s_1-s_2)^{\frac{1}{\gamma}}(1+s_2-s_3)^{\frac{1}{\gamma}}\left(1+\frac{1}{s_1-s_3}\right)\right\}$$

$$\cdot\left(\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+(1+s_3)^{-1}C^2_{g_1,T}C^\infty_{g_2,T}\right)ds_3ds_2ds_1$$

$$\lesssim \mathbb{A} \cdot \int_0^t \int_0^{s_1} \int_0^{s_2} (1+t-s_1)^{\frac{2-\gamma}{\gamma}} (1+s_1-s_2)^{\frac{1}{\gamma}} (1+s_2-s_3)^{\frac{1}{\gamma}} ds_3 ds_2 ds_1$$

$$+ \mathbb{A} \int_0^t \int_0^{s_1} \int_{s_3}^{s_1} (1+t-s_1)^{\frac{2-\gamma}{\gamma}} (1+s_1-s_2)^{\frac{1}{\gamma}} (1+s_2-s_3)^{\frac{1}{\gamma}} \frac{1}{s_1-s_3} ds_2 ds_3 ds_1$$

$$\lesssim \mathbb{A} \int_0^t \int_0^{s_1} \int_0^{s_2} (1+t-s_1)^{\frac{2-\gamma}{\gamma}} (1+s_1-s_2)^{\frac{1}{\gamma}} (1+s_2-s_3)^{\frac{1}{\gamma}} ds_3 ds_2 ds_1$$

$$+ \mathbb{A} \int_0^t \int_0^{s_1} (1+t-s_1)^{\frac{2-\gamma}{\gamma}} (1+s_1-s_3)^{\frac{1}{\gamma}} ds_3 ds_1$$

$$\lesssim \begin{cases} \mathbb{A}, & \text{if } -1<\gamma<0, \\ \mathbb{A}\,(1+t)^{\varsigma}, & \text{if } \gamma=-1, \\ \mathbb{A}(1+t)^{\frac{2}{\gamma}+2}, & \text{if } -2<\gamma<-1, \end{cases}$$

where $\mathbb{A} = (1+\delta M)\left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{g_1,T} C^\infty_{g_2,T}\right)$. Here, the third inequality holds since $\gamma < 0$ and $(1+s_1-s_2)(1+s_2-s_3) \geq 1+s_1-s_3$ for $s_3 \leq s_2 \leq s_1$.

For $\left\| \nabla_x^2 u^{(6)}(t,x,\xi) \right\|_{L^2}$, rewrite

$$u^{(6)} = \int_0^t \int_0^{s_1} \int_0^{s_2} \mathbb{M}_1 \mathbb{M}_2 \mathbb{M}_3 u^{(3)}(s_3,\cdot,\cdot) ds_3 ds_2 ds_1$$

$$= \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \int_0^{s_5} \mathbb{M}_1 \mathbb{M}_2 \mathbb{M}_3 \mathbb{M}_4 \mathbb{M}_5 \mathbb{M}_6 u^{(0)}(s_6,\cdot,\cdot)\, ds,$$

where $ds = ds_6 ds_5 ds_4 ds_3 ds_2 ds_1$, and then we can obtain

$$\left\| \nabla_x^2 u^{(6)}(t,x,\xi) \right\|_{L^2}$$

$$\lesssim \mathbb{A} \cdot \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_4} \int_0^{s_5} \int_0^{s_6} (1+t-s_1)^{\frac{2-\gamma}{\gamma}} (1+s_1-s_2)^{\frac{1}{\gamma}} \cdots (1+s_5-s_6)^{\frac{1}{\gamma}}$$

$$\cdot \left(1+\frac{1}{s_1-s_3}\right)\left(1+\frac{1}{s_4-s_6}\right) ds$$

$$\lesssim \begin{cases} \mathbb{A}, & \text{if } -1<\gamma<0, \\ \mathbb{A}\,(1+t)^{\varsigma}, & \text{if } \gamma=-1, \\ \mathbb{A}\,(1+t)^{5+\frac{5}{\gamma}}, & \text{if } -2<\gamma<-1, \end{cases}$$

by the same argument. The proof of this lemma is completed. □

3.3. Estimate of the remainder part

In this subsection, we return to deal with the remainder part $\mathcal{R}_{w_1}^{(6)}$.

Proposition 25 *(Regularization estimate on $\mathcal{R}_{w_1}^{(6)}$). Let ς be any positive number with $0 < \varsigma \ll 1$. Then*

$$\begin{aligned}
&\left\| \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi H^2_x} \\
&\lesssim (1+\delta M)\left(\eta \left\| f_{w_1 0}\right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T}\right) \cdot \begin{cases} (1+t)^2, & \text{if } -1 < \gamma < 0, \\ (1+t)^{2+\varsigma}, & \text{if } \gamma = -1, \\ (1+t)^{7+\frac{5}{\gamma}}, & \text{if } -2 < \gamma < -1. \end{cases}
\end{aligned}$$

Proof. In view of Lemma 13,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\| \mathcal{R}_{w_1}^{(6)} \right\|^2_{L^2_\xi H^2_x} &\lesssim \int\limits_{H_0^D \cup H_-^D} \sum_{|\alpha|\le 2} \left| \partial_x^\alpha \mathrm{P}_0 \mathcal{R}_{w_1}^{(6)} \right|^2 d\xi dx + \left\| \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi H^2_x} \left\| K_{w_1} u^{(6)} \right\|_{L^2_\xi H^2_x} \\
&\lesssim \sum_{|\alpha|\le 2} \left\| w_1^{-1} \partial_x^\alpha \mathcal{R}_{w_1}^{(6)} \right\|^2_{L^2} + \left\| \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi H^2_x} \left\| u^{(6)} \right\|_{L^2_\xi H^2_x} \\
&\lesssim \left\| w_1^{-1} \mathcal{R}_{w_1}^{(6)} \right\|^2_{L^2_\xi H^2_x} + \left\| \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi H^2_x} \left\| u^{(6)} \right\|_{L^2_\xi H^2_x},
\end{aligned}$$

the last inequality being valid since

$$\sum_{|\alpha|\le 2} \left\| w_1^{-1} \partial_x^\alpha \mathcal{R}_{w_1}^{(6)} \right\|_{L^2} \le C \left\| w_1^{-1} \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi H^2_x}$$

for some constant $C > 0$.

Now we need to estimate $\left\| w_1^{-1} \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi H^2_x}$. Let $z = w_1^{-1} \mathcal{R}_{w_1}^{(6)}$ and then z solves the equation

$$\partial_t z + \xi \cdot \nabla_x z = Lz + K\left(w_1^{-1} u^{(6)}\right). \tag{3.24}$$

By the energy estimate and (2.3), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|z\|^2_{L^2} &\le \int\limits_{\mathbb{R}^6} zLz d\xi dx + \left\| zK w_1^{-1} u^{(6)} \right\|_{L^2} \\
&\lesssim \|z\|_{L^2} \left\| w_1^{-1} u^{(6)} \right\|_{L^2} \lesssim \|z\|_{L^2} \left\| u^{(6)} \right\|_{L^2},
\end{aligned}$$

since $w_1(t,x,\xi) \geq 3\langle\xi\rangle_D^p \geq 1$. Therefore,

$$\frac{d}{dt}\|z\|_{L^2} \lesssim \left\|u^{(6)}\right\|_{L^2}. \tag{3.25}$$

Moreover, in view of (3.24),

$$\partial_t\left(\partial_{x_i}z\right)+\xi\cdot\nabla_x\left(\partial_{x_i}z\right)=L\left(\partial_{x_i}z\right)+K\left(\left(\partial_{x_i}w_1^{-1}\right)u^{(6)}\right)+K\left(w_1^{-1}\partial_{x_i}u^{(6)}\right),$$

$$\partial_t\left(\partial^2_{x_ix_k}z\right)+\xi\cdot\nabla_x\left(\partial^2_{x_ix_k}z\right)=L\left(\partial^2_{x_ix_k}z\right)+K\left(\left(\partial^2_{x_ix_k}w_1^{-1}\right)u^{(6)}\right)+K\left(\left(\partial_{x_i}w_1^{-1}\right)\partial_{x_k}u^{(6)}\right)$$
$$+K\left(\left(\partial_{x_k}w_1^{-1}\right)\partial_{x_i}u^{(6)}\right)+K\left(w_1^{-1}\partial^2_{x_ix_k}u^{(6)}\right).$$

By the energy estimate and (2.3) again, together with Lemma 9, we deduce that

$$\frac{d}{dt}\sum_{|\alpha|\leq 2}\left\|\partial_x^\alpha z\right\|_{L^2} \lesssim \left\|u^{(6)}\right\|_{L^2_\xi H^2_x}.$$

Since $z(0,x,\xi)=0$, it implies that

$$\sum_{|\alpha|\leq 2}\left\|w_1^{-1}\partial_x^\alpha \mathcal{R}^{(6)}_{w_1}\right\|_{L^2} \lesssim \sum_{|\alpha|\leq 2}\left\|\partial_x^\alpha z\right\|_{L^2} \lesssim \int_0^t \left\|u^{(6)}(\cdot,s)\right\|_{L^2_\xi H^2_x} ds.$$

According to Lemma 24, whenever $-1<\gamma<0$, we have

$$\begin{aligned}
&\int_0^t \left\|u^{(6)}(s,\cdot)\right\|_{L^2_\xi H^2_x} ds \\
&\lesssim \int_0^t (1+\delta M)\left[\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+(1+s)^{-1}C^2_{g_1,T}C^\infty_{h_1,T}\right]ds \\
&\lesssim (1+\delta M)(1+t)\left(\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+C^2_{g_1,T}C^\infty_{h_1,T}\right),
\end{aligned}$$

and thus

$$\frac{d}{dt}\left\|\mathcal{R}^{(6)}_{w_1}\right\|^2_{L^2_\xi H^2_x} \lesssim (1+t)^2\left[(1+\delta M)\left(\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+C^2_{g_1,T}C^\infty_{h_1,T}\right)\right]^2$$
$$+\left\|\mathcal{R}^{(6)}_{w_1}\right\|_{L^2_\xi H^2_x}(1+t)(1+\delta M)\left(\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x}+C^2_{g_1,T}C^\infty_{h_1,T}\right).$$

As a consequence,

$$\left\|\mathcal{R}_{w_1}^{(6)}\right\|_{L^2_\xi H^2_x} \lesssim (1+t)^2\,(1+\delta M)\left(\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x} + C^2_{g_1,T}C^\infty_{h_1,T}\right)$$

for $-1<\gamma<0$. The other cases $-2<\gamma<-1$ and $\gamma=-1$ can be obtained by the same argument and the proof of the proposition is completed. □

Therefore, in view of Proposition 25, the Sobolev inequality implies that

$$\begin{aligned}
&\left\|\mathcal{R}_{w_1}^{(6)}\right\|_{L^2_\xi L^\infty_x}\\
&\lesssim \left\|\mathcal{R}_{w_1}^{(6)}\right\|^{3/4}_{L^2_\xi H^2_x}\left\|\mathcal{R}_{w_1}^{(6)}\right\|^{1/4}_{L^2}\\
&\lesssim (1+\delta M)\left(\eta\left\|f_{w_10}\right\|_{L^\infty_{\xi,\beta}L^2_x} + C^2_{g_1,T}C^\infty_{h_1,T}\right)\cdot\begin{cases}(1+t)^2, & \text{if } -1<\gamma<0,\\ (1+t)^{2+\varsigma}, & \text{if } \gamma=-1,\\ (1+t)^{7+\frac{5}{\gamma}}, & \text{if } -2<\gamma<-1,\end{cases}
\end{aligned}\tag{3.26}$$

for any $0<\varsigma\ll 1$. Together with Lemma 21, we obtain the estimate of $\|u\|_{L^2_\xi L^\infty_x}$. Subsequently, we shall get the estimate for $\|u\|_{L^\infty_{\xi,\beta}L^\infty_x}$ via the bootstrap argument and the details are given in the proof of Theorem 18.

3.4. *Proof of Theorem 18*

Note that if $i\ge 2$, the solution u to (3.2) can be represented as

$$u(t,x,\xi)=W_{w_1}^{(i)}+\int_0^t \mathbb{S}_{w_1}(t,s)K_{w_1}\mathcal{R}_{w_1}^{(i-1)}(s)ds = W_{w_1}^{(i)}+\mathcal{R}_{w_1}^{(i)}.$$

In view of Lemma 21, it remains to estimate $\left\|\mathcal{R}_{w_1}^{(i)}\right\|_{L^\infty_{\xi,\beta}L^\infty_x}$ for some i in order to obtain the estimate for $\|u\|_{L^\infty_{\xi,\beta}L^\infty_x}$. To obtain the estimate for $\left\|\mathcal{R}_{w_1}^{(i)}\right\|_{L^\infty_{\xi,\beta}L^\infty_x}$, we consider γ in two different cases: $-2<\gamma\le -3/2$ and $-3/2<\gamma<0$.

In the case $-2<\gamma\le -3/2$, in view of (2.29), (2.30) and (3.13), together with the fact that

$$\left\|\mathbb{S}_{w_1}(s,s_1)q(s_1,x,\xi)\right\|_{L^4_\xi L^\infty_x} \lesssim (1+s-s_1)^{\frac{1-\gamma}{\gamma}}\left\|q(s_1,\cdot,\cdot)\right\|_{L^4_{\xi,1-\gamma}L^\infty_x},$$

we have

$$\left\|\mathcal{R}_{w_1}^{(8)}\right\|_{L^\infty_\xi L^\infty_x} = \left\|\int_0^t\int_0^s \mathbb{S}_{w_1}(t,s)\left[K_{w_1}\right]_s\mathbb{S}_{w_1}(s,s_1)\left[K_{w_1}\right]_{s_1}\mathcal{R}_{w_1}^{(6)}(s_1)ds_1ds\right\|_{L^\infty_\xi L^\infty_x}$$

$$\lesssim \int_0^t \int_0^s (1+t-s)^{\frac{7/4-\gamma}{\gamma}} (1+s-s_1)^{\frac{1-\gamma}{\gamma}} \left\| \mathcal{R}_{w_1}^{(6)}(s_1) \right\|_{L^2_\xi L^\infty_x} ds_1 ds$$

$$\lesssim \int_0^t \int_0^s (1+t-s)^{\frac{7/4-\gamma}{\gamma}} (1+s-s_1)^{\frac{1-\gamma}{\gamma}} (1+s_1)^{7+\frac{5}{\gamma}}$$

$$\cdot (1+\delta M) \left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T} \right) ds_1 ds$$

$$\lesssim (1+t)^{7+\frac{5}{\gamma}} (1+\delta M) \left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T} \right).$$

Combining this with Lemma 21, it follows

$$\begin{aligned} \|u\|_{L^\infty_\xi L^\infty_x} &\le \left\| W_{w_1}^{(8)} \right\|_{L^\infty_\xi L^\infty_x} + \left\| \mathcal{R}_{w_1}^{(8)} \right\|_{L^\infty_\xi L^\infty_x} \\ &\lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} + (1+t)^{-3/2+A+\varsigma} C^\infty_{g_1,T} C^\infty_{h_1,T} \\ &\quad + (1+t)^{7+\frac{5}{\gamma}} (1+\delta M) \left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T} \right). \end{aligned}$$

Note that

$$u(t,x,\xi) = u^{(0)}(t,x,\xi) + \int_0^t \mathbb{S}_{w_1}(t,s) K_{w_1} u(s) ds. \tag{3.27}$$

Hence, through (2.25), (3.13) and Lemma 21, we infer

$$\begin{aligned} \|u\|_{L^\infty_{\xi,\beta} L^\infty_x} &\lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} + (1+t)^{-3/2+A+\varsigma} C^\infty_{g_1,T} C^\infty_{h_1,T} \\ &\quad + (1+t)^{7+\frac{5}{\gamma}} (1+\delta M) \left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T} \right), \end{aligned}$$

by the bootstrap argument.

In the case $-3/2 < \gamma < 0$, we decompose u as $u = W_{w_1}^{(7)} + \mathcal{R}_{w_1}^{(7)}$. In view of (2.28) and (3.13),

$$\begin{aligned} \left\| \mathcal{R}_{w_1}^{(7)} \right\|_{L^\infty_\xi L^\infty_x} &= \left\| \int_0^t \mathbb{S}_{w_1}(t,s) \left[K_{w_1} \right]_s \mathcal{R}_{w_1}^{(6)}(s) ds \right\|_{L^\infty_\xi L^\infty_x} \\ &\lesssim \int_0^t (1+t-s)^{\frac{\frac{3}{2}-\gamma}{\gamma}} \left\| \mathcal{R}_{w_1}^{(6)} \right\|_{L^2_\xi L^\infty_x} ds. \end{aligned}$$

Hence, we obtain the estimate of $\|u\|_{L^\infty_\xi L^\infty_x}$ by using (3.26) and Lemma 21. Again, through (2.25), (3.13), (3.27), Lemma 21, we can conclude that

$$\|u\|_{L^\infty_{\xi,\beta}L^\infty_x} \lesssim \eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta}L^\infty_x} + (1+t)^{-\frac{3}{2}+A+\varsigma} C^\infty_{g_1,T} C^\infty_{h_1,T}$$

$$+ (1+\delta M)\left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta}L^2_x} + C^2_{g_1,T} C^\infty_{h_1,T}\right) \cdot \begin{cases} (1+t)^2, & \text{if } -1<\gamma<0, \\ (1+t)^{2+\varsigma}, & \text{if } \gamma=-1, \\ (1+t)^{7+\frac{5}{\gamma}}, & \text{if } \frac{-3}{2}<\gamma<-1, \end{cases}$$

by the bootstrap argument. This completes the proof of Theorem 18. □

3.5. The result for the exponential weight function w_2

For the exponential weight, we consider the inhomogeneous equation:

$$\begin{cases} \partial_t v + \xi \cdot \nabla_x v - \epsilon \left[\partial_t \rho + \xi \cdot \nabla_x \rho\right] v = L_{w_2} v + \Gamma_{w_2}(g_2, h_2), \\ v(0,x,\xi) = \eta f_{w_2 0}. \end{cases} \tag{3.28}$$

Let $T>0$, $\beta>3/2$, and $0<p\le 2$. Assume that $f_{w_2 0} \in L^\infty_{\xi,\beta}L^2_x \cap L^\infty_{\xi,\beta}L^\infty_x$. Also assume that g_2 and h_2 satisfy

$$\hat{C}^\infty_{g_2,T} = \sup_{0\le t\le T} (1+t)^{-B} \|g_2\|_{L^\infty_{\xi,\beta}L^\infty_x} < \infty, \quad \hat{C}^2_{g_2,T} = \sup_{0\le t\le T} \|g_2\|_{L^\infty_{\xi,\beta}L^2_x} < \infty, \tag{3.29}$$

for some constant $B>0$, and

$$\hat{C}^\infty_{h_2,T} = \sup_{0\le t\le T} (1+t)^{\frac{3}{2}} \left\| \langle\xi\rangle^p e^{\epsilon c_p \langle\xi\rangle^p} h_2 \right\|_{L^\infty_{\xi,\beta}L^\infty_x} < \infty. \tag{3.30}$$

Here the constant $c_p>0$ is the same as in Lemma 10.

Under these assumptions, following a similar argument as in the previous subsections, one can get the following theorem:

Theorem 26. *Let $\beta>3/2$ and $0<\varsigma\ll 1$. Assume that $f_{w_2 0} \in L^\infty_{\xi,\beta}L^2_x \cap L^\infty_{\xi,\beta}L^\infty_x$, and that g_2, h_2 satisfy (3.29) and (3.30), respectively. Then the solution v to (3.28) satisfies*

$$\|v\|_{L^\infty_{\xi,\beta}L^\infty_x} \lesssim \eta \left\| f_{w_2 0} \right\|_{L^\infty_{\xi,\beta}L^\infty_x} + (1+t)^{-\frac{3}{2}+B+\varsigma} \hat{C}^\infty_{g_2,T} \hat{C}^\infty_{h_2,T} \tag{3.31}$$

$$+ (1+\delta M)\left(\eta \left\| f_{w_2 0} \right\|_{L^\infty_{\xi,\beta}L^2_x} + \hat{C}^2_{g_2,T} \hat{C}^\infty_{h_2,T}\right) \cdot \begin{cases} (1+t)^2, & \text{if } -1<\gamma<0, \\ (1+t)^{2+\varsigma}, & \text{if } \gamma=-1, \\ (1+t)^{7+\frac{5}{\gamma}}, & \text{if } -2<\gamma<-1, \end{cases}$$

for $0\le t\le T$.

4. Proof of Theorem 2

To demonstrate Theorem 2, we also need to study the large time behavior of the solution of the Boltzmann equation (1.2) in suitable velocity weight. The result is stated in Theorem 1 but we postpone its proof to the next section. With the help of Theorems 1, 18, and 26, we are in the position to prove Theorem 2.

4.1. Proof of Theorem 2

In light of Theorem 18, we need to estimate $\|f_{w_1}\|_{L^\infty_{\xi,\beta}L^2_x}$ and $\|f_{w_2}\|_{L^\infty_{\xi,\beta}L^2_x}$. Let $T>0$. In view of (2.45) and $\langle\xi\rangle^{\gamma/2}\le 1$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_{w_1}\Gamma_{w_1}(f_{w_1},f)dxdv\\
&\lesssim \|f_{w_1}\|_{L^2_\sigma L^2_x}\left(\|f_{w_1}\|_{L^2_\sigma L^2_x}\|\langle\xi\rangle^p f\|_{L^\infty_\xi L^\infty_x}+\|f_{w_1}\|_{L^\infty_\xi L^2_x}\|\langle\xi\rangle^p f\|_{L^2_\sigma L^\infty_x}\right)\\
&\lesssim \left(\|\mathrm{P}_1 f_{w_1}\|^2_{L^2_\sigma L^2_x}+\|\mathrm{P}_0 f_{w_1}\|^2_{L^2_\sigma L^2_x}\right)\|\langle\xi\rangle^p f\|_{L^\infty_\xi L^\infty_x}+\|f_{w_1}\|_{L^2_\sigma L^2_x}\|f_{w_1}\|_{L^\infty_\xi L^2_x}\|\langle\xi\rangle^p f\|_{L^2_\sigma L^\infty_x}\\
&\lesssim \left\|\langle\xi\rangle^{p+\beta+\gamma/2}f(t)\right\|_{L^\infty_\xi L^\infty_x}\left(\|\mathrm{P}_1 f_{w_1}\|^2_{L^2_\sigma L^2_x}+\left[C^2_{f_{w_1},T}\right]^2\right),
\end{aligned}
$$

for $0\le t\le T$, where $C^2_{f_{w_1},T}:=\sup_{0\le t\le T}\|f_{w_1}\|_{L^\infty_{\xi,\beta}L^2_x}$. By Theorem 1 with $\hat{\varepsilon}=0$,

$$
\begin{aligned}
\left\|\langle\xi\rangle^{p+\beta+\gamma/2}f(t)\right\|_{L^\infty_\xi L^\infty_x}&\le\left\|\langle\xi\rangle^{p+\beta}f(t)\right\|_{L^\infty_\xi L^\infty_x}\\
&\lesssim\eta(1+t)^{-\frac{3}{2}}\left(\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^1_x}+\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\right)\\
&\lesssim\eta(1+t)^{-\frac{3}{2}}\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x},
\end{aligned}
$$

since $f_0(\cdot,\xi)$ has compact support contained in the unit ball centered at the origin for all ξ. After choosing $\delta,\eta>0$ sufficiently small, $D,M\ge 1$ sufficiently large with $\delta M\ll\nu_0$ and $D^{-1}\ll\delta M$, it follows from Lemma 13 and Theorem 1 that

$$
\begin{aligned}
\frac{d}{dt}\|f_{w_1}\|^2_{L^2}\le&-\left(\nu_0-C_1D^{-2}-C_2\delta-C_3\delta M-C_6\eta\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\right)\\
&\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\langle\xi\rangle^\gamma\left(\mathrm{P}_1 f_{w_1}\right)^2dxd\xi\\
&-\left(C_4\delta M-C_2\delta-C_1D^{-2}\right)\int_{H^D_+}[\delta(\langle x\rangle-Mt)]^{-1}\left|\mathrm{P}_0 f_{w_1}\right|^2dxd\xi\\
&+\left(C_1D^{-2}+C_2\delta+C_5\delta M\right)\int_{H^D_0}\left|\mathrm{P}_0 f_{w_1}\right|^2dxd\xi+C_1D^{-2}\int_{H^D_-}\left|\mathrm{P}_0 f_{w_1}\right|^2dxd\xi
\end{aligned}
$$

$$
\begin{aligned}
&+ C_6\eta\,(1+t)^{-\frac{3}{2}}\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\left[C^2_{f_{w_1},T}\right]^2\\
&\le C_6\eta\,(1+t)^{-\frac{3}{2}}\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\left[C^2_{f_{w_1},T}\right]^2 + C_8\,\|f\|^2_{L^2}\\
&\lesssim \eta\,(1+t)^{-\frac{3}{2}}\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\left[C^2_{f_{w_1},T}\right]^2 + \eta^2\,(1+t)^{-3/2}\,\|f_0\|^2_{L^\infty_{\xi,p+\beta+3j}L^\infty_x},
\end{aligned}
$$

the last inequality being valid since $\beta > \frac{3}{2}$ and $f_0(\cdot,\xi)$ has compact support contained in the unit ball centered at the origin for all ξ. Therefore, for $0 \le t \le T$,

$$
\begin{aligned}
\left\|f_{w_1}(t)\right\|_{L^2} &\lesssim \eta\left\|f_{w_10}\right\|_{L^2} + \eta^{1/2}\,\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\,C^2_{f_{w_1},T} + \eta\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} \\
&\lesssim \eta\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\,\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\,C^2_{f_{w_1},T}.
\end{aligned}
\tag{4.1}
$$

Next, in terms of the operator $\mathbb{S}_{w_1}(t;s)$, f_{w_1} can be rewritten as

$$
f_{w_1}(t) = \eta\mathbb{S}_{w_1}(t)f_{w_10} + \int_0^t \mathbb{S}_{w_1}(t;s)K_{w_1}f_{w_1}(s) + \mathbb{S}_{w_1}(t;s)\Gamma_{w_1}(f_{w_1},f)(s)ds,
\tag{4.2}
$$

for $0 \le t \le T$. In the sequel, we shall utilize this representation to establish the estimate for $\left\|f_{w_1}(t)\right\|_{L^\infty_{\xi,\beta}L^2_x}$ in two cases $-3/2 < \gamma < 0$ and $-2 < \gamma \le -3/2$ separately.

Case I: $-3/2 < \gamma < 0$**.** By (1.4), (2.28), (2.43), (3.14), and (4.1),

$$
\begin{aligned}
&\left\|f_{w_1}(t)\right\|_{L^\infty_\xi L^2_x}\\
&\le \eta\left\|\mathbb{S}_{w_1}(t)f_{w_10}\right\|_{L^\infty_\xi L^2_x} + \int_0^t \left\|\mathbb{S}_{w_1}(t;s)K_{w_1}f_{w_1}(s)\right\|_{L^\infty_\xi L^2_x} + \left\|\mathbb{S}_{w_1}(t;s)\Gamma_{w_1}(f_{w_1},f)(s)\right\|_{L^\infty_\xi L^2_x}ds\\
&\le \eta\left\|f_{w_10}\right\|_{L^\infty_\xi L^2_x} + C_{\gamma,p}\int_0^t (1+t-s)^{\frac{3/2-\gamma}{\gamma}}\left\|f_{w_1}(s)\right\|_{L^2}ds\\
&\quad + C_{\gamma,p}\int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}}\left\|f_{w_1}(s)\right\|_{L^\infty_\xi L^2_x}\left\|\langle\xi\rangle^p f(s)\right\|_{L^\infty_\xi L^\infty_x}ds\\
&\le \eta\left\|f_{w_10}\right\|_{L^\infty_\xi L^2_x} + C_{1,\gamma,p,\beta,j}\left[\eta\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\,\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\,C^2_{f_{w_1},T}\right]\\
&\quad + C_{1,\gamma,p,\beta,j}\int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}}\eta(1+s)^{-\frac{3}{2}}\left(\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\right)ds\cdot\sup_{0\le s\le T}\left\|f_{w_1}(s)\right\|_{L^\infty_\xi L^2_x}\\
&\le C'_{1,\gamma,p,\beta,j}\left[\eta\,\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\,\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\,C^2_{f_{w_1},T}\right]
\end{aligned}
$$

$$+ C'_{1,\gamma,p,\beta,j}\eta \left(\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} \right) \cdot \sup_{0\le s\le T} \left\| f_{w_1}(s) \right\|_{L^\infty_\xi L^2_x}.$$

Since $\eta > 0$ is sufficiently small such that $C'_{1,\gamma,p,\beta,j}\eta \left(\|f_0\|_{L^\infty_{\xi,p+3j}L^\infty_x} \right) < 1/2$, it follows that

$$\left\| f_{w_1}(t) \right\|_{L^\infty_\xi L^2_x} \le 2C'_{1,\gamma,p,\beta,j} \left[\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2} \|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T} \right].$$

In view of (4.2), we use a bootstrap argument to obtain

$$\left\| f_{w_1}(t) \right\|_{L^\infty_{\xi,\beta}L^2_x} \le C''_{1,\gamma,p,\beta,j} \left[\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2} \|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T} \right]$$

for $0 \le t \le T$, through (2.25), (2.43) and (3.14). Since $\eta > 0$ is sufficiently small, we have

$$\left\| f_{w_1}(t) \right\|_{L^\infty_{\xi,\beta}L^2_x} \le C^2_{f_{w_1},T} \le C'''_{1,\gamma,p,\beta,j}\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}.$$

Case II: $-2 < \gamma \le -3/2$. Utilizing (2.43) and (2.44) with $\beta > 3/2$ gives

$$\begin{aligned} &\left\| \exp\left(-\nu(\xi)(t-s)\right) \Gamma_{w_1}(f_{w_1}, f)(s) \right\|_{L^4_\xi L^2_x} \qquad (4.3) \\ &\le \left\| \exp\left(\frac{-\nu(\xi)(t-s)}{2} \right) \Gamma_{w_1}(f_{w_1}, f)(s) \right\|^{1/2}_{L^2_\xi L^2_x} \left\| \exp\left(\frac{-\nu(\xi)(t-s)}{2} \right) \Gamma_{w_1}(f_{w_1}, f)(s) \right\|^{1/2}_{L^\infty_\xi L^2_x} \\ &\le C_{\gamma,p,\beta}(1+t-s)^{-\frac{\gamma}{\gamma}} \left\| f_{w_1} \right\|_{L^\infty_{\xi,\beta}L^2_x} \left\| \langle\xi\rangle^p f \right\|_{L^\infty_{\xi,\beta}L^\infty_x}. \end{aligned}$$

Therefore, through (4.2), we have

$$\begin{aligned} &\left\| f_{w_1}(t) \right\|_{L^4_\xi L^2_x} \\ &\le \eta \left\| \mathbb{S}_{w_1}(t)\, f_{w_1 0} \right\|_{L^4_\xi L^2_x} + \int_0^t \left\| \mathbb{S}_{w_1}(t;s) K_{w_1} f_{w_1}(s) \right\|_{L^4_\xi L^2_x} + \left\| \mathbb{S}_{w_1}(t;s) \Gamma_{w_1}(f_{w_1}, f)(s) \right\|_{L^4_\xi L^2_x} ds \\ &\le \eta \left\| f_{w_1 0} \right\|_{L^4_\xi L^2_x} + C_{\gamma,p} \int_0^t (1+t-s)^{\frac{1-\gamma}{\gamma}} \left\| f_{w_1}(s) \right\|_{L^2_\xi L^2_x} ds \\ &\quad + C_{\gamma,p} \int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}} \left\| f_{w_1} \right\|_{L^\infty_{\xi,\beta}L^2_x} \left\| \langle\xi\rangle^p f \right\|_{L^\infty_{\xi,\beta}L^\infty_x} ds \\ &\le \eta \left\| f_{w_1 0} \right\|_{L^4_\xi L^2_x} + C_{1,\gamma,p,\beta,j} \left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2} \|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T} \right) \\ &\quad + C_{1,\gamma,p,\beta,j} \int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}} (1+s)^{-\frac{3}{2}} ds \cdot \left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} \right) C^2_{f_{w_1},T} \end{aligned}$$

$$\leq C'_{1,\gamma,p,\beta,j}\left[\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T} + \eta\left(\|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\right) C^2_{f_{w_1},T}\right]$$
$$\leq C''_{1,\gamma,p,\beta,j}\left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T}\right),$$

due to (1.4), (2.30), (4.1) and (4.3), whenever $\eta > 0$ is sufficiently small. That is,

$$\left\| f_{w_1}(t)\right\|_{L^4_\xi L^2_x} \leq C''_{1,\gamma,p,\beta,j}\left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T}\right). \tag{4.4}$$

Through (4.2) again, we infer

$$\begin{aligned}
&\left\| f_{w_1}(t)\right\|_{L^\infty_\xi L^2_x} \\
&\leq \eta \left\| f_{w_10}\right\|_{L^\infty_\xi L^2_x} + C_{\gamma,p}\int_0^t (1+t-s)^{\frac{7/4-\gamma}{\gamma}} \left\| f_{w_1}(s)\right\|_{L^4_\xi L^2_x} ds \\
&\quad + C_{\gamma,p}\int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}} \left\| f_{w_1}(s)\right\|_{L^\infty_\xi L^2_x} \left\| \langle\xi\rangle^p f(s)\right\|_{L^\infty_\xi L^\infty_x} ds \\
&\leq \eta \left\| f_{w_10}\right\|_{L^\infty_\xi L^2_x} + C_{\gamma,p}\int_0^t (1+t-s)^{\frac{7/4-\gamma}{\gamma}} \left\| f_{w_1}(s)\right\|_{L^4_\xi L^2_x} ds \\
&\quad + C_{1,\gamma,p,\beta,j}\int_0^t (1+t-s)^{-\frac{\gamma}{\gamma}}(1+s)^{-\frac{3}{2}} ds \cdot \left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\right) ds \cdot \sup_{0\leq t\leq T}\left\| f_{w_1}(s)\right\|_{L^\infty_\xi L^2_x} \\
&\leq C'_{1,\gamma,p,\beta,j}\left[\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T}\right] \\
&\quad + C'_{1,\gamma,p,\beta,j}\left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x}\right)\sup_{0\leq t\leq T}\left\| f_{w_1}(s)\right\|_{L^\infty_\xi L^2_x},
\end{aligned}$$

by using (1.4), (2.29), and (4.4). Since $\eta > 0$ is chosen sufficiently small, we get

$$\left\| f_{w_1}(t)\right\|_{L^\infty_\xi L^2_x} \leq 2C'_{1,\gamma,p,\beta,j}\left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T}\right).$$

Using the bootstrap argument, we eventually get

$$\left\| f_{w_1}(t)\right\|_{L^\infty_{\xi,\beta} L^2_x} \leq C''_{1,\gamma,p,\beta,j}\left(\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} + \eta^{1/2}\|f_0\|^{1/2}_{L^\infty_{\xi,p+\beta+3j}L^\infty_x} C^2_{f_{w_1},T}\right).$$

Since $\eta > 0$ is chosen sufficiently small, we get

$$\left\| f_{w_1}(t)\right\|_{L^\infty_{\xi,\beta} L^2_x} \leq C^2_{f_{w_1},T} \leq C'''_{1,\gamma,p,\beta,j}\eta \|f_0\|_{L^\infty_{\xi,p+\beta+3j}L^\infty_x},$$

for $0 \le t \le T$.

Gathering **Case I** and **Case II**, we obtain

$$\left\| f_{w_1}(t) \right\|_{L^\infty_{\xi,\beta} L^2_x} \le C^2_{f_{w_1},T} \le C'''_{1,\gamma,p,\beta,j} \eta \left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} \tag{4.5}$$

for $-2 < \gamma < 0$ and $p \ge 1$, where the constant $C'''_{1,\gamma,p,\beta,j} > 0$ independent of T. As for $\left\| f_{w_2}(t) \right\|_{L^\infty_{\xi,\beta} L^2_x}$, we choose a weight function w_2 satisfying ϵ, $\delta > 0$ sufficiently small, $M > 0$ sufficiently large such that $\epsilon c_p < \hat{\varepsilon}$, $\delta M > 0$ large and $\delta M \ll \epsilon^{-1}$. After that, by the energy estimate, we deduce that

$$\left\| f_{w_2}(t) \right\|_{L^2} \lesssim \eta \left\| f_{w3 0} \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} + \eta^{1/2} \left\| f_{w3 0} \right\|^{1/2}_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} C^2_{f_{w_2},T},$$

where $w_3 = \exp\left(\hat{\varepsilon} \left\langle \xi \right\rangle^p\right)$, $0 < p \le 2$, and $\hat{C}^2_{f_{w_2},T} = \sup_{0 \le t \le T} \left\| f_{w_2} \right\|_{L^\infty_{\xi,\beta} L^2_x}$. Following the above bootstrap argument, we as well have

$$\left\| f_{w_2}(t) \right\|_{L^\infty_{\xi,\beta} L^2_x} \le \hat{C}^2_{f_{w_2},T} \le C'''_{2,\gamma,p,\beta,j} \eta \left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x}, \tag{4.6}$$

for $-2 < \gamma < 0$ and $0 < p \le 2$, where the constant $C'''_{1,\gamma,p,\beta,j} > 0$ is independent of T.

Now, according to Theorem 18, if $-1 < \gamma < 0$,

$$\begin{aligned} (1+t)^{-A} \left\| f_{w_1} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} \lesssim (1+t)^{-A} &\left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} + (1+t)^{-\frac{3}{2}+A+\varsigma} C^\infty_{f_{w_1},T} C^\infty_{f,T} \right) \\ &+ (1+t)^{2-A} \left[(1+\delta M) \left(\eta \left\| f_{w_1 0} \right\|_{L^\infty_{\xi,\beta} L^2_x} + C^2_{f_{w_1},T} C^\infty_{f,T} \right) \right], \end{aligned}$$

$0 \le t \le T$. Taking $A = 2$ and choosing $\eta > 0$ sufficiently small, together with (4.5), we get

$$C^\infty_{f_{w_1},T} = \sup_{0 \le t \le T} (1+t)^{-2} \left\| f_{w_1} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} \lesssim \eta \left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} \left(1 + \eta \left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} \right),$$

since $C^\infty_{f,T} \lesssim \eta \left(\left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} \right)$ (due to (1.4)). It implies that

$$\left\| f_{w_1} \right\|_{L^\infty_{\xi,\beta} L^\infty_x} \lesssim (1+t)^2 \eta \left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} \left(1 + \eta \left\| f_0 \right\|_{L^\infty_{\xi,p+\beta+3j} L^\infty_x} \right),$$

for $0 \le t \le T$ and then for $0 \le t < \infty$ since T can be arbitrarily large. Note that for $\left\langle x \right\rangle > 2Mt$,

$$w_1(t,x,\xi) \gtrsim \left[\delta \left(\left\langle x \right\rangle - Mt \right) \right]^{\frac{p}{1-\gamma}} \text{ and } \left\langle x \right\rangle - Mt > \frac{\left\langle x \right\rangle}{3} + \frac{Mt}{3},$$

so that

$$\begin{aligned} &\left| f(t,x) \right|_{L^\infty_{\xi,\beta}} \\ &\lesssim \eta (1+t)^2 \left(\left\langle x \right\rangle + Mt \right)^{\frac{-p}{1-\gamma}} \left\| \left\langle \xi \right\rangle^{p+\beta+3j} f_0 \right\|_{L^\infty_\xi L^\infty_x} \left(1 + \eta \left\| \left\langle \xi \right\rangle^{p+\beta+3j} f_0 \right\|_{L^\infty_\xi L^\infty_x} \right), \end{aligned}$$

for $-1 < \gamma < 0$ and $p \geq 1$. Likewise, we can obtain the estimate of $|f(t,x)|_{L^\infty_{\xi,\beta}}$ in other cases by taking $A = 2 + \varsigma$ whenever $\gamma = -1$ and $A = 7 + \frac{5}{\gamma}$ whenever $-2 < \gamma < -1$ respectively in Theorem 18. This completes the proof of part (i). Imposing a certain exponential weight on the initial data f_0, we also note that for $\langle x \rangle > 2Mt$,

$$\rho(t,x,\xi) \gtrsim [\delta(\langle x \rangle - Mt)]^{\frac{p}{p+1-\gamma}} \text{ and } \langle x \rangle - Mt > \frac{\langle x \rangle}{3} + \frac{Mt}{3},$$

where $0 < p \leq 2$. Hence, part (ii) follows by taking $B = 2$ whenever $-1 < \gamma < 0$, $B = 2 + \varsigma$ whenever $\gamma = -1$, and $B = 7 + \frac{5}{\gamma}$ whenever $-2 < \gamma < -1$ respectively in Theorem 1, besides choosing $\eta > 0$ sufficiently small in each case. The proof of the theorem is completed. □

5. Proof of Theorem 1

This section is devoted to the large time behavior of the solution f to (1.2) in certain weighted normed spaces. Our strategy is to study the homogeneous/inhomogeneous linearized Boltzmann equation in the first two subsections, and then to demonstrate the large time behavior via an iteration scheme.

5.1. Linear Boltzmann equation

Let $\mathbb{G}^t$ be the solution operator of the linearized Boltzmann equation

$$\begin{cases} \partial_t g + \xi \cdot \nabla_x g = Lg, & (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ g(0,x,\xi) = g_0(x,\xi), & (x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3, \end{cases} \tag{5.1}$$

and let $\mathbb{S}^t$ be the solution operator of the damped transport equation

$$\begin{cases} \partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0, & (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ h(0,x,\xi) = h_0(x,\xi), & (x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{cases} \tag{5.2}$$

We will provide the estimate for the large time behavior of the solution g to (5.1).

Proposition 27. *Let $-2 < \gamma < 0$, $0 < p_1 \leq 2$, $p_2 > 3/2$, $\hat{\varepsilon} \geq 0$ sufficiently small, and let $j > 0$ be any sufficiently large number. Assume that $w_3 g_0 \in L^\infty_{\xi,p_2+j} L^\infty_x \cap L^\infty_{\xi,p_2+j} L^1_x$. Then there are positive constants $C_{i,\gamma,\hat{\varepsilon},p_1,p_2,j}$ and $\bar{C}_{i,\gamma,\hat{\varepsilon},p_1,p_2,j}$, $i = 1, 2$, such that the solution g to (5.1) satisfies*

$$\|w_3 g(t)\|_{L^\infty_{\xi,p_2} L^2_x} \leq C_{1,\gamma,\hat{\varepsilon},p_1,p_2,j} (1+t)^{-\frac{3}{4}} \left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right), \tag{5.3}$$

$$\|w_3 g(t)\|_{L^\infty_{\xi,p_2} L^\infty_x} \leq C_{2,\gamma,\hat{\varepsilon},p_1,p_2,j} (1+t)^{-\frac{3}{2}} \left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right). \tag{5.4}$$

Moreover,

$$\|w_3 g(t)\|_{L^\infty_{\xi,p_2+j}L^2_x} \le \bar{C}_{1,\gamma,\hat{\varepsilon},p_1,p_2,j}\left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j}L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j}L^\infty_x}\right), \tag{5.5}$$

$$\|w_3 g(t)\|_{L^\infty_{\xi,p_2+j}L^\infty_x} \le \bar{C}_{2,\gamma,\hat{\varepsilon},p_1,p_2,j}\left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j}L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j}L^\infty_x}\right). \tag{5.6}$$

Proof. By assumption, $g_0 \in L^\infty_{\xi,p_2+j}L^\infty_x \cap L^\infty_{\xi,p_2+j}L^1_x$. Then, following a similar argument as in [22, Propositions 7 and 15] (or see [29,30]), we see that

$$\begin{aligned}\|g(t)\|_{L^2_\xi L^2_x} &\lesssim (1+t)^{-\frac{3}{4}}\left(\left\|\langle\xi\rangle^j g_0\right\|_{L^2_\xi L^1_x} + \left\|\langle\xi\rangle^j g_0\right\|_{L^2_\xi L^2_x}\right) \\ &\lesssim (1+t)^{-\frac{3}{4}}\left(\left\|\langle\xi\rangle^j g_0\right\|_{L^\infty_{\xi,p_2}L^1_x} + \left\|\langle\xi\rangle^j g_0\right\|_{L^\infty_{\xi,p_2}L^\infty_x}\right),\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}\|g(t)\|_{L^2_\xi L^\infty_x} &\lesssim (1+t)^{-\frac{3}{2}}\left(\left\|\langle\xi\rangle^j g_0\right\|_{L^2_\xi L^2_x} + \left\|\langle\xi\rangle^j g_0\right\|_{L^2_\xi L^1_x} + \left\|\langle\xi\rangle^j g_0\right\|_{L^\infty_{\xi,p_2}L^\infty_x}\right) \\ &\lesssim (1+t)^{-\frac{3}{2}}\left(\left\|\langle\xi\rangle^j g_0\right\|_{L^\infty_{\xi,p_2}L^1_x} + \left\|\langle\xi\rangle^j g_0\right\|_{L^\infty_{\xi,p_2}L^\infty_x}\right).\end{aligned} \tag{5.8}$$

Now we prove (5.3) and the others are similar. In terms of the damped transport operator $\mathbb{S}^t$, g can be written as

$$g(t) = \mathbb{S}^t g_0 + \int_0^t \mathbb{S}^{t-s} K g(s) ds. \tag{5.9}$$

Let $T > 0$. For any $0 \le t \le T$,

$$w_3 |g(t)|_{L^2_x} \le w_3 \left|\mathbb{S}^t g_0\right|_{L^2_x} + w_3 \int_0^t \left|\mathbb{S}^{t-s} K g(s)\right|_{L^2_x} ds = I + II.$$

It is easy to see that

$$\begin{aligned}I &\le \sup_\xi \left(w_3 \left|\mathbb{S}^t g_0\right|_{L^2_x}\right) \le \left(\sup_\xi e^{-\nu(\xi)t}\langle\xi\rangle^{-j}\right)\|w_3 g_0\|_{L^\infty_{\xi,j}L^2_x} \\ &\le C_{\gamma,j}(1+t)^{\frac{j}{\gamma}}\|w_3 g_0\|_{L^\infty_{\xi,p_2+j}L^2_x} \le C_{\gamma,j}(1+t)^{-\frac{3}{4}}\|w_3 g_0\|_{L^\infty_{\xi,p_2+j}L^2_x},\end{aligned} \tag{5.10}$$

since j is sufficiently large. For II, it follows from (2.12), (2.13) and (2.25) that

$$
\begin{aligned}
& w_3 \left|\mathbb{S}^{t-s} K g(s)\right|_{L_x^2} = e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} e^{-(t-s)\nu(\xi)} \left|K g(s)\right|_{L_x^2} \\
& \leq e^{-(t-s)\nu(\xi)} \langle\xi\rangle^{\gamma-1} \left[\sup_{|\xi|\leq\tau} \left(e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} \langle\xi\rangle^{-\gamma+1} |K g(s)|_{L_x^2} \right) \right. \\
& \quad \left. + \sup_{|\xi|>\tau} \left(\langle\xi\rangle^{-1} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} \langle\xi\rangle^{2-\gamma} |K g(s)|_{L_x^2} \right) \right] \\
& \leq C_\gamma (1+t-s)^{\frac{1-\gamma}{\gamma}} \left(e^{\hat{\varepsilon}\langle\tau\rangle^{p_1}} \|K g(s)\|_{L^\infty_{\xi,1-\gamma} L_x^2} + (1+\tau)^{-1} \|w_3 K g(s)\|_{L^\infty_{\xi,2-\gamma} L_x^2} \right) \\
& \leq C_{\gamma,\hat{\varepsilon},p_1} (1+t-s)^{\frac{1-\gamma}{\gamma}} \cdot \begin{cases} \left(e^{\hat{\varepsilon}\tau^{p_1}} \|g(s)\|_{L^2_\xi L^2_x} + (1+\tau)^{-1} \|w_3 g(s)\|_{L^\infty_\xi L^2_x} \right), & \text{if } \frac{-3}{2} < \gamma < 0, \\ \left(e^{\hat{\varepsilon}\tau^{p_1}} \|g(s)\|_{L^4_\xi L^2_x} + (1+\tau)^{-1} \|w_3 g(s)\|_{L^\infty_\xi L^2_x} \right), & \text{if } -2 < \gamma \leq \frac{-3}{2}, \end{cases}
\end{aligned}
$$

for any $\tau > 0$. Whenever $-2 < \gamma \leq \frac{-3}{2}$, in view of (2.14), (5.7) and (5.9), we have

$$
\begin{aligned}
\|g(s)\|_{L^4_\xi L^2_x} & \leq \left(\sup_\xi e^{-s\nu(\xi)} \langle\xi\rangle^{-j} \right) \|g_0\|_{L^4_{\xi,j} L^2_x} \\
& \quad + \int_0^s \sup_\xi \left(e^{-\nu(\xi)(s-s')} \langle\xi\rangle^{-(1-\gamma)} \right) \left\| K g(s') \right\|_{L^4_{\xi,1-\gamma} L^2_x} ds' \qquad (5.11) \\
& \lesssim (1+s)^{\frac{j}{\gamma}} \|g_0\|_{L^4_{\xi,j} L^2_x} + \int_0^s \left(1+s-s'\right)^{\frac{1-\gamma}{\gamma}} \left\|g(s')\right\|_{L^2_\xi L^2_x} ds' \\
& \lesssim (1+s)^{\frac{j}{\gamma}} \|g_0\|_{L^4_{\xi,j} L^2_x} \\
& \quad + \int_0^s (1+s-s')^{\frac{1-\gamma}{\gamma}} (1+s')^{-\frac{3}{4}} \left(\|g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right) ds' \\
& \lesssim (1+s)^{-\frac{3}{4}} \left(\|g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right),
\end{aligned}
$$

the last inequality being valid since $j > 0$ is sufficiently large. Using (5.7) and (5.11), we deduce

$$
\begin{aligned}
II & \leq C_{\gamma,p_1,p_2,j} e^{\hat{\varepsilon}\langle\tau\rangle^{p_1}} \left(\|g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right) \int_0^t (1+t-s)^{\frac{1-\gamma}{\gamma}} (1+s)^{-\frac{3}{4}} ds \\
& \quad + C_{\gamma,p_1,p_2,j} (1+\tau)^{-1} \sup_{0\leq s\leq T} \left[(1+s)^{\frac{3}{4}} \|w_3 g(s)\|_{L^\infty_\xi L^2_x} \right] \cdot \int_0^t (1+t-s)^{\frac{1-\gamma}{\gamma}} (1+s)^{-\frac{3}{4}} ds \qquad (5.12) \\
& \leq C'_{\gamma,p_1,p_2,j} e^{\hat{\varepsilon}\langle\tau\rangle^{p_1}} (1+t)^{-\frac{3}{4}} \left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right)
\end{aligned}
$$

$$+ C'_{\gamma,p_1,p_2,j} (1+\tau)^{-1} (1+t)^{-\frac{3}{4}} \sup_{0\le s\le T} \left[(1+s)^{\frac{3}{4}} \|w_3 g(s)\|_{L^\infty_\xi L^2_x} \right].$$

After selecting $\tau > 0$ sufficiently large with $C'_{\gamma,p_1,p_2,j} (1+\tau)^{-1} < 1/2$, we obtain

$$\sup_{0\le t\le T} \left[(1+t)^{\frac{3}{4}} \|w_3 g(t)\|_{L^\infty_\xi L^2_x} \right] \le C_{1,\gamma,\hat{\varepsilon},p_1,p_2,j} \left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right),$$

due to (5.10) and (5.12). It implies that

$$\|w_3 g(t)\|_{L^\infty_\xi L^2_x} \le C_{1,\gamma,\hat{\varepsilon},p_1,p_2,j} (1+t)^{-\frac{3}{4}} \left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right)$$

for $0 \le t < \infty$, since $T > 0$ is arbitrary.

Finally, through the bootstrap argument, we get

$$\|w_3 g(t)\|_{L^\infty_{\xi,p_2} L^2_x} \le C_{1,\gamma,\hat{\varepsilon},p_1,p_2,j} (1+t)^{-\frac{3}{4}} \left(\|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 g_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right),$$

as desired. The proof of this proposition is completed. □

5.2. *The inhomogeneous Boltzmann equation*

In this section, we further consider the inhomogeneous Boltzmann equation

$$\begin{cases} \partial_t g + \xi \cdot \nabla_x g = Lg + \Gamma(h_1, h_2), \\ g(0,x,\xi) = g_0(x,\xi). \end{cases} \tag{5.13}$$

Now, let $0 < p_1 \le 2$, $p_2 > 3/2$, $\hat{\varepsilon} \ge 0$ sufficiently small, and $j > 0$ sufficiently large. We assume that g_0 satisfies

$$\|g_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^1_x} + \|g_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^\infty_x} \le b_0, \tag{5.14}$$

and h_i $(i = 1, 2)$ satisfies

$$\sup_t \left\{ (1+t)^{\frac{3}{4}} \|h_i(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^2_x}, \ (1+t)^{\frac{3}{4}} \|h_i(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^\infty_x}, \right.$$
$$\left. \|h_i(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^2_x}, \ \|h_i(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^\infty_x} \right\} \le b_i, \tag{5.15}$$

for some $b_0, b_1, b_2 > 0$. We will study the large time behavior of the solution g to (5.13) in some suitable norms (see Proposition 31).

Before proving Proposition 31, we need some preliminary results (Lemmas 28 and 30) regarding the nonlinear term Γ under the assumption (5.15).

Lemma 28. *Assume that h_1 and h_2 satisfy (5.15). Then*

$$\|\Gamma_{loss}(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \le C_1(1+t)^{-\frac{3}{4}}b_1b_2,$$

$$\|\Gamma_{loss}(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x} \le C_1(1+t)^{-\frac{3}{4}}b_1b_2,$$

$$\|\Gamma_{loss}(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \le C_1(1+t)^{-\frac{3}{4}}b_1b_2,$$

$$\left\|\Gamma_{gain}(h_1,h_2)(t)\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma+1}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X} \le C_2b_1b_2,$$

where $L_X = L^1_x$, L^2_x and L^∞_x.

Proof. By Lemma 15,

$$\begin{aligned}&|\Gamma_{loss}(h_1,h_2)\,(t)|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}\\ &\lesssim |h_1|_{L^\infty_\xi}|h_2|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)} + |h_2|_{L^\infty_\xi}|h_1|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)},\end{aligned}$$

and

$$\begin{aligned}&\left|\Gamma_{gain}(h_1,h_2)(t)\right|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma+1}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}\\ &\lesssim |h_1|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}|h_2|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)}.\end{aligned}$$

Therefore, according to the assumption (5.15) and that $p_2 > 0$, we obtain the desired estimates. □

To prove Lemma 30, we need an interpolation inequality:

Lemma 29.

$$\begin{aligned}&\|H(t,x,\xi)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X}\\ &\le 2\,\|H(t,x,\xi)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X}\,\|H(t,x,\xi)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X},\end{aligned}$$

where $L_X = L^1_x$ and L^∞_x.

Proof. For any $\xi_0 > 0$,

$$\begin{aligned}&\|H(t,x,\xi)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X}\\ &\le \langle\xi_0\rangle^{j}\sup_{|\xi|\le\xi_0}\left|\langle\xi\rangle^{p_2}\,e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}H(t,x,\xi)\right|_{L_X} + \langle\xi_0\rangle^{-j}\sup_{|\xi|>\xi_0}\left|\langle\xi\rangle^{p_2+2j}\,e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}H(t,x,\xi)\right|_{L_X}\end{aligned}$$

$$\leq \langle\xi_0\rangle^j \|H(t,x,\xi)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X} + \langle\xi_0\rangle^{-j} \|H(t,x,\xi)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X}.$$

We can get the desired result by choosing $\xi_0 > 0$ such that

$$\langle\xi_0\rangle^j = \|H(t,x,\xi)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X} \|H(t,x,\xi)\|^{-\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L_X}. \qquad \square$$

Lemma 30. *Assume that h_1 and h_2 satisfy (5.15). Then there exists a positive constant $C_{\gamma,\hat{\varepsilon},p_1,p_2,j}$ such that*

$$\|\Gamma(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \leq C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(1+t)^{-\frac{3}{4}} b_1 b_2,$$

$$\|\Gamma(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \leq C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(1+t)^{-\frac{3}{4}} b_1 b_2.$$

Proof. It readily follows from Lemma 15 that

$$\|\Gamma_{loss}(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \leq C_1(1+t)^{-\frac{3}{2}} b_1 b_2,$$

$$\|\Gamma_{loss}(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \leq C_1(1+t)^{-\frac{3}{2}} b_1 b_2,$$

$$\left\|\Gamma_{gain}(h_1,h_2)(t)\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2-\gamma+1} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \leq C_2(1+t)^{-\frac{3}{2}} b_1 b_2,$$

$$\left\|\Gamma_{gain}(h_1,h_2)(t)\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2-\gamma+1} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \leq C_2(1+t)^{-\frac{3}{2}} b_1 b_2.$$

Combining this with Lemmas 28 and 29, we obtain

$$\begin{aligned}
&\|\Gamma(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \\
&\leq 2\,\|\Gamma(h_1,h_2)(t)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \|\Gamma(h_1,h_2)(t)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} \\
&\leq C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(1+t)^{-\frac{3}{4}} b_1 b_2,
\end{aligned}$$

and

$$\begin{aligned}
&\|\Gamma(h_1,h_2)(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \\
&\leq 2\,\|\Gamma(h_1,h_2)(t)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \|\Gamma(h_1,h_2)(t)\|^{\frac{1}{2}}_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \\
&\leq C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(1+t)^{-\frac{3}{4}} b_1 b_2. \qquad \square
\end{aligned}$$

Proposition 31. *Assume that g_0 satisfies (5.14) and that h_1 and h_2 satisfy (5.15). Then there exists a number $C_{\gamma,\hat{\varepsilon},p_1,p_2,j} > 0$ such that the solution g to (5.13) satisfies*

$$\max\left\{(1+t)^{\frac{3}{4}}\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x},(1+t)^{\frac{3}{4}}\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x},\right.$$
$$\left.\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x},\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}\right\}\le C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(b_0+b_1b_2).$$

Proof. By Duhamel's principle, g can be written as

$$g(t)=\mathbb{G}^t g_0+\int_0^t \mathbb{G}^{t-s}\Gamma(h_1,h_2)(s)ds.$$

Hence, in view of Proposition 27 and Lemma 30,

$$\begin{aligned}
&\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}\\
&\le\left\|\mathbb{G}^t g_0\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}+\int_0^t\left\|\mathbb{G}^{t-s}\Gamma(h_1,h_2)(s)\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}ds\\
&\lesssim(1+t)^{-\frac{3}{2}}\left(\|g_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x}+\|g_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}\right) \qquad (5.16)\\
&\quad+\int_0^t(1+t-s)^{-\frac{3}{2}}\|\Gamma(h_1,h_2)(s)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x}ds\\
&\quad+\int_0^t(1+t-s)^{-\frac{3}{2}}\|\Gamma(h_1,h_2)(s)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}ds\\
&\lesssim(1+t)^{-\frac{3}{2}}b_0+\int_0^t(1+t-s)^{-\frac{3}{2}}\left((1+s)^{-\frac{3}{4}}+(1+s)^{-\frac{3}{4}}\right)b_1b_2ds\\
&\lesssim(1+t)^{-\frac{3}{4}}(b_0+b_1b_2),
\end{aligned}$$

i.e.,

$$\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}\lesssim(1+t)^{-\frac{3}{4}}(b_0+b_1b_2).$$

This completes the estimate for $\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}$, and the estimate for $\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x}$ can be obtained via the same argument.

On the other hand, regarding (5.13) as the damped transport equation with the source term $Kg + \Gamma(h_1, h_2)$, g can be rewritten as

$$g(t) = \mathbb{S}^t g_0 + \int_0^t \mathbb{S}^{t-s} \left[Kg(s) + \Gamma(h_1, h_2)(s)\right] ds.$$

Let $T > 0$. Hence, for any $0 \le t \le T$,

$$\begin{aligned} |g(t)|_{L^\infty_x} &\le \left|\mathbb{S}^t g_0\right|_{L^\infty_x} + \int_0^t \left|\mathbb{S}^{t-s}\left(Kg(s) + \Gamma(h_1, h_2)(s)\right)\right|_{L^\infty_x} ds \\ &\le e^{-\nu(\xi)t} |g_0|_{L^\infty_x} + \int_0^t e^{-\nu(\xi)(t-s)} |Kg(s) + \Gamma(h_1, h_2)(s)|_{L^\infty_x} ds. \end{aligned}$$

By assumption (5.14), it immediately follows that

$$e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} \langle\xi\rangle^{p_2+2j} e^{-\nu(\xi)t} |g_0|_{L^\infty_x} \le b_0. \tag{5.17}$$

Next, in view of (2.9) and (2.25),

$$\begin{aligned} & e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} \langle\xi\rangle^{p_2+2j} e^{-\nu(\xi)(t-s)} |Kg(s)|_{L^\infty_x} \\ \le & \sup_{|\xi|\le\tau} \left[e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} e^{-\nu(\xi)(t-s)} \langle\xi\rangle^{\gamma-1} \langle\xi\rangle^{1-\gamma} \langle\xi\rangle^{p_2+2j} |Kg(s)|_{L^\infty_x}\right] \\ & + \sup_{|\xi|>\tau} \left[e^{-\nu(\xi)(t-s)} \langle\xi\rangle^{\gamma-1} \langle\xi\rangle^{-1} \left(\langle\xi\rangle^{2-\gamma} \langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} |Kg(s)|_{L^\infty_x}\right)\right] \\ \le & C_1(1+t-s)^{\frac{1-\gamma}{\gamma}} \Big(\langle\tau\rangle^{2j} e^{\hat{\varepsilon}\langle\tau\rangle^{p_1}} \|Kg(s)\|_{L^\infty_{\xi,p_2+1-\gamma} L^\infty_x} \\ & + (1+\tau)^{-1} \|Kg(s)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j+2-\gamma} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^\infty_x}\Big) \\ \le & C_1'(1+t-s)^{\frac{1-\gamma}{\gamma}} \left(\langle\tau\rangle^{2j} e^{\hat{\varepsilon}\langle\tau\rangle^{p_1}} \|g(s)\|_{L^\infty_{\xi,p_2} L^\infty_x} + (1+\tau)^{-1} \|g(s)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^\infty_x}\right) \end{aligned}$$

for any $\tau > 0$. Picking $\tau > 0$ such that $(1+\tau)^{-1} C_1' < \frac{1}{4}$, we get

$$\begin{aligned} & \int_0^t e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}} \langle\xi\rangle^{p_2+2j} e^{-\nu(\xi)(t-s)} |Kg(s)|_{L^\infty_x} ds \qquad (5.18) \\ \le & \int_0^t (1+t-s)^{\frac{1-\gamma}{\gamma}} \left(C_2 \|g(s)\|_{L^\infty_{\xi,p_2} L^\infty_x} + \frac{1}{4} \sup_{0\le s\le T} \|g(s)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right) L^\infty_x}\right) ds \end{aligned}$$

$$\le C_3\left(b_0+b_1b_2\right)+\frac{1}{2}\sup_{0\le s\le T}\|g(s)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x},$$

here the estimate for $\|g(s)\|_{L^\infty_{\xi,p_2}L^\infty_x}$ following the same argument as (5.16). Finally, we split Γ into two parts Γ_{loss} and Γ_{gain}. Then, it follows from Lemma 28 that

$$\begin{aligned}
&\int_0^t e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\langle\xi\rangle^{p_2+2j}e^{-\nu(\xi)(t-s)}\left|\Gamma_{gain}(h_1,h_2)(s)\right|_{L^\infty_x}ds \qquad (5.19)\\
&\le\int_0^t C_4(1+t-s)^{\frac{1-\gamma}{\gamma}}\left\|\Gamma_{gain}(h_1,h_2)(s)\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j+1-\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}ds\\
&\le\int_0^t C_5(1+t-s)^{\frac{1-\gamma}{\gamma}}b_1b_2ds\le C_5b_1b_2
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\langle\xi\rangle^{p_2+2j}e^{-\nu(\xi)(t-s)}\left|\Gamma_{loss}(h_1,h_2)(s)\right|_{L^\infty_x}ds \qquad (5.20)\\
&\le\int_0^t(1+t-s)^{-1}\left\|\Gamma_{loss}(h_1,h_2)(s)\right\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j-\gamma}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}ds\\
&\le C_6b_1b_2\int_0^t(1+t-s)^{-1}(1+s)^{-\frac{3}{4}}ds\le C_6b_1b_2.
\end{aligned}$$

Combining (5.17)-(5.20) gives

$$\sup_{0\le t\le T}\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}\le C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(b_0+b_1b_2),$$

which implies that

$$\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}\le C_{\gamma,\hat{\varepsilon},p_1,p_2,j}(b_0+b_1b_2)$$

for $0\le t<\infty$, since $T>0$ is arbitrary. The estimate for $\|g(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j}e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x}$ can be obtained via the same argument as well. The proof of this proposition is completed. □

5.3. Proof of Theorem 1

Define a norm as

$$|||h||| \equiv \sup_t \Bigg\{ (1+t)^{\frac{3}{4}} \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x}, (1+t)^{\frac{3}{4}} \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x}, \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^2_x}, \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \Bigg\}.$$

Now, we consider the iteration $\left\{f^{(i)}\right\}$ for which $f^{(0)}(t,x,\xi) \equiv 0$ and $f^{(i+1)}$, $i \in \mathbb{N} \cup \{0\}$, is a solution to the equation

$$\begin{cases} \partial_t f^{(i+1)} + \xi \cdot \nabla_x f^{(i+1)} = L f^{(i+1)} + \Gamma(f^{(i)}, f^{(i)}), \\ f^{(i+1)}(0,x,\xi) = \eta f_0(x,\xi), \end{cases} \tag{5.21}$$

where $\eta > 0$ is sufficiently small such that

$$\left(1 + C_{\gamma,\hat{\varepsilon},p_1,p_2,j}\right)^2 \eta \left(\|f_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} + \|f_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \right) < 1/8.$$

Denote

$$b_0 := \eta \left(\|f_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^1_x} + \|f_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{p_2+2j} e^{\hat{\varepsilon}\langle\xi\rangle^{p_1}}\right)L^\infty_x} \right).$$

According to Proposition 31,

$$|||f^{(1)}||| \le C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0 \le 2C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0,$$

and then

$$\begin{aligned} |||f^{(2)}||| &\le C_{\gamma,\hat{\varepsilon},p_1,p_2,j} \left[b_0 + \left(2C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0\right)^2 \right] \\ &\le 2C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0 \left[\frac{1}{2} + 2C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0 \right] \\ &\le 2C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0. \end{aligned}$$

By induction on i, we get

$$|||f^{(i+1)}||| \le 2C_{\gamma,\hat{\varepsilon},p_1,p_2,j} b_0,$$

through Proposition 31. Hence, we get the boundedness of $\left\{f^{(i)}\right\}$ in the norm $|||\cdot|||$.

Next, we demonstrate the convergence of $\left\{f^{(i)}\right\}$ and uniqueness of its limit. Set $h^{(i+1)} = f^{(i+1)} - f^{(i)}$ and then $h^{(i+1)}$ satisfies the equation

$$\begin{cases} \partial_t h^{(i+1)} + \xi \cdot \nabla_x h^{(i+1)} = L h^{(i+1)} + \Gamma(h^{(i)}, f^{(i)}) + \Gamma(f^{(i-1)}, h^{(i)}), \\ h^{(i+1)}(0, x, \xi) = 0. \end{cases} \tag{5.22}$$

According to Proposition 31, we get

$$|||h^{(i+1)}||| \le 4\left(C_{\gamma,\hat{\varepsilon},p_1,p_2,j}\right)^2 b_0 |||h^{(i)}|||$$

for all $i \in \mathbb{N}$. Since $4\left(C_{\gamma,\hat{\varepsilon},p_1,p_2,j}\right)^2 b_0 < 1/2$, $\left\{f^{(i)}\right\}$ is a Cauchy sequence in the norm $|||\cdot|||$, so that it converges and its limit f will satisfy

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2} L^2_x} \le \eta C_1 (1+t)^{-\frac{3}{4}} \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^\infty_x} \right), \tag{5.23}$$

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2} L^\infty_x} \le \eta C_2 (1+t)^{-\frac{3}{4}} \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^\infty_x} \right), \tag{5.24}$$

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2+2j} L^2_x} \le \eta \bar{C}_1 \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^\infty_x} \right), \tag{5.25}$$

$$\|w_3 f(t)\|_{L^\infty_{\xi,p_2+2j} L^\infty_x} \le \eta \bar{C}_2 \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+2j} L^\infty_x} \right). \tag{5.26}$$

Therefore, (1.4), (1.5), (1.6) are obtained.

Finally, we will use a bootstrap argument to improve the estimate (5.24). Write f as

$$f = \eta \mathbb{G}^t f_0 + \int_0^t \mathbb{G}^{t-s} \Gamma(f, f)(s) ds,$$

and then we have

$$\begin{aligned}
&\|w_3 f(t)\|_{L^\infty_{\xi,p_2} L^\infty_x} \\
&\le \eta \left\| w_3 \mathbb{G}^t f_0 \right\|_{L^\infty_{\xi,p_2} L^\infty_x} + \int_0^t \left\| w_3 \mathbb{G}^{t-s} \Gamma(f, f)(s) \right\|_{L^\infty_{\xi,p_2} L^\infty_x} ds \\
&\lesssim \eta (1+t)^{-\frac{3}{2}} \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right) \\
&\quad + \int_0^t (1+t-s)^{-\frac{3}{2}} \left(\|w_3 \Gamma(f, f)(s)\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 \Gamma(f, f)(s)\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right) ds \\
&\lesssim \eta (1+t)^{-\frac{3}{2}} \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right) \\
&\quad + \int_0^t (1+t-s)^{-\frac{3}{2}} \left(\|w_3 f\|^2_{L^\infty_{\xi,p_2+j} L^2_x} + \|w_3 f\|^2_{L^\infty_{\xi,p_2+j} L^\infty_x} \right) ds \\
&\lesssim \eta (1+t)^{-\frac{3}{2}} \left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+j} L^1_x} + \|w_3 f_0\|_{L^\infty_{\xi,p_2+j} L^\infty_x} \right)
\end{aligned}$$

$$+\eta\int_0^t (1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}\,ds\cdot\left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+3j}L^1_x}+\|w_3 f_0\|_{L^\infty_{\xi,p_2+3j}L^\infty_x}\right)$$

$$\lesssim \eta(1+t)^{-\frac{3}{2}}\left(\|w_3 f_0\|_{L^\infty_{\xi,p_2+3j}L^1_x}+\|w_3 f_0\|_{L^\infty_{\xi,p_2+3j}L^\infty_x}\right),$$

by using (2.49), Proposition 27, and (5.24). This completes the proof.

6. Appendix

6.1. Proof of (2.41) and (2.42)

We claim that

$$\left(\int\left|\nu^{-1/2}(\xi)\Gamma(g,h)(\xi)\right|^2 d\xi\right)^{1/2}\lesssim |g|_{L^\infty_\xi}|h|_{L^2_\sigma}+|g|_{L^2_\sigma}|h|_{L^\infty_\xi}, \tag{6.1}$$

$$\left(\int\left|\nu^{-1}(\xi)\Gamma(g,h)(\xi)\right|^2 d\xi\right)^{1/2}\lesssim |g|_{L^\infty_\xi}|h|_{L^2_\xi}+|g|_{L^2_\xi}|h|_{L^\infty_\xi}. \tag{6.2}$$

Recall that we split Γ into two parts Γ_{gain} and Γ_{loss} as below:

$$\begin{aligned}\Gamma(g,h) &\equiv \Gamma_{gain}(g,h)-\Gamma_{loss}(g,h)\\ &=\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2} B(\vartheta)|\xi-\xi_*|^\gamma \mathcal{M}_*^{1/2}\left[g'_*h'+g'h'_*\right]d\xi_* d\omega\\ &\quad-\frac{1}{2}\int_{\mathbb{R}^3\times\mathbb{S}^2} B(\vartheta)|\xi-\xi_*|^\gamma \mathcal{M}_*^{1/2}\left[g_*h+gh_*\right]d\xi_* d\omega.\end{aligned}$$

In the sequel, we shall estimate Γ_{gain} and Γ_{loss} individually.

Estimate on $\Gamma_{loss}(g,h)$. It readily follows from Lemma 14 that

$$|\Gamma_{loss}(g,h)|\lesssim \nu(\xi)\left(|g|_{L^\infty_\xi}|h|+|g|\,|h|_{L^\infty_\xi}\right). \tag{6.3}$$

Therefore, we have

$$\left(\int\left|\nu^{-1/2}(\xi)\Gamma_{loss}(g,h)(\xi)\right|^2 d\xi\right)^{1/2}\lesssim |g|_{L^\infty_\xi}|h|_{L^2_\sigma}+|g|_{L^2_\sigma}|h|_{L^\infty_\xi}, \tag{6.4}$$

$$\left(\int\left|\nu^{-1}(\xi)\Gamma_{loss}(g,h)(\xi)\right|^2 d\xi\right)^{1/2}\lesssim |g|_{L^\infty_\xi}|h|_{L^2_\xi}+|g|_{L^2_\xi}|h|_{L^\infty_\xi}. \tag{6.5}$$

Estimate on $\Gamma_{gain}(g,h)$. By the Cauchy-Schwartz inequality and Lemma 14,

$$\left|\Gamma_{gain}(g,h)(\xi)\right|^2$$
$$\lesssim \nu(\xi)\left(\int_{\mathbb{R}^3\times\mathbb{S}^2} |B(\vartheta)|\,|\xi-\xi_*|^\gamma \exp\left(-\frac{|\xi_*|^2}{4}\right)\left[\left|g_*'\right|^2\left|h'\right|^2+\left|g'\right|^2\left|h_*'\right|^2\right]d\xi_* d\omega\right), \quad (6.6)$$

so that

$$\int_{\mathbb{R}^3}\left|\nu^{-1}(\xi)\Gamma_{gain}(g,h)(\xi)\right|^2 d\xi$$
$$\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{\mathbb{R}^3\times\mathbb{R}^3} \nu(\xi)^{-1}|\xi-\xi_*|^\gamma \exp(-\frac{|\xi_*|^2}{4})\left[\left|g_*'\right|^2\left|h'\right|^2+\left|g'\right|^2\left|h_*'\right|^2\right]d\xi_* d\xi\right)d\omega.$$

We split the (ξ_*,ξ)-space into three regions: $I_1=\left\{|\xi_*|\ge\frac{|\xi|}{2}\right\}$, $I_2=\left\{|\xi_*|<\frac{|\xi|}{2}, |\xi|\le 1\right\}$, and $I_3=\left\{|\xi_*|<\frac{|\xi|}{2}, |\xi|>1\right\}$.

Case 1: On $I_1\equiv\{|\xi_*|\ge|\xi|/2\}$. Since $|\xi-\xi_*|=|\xi'-\xi_*'|$, $|\xi|^2+|\xi_*|^2=\left|\xi'\right|^2+\left|\xi_*'\right|^2$, and $\nu^{-1}\approx\langle\xi\rangle^{-\gamma}\lesssim\left\langle\xi'\right\rangle^{-\gamma}\left\langle\xi_*'\right\rangle^{-\gamma}$, we have

$$\begin{aligned}
&\int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_1} \nu(\xi)^{-1}|\xi-\xi_*|^\gamma \exp(-\frac{|\xi_*|^2}{4})\left[\left|g_*'\right|^2\left|h'\right|^2+\left|g'\right|^2\left|h_*'\right|^2\right]d\xi_* d\xi\right)d\omega \qquad (6.7)\\
&\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_1} \nu(\xi)^{-1}|\xi-\xi_*|^\gamma \exp(-\frac{|\xi_*|^2}{8}-\frac{|\xi|^2}{32})\left[\left|g_*'\right|^2\left|h'\right|^2+\left|g'\right|^2\left|h_*'\right|^2\right]d\xi_* d\xi\right)d\omega\\
&\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{\psi(I_1)} |\xi'-\xi_*'|^\gamma \exp\left(-\left(\frac{\left|\xi_*'\right|^2}{32}+\frac{\left|\xi'\right|^2}{32}\right)\right)\left[\left|g_*'\right|^2\left|h'\right|^2+\left|g'\right|^2\left|h_*'\right|^2\right]d\xi_*' d\xi'\right)d\omega\\
&\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_1} |\xi-\xi_*|^\gamma \exp\left(-\left(\frac{|\xi_*|^2}{32}+\frac{|\xi|^2}{32}\right)\right)\left[|g_*|^2|h|^2+|g|^2|h_*|^2\right]d\xi_* d\xi\right)d\omega\\
&\lesssim |g|^2_{L^\infty_\xi}|h|^2_{L^2_\sigma}+|g|^2_{L^2_\sigma}|h|^2_{L^\infty_\xi},
\end{aligned}$$

by change of the variables $(\xi_*,\xi)\xrightarrow{\psi}(\xi_*',\xi')$ and Lemma 14.

Case 2: On $I_2=\{|\xi_*|<|\xi|/2, |\xi|\le 1\}$. In this region, $|\xi-\xi_*|>|\xi|/2$, $1\le\nu^{-1}(\xi)\le 2$, and

$$\left|\xi_*'\right|, \left|\xi'\right|\le\sqrt{2}\left(\left|\xi'\right|^2+\left|\xi_*'\right|^2\right)^{1/2}=\sqrt{2}\left(|\xi|^2+|\xi_*|^2\right)^{1/2}\le 2|\xi|<2.$$

Hence,

$$\int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_2} \nu(\xi)^{-1}|\xi-\xi_*|^\gamma \exp(-\frac{|\xi_*|^2}{4})\left[|g'_*|^2|h'|^2+|g'|^2|h'_*|^2\right]d\xi_* d\xi\right)d\omega \quad (6.8)$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_2} |\xi|^\gamma \exp(-\frac{|\xi_*|^2}{4})\left[|g'_*|^2|h'|^2+|g'|^2|h'_*|^2\right]d\xi_* d\xi\right)d\omega$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_2} |\xi'_*|^\gamma \left[|g'_*|^2|h'|^2+|g'|^2|h'_*|^2\right]d\xi_* d\xi\right)d\omega$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{\psi(I_2)} |\xi'_*|^\gamma \left[|g'_*|^2|h'|^2+|g'|^2|h'_*|^2\right]d\xi'_* d\xi'\right)d\omega$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_2} |\xi_*|^\gamma \left[|g_*|^2|h|^2+|g|^2|h_*|^2\right]d\xi_* d\xi\right)d\omega$$

$$\lesssim |g|^2_{L^\infty_\xi}|h|^2_{L^2_\sigma}+|g|^2_{L^2_\sigma}|h|^2_{L^\infty_\xi}.$$

The last inequality is valid since $\int_{|\xi_*|<1/2}|\xi_*|^\gamma d\xi_* < \infty$,

$$\int_{|\xi|\le 1}|h|^2 d\xi = \int_{|\xi|\le 1}\nu^{-1}(\xi)\,\nu(\xi)\,|h|^2 d\xi \le C_\gamma \int_{|\xi|\le 1}\nu(\xi)\,|h|^2 d\xi \le C_r\,|h|^2_{L^2_\sigma},$$

$$\int_{|\xi|\le 1}|g|^2 d\xi = \int_{|\xi|\le 1}\nu^{-1}(\xi)\,\nu(\xi)\,|g|^2 d\xi \le C_\gamma \int_{|\xi|\le 1}\nu(\xi)\,|h|^2 d\xi \le C_r\,|g|,$$

for some $C_\gamma > 0$.

Case 3: On $I_3 = \{|\xi_*| < |\xi|/2, |\xi| > 1\}$. In this region, $|\xi-\xi_*| > |\xi|/2 > (1+|\xi|)/4$; moreover,

$$\frac{\langle\xi'\rangle}{\sqrt{2}}, \frac{\langle\xi'_*\rangle}{\sqrt{2}} \le \left(\frac{1+|\xi'|^2}{2}+\frac{1+|\xi'_*|^2}{2}\right)^{1/2} = \left(1+\frac{|\xi|^2+|\xi_*|^2}{2}\right)^{1/2} \le \left(1+|\xi|^2\right)^{1/2},$$

which implies that $\nu(\xi) \lesssim \langle\xi'\rangle^\gamma, \langle\xi'_*\rangle^\gamma$ (used in the proof of (6.12)). Hence,

$$\int_{\mathbb{S}^2} B(\vartheta)\left(\int_{I_3} \nu(\xi)^{-1}|\xi-\xi_*|^\gamma \exp(-\frac{|\xi_*|^2}{4})\left[|g'_*|^2|h'|^2+|g'|^2|h'_*|^2\right]d\xi_* d\xi\right)d\omega \quad (6.9)$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta) \left(\int_{I_3} \left[|g_*'|^2 |h'|^2 + |g'|^2 |h_*'|^2 \right] d\xi_* d\xi \right) d\omega$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta) \left(\int_{\psi(I_3)} \left[|g_*'|^2 |h'|^2 + |g'|^2 |h_*'|^2 \right] d\xi_*' d\xi' \right) d\omega$$

$$\lesssim \int_{\mathbb{S}^2} B(\vartheta) \left(\int_{I_3} \left[|g_*|^2 |h|^2 + |g|^2 |h_*|^2 \right] d\xi_* d\xi \right) d\omega$$

$$\lesssim |g|^2_{L^\infty_\xi} |h|^2_{L^2_\xi} + |g|^2_{L^2_\xi} |h|^2_{L^\infty_\xi} . \tag{6.10}$$

Gathering (6.7)-(6.9) yields

$$\left(\int \left| \nu^{-1}(\xi) \Gamma_{gain}(g,h)(\xi) \right|^2 d\xi \right)^{1/2} \lesssim |g|_{L^\infty_\xi} |h|_{L^2_\xi} + |g|_{L^2_\xi} |h|_{L^\infty_\xi} . \tag{6.11}$$

Similarly, we have

$$\left(\int \left| \nu^{-1/2}(\xi) \Gamma_{gain}(g,h)(\xi) \right|^2 d\xi \right)^{1/2} \tag{6.12}$$

$$\lesssim \left(\int_{\mathbb{S}^2} B(\vartheta) \int_{I_1 \cup I_2 \cup I_3} |\xi - \xi_*|^\gamma \exp(-\frac{|\xi_*|^2}{4}) \left[|g_*'|^2 |h'|^2 + |g'|^2 |h_*'|^2 \right] d\xi_* d\xi d\omega \right)^{1/2}$$

$$\lesssim |g|_{L^\infty_\xi} |h|_{L^2_\sigma} + |g|_{L^2_\sigma} |h|_{L^\infty_\xi} ,$$

by following the same argument.

As a consequence, combining (6.5) and (6.11), we obtain (6.2). Combining (6.4) and (6.12), we obtain (6.1) and thus

$$\left| \langle f, \Gamma(g,h) \rangle_\xi \right| \le \left(\int_{\mathbb{R}^3} \nu(\xi) |f|^2 d\xi \right)^{1/2} \left(\int_{\mathbb{R}^3} \nu^{-1}(\xi) |\Gamma(g,h)|^2 d\xi \right)^{1/2}$$

$$\lesssim |f|_{L^2_\sigma} \left(|g|_{L^\infty_\xi} |h|_{L^2_\sigma} + |g|_{L^2_\sigma} |h|_{L^\infty_\xi} \right) .$$

The proof is completed. □

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