COMMUN. MATH. SCI.
Vol. 16, No. 8, pp. 2261–2300

SMOOTHING EFFECTS AND DECAY ESTIMATE OF THE SOLUTION OF THE LINEARIZED TWO SPECIES LANDAU EQUATION*

YU-CHU LIN†, HAITAO WANG‡, AND KUNG-CHIEN WU§

Abstract. We study the Landau equation for a mixture of two species in the whole space, with initial condition of one species near a vacuum and the other near a Maxwellian equilibrium state. For the linearized level, without any smoothness assumption on the initial data, it is shown that the solution becomes smooth instantaneously in both the space and momentum variables. Moreover, the large-time behavior of the solution is also obtained.

Keywords. Landau equation; Large-time behavior; Smoothing effects.

AMS subject classifications. 35Q20; 82C40.

1. Introduction

1.1. The model. This paper is concerned with the Cauchy problem for a system of Landau equations describing collisions in an ideal plasma mixture. The mixture is constituted by two species with mass (m_A, m_B) and is described by density functions $(F_A(t,x,p), F_B(t,x,p))$ defined in the phase-space of position and momentum. It takes the form of the following system:

$$\begin{cases} \partial_t F_A + \dfrac{1}{m_A} p \cdot \nabla_x F_A = Q^{AA}(F_A, F_A) + Q^{AB}(F_A, F_B), \\ \partial_t F_B + \dfrac{1}{m_B} p \cdot \nabla_x F_B = Q^{BB}(F_B, F_B) + Q^{BA}(F_B, F_A). \end{cases} \tag{1.1}$$

The right-hand side consists of the usual collision terms which for $X, Y \in \{A, B\}$ are given by:

$$\begin{aligned} &Q^{XY}(F_X, F_Y) \\ &= \nabla_p \cdot \left\{ \int_{\mathbb{R}^3} \Phi^{X,Y}\left(\frac{p}{m_X} - \frac{p_*}{m_Y}\right) \left[F_Y(p_*)\nabla_p F_X(p) - F_X(p)\nabla_{p_*} F_Y(p_*)\right] dp_* \right\}, \end{aligned}$$

here

$$\Phi^{X,Y}(z) = \frac{m_X m_Y}{m_X + m_Y}\left[I_3 - \frac{z \otimes z}{|z|^2}\right]\varphi(|z|),$$

the potential $\varphi(|z|) = |z|^{\gamma+2}$. We assume throughout this paper that $\gamma \in [-2, 1]$.

---

*Received: March 22, 2018; Accepted (in revised form): August 22, 2018. Communicated by Irene Gamba.

†Department of Mathematics, National Cheng Kung University, Dasyue Road, Tainan City, Taiwan (yuchu@mail.ncku.edu.tw).

‡Institute of Natural Sciences and School of Mathematical Sciences, MOE-LSC, Shanghai Jiao Tong University, Shanghai, China (haitallica@sjtu.edu.cn).

§Department of Mathematics, National Cheng Kung University, Tainan City, Taiwan and National Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan (kungchienwu@gmail.com).

We can find the following invariant properties of the collision operator (see [11]):

$$\begin{cases} \int_{\mathbb{R}^3} \{1, p, |p|^2\} Q^{XX}(F_X, F_X) dp = 0 \quad \text{for} \quad X \in \{A, B\}, \\ \int_{\mathbb{R}^3} Q^{AB}(F_A, F_B) dp = \int_{\mathbb{R}^3} Q^{BA}(F_B, F_A) dp = 0, \\ \int_{\mathbb{R}^3} p \left[Q^{AB}(F_A, F_B) + Q^{BA}(F_B, F_A) \right] dp = 0, \\ \int_{\mathbb{R}^3} \frac{|p|^2}{2} \left[\frac{1}{m_A} Q^{AB}(F_A, F_B) + \frac{1}{m_B} Q^{BA}(F_B, F_A) \right] dp = 0. \end{cases} \tag{1.2}$$

This means that the system (1.1) conserves mass, total momentum and total energy:

$$\begin{cases} \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} F_X dp dx = 0 \quad \text{for} \quad X \in \{A, B\}, \\ \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} p [F_A + F_B] dp dx = 0, \\ \frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|p|^2}{2} \left[\frac{1}{m_A} F_A + \frac{1}{m_B} F_B \right] dp dx = 0. \end{cases} \tag{1.3}$$

1.2. The linearized problem. Based on the idea introduced by [18], we consider the following perturbation:

$$\begin{cases} F_A = \sqrt{M_A} f_A, \\ F_B = M_B + \sqrt{M_B} f_B, \end{cases}$$

with the initial conditions of F_A and F_B satisfying

$$\begin{cases} F_A(0, x, p) = \sqrt{M_A} g_{in}(x, p), \\ F_B(0, x, p) = M_B + \sqrt{M_B} h_{in}(x, p), \end{cases}$$

where M_A and M_B are the Maxwellian states. Here, we fix the Maxwellian state of species B as the standard Gaussian, i.e.,

$$M_B(p) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|p|^2}{2} \right),$$

and then M_A, the Maxwellian state of species A, is chosen uniquely as

$$M_A(p) = \frac{1}{(2\pi m_A/m_B)^{3/2}} \exp\left(-\frac{m_A}{m_B} \frac{|p|^2}{2} \right),$$

for which the condition

$$Q^{AB}(M_A, M_B) = Q^{BA}(M_B, M_A) = 0$$

is valid. Therefore, under this choice, the two Maxwellian states satisfy

$$\begin{cases} Q^{AA}(M_A, M_A) = 0, \quad Q^{BB}(M_B, M_B) = 0, \\ Q^{AB}(M_A, M_B) = 0, \quad Q^{BA}(M_B, M_A) = 0. \end{cases} \tag{1.4}$$

It is easy to check that the system for (f_A, f_B) satisfies

$$\begin{cases} \partial_t f_A + \dfrac{1}{m_A} p \cdot \nabla_x f_A = L_{AB} f_A + \Gamma^{AA}(f_A, f_A) + \Gamma^{AB}(f_A, f_B), \\ \partial_t f_B + \dfrac{1}{m_B} p \cdot \nabla_x f_B = L_{BB} f_B + \Gamma^{BB}(f_B, f_B) + L_{BA} f_A + \Gamma^{BA}(f_B, f_A), \end{cases}$$

where

$$\begin{cases} L_{AB} f_A = \dfrac{1}{\sqrt{M_A}} Q^{AB}(f_A \sqrt{M_A}, M_B), \\ L_{BA} f_A = \dfrac{1}{\sqrt{M_B}} Q^{BA}(M_B, f_A \sqrt{M_A}), \\ L_{BB} f_B = \dfrac{1}{\sqrt{M_B}} \Big(Q^{BB}(M_B, f_B \sqrt{M_B}) + Q^{BB}(f_B \sqrt{M_B}, M_B) \Big), \\ \Gamma^{XY}(f_X, f_Y) = \dfrac{1}{\sqrt{M_X}} Q^{XY}(f_X \sqrt{M_X}, f_Y \sqrt{M_Y}). \end{cases}$$

If we assume the total mass of species A and the perturbations in species B are sufficiently small, in general, the basic time asymptotic behavior of the nonlinear solution is governed primarily by the linearized equation (up to a large time scale), since the nonlinear term is quadratic. In this regard, we will consider the linearized problem as follows:

$$\begin{cases} \partial_t g + \dfrac{1}{m_A} p \cdot \nabla_x g = L_{AB} g, \\ \partial_t h + \dfrac{1}{m_B} p \cdot \nabla_x h = L_{BB} h + L_{BA} g. \end{cases} \tag{1.5}$$

In this paper, we shall study the regularization effect and large-time behavior of the system (1.5).

1.3. Review of previous works. The studies of gas mixtures in terms of the Boltzmann equations were firstly introduced by [21]. Many interesting physical problems such as "ghost effects" and "Knudsen layer" for gas mixtures have been investigated in [3, 17]. From a mathematical point of view, if we assume that all species are close to equilibrium, the explicit spectral gap estimate and the hypocoercivity for a linearized multi-species Boltzmann system in the torus can be found in [8], and exponential decay towards equilibrium in general function spaces was done by Briant [4] and Briant and Daus [5]. Moreover, the global existence and stability of mild solutions to the Boltzmann system were completed by Ha, Noh and Yun in [13]. This research is also interested in phenomena related to vapor-vapor mixtures. This corresponds to the mathematical formulation that one species is near a vacuum and the other is near a Maxwellian equilibrium state. A qualitative-quantitative mathematical analysis in the whole space case was introduced by Sotirov and Yu [18], and the torus case can be found in [22].

For the mono-species Landau equation, we refer to Alexandre and Villani [2] for existence of renormalized solutions, to Desvillette and Villani [10] for conditional almost exponential convergence towards equilibrium and to a recent work by Carrapatoso, Tristani and Wu [6] for exponential decay towards equilibrium when initial data are close enough to equilibrium. Moreover, Guo [12] and Strain and Guo [19, 20] developed an existence and convergence towards equilibrium theory based on energy methods for initial data close to the equilibrium state in some Sobolev norms. Recently the set of initial data for which this theory is valid has been enlarged by Carrapatoso and Mischler [7] via a linearization method.

For the multi-species Landau equation, we refer to Chapter 4 in [14] for a physical derivation and discussion. However, on the mathematical side, there are very few results. One can only find the work done by Gualdani and Zamponi [11], in which all species are assumed to be close to equilibrium; the explicit spectral gap estimate and the hypocoercivity for a linearized multi-species Landau system in the torus were obtained.

In this paper, we are concerned with the Landau equation in the whole space, with initial condition of one species near a vacuum and the other near a Maxwellian equilibrium state, and study the regularization effect and large-time behavior of solutions of this Cauchy problem.

1.4. Main theorem. Before the presentation of the main theorem, let us define some notation in this paper. We denote $\langle p\rangle^s=(1+|p|^2)^{s/2}$, $s\in\mathbb{R}$. For the microscopic variable p, we denote

$$|f|_{L^2_p}=\Big(\int_{\mathbb{R}^3}|f|^2dp\Big)^{1/2},$$

and the weighted norms $|f|_{L^2_p(m)}$ and $|f|_{L^2_\theta}$ respectively by

$$|f|_{L^2_p(m)}=\Big(\int_{\mathbb{R}^3}|f|^2m dp\Big)^{1/2},\quad |f|_{L^2_\theta}=\Big(\int_{\mathbb{R}^3}|f|^2\langle p\rangle^{2\theta}dp\Big)^{1/2},$$

where $m=m(p)$ is a weight function. The L^2_p inner product in $\mathbb{R}^3$ will be denoted by $\big\langle\cdot,\cdot\big\rangle_p$. For the space variable x, we have similar notation. In fact, L^2_x is the classical Hilbert space with norm

$$|f|_{L^2_x}=\Big(\int_{\mathbb{R}^3}|f|^2dx\Big)^{1/2}.$$

We denote the sup norm as

$$|f|_{L^\infty_x}=\sup_{x\in\mathbb{R}^3}|f(x)|.$$

The standard inner product in $\mathbb{R}^3$ will be denoted by $(\cdot,\cdot)$. For any vector function $u\in L^2_p$, $\mathbb{P}(p)u$ denotes the orthogonal projection to the direction of vector p, i.e.,

$$\mathbb{P}(p)u=\frac{u\cdot p}{|p|^2}p.$$

For the Landau equation, the natural norm in p is $|\cdot|_{H^1_\sigma}$, which is defined by

$$|f|^2_{H^1_\sigma}=|\langle p\rangle^{\frac{\gamma+2}{2}}f|^2_{L^2_p}+|\langle p\rangle^{\frac{\gamma}{2}}\mathbb{P}(p)\nabla_pf|^2_{L^2_p}+\big|\langle p\rangle^{\frac{\gamma+2}{2}}\big[I_3-\mathbb{P}(p)\big]\nabla_pf\big|^2_{L^2_p},$$

and the weighted norms are defined as

$$|f|^2_{H^1_\sigma(m)}=|m^{1/2}\langle p\rangle^{\frac{\gamma+2}{2}}f|^2_{L^2_p}+|m^{1/2}\langle p\rangle^{\frac{\gamma}{2}}\mathbb{P}(p)\nabla_pf|^2_{L^2_p}+\big|m^{1/2}\langle p\rangle^{\frac{\gamma+2}{2}}\big[I_3-\mathbb{P}(p)\big]\nabla_pf\big|^2_{L^2_p},$$

and

$$|f|^2_{H^1_{\sigma,\theta}}=|\langle p\rangle^{\frac{\gamma+2}{2}+\theta}f|^2_{L^2_p}+|\langle p\rangle^{\frac{\gamma}{2}+\theta}\mathbb{P}(p)\nabla_pf|^2_{L^2_p}+\big|\langle p\rangle^{\frac{\gamma+2}{2}+\theta}\big[I_3-\mathbb{P}(p)\big]\nabla_pf\big|^2_{L^2_p}.$$

Moreover, we define

$$\|f\|_{L^2}^2=\int_{\mathbb{R}^3}|f|_{L^2_p}^2dx,\quad \|f\|_{L^2(m)}^2=\int_{\mathbb{R}^3}|f|_{L^2_p(m)}^2dx,\quad \|f\|_{L^2_\theta}^2=\int_{\mathbb{R}^3}|f|_{L^2_\theta}^2dx,$$

$$\|f\|_{H^1_\sigma}^2=\int_{\mathbb{R}^3}|f|_{H^1_\sigma}^2dx,\quad \|f\|_{H^1_\sigma(m)}=\int_{\mathbb{R}^3}|f|_{H^1_\sigma(m)}^2dx,\quad \|f\|_{H^1_{\sigma,\theta}}^2=\int_{\mathbb{R}^3}|f|_{H^1_{\sigma,\theta}}^2dx,$$

and

$$\|f\|_{L^\infty_xL^2_p}=\sup_{x\in\mathbb{R}^3}|f|_{L^2_p},\quad \|f\|_{L^1_xL^2_p}=\int_{\mathbb{R}^3}|f|_{L^2_p}dx.$$

Finally, we define the high order Sobolev norm: Let k be a non-negative integer and k_1,k_2 be multi-indexes,

$$\|f\|_{H^k(m)}=\sum_{|k_1|+|k_2|\le k}\left\|\partial_x^{k_1}\partial_p^{k_2}f\right\|_{L^2(m)}.$$

The domain decomposition plays an important role for the coercivity of the collision operators, hence we define a cut-off function $\chi:\mathbb{R}\to\mathbb{R}$, which is a smooth non-increasing function, $\chi(s)=1$ for $s\le 1$, $\chi(s)=0$ for $s\ge 2$ and $0\le\chi\le 1$. Moreover, we define $\chi_R(s)=\chi(s/R)$.

For simplicity of notation, hereafter, we abbreviate " $\le C$ " to " $\lesssim$ ", where C is a positive constant depending only on fixed numbers.

In the following, we state our main result.

THEOREM 1.1. *Let k and ℓ be non-negative integers, and assume that the initial conditions $(g_{in},h_{in})\in L^1_xL^2_p\cap L^2(w_{k+\ell+2})$. Then the solutions of (1.5) have the following estimates for $t\ge 1$:*

$$\|\nabla_p^\ell\nabla_x^k g\|_{L^2}\lesssim(1+t)^{-(3+k)/2}\left(\|g_{in}\|_{L^1_xL^2_p}+\|g_{in}\|_{L^2(w_{k+\ell+2})}\right),$$

and

$$\|\nabla_p^\ell\nabla_x^k h\|_{L^2}\lesssim(1+t)^{-(3+k)/2}\left(\|(g_{in},h_{in})\|_{L^1_xL^2_p}+\|(g_{in},h_{in})\|_{L^2(w_{k+\ell+2})}\right).$$

Here

$$w_n\equiv\begin{cases}1, & \gamma\in[0,1],\\ \langle p\rangle^{|\gamma|n}, & \gamma\in[-2,0).\end{cases}$$

Let us remark that this theorem is a simplified version, and it is covered in more detail in Theorems 3.1 and 4.1.

1.5. Method of proof and plan of the paper. The basic principles used to develop the theory are the "separation of scales" and "regularization estimate". The separation of scales is a concept created in [15] for a 1D Boltzmann equation with hard sphere in the whole space. The regularization estimate for the Landau equation was constructed in [6] initially.

In this paper, a long wave-short wave decomposition is introduced. Under this decomposition, one can follow the spectrum analysis of the Landau collision operator in [23] to decompose the solution to the linearized Landau equation as the long wave-fluid part, the long wave-nonfluid part and the short wave part, which separate the scales, and hence get different decay rates for each part.

On the other hand, the regularization estimate plays an important role in this paper, which leads us to remove the regularity assumption on the initial condition. Based on the basic regularization estimate established in [6], we are able to construct new functionals for high order derivatives, and then by using a bootstrap process to get the smoothing estimates in both the space variable x and momentum variable p.

The separation of scales helps us get different decay rates in each part, but this idea cannot improve regularity. Contrarily, the regularization estimate helps us improve regularity, but this cannot give time decay estimates. Fortunately, putting these two ideas together, by interpolation argument, one can get the smoothing effect and time decay simultaneously. To the best of our knowledge, this thinking is itself new for Landau kinetic equations.

In the following we point out some significant points of our method:

- The problem we consider here is the whole space case, and hence we expect the algebraic decay estimate of the solutions rather than the exponential decay in the torus case [4–6,8,11,19,20]. Under our linearization (one species near a vacuum and the other near equilibrium), the system is decoupled, so that one can estimate each equation (g,h) separately. For solution g, as we mentioned above, one can get smoothing effect and time decay simultaneously by interpolation argument.

  For solution h, we solve it by using the Duhamel principle and treating g as a source term. This method usually results in that the decay rate of h is slower than that of g. Indeed, the characteristics for the macroscopic equations for h coincide with the direction of mass diffusion for g. Thus, there is a possible wave resonance, which leads to the slower decay of solution h. On the microscopic level, the resonance is from the small divisor caused by the closeness between different wave frequencies (i.e., $\lambda(\eta)$ in Lemma 2.5 and $\sigma_j(\eta)$, $j=2,3,4$ in Lemma 2.6). It was shown in [18] that for a 1D Boltzmann gas mixture there is only one characteristic for h coinciding with g, which is the diffusion sound wave carrying mass and energy. The resonance problem was resolved by considering a microscopic cancellation representing the conservation of mass and total energy. For our 3D case, there are other two characteristics for h having the same direction as mass diffusion for g in addition to the diffusion sound wave. Their resonances are resolved by other microscopic cancellations from the conservation of total momentum and orthogonality between propagating directions (see the proof of Proposition 4.1 for details). These crucial cancellation properties lead h to have the same decay rate as g. It would also be interesting to understand the detailed wave propagation of (g,h), and they are postponed to our future studies.
- The method to prove high order regularization estimates in this paper is interesting and itself new. In the paper [6], a first order regularization estimate for the semigroup $e^{t\mathcal{L}_{AB}}$ in both the x and p variables is obtained (see Lemma 3.1), where $\mathcal{L}_{AB}$ contains the transport part $p\cdot\nabla_x$ and the ellipticity part Λ_{AB}, and the regularization mechanism is their combined effect. In order to improve the high order x and p regularities based on this result, we employ different arguments respectively. For the space variable x, we design a Picard-type iteration, which treats $\mathcal{L}_{AB}$ as an anstaz and the operator K_{AB} as a source term. The

first several terms in the iteration contain the most singular part of the solution. Thereupon, after finite iterations (depending on the differentiation order), we can extract the singular waves and then the remainder part will become smooth in x variable for all time (see Lemma 3.2). For the momentum p, we construct a new functional including high order p derivatives and all lower order x, p and mixed derivatives, where the lower order derivatives are used to take care of the nonzero commutator $[p\cdot\nabla_x,\nabla_p]$. By making use of x regularity obtained previously, and choosing suitable combinations of derivatives, one can get high order regularization estimates in p variable for the semigroup $e^{t\mathcal{L}_{AB}}$. Then the smoothing effect of the solution in the p variable will be obtained inductively (see Subsection 3.2.1).

The paper is organized as follows: we list some properties of the linearized collision operator in section 2; then we prove the estimates of g and h in Sections 3 and 4 respectively.

2. Preliminaries

The basic structures of the linearized Landau collision operator are well known [9]. Here we take a closer look at these structures with more modifications (see Lemma 2.1-Lemma 2.3). After that, Lemma 2.4 is done by Guo [12], which would be of great help to us regarding the regularization estimate. On the other hand, the separation of scales relies on the spectrum analysis of the operator $L^{\eta}_{XY}=-\frac{i\eta\cdot p}{m_X}+L_{XY}$ in the classical Hilbert space L^2_p, where $(X,Y)=(A,B)$ or (B,B); this is why our argument cannot include the case of very soft potentials and Coulomb potential. Following the systematic procedure established by Yang and Yu [23], we can easily derive the spectrum of the operator L^{η}_{XY}. Moreover, we demonstrate that when $|\eta|$ is small, the corresponding eigenfunctions are not only smooth but also decay faster than any polynomial in p.

Lemma 2.1 (The null spaces of L_{BB} and L_{AB}). *The null space of L_{BB} is a five-dimensional vector space with an orthonormal basis $\{\chi_i\}_{i=0}^4$, where*

$$\{\chi_0,\chi_j,\chi_4\}=\left\{\sqrt{M_B},p_j\sqrt{M_B},\frac{1}{\sqrt{6}}(|p|^2-3)\sqrt{M_B}\right\},\quad j=1,2,3.$$

On the other hand, the null space of L_{AB} is a one-dimensional vector space $\{E_D\}$, where $E_D=M_A^{1/2}$.

Lemma 2.2 (Decomposition). *Let $(X,Y)=(B,B)$ or (A,B). Then*
(i) the collision operator L_{XY} consists of an elliptic-type operator $\widetilde{\Lambda}^{XY}$ and an integral operator $\widetilde{K}^{XY}$:

$$L_{XY}f=-\widetilde{\Lambda}^{XY}f+\widetilde{K}^{XY}f,$$

where

$$\widetilde{\Lambda}^{XY}f=-\nabla_p\cdot\left[\sigma^{XY}\nabla_p f\right]+\left(\frac{1}{4}\frac{m_Y^2}{m_X^2}(\sigma^{XY}p,p)-\frac{1}{2}\frac{m_Y}{m_X}\nabla_p\cdot\left[\sigma^{XY}p\right]\right)f,$$

$$\widetilde{K}^{XY}f=\begin{cases}\int_{\mathbb{R}^3}M_B^{-1/2}(p)M_B^{-1/2}(p_*)\nabla_p\cdot\nabla_{p_*}\cdot Z^{XY}(p,p_*)f(p_*)\,dp_*, & (X,Y)=(B,B),\\ 0, & (X,Y)=(A,B),\end{cases}$$

with

$$\sigma^{XY}(p)=\int_{\mathbb{R}^3}\Phi^{X,Y}\left(\frac{p}{m_X}-\frac{p_*}{m_Y}\right)M_Y(p_*)\,dp_*,$$
$$Z^{XY}(p,p_*)=M_X(p)\,M_Y(p_*)\,\Phi^{X,Y}\left(\frac{p}{m_X}-\frac{p_*}{m_Y}\right).$$

(ii) The spectrum of $\sigma^{XY}(p)$ consists of a simple eigenvalue $\lambda_1(p)$ associated with the eigenvector p, and a double eigenvalue $\lambda_2(p)$ associated with the eigenspace $p^{\perp}$. Furthermore

$$\lambda_1^{XY}(p)=\frac{m_Xm_Y}{m_X+m_Y}\frac{1}{m_Y^{\gamma+2}}\int_{\mathbb{R}^3}\left(1-\left(\frac{p}{|p|},\frac{p_*}{|p_*|}\right)^2\right)M_Y\left(\frac{m_Y}{m_X}p-p_*\right)|p_*|^{\gamma+2}\,dp_*,$$
$$\lambda_2^{XY}(p)=\frac{m_Xm_Y}{m_X+m_Y}\frac{1}{m_Y^{\gamma+2}}\int_{\mathbb{R}^3}\left(1+\left(\frac{p}{|p|},\frac{p_*}{|p_*|}\right)^2\right)M_Y\left(\frac{m_Y}{m_X}p-p_*\right)|p_*|^{\gamma+2}\,dp_*,$$

with

$$\liminf_{|p|\to 0}\lambda_1^{XY}(p)>0,\quad \liminf_{|p|\to 0}\lambda_2^{XY}(p)>0,$$

and as $|p|\to\infty$,

$$\lambda_1^{XY}(p)\sim\frac{m_Xm_Y}{m_X+m_Y}\frac{2}{m_Y^{\gamma+2}}\left(\frac{m_Y}{m_X}|p|\right)^{\gamma},\qquad \lambda_2^{XY}(p)\sim\frac{m_Xm_Y}{m_X+m_Y}\frac{1}{m_Y^{\gamma+2}}\left(\frac{m_Y}{m_X}|p|\right)^{\gamma+2}.$$

Immediately, we have

$$\nabla_p\cdot\sigma^{XY}(p)=-\frac{m_Y^2}{m_X^2}\sigma^{XY}(p)p=-\frac{m_Y^2}{m_X^2}\lambda_1^{XY}(p)p,\quad (p,\sigma^{XY}(p)p)=\lambda_1^{XY}(p)|p|^2,$$
$$(u,\sigma^{XY}(p)u)=\lambda_1^{XY}(p)|\mathbb{P}(p)u|^2+\lambda_2^{XY}(p)\left|\left[I_3-\mathbb{P}(p)\right]u\right|^2$$
$$\geq c_0\left\{\langle p\rangle^{\gamma}|\mathbb{P}(p)u|^2+\langle p\rangle^{\gamma+2}|[I_3-\mathbb{P}(p)]u|^2\right\}.$$

(iii) For any $k\in\mathbb{N}$, we have

$$|\nabla_p^k\sigma^{XY}(p)|\lesssim\langle p\rangle^{\gamma+2-k},\quad |\nabla_p^k(\sigma^{XY}(p)p)|\lesssim\langle p\rangle^{\gamma+1-k},$$

and as $|p|\to\infty$,

$$|\nabla_p^k\lambda_1^{XY}(p)|\lesssim\langle p\rangle^{\gamma-k},\quad |\nabla_p^k\lambda_2^{XY}(p)|\lesssim\langle p\rangle^{\gamma+2-k}.$$

(iv) (Coercivity) Rewrite

$$\widetilde{\Lambda}^{XY}f=-\nabla_p\cdot\left[\sigma^{XY}\nabla_pf\right]+\left(\frac{1}{4}\frac{m_Y^2}{m_X^2}(\sigma^{XY}p,p)-\frac{1}{2}\frac{m_Y}{m_X}\left((\nabla_p\cdot\sigma^{XY},p)+Tr\sigma^{XY}\right)\right)f$$
$$=-\nabla_p\cdot\left[\sigma^{XY}\nabla_pf\right]+\left\{\frac{1}{4}\lambda_1^{XY}(p)\left(\frac{m_Y}{m_X}|p|\right)^2+\frac{1}{2}\lambda_1^{XY}(p)\frac{m_Y}{m_X}\left(\frac{m_Y}{m_X}|p|\right)^2\right.$$
$$\left.-\frac{m_Y}{m_X}\lambda_2^{XY}(p)-\frac{1}{2}\frac{m_Y}{m_X}\lambda_1^{XY}(p)\right\}f.$$

Let

$$-\Lambda^{XY}f = -\widetilde{\Lambda}^{XY}f - \varpi\chi_R f,\ K^{XY}f = \widetilde{K}^{XY}f + \varpi\chi_R f,$$

where ϖ, R *are positive constants large enough. Then*

$$\left\langle \Lambda^{XY}f, f\right\rangle_p \geq c_0 |f|^2_{H^1_\sigma},$$

for some $c_0 > 0$, *and*

$$\left\langle K^{XY}f, f\right\rangle_p \leq C|f|^2_{L^2_p}.$$

The behavior of the operator L_{BA} is different from L_{BB} and L_{AB}, but similar to K^{BB}.

LEMMA 2.3. *The operator* L_{BA} *is a bounded integral operator:*

$$L_{BA}f = \int_{\mathbb{R}^3} M_B^{-1/2}(p) M_A^{-1/2}(p_*) \nabla_p \cdot \nabla_{p_*} \cdot Z^{BA}(p,p_*) f(p_*) dp_*,$$

where

$$Z^{BA}(p,p_*) = M_B(p) M_A(p_*) \Phi^{B,A}\left(\frac{p}{m_B} - \frac{p_*}{m_A}\right).$$

LEMMA 2.4 (Lemma 6, [12]). *Let* $(X,Y) = (B,B)$ *or* (A,B), *assume* $|\alpha| \geq 0$, *for small* $\delta > 0$, *we have*

$$\langle \langle p\rangle^{2\theta} \partial_p^\alpha \left(\widetilde{\Lambda}^{XY} f\right), \partial_p^\alpha \overline{f}\rangle_p \geq c_0 |\partial_p^\alpha f|^2_{H^1_{\sigma,\theta}} - \delta \sum_{|\overline{\alpha}| \leq |\alpha|} |\partial_p^{\overline{\alpha}} f|^2_{H^1_{\sigma,\theta}} - C_\delta |\chi_R f|^2_{L^2_p}$$

and

$$\left|\langle \langle p\rangle^{2\theta} \partial_p^\alpha \left(\widetilde{K}^{XY} f_1\right), \partial_p^\alpha \overline{f}_2\rangle_p\right| \leq |\partial_p^\alpha f_2|_{H^1_{\sigma,\theta}} \left\{\delta \sum_{|\overline{\alpha}| \leq |\alpha|} |\partial_p^{\overline{\alpha}} f_1|_{H^1_{\sigma,\theta}} + C_\delta |\chi_R f_1|_{L^2_p}\right\}.$$

In particular,

$$\langle \langle p\rangle^{2\theta} \partial_p^\alpha (L_{XY} f), \partial_p^\alpha \overline{f}\rangle_p \leq -c_0 |\partial_p^\alpha f|^2_{H^1_{\sigma,\theta}} + \delta \sum_{|\overline{\alpha}| \leq |\alpha|} |\partial_p^{\overline{\alpha}} f|^2_{H^1_{\sigma,\theta}} + C_\delta |\chi_R f|^2_{L^2_p}. \tag{2.1}$$

Taking the Fourier transform in the space variable x, we consider the operator $L^\eta_{XY} = -\frac{i\eta\cdot p}{m_X} + L_{XY}$, where $(X,Y) = (A,B)$ or (B,B). Then the corresponding spectrums $\text{Spec}_{XY}(\eta)$ in the classical Hilbert space L^2_p are derived as follows.

LEMMA 2.5 (Spectrum of L^η_{AB} [23]). *Given* $\delta > 0$
(i) there exists $\tau_2 = \tau_2(\delta) > 0$ *such that if* $|\eta| > \delta$,

$$\text{Spec}(L^\eta_{AB}) \subset \{z \in \mathbb{C} : \text{Re}(z) < -\tau_2\}. \tag{2.2}$$

(ii) If $\eta = |\eta|\omega$ *and* $|\eta| < \delta$, *the spectrum within the region* $\{z \in \mathbb{C} : Re(z) > -\tau_2\}$ *is consisting of exactly one eigenvalue* $\{\lambda(\eta)\}$,

$$\text{Spec}(L^\eta_{AB}) \cap \{z \in \mathbb{C} : \text{Re}(z) > -\tau_2\} = \{\lambda(\eta)\}, \tag{2.3}$$

associated with its eigenfunction $\{e_D(\eta)\}$. *They have the expansions*

$$\begin{aligned} \lambda(\eta) &= -a_2|\eta|^2 + O(|\eta|^3), \\ e_D(\eta) &= E_D + iE_{D,1}|\eta| + O(|\eta|^2), \end{aligned} \tag{2.4}$$

with $a_2 = -\left\langle L_{AB}^{-1}(p\cdot\omega/m_A)E_D, (p\cdot\omega/m_A)E_D\right\rangle_p > 0$ *and* $E_{D,1} = L_{AB}^{-1}(p\cdot\omega/m_A)E_D$. *Here* $\{e_D(\eta)\}$ *can be normalized by* $\left\langle e_D(-\eta), e_D(\eta)\right\rangle_p = 1$.

More precisely, the semigroup $e^{(-i\eta\cdot p + L_{AB})t}$ *can be decomposed as*

$$e^{(-i\eta\cdot p+L_{AB})t} f = e^{(-i\eta\cdot p+L_{AB})t}\Pi_\eta^{D\perp} f + \chi_{\{|\eta|<\delta\}} e^{\lambda(\eta)t}\left\langle e_D(-\eta), f\right\rangle_p e_D(\eta), \tag{2.5}$$

and there exist $a(\tau_2) > 0$, $\overline{a}_2 > 0$ *such that* $|e^{(-i\eta\cdot p+L_{AB})t}\Pi_\eta^{D\perp}|_{L^2_p} \lesssim e^{-a(\tau_2)t}$ *and* $|e^{\lambda(\eta)t}| \le e^{-\overline{a}_2|\eta|^2 t}$.

Moreover, the eigenfunction $e_D(p,\eta)$ *is smooth and decays faster than any polynomial in* p, *i.e., for any* $\theta \ge 0$, $|\alpha| \ge 0$, $|\langle p\rangle^\theta \partial_p^\alpha e_D(p,\eta)|_{L^2_p} \le C_{\theta,\alpha}|e_D(p,\eta)|_{L^2_p}$.

Proof. The spectrum analysis can be found in [23]. Here it is sufficient to show that $e_D(p,\eta)$ as a function of p is in Schwartz space. Note that

$$\left(L_{AB} - \frac{i\eta\cdot p}{m_A}\right) e_D(p,\eta) = \lambda(\eta) e_D(p,\eta).$$

Let α be a multi-index with $|\alpha| \ge 0$. Taking ∂_p^α, we get

$$\lambda(\eta)\partial_p^\alpha e_D(p,\eta) = \partial_p^\alpha\left(L_{AB}e_D(p,\eta)\right) - \frac{i\eta\cdot p}{m_A}\partial_p^\alpha e_D(p,\eta) - \sum_{\substack{\beta<\alpha\\ |\beta|=|\alpha|-1}} C_\alpha^\beta \partial_p^{\alpha-\beta}\left(\frac{i\eta\cdot p}{m_A}\right)\partial_p^\beta e_D(p,\eta),$$

here $C_\alpha^\beta = \frac{\alpha!}{(\alpha-\beta)!\beta!}$. Take the inner product to yield

$$\begin{aligned} &\left\langle \langle p\rangle^{2\theta}\partial_p^\alpha\left(L_{AB}e_D(p,\eta)\right), \partial_p^\alpha e_D(p,-\eta)\right\rangle \\ &= \left\langle \langle p\rangle^{2\theta}\frac{i\eta\cdot p}{m_A}\partial_p^\alpha e_D(p,\eta), \partial_p^\alpha e_D(p,-\eta)\right\rangle + \lambda(\eta)\left|\langle p\rangle^{2\theta}\partial_p^\alpha e_D(p,\eta)\right|^2 \\ &\quad + \sum_{\beta<\alpha, |\beta|=|\alpha|-1} C_\alpha^\beta \left\langle \langle p\rangle^{2\theta}\partial_p^{\alpha-\beta}\left(\frac{i\eta\cdot p}{m_A}\right)\partial_p^\beta e_D(p,\eta), \partial_p^\alpha e_D(p,-\eta)\right\rangle, \end{aligned}$$

and

$$\begin{aligned} &\left\langle \langle p\rangle^{2\theta}\partial_p^\alpha\left(L_{AB}e_D(p,-\eta)\right), \partial_p^\alpha e_D(p,\eta)\right\rangle \\ &= \left\langle \langle p\rangle^{2\theta}\frac{-i\eta\cdot p}{m_A}\partial_p^\alpha e_D(p,-\eta), \partial_p^\alpha e_D(p,\eta)\right\rangle + \lambda(-\eta)\left|\langle p\rangle^{2\theta}\partial_p^\alpha e_D(p,\eta)\right|^2 \\ &\quad + \sum_{\beta<\alpha, |\beta|=|\alpha|-1} C_\alpha^\beta \left\langle \langle p\rangle^{2\theta}\partial_p^{\alpha-\beta}\left(\frac{-i\eta\cdot p}{m_A}\right)\partial_p^\beta e_D(p,-\eta), \partial_p^\alpha e_D(p,\eta)\right\rangle. \end{aligned}$$

Adding the above two equations, and in view of (2.1), we have

$$\begin{aligned} c_0|\partial_p^\alpha e_D(p,\eta)|^2_{H^1_{\sigma,\theta}} \le{}& \frac{1}{2}(|\lambda(\eta)| + |\lambda(-\eta)|)|\langle p\rangle^\theta \partial_p^\alpha e_D(p,\eta)|^2_{L^2_p} \\ &+ \left(\kappa + C|\eta|\right)\sum_{|\overline{\alpha}|\le|\alpha|} |\partial_p^{\overline{\alpha}} e_D(p,\eta)|^2_{H^1_{\sigma,\theta}} + C_\kappa|\chi_R e_D(p,\eta)|^2_{L^2_p}. \end{aligned}$$

Taking summation over $|\alpha|\le k$, for $|\eta|$ small and choosing κ sufficiently small, we have

$$\sum_{|\alpha|\le k}|\partial_p^\alpha e_D(p,\eta)|^2_{H^1_{\sigma,\theta}}\le C|\chi_R e_D(p,\eta)|^2_{L^2_p}.$$

Since this holds for any k and θ, the proof is complete. ◻

LEMMA 2.6 (Spectrum of L^η_{BB} [23]). *Given $\delta>0$,*
(i) there exists $\tau_1=\tau_1(\delta)>0$ such that if $|\eta|>\delta$,

$$\mathrm{Spec}(L^\eta_{BB})\subset\{z\in\mathbb{C}:\mathrm{Re}(z)<-\tau_1\}. \tag{2.6}$$

(ii) If $\eta=|\eta|\omega$ and $|\eta|<\delta$, the spectrum within the region $\{z\in\mathbb{C}:Re(z)>-\tau_1\}$ consists of exactly five eigenvalues $\{\sigma_j(\eta)\}_{j=0}^4$,

$$\mathrm{Spec}(L^\eta_{BB})\cap\{z\in\mathbb{C}:\mathrm{Re}(z)>-\tau_1\}=\{\sigma_j(\eta)\}_{j=0}^4, \tag{2.7}$$

associated with the corresponding eigenfunctions $\{e_j(\eta)\}_{j=0}^4$. They have the expansions

$$\begin{aligned}&\sigma_j(\eta)=ia_{j,1}|\eta|-a_{j,2}|\eta|^2+O(|\eta|^3),\\&e_j(\eta)=E_j+O(|\eta|),\end{aligned} \tag{2.8}$$

with $a_{j,2}=\left\langle L^{-1}_{BB}\mathbb{P}_1(p\cdot\omega/m_B)E_j,\mathbb{P}_1(p\cdot\omega/m_B)E_j\right\rangle_p>0$ and

$$\begin{cases}a_{01}=\frac{1}{m_B}\sqrt{\frac{5}{3}},\quad a_{11}=-\frac{1}{m_B}\sqrt{\frac{5}{3}},\quad a_{21}=a_{31}=a_{41}=0,\\ E_0=\sqrt{\frac{3}{10}}\chi_0+\sqrt{\frac{1}{2}}\omega\cdot\Psi+\sqrt{\frac{1}{5}}\chi_4,\\ E_1=\sqrt{\frac{3}{10}}\chi_0-\sqrt{\frac{1}{2}}\omega\cdot\Psi+\sqrt{\frac{1}{5}}\chi_4,\\ E_2=-\sqrt{\frac{2}{5}}\chi_0+\sqrt{\frac{3}{5}}\chi_4,\\ E_3=\omega_1^\perp\cdot\Psi,\\ E_4=\omega_2^\perp\cdot\Psi,\end{cases} \tag{2.9}$$

where $\Psi=(\chi_1,\chi_2,\chi_3)$, $\{\omega,\omega_1^\perp,\omega_2^\perp\}$ is an orthonormal basis of $\mathbb{R}^3$, $\{e_j(\eta)\}_{j=0}^4$ can be normalized by $\left\langle e_j(-\eta),e_l(\eta)\right\rangle_p=\delta_{jl}$, $0\le j,l\le 4$.

More precisely, the semigroup $e^{(-i\eta\cdot p+L_{BB})t}$ can be decomposed as

$$\begin{aligned}e^{(-i\eta\cdot p+L_{BB})t}f&=e^{(-i\eta\cdot p+L_{BB})t}\Pi_\eta^\perp f\\&\quad+\chi_{\{|\eta|<\delta\}}\sum_{j=1}^3 e^{\sigma_j(\eta)t}\left\langle e_j(-\eta),f\right\rangle_p e_j(\eta),\end{aligned} \tag{2.10}$$

and there exist $a(\tau_1)>0$, $\overline{a}_1>0$ such that $|e^{(-i\eta\cdot p+L_{BB})t}\Pi_\eta^\perp|_{L^2_p}\lesssim e^{-a(\tau_1)t}$ and $|e^{\sigma_j(\eta)t}|\le e^{-\overline{a}_1|\eta|^2t}$ for all $0\le j\le 4$.

Moreover, the eigenfunctions $e_j(p,\eta)$, $0\le j\le 4$ are smooth and decay faster than any polynomial in p, i.e., for any $\theta\ge 0$, $|\alpha|\ge 0$, $|\langle p\rangle^\theta\partial_p^\alpha e_j(p,\eta)|_{L^2_p}\le C_{\theta,\alpha}|e_j(p,\eta)|_{L^2_p}$.

3. The result for g

In accordance with the theory of the semi-groups, the solution of the linearized Landau equation

$$\partial_t g + \frac{1}{m_A} p \cdot \nabla_x g = L_{AB} g, \quad g(0,x,p) = g_{in}(x,p),$$

can be represented by

$$g(t,x,p) = \mathbb{G_{AB}}^t g_{in} = \int_{\mathbb{R}^3} e^{i\eta\cdot x + (-ip\cdot\eta/m_A + L_{AB})t} \hat{g_{in}}(\eta,p)\, d\eta,$$

where $\mathbb{G_{AB}}^t$ is the solution operator and $\hat{g}$ means the Fourier transform in the space variable x. Furthermore, we introduce a long wave-short wave decomposition for the solution g:

$$g = g_L + g_S,$$

where

$$g_L = \int_{|\eta|<\delta} e^{i\eta\cdot x + (-ip\cdot\eta/m_A + L_{AB})t} \hat{g_{in}}(\eta,p)\, d\eta,$$

$$g_S = \int_{|\eta|\geq\delta} e^{i\eta\cdot x + (-ip\cdot\eta/m_A + L_{AB})t} \hat{g_{in}}(\eta,p)\, d\eta.$$

In order to study the long wave part g_L, we need further to decompose it into the fluid part and non-fluid part, i.e., $g_L = g_{L,0} + g_{L,\perp}$, where

$$g_{L,0} = \int_{|\eta|<\delta} e^{\lambda(\eta)t} e^{i\eta\cdot x} \left\langle e_D(-\eta), \hat{g_{in}} \right\rangle_p e_D(\eta) d\eta,$$

$$g_{L,\perp} = \int_{|\eta|<\delta} e^{i\eta\cdot x + (-ip\cdot\eta/m_A + L_{AB})t} \Pi_\eta^{D\perp} \hat{g_{in}}(\eta,p)\, d\eta.$$

The main result for the solution g is stated as below.

THEOREM 3.1. *Let k and ℓ be non-negative integers. Then for $t \geq 1$,*

$$\left\| \nabla_p^\ell \nabla_x^k g_{L,0} \right\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \left\| g_{in} \right\|_{L_x^1 L_p^2},$$

$$\left\| \nabla_p^\ell \nabla_x^k g_{L,\perp} \right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \left(\left\| g_{in} \right\|_{L^2(w_{\ell+1})} + \left\| g_{in} \right\|_{L_x^1 L_p^2} \right),$$

and

$$\left\| \nabla_p^\ell \nabla_x^k g_S \right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \left\| g_{in} \right\|_{L^2(w_{k+\ell+2})},$$

the constant $C>0$ depending only upon k and ℓ. Here

$$w_n \equiv \begin{cases} 1, & \gamma \in [0,1], \\ \langle p \rangle^{|\gamma| n}, & \gamma \in [-2,0). \end{cases}$$

In what follows, we prove this theorem in several stages. To begin with, we establish the x-regularity of g in Subsection 3.1. After that, we take advantage of it to improve the p-regularity via certain energy functionals and Duhamel's principle in the next subsection. On the basis of these regularization estimates, we finally obtain the decay rate by invoking Sobolev's inequality and interpolation inequality.

3.1. Improvement of the x-regularity. It immediately follows from the Fourier transform in x and the p-regularity of the eigenfunction $e_D(\eta)$ that

$$\left\|\nabla_p^\ell \nabla_x^k g_{L,0}\right\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2}.$$

On the other hand, from the spectral gap it follows that

$$\left\|\nabla_x^k g_{L,\perp}\right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \|g_{in}\|_{L_x^1 L_p^2},$$

and

$$\|g_S\|_{L^2} \lesssim e^{-Ct} \|g_{in}\|_{L^2}.$$

Noticing that $g_S = g - g_L$, we need the regularization estimate of g in x further. To that end, we introduce a new operator $\mathcal{L}_{AB}$:

$$\mathcal{L}_{AB} = -\frac{1}{m_A} p\cdot\nabla_x - \Lambda^{AB}.$$

Let n be a non-negative integer and let $\theta \geq 0$. Hereafter, we define $m_0 = m_\theta = \langle p\rangle^\theta$ and

$$m_n \equiv \begin{cases} m_\theta, & \gamma\in[0,1], \\ \langle p\rangle^{|\gamma| n} m_\theta, & \gamma\in[-2,0). \end{cases}$$

Recall the energy estimate and the regularization property of the operator $e^{t\mathcal{L}_{AB}}$ in [6]:

Lemma 3.1. *If u solves the equation*

$$\begin{cases} \partial_t u = \mathcal{L}_{AB} u, \\ u(0,x,p) = u_0(x,p), \end{cases} \tag{3.1}$$

then there exists $C>0$ such that

$$\left\|e^{t\mathcal{L}_{AB}} u_0\right\|_{L^2(m_\theta)} \leq e^{-Ct} \|u_0\|_{L^2(m_\theta)}.$$

Moreover, for $0<t\leq 1$, we have

$$\int |\nabla_p e^{t\mathcal{L}_{AB}} u_0|^2 m_\theta dx dp = O(t^{-1}) \int |u_0|^2 m_1 dx dp,$$

and

$$\int |\nabla_x e^{t\mathcal{L}_{AB}} u_0|^2 m_\theta dx dp = O(t^{-3}) \int |u_0|^2 m_1 dx dp.$$

Utilizing this lemma, we establish the regularity in x as below. Worthy to be mentioned here, the proof for the case $k=1$ is crucial. For the x-derivatives, from Lemma 3.1, $t^{-3/2}$ is not integrable as t is small. It appears to be harmful whenever we use the Duhamel principle. In fact, one can see that the $L_x^2 L_p^2(m_\theta)$ norms of the integrands in (3.8) and (3.10) are integrable in t if we appropriately couple the operator $t\nabla_x$ with $e^{t\mathcal{L}_{AB}}$.

Lemma 3.2. *Let f be a solution of the linearized Landau equation*

$$\begin{cases} \partial_t f + \frac{1}{m_A} p \cdot \nabla_x f = L_{AB} f, \\ f(0,x,p) = f_{in}(x,p). \end{cases} \tag{3.2}$$

Let $k \in \mathbb{N} \cup \{0\}$. Then

(i) for $0 < t \leq 1$

$$\left\|\nabla_x^k f\right\|_{L^2(m_\theta)} \lesssim t^{-\frac{3}{2}k} \|f_{in}\|_{L^2(m_k)}.$$

(ii) For $t \geq 1$,

$$\left\|\nabla_x^k f\right\|_{L^2(m_\theta)} \lesssim \|f_{in}\|_{L^2(m_k)}.$$

Proof. For $k=0$, from Lemma 2.4 and the fact that $\|f\|_{L^2} \leq \|f_{in}\|_{L^2}$, it follows that

$$\frac{d}{dt}\|f\|^2_{L^2(m_\theta)} \lesssim -\|f\|^2_{H^1_\sigma(m_\theta)} + c\|f\|^2_{L^2} \lesssim -\|f\|^2_{L^2(m_\theta)} + c\|f_{in}\|^2_{L^2},$$

which implies that there exists a universal constant $C > 0$ such that

$$\|f\|_{L^2(m_\theta)} \leq C\|f_{in}\|_{L^2(m_\theta)}. \tag{3.3}$$

Moreover, since the operator ∂_x^α is commutative with the Equation (3.2),

$$\|\partial_x^\alpha f(s_2)\|_{L^2(m_\theta)} \leq C\|\partial_x^\alpha f(s_1)\|_{L^2(m_\theta)}, \quad 0 < s_1 < s_2, \tag{3.4}$$

as well.

To prove this lemma for $k \geq 1$, we design a Picard-type iteration, treating K^{AB} as a source term, as below: The zeroth order approximation $f^{(0)}$ is defined by

$$\begin{cases} \partial_t f^{(0)} = \mathcal{L}_{AB} f^{(0)}, \\ f^{(0)}(0,x,p) = f_{in}(x,p), \end{cases} \tag{3.5}$$

and thus the difference $f - f^{(0)}$ satisfies

$$\begin{cases} \partial_t (f - f^{(0)}) = \mathcal{L}_{AB}(f - f^{(0)}) + K^{AB}(f - f^{(0)}) + K^{AB} f^{(0)}, \\ (f - f^{(0)})(0,x,p) = 0. \end{cases}$$

Therefore, we define the first order approximation $f^{(1)}$ by

$$\begin{cases} \partial_t f^{(1)} = \mathcal{L}_{AB} f^{(1)} + K^{AB} f^{(0)}, \\ f^{(1)}(0,x,p) = 0. \end{cases} \tag{3.6}$$

Inductively, we can define the j^{th} order approximation $f^{(j)}$, $j \geq 1$, as

$$\begin{cases} \partial_t f^{(j)} = \mathcal{L}_{AB} f^{(j)} + K^{AB} f^{(j-1)}, \\ f^{(j)}(0,x,p) = 0. \end{cases} \tag{3.7}$$

Now, rewrite f as

$$f=f^{(0)}+f^{(1)}+f^{(2)}+\cdots+f^{(2k)}+\mathcal{R}^{(k)},$$

here the remainder $\mathcal{R}^{(k)}$ solves the equation

$$\begin{cases}\partial_t\mathcal{R}^{(k)}+\frac{1}{m_A}p\cdot\nabla_x\mathcal{R}^{(k)}=L_{AB}\mathcal{R}^{(k)}+K^{AB}f^{(2k)},\\ \mathcal{R}^{(k)}(0,x,p)=0.\end{cases}$$

Under this decomposition, we shall tackle the x-regularity for each component in the cases of $k=1$ and $k=2$. The case of $k\geq 3$ follows the same argument as $k=2$.

Case 1: $k=1$, $f=\sum_{j=0}^{2}f^{(j)}+\mathcal{R}^{(1)}$.

Step 1: The first x-derivative in small time. We claim that for $0<t\leq 1$ and $0\leq j\leq 2$,

$$\|\nabla_x f^{(j)}\|_{L^2(m_\theta)}\lesssim t^{-\frac{3}{2}+j}\|f_{in}\|_{L^2(m_1)},$$

and

$$\|\nabla_x\mathcal{R}^{(1)}\|_{L^2(m_\theta)}\lesssim\|f_{in}\|_{L^2(m_1)}.$$

The estimate of $f^{(0)}$ is obvious from Lemma 3.1. Notice that

$$f^{(1)}=\int_0^t e^{(t-s)\mathcal{L}_{AB}}K^{AB}e^{s\mathcal{L}_{AB}}f_{in}ds.$$

Hence,

$$\begin{aligned}\nabla_x f^{(1)}&=\int_0^t\frac{(t-s)+s}{t}\nabla_x e^{(t-s)\mathcal{L}_{AB}}K^{AB}e^{s\mathcal{L}_{AB}}f_{in}ds\\&=\int_0^t\frac{1}{t}(t-s)\nabla_x e^{(t-s)\mathcal{L}_{AB}}K^{AB}e^{s\mathcal{L}_{AB}}f_{in}ds\\&\quad+\int_0^t\frac{1}{t}e^{(t-s)\mathcal{L}_{AB}}K^{AB}\left(s\nabla_x e^{s\mathcal{L}_{AB}}f_{in}\right)ds.\end{aligned}\tag{3.8}$$

By Lemma 3.1,

$$\begin{aligned}\left\|\nabla_x f^{(1)}\right\|_{L^2(m_\theta)}&\lesssim\int_0^t t^{-1}\left[(t-s)^{-1/2}+s^{-1/2}\right]ds\|f_{in}\|_{L^2(m_1)}\\&\lesssim t^{-1/2}\|f_{in}\|_{L^2(m_1)}.\end{aligned}\tag{3.9}$$

Likewise,

$$f^{(2)}=\int_0^t\int_0^{s_1}e^{(t-s_1)\mathcal{L}_{AB}}K^{AB}e^{(s_1-s_2)\mathcal{L}_{AB}}K^{AB}e^{s_2\mathcal{L}_{AB}}f_{in}ds_2ds_1,$$

and

$$\begin{aligned}\nabla_x f^{(2)}&=\int_0^t\int_0^{s_1}\frac{(s_1-s_2)+s_2}{s_1}\nabla_x e^{(t-s_1)\mathcal{L}_{AB}}K^{AB}e^{(s_1-s_2)\mathcal{L}_{AB}}K^{AB}e^{s_2\mathcal{L}_{AB}}f_{in}ds_2ds_1\\&=\int_0^t\int_0^{s_1}\frac{1}{s_1}e^{(t-s_1)\mathcal{L}_{AB}}K^{AB}\left[(s_1-s_2)\nabla_x e^{(s_1-s_2)\mathcal{L}_{AB}}K^{AB}e^{s_2\mathcal{L}_{AB}}f_{in}\right]ds_2ds_1,\end{aligned}$$

$$+\int_0^t\int_0^{s_1}\frac{1}{s_1}e^{(t-s_1)\mathcal{L}_{AB}}K^{AB}e^{(s_1-s_2)\mathcal{L}_{AB}}K^{AB}\left[s_2\nabla_x e^{s_2\mathcal{L}_{AB}}f_{in}\right]ds_2ds_1. \quad (3.10)$$

By Lemma 3.1 again,

$$\begin{aligned}\left\|\nabla_x f^{(2)}\right\|_{L^2(m_\theta)} &\le \int_0^t\int_0^{s_1}s_1^{-1}\left[(s_1-s_2)^{-1/2}+s_2^{-1/2}\right]ds_2ds_1\|f_{in}\|_{L^2(m_1)}\\ &\lesssim t^{1/2}\|f_{in}\|_{L^2(m_1)}.\end{aligned} \quad (3.11)$$

Here we remark that $\nabla_x f^{(2)}$ is regular when t is close to 0.

For the remainder $\mathcal{R}^{(1)}$, since

$$\nabla_x\mathcal{R}^{(1)}=\int_0^t \mathbb{G}_{\mathbb{AB}}{}^{t-s}K^{AB}\nabla_x f^{(2)}(s)ds,$$

we deduce

$$\left\|\nabla_x\mathcal{R}^{(1)}\right\|_{L^2(m_\theta)}\lesssim\int_0^t\left\|\nabla_x f^{(2)}(s)\right\|_{L^2(m_\theta)}ds\lesssim t^{3/2}\|f_{in}\|_{L^2(m_1)},$$

by (3.3) and (3.11). In conclusion, for $0<t\le 1$,

$$\|\nabla_x f\|_{L^2(m_\theta)}\lesssim t^{-3/2}\|f_{in}\|_{L^2(m_1)}. \quad (3.12)$$

Step 2: The first x-derivative in large time. It readily follows from (3.4) and (3.12) that for $t>1$,

$$\|\nabla_x f(t)\|_{L^2(m_\theta)}\lesssim\|\nabla_x f(1)\|_{L^2(m_\theta)}\lesssim\|f_{in}\|_{L^2(m_1)}.$$

Case 2: $k=2$, $f=\sum_{j=0}^4 f^{(j)}+\mathcal{R}^{(2)}$.

Step 1: The second x-derivative in small time. Let $0<t\le 1$. We shall show that

$$\|\nabla_x^2 f^{(j)}\|_{L^2(m_\theta)}\lesssim t^{-3+j}\|f_{in}\|_{L^2(m_2)},\quad 0\le j\le 4, \quad (3.13)$$

and

$$\|\nabla_x^2\mathcal{R}^{(2)}\|_{L^2(m_\theta)}\lesssim t^2\|f_{in}\|_{L^2(m_2)}.$$

For the first term, we only give the estimates of $f^{(0)}$ and $f^{(1)}$ and the others are similar.

Let $0<t_0\le 1$ and $t_0/2<t\le t_0$. Since

$$\nabla_x f^{(0)}(t)=e^{(t-t_0/2)\mathcal{L}_{AB}}\nabla_x f^{(0)}(t_0/2),$$

we have

$$\left\|\nabla_x^2 f^{(0)}(t)\right\|_{L^2(m_\theta)}\lesssim(t-t_0/2)^{-3/2}(t_0/2)^{-3/2}\|f_{in}\|_{L^2(m_2)},$$

by Lemma 3.1. Taking $t=t_0$ gives

$$\left\|\nabla_x^2 f^{(0)}(t_0)\right\|_{L^2(m_\theta)}\lesssim t_0^{-3}\|f_{in}\|_{L^2(m_2)}. \quad (3.14)$$

Since $t_0 \in (0,1]$ is arbitrary, this completes the proof of $f^{(0)}$. Similarly, we have

$$\nabla_x f^{(1)}(t) = e^{(t-t_0/2)\mathcal{L}_{AB}} \nabla_x f^{(1)}(t_0/2) + \int_{t_0/2}^{t} e^{(t-s)\mathcal{L}_{AB}} K^{AB} \nabla_x f^{(0)}(s) ds,$$

and thus it follows from Lemma 3.1, (3.9) and (3.14) that

$$\left\| \nabla_x^2 f^{(1)}(t) \right\|_{L^2(m_\theta)} \lesssim (t-t_0/2)^{-3/2}(t_0/2)^{-1/2} \|f_{in}\|_{L^2(m_2)} + \int_{t_0/2}^{t} s^{-3} \|f_{in}\|_{L^2(m_2)} ds.$$

Plugging $t = t_0$ into the above inequality yields

$$\left\| \nabla_x^2 f^{(1)}(t_0) \right\|_{L^2(m_\theta)} \lesssim t_0^{-2} \|f_{in}\|_{L^2(m_2)}, \tag{3.15}$$

as desired.

For the remainder $\mathcal{R}^{(2)}$, since

$$\nabla_x^2 \mathcal{R}^{(2)} = \int_0^t \mathbb{G}_{\mathbb{AB}}{}^{t-s} K^{AB} \nabla_x^2 f^{(4)}(s) ds,$$

we deduce

$$\left\| \nabla_x^2 \mathcal{R}^{(2)} \right\|_{L^2(m_\theta)} \lesssim \int_0^t \left\| \nabla_x^2 f^{(4)}(s) \right\|_{L^2(m_\theta)} ds \lesssim t^2 \|f_{in}\|_{L^2(m_2)},$$

by (3.3) and (3.13). Consequently, for $0 < t \le 1$,

$$\left\| \nabla_x^2 f \right\|_{L^2(m_\theta)} \lesssim t^{-3} \|f_{in}\|_{L^2(m_2)}.$$

Step 2: The second x-derivative in large time. For $t > 1$, we have

$$\|\nabla_x^2 f\|_{L^2(m_\theta)} \lesssim \|\nabla_x^2 f(1)\|_{L^2(m_\theta)} \lesssim \|f_{in}\|_{L^2(m_2)},$$

due to (3.4). ∎

We now return to the x-regularity problem for g_S. By Lemma 3.2 (ii) with initial condition $f_{in}(x,p) = g_{in}(x,p)$ and taking $m_\theta = 1$,

$$\left\| \nabla_x^k g \right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_k)}, \quad k \in \mathbb{N} \cup \{0\},$$

for $t \ge 1$. Therefore,

$$\left\| \nabla_x^k g_S \right\|_{L^2} \le \left\| \nabla_x^k g \right\|_{L^2} + \left\| \nabla_x^k g_L \right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_k)},$$

for each $k \in \mathbb{N} \cup \{0\}$. On the other hand, the interpolation inequality says that

$$\left\| \nabla_x^k g_S \right\|_{L^2} \lesssim \left\| \nabla_x^{k-1} g_S \right\|_{L^2}^{1/2} \left\| \nabla_x^{k+1} g_S \right\|_{L^2}^{1/2}, \quad k \in \mathbb{N}.$$

Hence,

$$\left\| \nabla_x^k g_S \right\|_{L^2} \le \begin{cases} \|g_S\|_{L^2}^{1/2} \left\| \nabla_x^2 g_S \right\|_{L^2}^{1/2}, & k = 1 \\ \|g_S\|_{L^2}^{2^{-k}} \left(\prod_{j=2}^{k} \left\| \nabla_x^j g_S \right\|_{L^2}^{2^{-(k-j+2)}} \right) \left\| \nabla_x^{k+1} g_S \right\|_{L^2}^{1/2}, & k \ge 2 \end{cases}$$

$$\lesssim \|g_S\|_{L^2}^{2^{-k}} \|g_{in}\|_{L^2(w_{k+1})}^{1-2^{-k}}$$
$$\lesssim e^{-2^{-k}Ct} \|g_{in}\|_{L^2(w_{k+1})}. \tag{3.16}$$

Combining this with the Sobolev inequality [1, Theorem 5.8], yields

$$\left\|\nabla_x^k g_S\right\|_{L_x^\infty L_p^2} \le \left\|\nabla_x^k g_S\right\|_{L_p^2 L_x^\infty} \lesssim \left\|\nabla_x^k g_S\right\|_{H_x^2 L_p^2}^{3/4} \left\|\nabla_x^k g_S\right\|_{L^2}^{1/4}$$
$$\lesssim \left(e^{-2^{-k}Ct} \|g_{in}\|_{L^2(w_{k+1})}\right)^{1/4} \|g_{in}\|_{L^2(w_{k+2})}^{3/4}$$
$$\lesssim e^{-2^{-k-2}Ct} \|g_{in}\|_{L^2(w_{k+2})}.$$

Summing up, we conclude the following proposition.

Proposition 3.1. *Let $k,\ \ell \in \mathbb{N}\cup\{0\}$. Then for $t \ge 1$,*

$$\left\|\nabla_p^\ell \nabla_x^k g_{L,0}\right\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2},$$

$$\left\|\nabla_x^k g_{L,\perp}\right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \|g_{in}\|_{L_x^1 L_p^2},$$

and

$$\left\|\nabla_x^k g_S\right\|_{L_x^\infty L_p^2} \le e^{-C_k t} \|g_{in}\|_{L^2(w_{k+2})},$$

where $C>0,\ C_k>0$. Here

$$w_n \equiv \begin{cases} 1, & \gamma \in [0,1], \\ \langle p\rangle^{|\gamma| n}, & \gamma \in [-2,0). \end{cases}$$

3.2. Improvement of the p-regularity (general discussion). In this subsection, we will study the p-regularity of the linearized Landau equation

$$\begin{cases} \partial_t f + \dfrac{1}{m_A} p \cdot \nabla_x f = L_{AB} f, \\ f(0,x,p) = f_{in}(x,p). \end{cases} \tag{3.17}$$

3.2.1. Improvement of the p-regularity in finite time. Assume that $0<t\le 1$. In order to improve the p-regularity of f in finite time, we first establish the p-regularity of the operator $e^{t\mathcal{L}_{AB}}$. Specifically, let u be a solution of the equation $\partial_t u = \mathcal{L}_{AB} u$, $u(0,x,p) = u_0(x,p)$, and then we prove that for each $\ell \in \mathbb{N}$, there exists a functional $\mathcal{F}_\ell$ such that

$$t\left\|\nabla_p^\ell u\right\|_{L^2(m_\theta)}^2 \lesssim \mathcal{F}_\ell(0,u), \quad 0 \le t \le 1,$$

where $\mathcal{F}_\ell(0,u)$ involves only the other derivatives of the initial data u_0 with differentiation order less than or equal to ℓ. With the aid of the operator $e^{t\mathcal{L}_{AB}}$ and the Duhamel principle, we use a bootstrap process to get the regularization estimate of f in both the space variable x and momentum variable p simultaneously.

Before proceeding, we derive some useful inequalities to simplify tedious computations regarding the evolutions we will consider later on. Besides, these inequalities clue us in on the form of the desired functional $\mathcal{F}_\ell$.

Lemma 3.3. *Let u be a solution to the equation $\partial_t u = \mathcal{L}_{AB} u$. Then for $k,\ell \in \mathbb{N}\cup\{0\}$, there exist $c_0>0$ independent of k and ℓ, and $C>0$ such that*

$$\frac{d}{dt}\left\|\nabla_x^k u\right\|_{L^2(m_\theta)}^2 \le -c_0\left\|\nabla_x^k u\right\|_{H_\sigma^1(m_\theta)}^2, \tag{3.18}$$

$$\begin{aligned}\frac{d}{dt}\left\|\nabla_p^\ell u\right\|_{L^2(m_\theta)}^2 \le & -c_0\left\|\nabla_p^\ell u\right\|_{H_\sigma^1(m_\theta)}^2\\ &+C\left(\sum_{j=0}^{\ell-1}\left\|\nabla_p^j u\right\|_{H_\sigma^1(m_\theta)}^2+\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x u\right\|_{L^2(m_\theta)}^2+\|u\|_{L^2}^2\right),\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\frac{d}{dt}\left\|\nabla_p^\ell\nabla_x^k u\right\|_{L^2(m_\theta)}^2 \le & -c_0\left\|\nabla_p^\ell\nabla_x^k u\right\|_{H_\sigma^1(m_\theta)}^2\\ &+C\left(\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x^k u\right\|_{H_\sigma^1(m_\theta)}^2+\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x^{k+1} u\right\|_{L^2(m_\theta)}^2+\left\|\nabla_x^k u\right\|_{L^2}^2\right).\end{aligned} \tag{3.20}$$

Proof. Let α and β be any multi-indexes. Direct computation shows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u\|_{L^2(m_\theta)}^2 =& \frac{1}{2}\int\nabla_x\cdot\left(-\frac{p}{m_A}|\partial_x^\alpha u|^2\right)m_\theta dxdp\\ &+\int\left(\partial_x^\alpha u\nabla_p\cdot\left[\sigma^{AB}\nabla_p\partial_x^\alpha u\right]-\varphi(p)|\partial_x^\alpha u|^2\right)m_\theta dxdp\\ =&-\int\left(\sigma^{AB}\nabla_p\partial_x^\alpha u,\nabla_p\partial_x^\alpha u\right)m_\theta dxdp\\ &-\int\left[\varphi(p)|\partial_x^\alpha u|^2 m_\theta+\frac{1}{2}\left(\sigma^{AB}\nabla_p|\partial_x^\alpha u|^2,\nabla_p m_\theta\right)\right]dxdp\\ =&-\int\left(\sigma^{AB}\nabla_p\partial_x^\alpha u,\nabla_p\partial_x^\alpha u\right)m_\theta dxdp\\ &-\int\left[\varphi(p)-\frac{1}{2m_\theta}\sum_{i,j=1}^3\left(\partial_j\sigma_{ij}^{AB}\partial_i m_\theta+\sigma_{ij}^{AB}\partial_{ij}^2 m_\theta\right)\right]|\partial_x^\alpha u|^2 m_\theta dxdp,\end{aligned}$$

where $\partial_j=\partial_{p_j}$, $\partial_{ij}^2=\partial_{p_ip_j}^2$ and

$$\varphi(p)=\frac{1}{4}\frac{m_B^2}{m_A^2}\left(\sigma^{AB}p,p\right)-\frac{1}{2}\frac{m_B}{m_A}\nabla_p\cdot\left(\sigma^{AB}p\right)+\varpi\chi_R.$$

By Lemma 2.2 (ii) and (iii),

$$\left|\frac{1}{2m_\theta}\sum_{i,j=1}^3\left(\partial_j\sigma_{ij}^{AB}\partial_i m_\theta+\sigma_{ij}^{AB}\partial_{ij}m_\theta\right)\right|\le c\langle p\rangle^\gamma,$$

which implies that

$$\varphi(p)-\frac{1}{2m_\theta}\sum_{i,j=1}^3\left(\partial_j\sigma_{ij}^{AB}\partial_i m_\theta+\sigma_{ij}^{AB}\partial_{ij}m_\theta\right)$$

$$
\begin{aligned}
&\geq \frac{1}{4}\lambda_1^{AB}(p)\left(\frac{m_B}{m_A}|p|\right)^2+\frac{1}{2}\lambda_1^{AB}(p)\frac{m_B}{m_A}\left(\frac{m_B}{m_A}|p|\right)^2\\
&\qquad -\frac{m_B}{m_A}\lambda_2^{AB}(p)-\frac{1}{2}\frac{m_B}{m_A}\lambda_1^{AB}(p)+\varpi\chi_R-c\langle p\rangle^{\gamma}\\
&\geq c_1\langle p\rangle^{\gamma+2},
\end{aligned}
$$

provided ϖ and R are sufficiently large. Owing to Lemma 2.2 (ii), there exists $c_2>0$ independent of α such that

$$
\frac{d}{dt}\|\partial_x^\alpha u\|^2_{L^2(m_\theta)}\leq -c_2\|\partial_x^\alpha u\|^2_{H^1_\sigma(m_\theta)},
$$

as required.

For $|\alpha|\geq 1$, compute the evolution

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\partial_p^\alpha u\right\|^2_{L^2(m_\theta)}\\
&=\int \partial_p^\alpha u\partial_p^\alpha\left(-\frac{p}{m_A}\cdot\nabla_x u-\widetilde{\Lambda}^{AB}u-\varpi\chi_R u\right)m_\theta dxdp\\
&=\frac{1}{2}\int\nabla_x\cdot\left(-\frac{p}{m_A}\left|\partial_p^\alpha u\right|^2\right)m_\theta dxdp\\
&\quad -\int\left[\sum_{\beta<\alpha,\ |\beta|=|\alpha|-1}C_\alpha^\beta\partial_p^{\alpha-\beta}\left(\frac{p}{m_A}\right)\cdot\nabla_x\partial_p^\beta u\right]\partial_p^\alpha u m_\theta dxdp\\
&\quad -\int\left(\partial_p^\alpha\left(\widetilde{\Lambda}^{AB}u\right)\right)\partial_p^\alpha u m_\theta dxdp-\varpi\int\partial_p^\alpha u\partial_p^\alpha(\chi_R u)m_\theta dxdp\\
&=-\int\left[\sum_{\beta<\alpha,\ |\beta|=|\alpha|-1}C_\alpha^\beta\partial_p^{\alpha-\beta}\left(\frac{p}{m_A}\right)\cdot\nabla_x\partial_p^\beta u\right]\partial_p^\alpha u m_\theta dxdp\\
&\quad -\int\left(\partial_p^\alpha\left(\widetilde{\Lambda}^{AB}u\right)\right)\partial_p^\alpha u m_\theta dxdp\\
&\quad -\varpi\int\chi_R\left|\partial_p^\alpha u\right|^2 m_\theta dxdp-\varpi\int\sum_{\beta<\alpha}C_\alpha^\beta\partial_p^\alpha u\partial_p^{\alpha-\beta}\chi_R\partial_p^\beta u m_\theta dxdp.
\end{aligned}
$$

By choosing $0<\delta\ll 1$ and the Cauchy inequality,

$$
\begin{aligned}
&\left|\int\left[\sum_{\substack{\beta<\alpha,\\ |\beta|=|\alpha|-1}}C_\alpha^\beta\partial_p^{\alpha-\beta}\left(\frac{p}{m_A}\right)\cdot\nabla_x\partial_p^\beta u\right]\partial_p^\alpha u m_\theta dxdp\right|\\
&\leq\delta\left\|\partial_p^\alpha u\right\|^2_{L^2(m_\theta)}+C_\delta\sum_{|\overline{\alpha}|<|\alpha|}\left\|\nabla_x\partial_p^{\overline{\alpha}}u\right\|^2_{L^2(m_\theta)},
\end{aligned}
$$

and

$$
\left|\varpi\int\sum_{\beta<\alpha}C_\alpha^\beta\partial_p^\alpha u\partial_p^{\alpha-\beta}\chi_R\partial_p^\beta u m_\theta dxdp\right|\leq\delta\left\|\partial_p^\alpha u\right\|^2_{L^2(m_\theta)}+C_\delta\sum_{|\overline{\alpha}|<|\alpha|}\left\|\partial_p^{\overline{\alpha}}u\right\|^2_{L^2(m_\theta)}.
$$

From Lemma 2.4,

$$
\begin{aligned}
&-\int \left(\partial_p^\alpha\left(\widetilde{\Lambda}^{AB}u\right)\right)\partial_p^\alpha u m_\theta dxdp\\
&\leq -c_1\left\|\partial_p^\alpha u\right\|^2_{H^1_\sigma(m_\theta)}+\delta\sum_{|\overline{\alpha}|\leq|\alpha|}\left\|\partial_p^{\overline{\alpha}}u\right\|^2_{H^1_\sigma(m_\theta)}+C_\delta\|\chi_R u\|^2_{L^2}.
\end{aligned}
$$

Together with $\left\|\partial_p^\alpha u\right\|^2_{L^2(m_\theta)}\leq\left\|\partial_p^\alpha u\right\|^2_{H^1_\sigma(m_\theta)}$, we therefore have

$$
\begin{aligned}
\frac{d}{dt}\left\|\nabla_p^\ell u\right\|^2_{L^2(m_\theta)}\leq &-\frac{c_1}{2}\left\|\partial_p^\ell u\right\|^2_{H^1_\sigma(m_\theta)}\\
&+C\left(\sum_{j=0}^{\ell-1}\left\|\nabla_p^j u\right\|^2_{H^1_\sigma(m_\theta)}+\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x u\right\|^2_{L^2(m_\theta)}+\|u\|^2_{L^2}\right),
\end{aligned}
$$

for $\ell\in\mathbb{N}$. Since ∂_x^α is commutative with $\mathcal{L}_{AB}$, (3.20) is a consequence of (3.19). ∎

REMARK 3.1. The simplified case that there is no weight function and no p-derivatives has been considered by Mouhot and Neumann in [16].

Now, we embark on the p-regularity estimate for $e^{t\mathcal{L}_{AB}}$ and f in turn. For clarification, we first elaborate our procedure in the cases of $\ell=1$ and $\ell=2$. For general ℓ, we provide the explicit form of the desired functional $\mathcal{F}_\ell$ and complete the proof inductively on ℓ.

Step 1: The estimate of $\nabla_p\nabla_x^k f$ (i.e., $\ell=1$). Define the functional

$$
\mathcal{F}_1(t,u)\equiv\int u^2 m_1 dxdp+\int|\nabla_x u|^2 m_\theta dxdp+\kappa t\int|\nabla_p u|^2 m_\theta dxdp.
$$

In view of Lemma 3.3,

$$
\frac{d}{dt}\int u^2 m_1 dxdp\lesssim-\|u\|^2_{H^1_\sigma(m_1)},\qquad \frac{d}{dt}\int|\nabla_x u|^2 m_\theta dxdp\lesssim-\|\nabla_x u\|^2_{H^1_\sigma(m_\theta)},
$$

$$
\frac{d}{dt}\int|\nabla_p u|^2 m_\theta dxdp\lesssim-\|\nabla_p u\|^2_{H^1_\sigma(m_\theta)}+C\left(\|\nabla_x u\|^2_{H^1_\sigma(m_\theta)}+\|u\|^2_{H^1_\sigma(m_\theta)}\right).
$$

Collecting terms gives

$$
\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t,u)\lesssim&\|u\|^2_{H^1_\sigma(m_1)}(-1+C\kappa t)+C\kappa\int|\nabla_p u|^2 m_\theta dxdp\\
&+\|\nabla_x u\|^2_{H^1_\sigma(m_\theta)}(-1+C\kappa t)+\|\nabla_p u\|^2_{H^1_\sigma(m_\theta)}(-\kappa t).
\end{aligned}
$$

Let $\kappa=\varepsilon^2$. Choosing $\varepsilon>0$ sufficiently small, and noting that the second term can be dominated by the first one from the definition of $H^1_\sigma(m_1)$, we thereby obtain

$$
\frac{d}{dt}\mathcal{F}_1(t,u)<0\quad\text{for}\quad 0<t<1.
$$

It implies that for $0\leq t\leq 1$,

$$
t\|\nabla_p u(t)\|^2_{L^2(m_\theta)}\lesssim\mathcal{F}_1(0,u)=\|u(0)\|^2_{L^2(m_1)}+\|\nabla_x u(0)\|^2_{L^2(m_\theta)}. \tag{3.21}
$$

Next, we return to the estimate for f. Let $0<t_0\leq 1$ be given. By Duhamel's principle,

$$f(t)=e^{(t-t_0/2)\mathcal{L}_{AB}}f(t_0/2)+\int_{t_0/2}^{t}e^{(t-s)\mathcal{L}_{AB}}K^{AB}f(s)\,ds,$$

for $0<t_0/2<t\leq 1$. Hence, we have

$$\begin{aligned}\|\nabla_p f(t)\|_{L^2(m_\theta)}\leq &C(t-t_0/2)^{-\frac{1}{2}}\left(\|f(t_0/2)\|_{L^2(m_1)}+\|\nabla_x f(t_0/2)\|_{L^2(m_\theta)}\right)\\&+C\int_{t_0/2}^{t}(t-s)^{-\frac{1}{2}}\left(\|f(s)\|_{L^2(m_1)}+\|\nabla_x f(s)\|_{L^2(m_\theta)}\right)ds\\\leq &C(t-t_0/2)^{-\frac{1}{2}}t_0^{-\frac{3}{2}}\|f_{in}\|_{L^2(m_1)},\end{aligned}\tag{3.22}$$

due to Lemma 3.2 (i), (3.3) and (3.21). Consequently,

$$\|\nabla_p f(t)\|_{L^2(m_\theta)}\lesssim t^{-\frac{1}{2}-\frac{3}{2}}\|f_{in}\|_{L^2(m_1)},\qquad 0<t\leq 1.$$

On the other hand, since ∂_x^α commutes with the Equation (3.17), we can improve the mixed regularity simultaneously. Precisely, replacing f by $\partial_x^\alpha f$ in (3.22) gives

$$\begin{aligned}\left\|\nabla_p\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq &C(t-t_0/2)^{-\frac{1}{2}}\left(\left\|\nabla_x^k f(t_0/2)\right\|_{L^2(m_1)}+\left\|\nabla_x^{k+1}f(t_0/2)\right\|_{L^2(m_\theta)}\right)\\&+C\int_{t_0/2}^{t}(t-s)^{-\frac{1}{2}}\left(\left\|\nabla_x^k f(s)\right\|_{L^2(m_1)}+\left\|\nabla_x^{k+1}f(s)\right\|_{L^2(m_\theta)}\right)ds.\end{aligned}$$

By Lemma 3.2 (i),

$$\left\|\nabla_p\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq Ct^{-\frac{1}{2}-\frac{3}{2}(k+1)}\|f_{in}\|_{L^2(m_{k+1})}.\tag{3.23}$$

Step 2: The estimate of $\nabla_p^2\nabla_x^k f$ (i.e., $\ell=2$). Define the functional

$$\begin{aligned}\mathcal{F}_2(t,u)\equiv&\int(u^2+|\nabla_x u|^2+\kappa_1|\nabla_p u|^2)m_1\,dxdp\\&+\int(\left|\nabla_x^2 u\right|^2+\kappa_2|\nabla_p\nabla_x u|^2)m_\theta\,dxdp\\&+\kappa_3 t\int\left|\nabla_p^2 u\right|^2m_\theta\,dxdp.\end{aligned}$$

By Lemma 3.3,

$$\frac{d}{dt}\int(u^2+|\nabla_x u|^2)m_1\,dxdp\lesssim-\|u\|^2_{H^1_\sigma(m_1)}-\|\nabla_x u\|^2_{H^1_\sigma(m_1)},$$

$$\frac{d}{dt}\int|\nabla_p u|^2m_1\,dxdp\lesssim-\|\nabla_p u\|^2_{H^1_\sigma(m_1)}+C\left(\|\nabla_x u\|^2_{H^1_\sigma(m_1)}+\|u\|^2_{H^1_\sigma(m_1)}\right),$$

$$\frac{d}{dt}\int\left|\nabla_x^2 u\right|^2m_\theta\,dxdp\lesssim-\|\nabla_x^2 u\|^2_{H^1_\sigma(m_\theta)},$$

$$\frac{d}{dt}\int|\nabla_p\nabla_x u|^2m_\theta\,dxdp\lesssim-\|\nabla_p\nabla_x u\|^2_{H^1_\sigma(m_\theta)}+C\left(\|\nabla_x^2 u\|^2_{H^1_\sigma(m_\theta)}+\|\nabla_x u\|^2_{H^1_\sigma(m_\theta)}\right),$$

$$\frac{d}{dt}\int\left|\nabla_p^2 u\right|^2 m_\theta dxdp$$
$$\lesssim -\|\nabla_p^2 u\|^2_{H^1_\sigma(m_\theta)}+C\left(\|\nabla_p\nabla_x u\|^2_{H^1_\sigma(m_\theta)}+\|\nabla_x u\|^2_{H^1_\sigma(m_\theta)}+\|\nabla_p u\|^2_{H^1_\sigma(m_\theta)}+\|u\|^2_{H^1_\sigma(m_\theta)}\right).$$

Hence,

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t,u)\lesssim&\|u\|^2_{H^1_\sigma(m_1)}(-1+C\kappa_1+C\kappa_3 t)+\|\nabla_x u\|^2_{H^1_\sigma(m_1)}(-1+C\kappa_1+C\kappa_2+C\kappa_3 t)\\ &+\|\nabla_p u\|^2_{H^1_\sigma(m_1)}(-\kappa_1+C\kappa_3 t)+C\kappa_3\|\nabla_p^2 u\|^2_{L^2(m_\theta)}\\ &+\|\nabla_x^2 u\|^2_{H^1_\sigma(m_\theta)}(-1+C\kappa_2)+\|\nabla_p\nabla_x u\|^2_{H^1_\sigma(m_\theta)}(-\kappa_2+C\kappa_3 t)\\ &+\|\nabla_p^2 u\|^2_{H^1_\sigma(m_\theta)}(-\kappa_3 t).\end{aligned}$$

Set $\kappa_i=\varepsilon^i$. Choosing $\varepsilon>0$ sufficiently small, we find

$$\frac{d}{dt}\mathcal{F}_2(t,u)\le 0,\quad \text{for}\quad 0<t<1,$$

so that

$$t\left\|\nabla_p^2 u(t)\right\|^2_{L^2(m_\theta)}\lesssim\mathcal{F}_2(0,u)=\|u(0)\|^2_{H^1(m_1)}+\left\|\nabla_x^2 u(0)\right\|^2_{L^2(m_\theta)}+\|\nabla_p\nabla_x u(0)\|^2_{L^2(m_\theta)},\tag{3.24}$$

for $0\le t\le 1$.

Let $0<t_0\le 1$. By Duhamel's principle,

$$f(t)=e^{(t-t_0/2)\mathcal{L}_{AB}}f(t_0/2)+\int_{t_0/2}^t e^{(t-s)\mathcal{L}_{AB}}K^{AB}f(s)\,ds,$$

for $0<t_0/2<t\le 1$. Using Lemma 3.2, (3.3),(3.23) and (3.24), we obtain

$$\begin{aligned}&\left\|\nabla_p^2 f(t)\right\|_{L^2(m_\theta)}\\ \le& C(t-t_0/2)^{-\frac{1}{2}}\left(\|f(t_0/2)\|_{H^1(m_1)}+\left\|\nabla_x^2 f(t_0/2)\right\|_{L^2(m_\theta)}+\|\nabla_p\nabla_x f(t_0/2)\|_{L^2(m_\theta)}\right)\\ &+C\int_{t_0/2}^t(t-s)^{-\frac{1}{2}}\left(\|f(s)\|_{H^1(m_1)}+\left\|\nabla_x^2 f(s)\right\|_{L^2(m_\theta)}+\|\nabla_p\nabla_x f(s)\|_{L^2(m_\theta)}\right)ds\\ \le& C(t-t_0/2)^{-\frac{1}{2}}t_0^{-\frac{1}{2}-\frac{3}{2}\times 2}\|f_{in}\|_{L^2(m_2)}.\end{aligned}\tag{3.25}$$

Namely,

$$\left\|\nabla_p^2 f(t)\right\|_{L^2(m_\theta)}\lesssim t^{-\frac{2}{2}-\frac{3}{2}\times 2}\|f_{in}\|_{L^2(m_2)},\qquad 0<t\le 1.$$

Again, since ∂_x^α is commutative with the Equation (3.17), replacing f by $\partial_x^\alpha f$ in (3.25) gives

$$\begin{aligned}&\left\|\nabla_p^2\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\\ \le& C(t-t_0/2)^{-\frac{1}{2}}\Bigg(\left\|\nabla_x^k f(t_0/2)\right\|_{H^1(m_1)}\\ &+\left\|\nabla_x^{k+2}f(t_0/2)\right\|_{L^2(m_\theta)}+\left\|\nabla_p\nabla_x^{k+1}f(t_0/2)\right\|_{L^2(m_\theta)}\Bigg)\end{aligned}$$

$$+C\int_{t_0/2}^{t}(t-s)^{-\frac{1}{2}}\left(\left\|\nabla_x^k f(s)\right\|_{H^1(m_1)}+\left\|\nabla_x^{k+2} f(s)\right\|_{L^2(m_\theta)}+\left\|\nabla_p\nabla_x^{k+1} f(s)\right\|_{L^2(m_\theta)}\right)ds$$
$$\leq C(t-t_0/2)^{-\frac{1}{2}}t_0^{-\frac{1}{2}-\frac{3}{2}(k+2)}\|f_{in}\|_{L^2(m_{k+2})}.$$

That is,

$$\left\|\nabla_p^2\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq Ct^{-\frac{2}{2}-\frac{3}{2}(k+2)}\|f_{in}\|_{L^2(m_{k+2})}.$$

Step 3: The estimate of $\nabla_p^\ell\nabla_x^k f$. For general $\ell\in\mathbb{N}$, consider the functional

$$\mathcal{F}_\ell(t,u)=\int\left(\sum_{j=0}^{\ell-1}\left|\nabla_x^j u\right|^2+\sum_{j=1}^{\ell-1}\sum_{q=1}^{j}\varepsilon^q\left|\nabla_p^q\nabla_x^{j-q}u\right|^2\right)m_1 dxdp$$
$$+\int\left(\left|\nabla_x^\ell u\right|^2+\sum_{j=1}^{\ell-1}\varepsilon^{j+1}\left|\nabla_p^j\nabla_x^{\ell-j}u\right|^2\right)m_\theta dxdp$$
$$+\varepsilon^{\ell+1}t\int\left|\nabla_p^\ell u\right|^2 m_\theta dxdp,$$

for $\varepsilon>0$ sufficiently small. Applying the above argument inductively on ℓ, we obtain high order p-regularity in small time and conclude our result as below.

PROPOSITION 3.2. *Let k, $\ell\in\mathbb{N}\cup\{0\}$. Then for $0<t\leq 1$,*

$$\left\|\nabla_p^\ell\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq Ct^{-\frac{\ell}{2}-\frac{3}{2}(k+\ell)}\|f_{in}\|_{L^2(m_{k+\ell})}.$$

3.2.2. Improvement of the p regularity in large time. In what follows, we establish the p-regularity in large time through the Grönwall-type inequalities.

PROPOSITION 3.3. *Let f be a solution to Equation (3.17) and let k, $\ell\in\mathbb{N}\cup\{0\}$. Then for $t\geq 1$,*

$$\left\|\nabla_p^\ell\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq C\|f_{in}\|_{L^2(m_{k+\ell})},$$

the constant C depending only upon k and ℓ.

Proof. Let $n\in\mathbb{N}$. Define

$$H^n[f](t)=\sum_{j=0}^{n}\left\|\nabla_p^j f(t)\right\|_{L^2(m_\theta)}^2,\quad H_x^n[f](t)=\sum_{j=0}^{n}\left\|\nabla_p^j\nabla_x f(t)\right\|_{L^2(m_\theta)}^2.$$

We first claim that

$$\frac{d}{dt}H^n[f](t)\leq -cH^n[f](t)+C\left(H_x^{n-1}[f](t)+\|f(t)\|_{L^2}^2\right),$$

for some constants $c>0$ small and $C>0$ large. To confirm this, in view of Lemma 2.4, we find

$$\frac{d}{dt}\left\|\nabla_p^\ell f\right\|_{L^2(m_\theta)}^2\lesssim-\left\|\nabla_p^\ell f\right\|_{H_\sigma^1(m_\theta)}^2$$
$$+C\left[\sum_{j=0}^{\ell-1}\left\|\nabla_p^j f\right\|_{H_\sigma^1(m_\theta)}^2+\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x f\right\|_{L^2(m_\theta)}^2+\|f\|_{L^2}^2\right],$$

so that

$$\frac{d}{dt}\sum_{j=0}^{n}\varepsilon^j\left\|\nabla_p^j f\right\|_{L^2(m_\theta)}^2, \lesssim -\varepsilon^n\left\|\nabla_p^n f\right\|_{H^1_\sigma(m_\theta)}^2-\sum_{j=0}^{n-1}\left[\varepsilon^j-C\left(\varepsilon^{j+1}+\cdots+\varepsilon^n\right)\right]\left\|\nabla_p^j f\right\|_{H^1_\sigma(m_\theta)}^2$$
$$+nC\left(\sum_{j=0}^{n-1}\left\|\nabla_p^j\nabla_x f\right\|_{L^2(m_\theta)}^2+\|f\|_{L^2}^2\right)$$
$$\lesssim -\varepsilon^n\left\|\nabla_p^n f\right\|_{L^2(m_\theta)}^2-\sum_{j=0}^{n-1}\varepsilon^j\left[1-nC\varepsilon\right]\left\|\nabla_p^j f\right\|_{L^2(m_\theta)}^2$$
$$+nC\left(H_x^{n-1}[f](t)+\|f(t)\|_{L^2}^2\right),$$

whenever we choose $0<\varepsilon\ll 1$ with $1-nC\varepsilon\geq 1/2$. Hence,

$$\frac{d}{dt}H^n[f](t)\lesssim -\left\|\nabla_p^n f\right\|_{L^2(m_\theta)}^2-\sum_{j=0}^{n-1}\frac{\varepsilon^{j-n}}{2}\left\|\nabla_p^j f\right\|_{L^2(m_\theta)}^2+\frac{nC}{\varepsilon^n}\left(H_x^{n-1}[f](t)+\|f(t)\|_{L^2}^2\right)$$
$$\lesssim -H^n[f](t)+C'\left(H_x^{n-1}[f](t)+\|f(t)\|_{L^2}^2\right),$$

as required. Now, for $t\geq 1$, we have

$$H^n[f](t)\leq e^{-c(t-1)}H^n[f](1)+C\int_1^t e^{-c(t-s)}\left(H_x^{n-1}[f](s)+\|f(s)\|_{L^2}^2\right)ds$$
$$\leq e^{-c(t-1)}\sum_{j=0}^{n}\left\|\nabla_p^j f(1)\right\|_{L^2(m_\theta)}^2+C'\left(1-e^{-c(t-1)}\right)\|f(0)\|_{L^2}^2$$
$$+C\int_1^t e^{-c(t-s)}H_x^{n-1}[f](s)\,ds$$
$$\lesssim \|f_{in}\|_{L^2(m_n)}^2+\int_1^t e^{-c(t-s)}H_x^{n-1}[f](s)\,ds,$$

the last inequality being true due to Proposition 3.2.

Next, we shall claim that

$$H_x^{n-1}[f](t)\leq C\|f_{in}\|_{L^2(m_n)}^2\ ,\quad t\geq 1,$$

for some constant $C>0$ depending only upon n. When $n=1$, it is from Lemma 3.2 (ii). When $n\geq 2$, consider the functional

$$\mathcal{A}_N(t,f)\equiv\int\left(\sum_{j=1}^{N+1}\left|\nabla_x^j f(t)\right|^2+\sum_{q=1}^{N}\sum_{j=1}^{N+1-q}\varepsilon^{j+N-q}\left|\nabla_p^j\nabla_x^q f(t)\right|^2\right)m_\theta\,dx\,dp,$$

for $\varepsilon>0$ sufficiently small. By Lemma 2.4 again, we have

$$\frac{d}{dt}\left\|\nabla_x^k f\right\|_{L^2(m_\theta)}^2\lesssim -\left\|\nabla_x^k f\right\|_{H^1_\sigma(m_\theta)}^2+C\left\|\nabla_x^k f\right\|_{L^2}^2,$$

and

$$\frac{d}{dt}\left\|\nabla_p^\ell\nabla_x^k f\right\|_{L^2(m_\theta)}^2\lesssim -\left\|\nabla_p^\ell\nabla_x^k f\right\|_{H^1_\sigma(m_\theta)}^2$$
$$+C\left[\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x^k f\right\|_{H^1_\sigma(m_\theta)}^2+\sum_{j=0}^{\ell-1}\left\|\nabla_p^j\nabla_x^{k+1} f\right\|_{L^2(m_\theta)}^2+\left\|\nabla_x^k f\right\|_{L^2}^2\right].$$

Hence, following the same argument as $H^n[f]$, we deduce

$$\frac{d}{dt}\mathcal{A}_N(t,f) \le -c\mathcal{A}_N(t,f) + C\sum_{j=1}^{N+1}\left\|\nabla_x^j f(t)\right\|_{L^2}^2,$$

for some constants $c>0$ small and $C>0$ large as well. Therefore, for $t\ge 1$

$$\begin{aligned}\mathcal{A}_N(t,f) &\le e^{-c(t-1)}\mathcal{A}_N(1,f) + C\int_1^t e^{-c(t-s)}\sum_{j=1}^{N+1}\left\|\nabla_x^j f(s)\right\|_{L^2}^2 ds\\ &\le e^{-c(t-1)}\mathcal{A}_N(1,f) + C'\int_1^t e^{-c(t-s)}\sum_{j=1}^{N+1}\left\|\nabla_x^j f(1)\right\|_{L^2}^2 ds\\ &\lesssim \mathcal{A}_N(1,f) + \sum_{j=1}^{N+1}\left\|\nabla_x^j f(1)\right\|_{L^2}^2 \lesssim \|f_{in}\|_{L^2(m_{N+1})}^2,\end{aligned}$$

due to Proposition 3.2. It implies that for $t\ge 1$,

$$H_x^{n-1}[f](t) \lesssim \mathcal{A}_{n-1}(t,f) \lesssim \|f_{in}\|_{L^2(m_n)}^2.$$

As a consequence, we have

$$H^n[f](t) \lesssim \|f_{in}\|_{L^2(m_n)}^2, \qquad H_x^n[f](t) \lesssim \|f_{in}\|_{L^2(m_{n+1})}^2.$$

Therefore, for $\ell\in\mathbb{N}$,

$$\left\|\nabla_p^\ell f(t)\right\|_{L^2(m_\theta)} \lesssim \|f_{in}\|_{L^2(m_\ell)}, \qquad t\ge 1.$$

In fact, it holds that for any k, $\ell\in\mathbb{N}$

$$\left\|\nabla_p^\ell\nabla_x^k f(t)\right\|_{L^2(m_\theta)}^2 \lesssim \mathcal{A}_{k+\ell-1}(t,f) \lesssim \|f_{in}\|_{L^2(m_{k+\ell})}^2, \quad t\ge 1.$$

∎

3.3. Proof of Theorem 3.1. Recall that

$$g_{L,0} = \int_{|\eta|<\delta} e^{\lambda(\eta)t}e^{i\eta\cdot x}\left\langle e_D(-\eta),\hat{g_{in}}\right\rangle_p e_D(\eta)d\eta,$$

Its regularity has been done in Proposition 3.1. Therefore, it remains to demonstrate that for any k, $\ell\in\mathbb{N}\cup\{0\}$,

$$\left\|\nabla_p^\ell\nabla_x^k g_{L,\perp}\right\|_{L_x^\infty L_p^2} \lesssim e^{-2^{-(\ell+2)}Ct}\left(\|g_{in}\|_{L^2(w_{\ell+1})} + \|g_{in}\|_{L_x^1L_p^2}\right)$$

and

$$\left\|\nabla_p^\ell\nabla_x^k g_S\right\|_{L_x^\infty L_p^2} \lesssim e^{-C_{k,\ell}t}\|g_{in}\|_{L^2(w_{k+\ell+2})},$$

whenever $t\ge 1$. Here $C_{k,\ell}>0$ depends only upon k and ℓ. Among them, the case in which $k\in\mathbb{N}\cup\{0\}$ and $\ell=0$ is a consequence of Proposition 3.1 and thus we may assume that $k\in\mathbb{N}\cup\{0\}$ and $\ell\in\mathbb{N}$ in the following discussion.

Firstly, we prove that for $t \geq 1$, the L^2-norms of any derivatives of g_L and g_S are bounded above uniformly in t by multiples of certain weighted L^2-norms of the initial data g_{in}. To see this, note that the long wave part g_L satisfies

$$\begin{cases} \partial_t g_L + \dfrac{1}{m_A} p \cdot \nabla_x g_L = L_{AB} g_L, & (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ g_L(0,x,p) = (g_L)_{in}(x,p), \end{cases} \tag{3.26}$$

and the short wave part g_S satisfies

$$\begin{cases} \partial_t g_S + \dfrac{1}{m_A} p \cdot \nabla_x g_S = L_{AB} g_S, & (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ g_S(0,x,p) = (g_S)_{in}(x,p), \end{cases} \tag{3.27}$$

respectively, where

$$(g_L)_{in}(x,p) = \int_{|\eta|<\delta} e^{i\eta \cdot x} \hat{g_{in}}(\eta,p)\, d\eta, \qquad (g_S)_{in}(x,p) = \int_{|\eta|\geq\delta} e^{i\eta \cdot x} \hat{g_{in}}(\eta,p)\, d\eta.$$

Hence, from Proposition 3.3 (with $m_\theta = 1$), it readily follows that

$$\left\|\nabla_p^\ell \nabla_x^k g_L(t)\right\|_{L^2},\ \left\|\nabla_p^\ell \nabla_x^k g_S(t)\right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_{k+\ell})}.$$

However,

$$\left\|\nabla_x^k g_L(t)\right\|_{L^2} \lesssim \|g_{in}\|_{L^2}$$

for all $k \in \mathbb{N} \cup \{0\}$. In the course of the proof of Propositions 3.2 and 3.3, we can improve the regularity of g_L to obtain

$$\left\|\nabla_p^\ell \nabla_x^k g_L(t)\right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_\ell)},\ t \geq 1.$$

In sum, we summarize the L^2-norm bounds for derivatives as the following:

Proposition 3.4. *Let $k, \ell \in \mathbb{N} \cup \{0\}$. Then for $t \geq 1$*

$$\left\|\nabla_p^\ell \nabla_x^k g_L(t)\right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_\ell)},$$

and

$$\left\|\nabla_p^\ell \nabla_x^k g_S(t)\right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_{k+\ell})}.$$

Thereupon, write $g_{L,\perp} = g_L - g_{L,0}$ and then we find

$$\begin{aligned} \left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}\right\|_{L^2} &\leq \left\|\nabla_p^\ell \nabla_x^k g_L\right\|_{L^2} + \left\|\nabla_p^\ell \nabla_x^k g_{L,0}\right\|_{L^2} \\ &\lesssim \|g_{in}\|_{L^2(w_\ell)} + (1+t)^{-(3+2k)/4} \|g_{in}\|_{L^1_x L^2_p} \\ &\lesssim \|g_{in}\|_{L^2(w_\ell)} + \|g_{in}\|_{L^1_x L^2_p} \end{aligned} \tag{3.28}$$

for $t \geq 1$, and $k,\ \ell \in \mathbb{N} \cup \{0\}$.

Further, to obtain the decay rate, we employ the Sobolev inequality and the interpolation inequality to attain this end. Specifically, by the Sobolev inequality [1], we find

$$\left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}\right\|_{L_x^\infty L_p^2} \lesssim \left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}\right\|_{H_x^2 L_p^2}^{3/4} \left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}\right\|_{L^2}^{1/4}.$$

On the other hand, the interpolation inequality says that

$$\left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}\right\|_{L^2} \lesssim \begin{cases} \left\|\nabla_x^k g_{L,\perp}\right\|_{L^2}^{1/2} \left\|\nabla_p^2 \nabla_x^k g_{L,\perp}\right\|_{L^2}^{1/2}, & \ell = 1 \\ \left\|\nabla_x^k g_{L,\perp}\right\|_{L^2}^{2^{-\ell}} \left(\prod_{j=2}^{\ell} \left\|\nabla_p^j \nabla_x^k g_{L,\perp}\right\|_{L^2}^{2^{-(\ell-j+2)}}\right) \left\|\nabla_p^{\ell+1} \nabla_x^k g_{L,\perp}\right\|_{L^2}^{1/2}, & \ell \geq 2 \end{cases}$$

and

$$\begin{aligned} \left\|\nabla_x^k g_{L,\perp}\right\|_{L^2} &\lesssim \begin{cases} \|g_{L,\perp}\|_{L^2}^{1/2} \left\|\nabla_x^2 g_{L,\perp}\right\|_{L^2}^{1/2}, & k = 1 \\ \|g_{L,\perp}\|_{L^2}^{2^{-k}} \left(\prod_{j=2}^{k} \left\|\nabla_x^j g_{L,\perp}\right\|_{L^2}^{2^{-(k-j+2)}}\right) \left\|\nabla_x^{k+1} g_{L,\perp}\right\|_{L^2}^{1/2}, & k \geq 2 \end{cases} \\ &\lesssim e^{-Ct} \|g_{in}\|_{L^2} \end{aligned}$$

for all $t \geq 1$. Accordingly, combining these with (3.28) yields

$$\left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}\right\|_{L_x^\infty L_p^2} \lesssim e^{-2^{-(\ell+2)}Ct} \left(\|g_{in}\|_{L^2(w_{\ell+1})} + \|g_{in}\|_{L_x^1 L_p^2}\right),$$

for all $t \geq 1$. Meanwhile, with the same argument, we also obtain

$$\begin{aligned} \left\|\nabla_p^\ell \nabla_x^k g_S\right\|_{L_x^\infty L_p^2} &\lesssim \left\|\nabla_p^\ell \nabla_x^k g_S\right\|_{H_x^2 L_p^2}^{3/4} \left\|\nabla_p^\ell \nabla_x^k g_S\right\|_{L^2}^{1/4} \\ &\lesssim e^{-2^{-(k+\ell+2)}Ct} \|g_{in}\|_{L^2(w_{k+\ell+2})}, \end{aligned}$$

by (3.16) and Proposition 3.4. This completes the proof of Theorem 3.1.

4. The result for h

Recall the equation

$$\begin{cases} \partial_t h + \dfrac{1}{m_B} p \cdot \nabla_x h = L_{BB} h + L_{BA} g, & (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ h(0,x,p) = h_{in}(x,p). \end{cases} \tag{4.1}$$

In previous sections, the regularity of the solution g has been investigated well. Now, regarding g as a source term, we shall devote to establish the regularization estimate for the solution h in the rest of the article. Especially, we demonstrate that h has the same decay rate as g (see Theorem 1.1).

Let ${\mathbb{G}_{\mathbb{BB}}}^t$ be the solution operator of

$$\partial_t f + \frac{1}{m_B} p \cdot \nabla_x f = L_{BB} f,$$

and then the homogeneous part of h is $\widetilde{h} = {\mathbb{G}_{\mathbb{BB}}}^t h_{in}$. Concerning estimates on the homogeneous part, it is the same as those on g_L and g_S in previous sections and hence we only state the results without proofs. As before, we decompose $\widetilde{h}$ into the long wave-fluid part $\widetilde{h}_{L,0}$, long wave-nonfluid part $\widetilde{h}_{L,\perp}$ and short wave part $\widetilde{h}_S$. Then they respectively satisfy

$$\left\|\nabla_p^\ell \nabla_x^k \widetilde{h}_{L,0}\right\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \|h_{in}\|_{L_x^1 L_p^2},$$

$$\left\|\nabla_p^\ell \nabla_x^k \widetilde{h}_{L,\perp}\right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct}\left(\|h_{in}\|_{L^2(w_{\ell+1})} + \|h_{in}\|_{L_x^1 L_p^2}\right),$$

and

$$\left\|\nabla_p^\ell \nabla_x^k \widetilde{h}_S\right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \|h_{in}\|_{L^2(w_{k+\ell+2})},$$

with the constant $C>0$ depending on k and ℓ, and

$$w_n \equiv \begin{cases} 1, & \gamma \in [0,1], \\ \langle p\rangle^{|\gamma| n}, & \gamma \in [-2,0). \end{cases}$$

Thereupon, it remains to control the inhomogeneous part. Hereafter, we may assume $h_{in}=0$.

We now decompose the solution h into the long wave part h_L and the short wave part h_S, which respectively satisfy

$$\begin{cases} \partial_t h_L + \dfrac{1}{m_B} p\cdot\nabla_x h_L = L_{BB} h_L + L_{BA} g_L, & (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ h_L(0,x,p) = 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \partial_t h_S + \dfrac{1}{m_B} p\cdot\nabla_x h_S = L_{BB} h_S + L_{BA} g_S, & (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ h_S(0,x,p) = 0. \end{cases} \tag{4.3}$$

It is easy to see that

$$\begin{aligned} h_L(t,x,p) &= \int_0^t \int_{|\eta|<\delta} e^{i\eta\cdot x} e^{(-ip\cdot\eta/m_B + L_{BB})(t-s)} L_{BA} e^{(-ip\cdot\eta/m_A + L_{AB})s} \hat{g}_{in} d\eta ds \\ &= h_{00}^L + h_{0\perp}^L + h_{\perp 0}^L + h_{\perp\perp}^L, \end{aligned}$$

where

$$h_{00}^L = \sum_{j=0}^4 \int_0^t \int_{|\eta|<\delta} e^{i\eta\cdot x} e^{\sigma_j(\eta)(t-s) + \lambda(\eta)s} \big\langle e_j(-\eta), L_{BA} e_D(\eta)\big\rangle_p \big\langle \hat{g}_{in}, e_D(-\eta)\big\rangle_p e_j(\eta) d\eta ds,$$

$$h_{0\perp}^L = \sum_{j=0}^4 \int_0^t \int_{|\eta|<\delta} e^{i\eta\cdot x} e^{\sigma_j(\eta)(t-s)} \big\langle e_j(-\eta), L_{BA} e^{(-ip\cdot\eta/m_A + L_{AB})s} \Pi_\eta^{D\perp} \hat{g}_{in}\big\rangle_p e_j(\eta) d\eta ds,$$

$$h_{\perp 0}^L = \int_0^t \int_{|\eta|<\delta} e^{i\eta\cdot x} e^{\lambda(\eta)s} e^{(-ip\cdot\eta/m_B + L_{BB})(t-s)} \Pi_\eta^\perp L_{BA} e_D(\eta) \big\langle e_D(-\eta), \hat{g}_{in}\big\rangle_p d\eta ds,$$

$$h_{\perp\perp}^L = \int_0^t \int_{|\eta|<\delta} e^{i\eta\cdot x} e^{(-ip\cdot\eta/m_B + L_{BB})(t-s)} \Pi_\eta^\perp L_{BA} e^{(-ip\cdot\eta/m_A + L_{AB})s} \Pi_\eta^{D\perp} \hat{g}_{in} d\eta ds.$$

Here the eigenfunctions $\{e_j(\eta)\}_{j=0}^4$ and $e_D(\eta)$ are defined as in (2.8) and (2.4) with $\langle e_j(-\eta),e_l(\eta)\rangle_p = \delta_{jl}$ and $\langle e_D(-\eta),e_D(\eta)\rangle_p = 1$. The main result for the solution h is stated as below.

THEOREM 4.1 (Main result of h). *Let k and ℓ be non-negative integers. Then for $t \geq 1$,*

$$\|\nabla_p^\ell \nabla_x^k (h_{00}^L + h_{0\perp}^L)\|_{L_x^\infty L_p^2} \leq (1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2},$$

$$\left\|\nabla_p^\ell \nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \left(\|g_{in}\|_{L^2} + \|g_{in}\|_{L_x^1 L_p^2} \right),$$

and

$$\left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L\right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \|g_{in}\|_{L^2(w_{\ell+1})}.$$

On the other hand,

$$\left\|\nabla_p^\ell \nabla_x^k h_S\right\|_{L_x^\infty L_p^2} \lesssim e^{-Ct} \|g_{in}\|_{L^2(w_{k+\ell+2})},$$

the constant $C > 0$ depending on k and ℓ. Here

$$w_n \equiv \begin{cases} 1, & \gamma \in [0,1], \\ \langle p \rangle^{|\gamma| n}, & \gamma \in [-2,0). \end{cases}$$

4.1. Improvement of the x-regularity. In this subsection, we deal with the x-regularization estimate for the long wave part h_L. Owing to the fact that the system (1.5) is decoupled, we naturally use the Duhamel principle to solve h through treating g as a source term. However, there are some possible wave resonances between g and h as mentioned in the Introduction, and thus this method usually leads the solution h to a lower decay rate than g. To remedy this, further physical properties of the collision operators, namely the microscopic cancellations representing the conservation of mass, total momentum and total energy, are employed in the following discussion.

PROPOSITION 4.1 (Long wave h_L, improvement of the x regularity). *For $k,\ell \in \mathbb{N} \cup \{0\}$*

$$\|\nabla_p^\ell \nabla_x^k (h_{00}^L + h_{0\perp}^L)\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2},$$

$$\|\nabla_x^k h_{\perp 0}^L\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(4+k)/2} \|g_{in}\|_{L_x^1 L_p^2}, \tag{4.4}$$

and

$$\|\nabla_x^k h_{\perp\perp}^L\|_{L_x^\infty L_p^2} \lesssim t e^{-a(\tau)t} \|g_{in}\|_{L_x^1 L_p^2}. \tag{4.5}$$

Proof. We first split $h_{00}^L = I_0 + I_1 + I_2 + I_3 + I_4$, where each I_j is the orthogonal projection of h_{00}^L along e_j, i.e.,

$$I_j = \int_0^t \int_{|\eta|<\delta} e^{i\eta \cdot x} e^{\sigma_j(\eta)(t-s)+\lambda(\eta)s} \langle e_j(-\eta), L_{BA} e_D(\eta) \rangle_p \langle \hat{g}_{in}, e_D(-\eta) \rangle_p e_j(\eta) d\eta ds.$$

Notice that $\lambda(\eta)-\sigma_j(\eta)\neq 0$ for $j=0$ and 1. It is natural to evaluate I_j $(j=0,1)$ by the fundamental theorem of calculus, that is,

$$\int_0^t e^{\sigma_j(\eta)(t-s)+\lambda(\eta)s}ds=\frac{1}{\lambda(\eta)-\sigma_j(\eta)}\left[e^{\lambda(\eta)t}-e^{\sigma_j(\eta)t}\right],\quad j=0,1. \tag{4.6}$$

On the other hand, we have such a "microscopic cancellation" $L_{BA}E_D=0$ from $Q_{BA}(M_B,M_A)=0$ that

$$\begin{aligned}\langle e_j(-\eta),L_{BA}e_D(\eta)\rangle_p=&\frac{1}{\|E_D+iE_{D,1}\eta+O(|\eta|^2)\|_{L^2_p}}\left\langle e_j(-\eta),iL_{BA}E_{D,1}\eta+O(|\eta|^2)\right\rangle_p\\ =&O(|\eta|).\end{aligned}$$

Therefore,

$$\|\nabla_x^k I_j\|_{L^\infty_x L^2_p}\le C\int_{|\eta|<\delta}\frac{|\eta|^{k+1}}{|\lambda(\eta)-\sigma_j(\eta)|}\left|e^{\lambda(\eta)t}+e^{\sigma_j(\eta)t}\right||\hat{g}_{in}|_{L^2_p}d\eta,$$

for all $k\in\mathbb{N}\cup\{0\}$. Further, since

$$\frac{|\eta|}{|\lambda(\eta)-\sigma_j(\eta)|}=\frac{1}{|(-a_2+a_{j,2})\eta+ia_{j,1}+O(|\eta|^2)|}=O(1),\quad j=0,1, \tag{4.7}$$

we deduce

$$\begin{aligned}\|\nabla_x^k I_0\|_{L^\infty_x L^2_p},\ \|\nabla_x^k I_1\|_{L^\infty_x L^2_p}\le& C\int_{|\eta|<\delta}|\eta|^k\left|e^{\lambda(\eta)t}+e^{\sigma_j(\eta)t}\right|d\eta\|g_{in}\|_{L^1_x L^2_p}\\ \lesssim&(1+t)^{-(k+3)/2}\|g_{in}\|_{L^1_x L^2_p}.\end{aligned}$$

In contrast with I_0 and I_1, we lose (4.6) or (4.7) in the case of I_j, $j=1,2$ and 3. To maintain the decay rate of h, we come up with further microscopic cancellations as well as $L_{BA}E_D=0$. In the light of the conservation of mass and total energy, the operator

$$L_{AB}J+\frac{m_A}{m_B}\frac{\sqrt{M_B}}{\sqrt{M_A}}L_{BA}J$$

is orthogonal to the collision invariants $\sqrt{M_A}$ and $|p|^2\sqrt{M_A}$. With this applied to $J=E_{D,1}=L_{AB}^{-1}(p\cdot\omega/m_A)E_D$ (where we recall that $E_D=\sqrt{M_A}$) it follows

$$\int_{\mathbb{R}^3}\left(L_{AB}[L_{AB}^{-1}(p\cdot\omega/m_A)E_D]+\frac{m_A}{m_B}\frac{\sqrt{M_B}}{\sqrt{M_A}}L_{BA}[L_{AB}^{-1}(p\cdot\omega/m_A)E_D]\right)\Phi_1\sqrt{M_A}dp=0,$$

for any linear combination Φ_1 of 1 and $|p|^2$, so that

$$\frac{m_A}{m_B}\int_{\mathbb{R}^3}\Phi_1\sqrt{M_B}L_{BA}[L_{AB}^{-1}(p\cdot\omega/m_A)E_D]dp=-\int_{\mathbb{R}^3}\Phi_1(p\cdot\omega/m_A)M_Adp.$$

Choosing $\Phi_1=E_2/\sqrt{M_B}=\sqrt{\frac{1}{10}}(-5+|p|^2)$, we have

$$\int_{\mathbb{R}^3}E_2L_{BA}[L_{AB}^{-1}(p\cdot\omega/m_A)E_D]dp=-\frac{m_B}{m_A}\int_{\mathbb{R}^3}\sqrt{\frac{1}{10}}(-5+|p|^2)(p\cdot\omega/m_A)M_Adp=0,$$

since the integrand is odd. It turns out that $\left\langle e_2(-\eta), L_{BA} e_D(\eta)\right\rangle_p = O(|\eta|^2)$. Hence

$$\begin{aligned}
\|\nabla_x^k I_2\|_{L_x^\infty L_p^2} &\le C\int_0^t \int_{|\eta|<\delta} |\eta|^{2+k} \left| e^{\sigma_2(\eta)(t-s)+\lambda(\eta)s} \right| |\hat{g}_{in}|_{L_p^2} d\eta ds \\
&\le C\int_0^t \int_{|\eta|<\delta} |\eta|^{2+k} e^{-\overline{a}|\eta|^2 t} d\eta ds \|g_{in}\|_{L_x^1 L_p^2} \\
&\le C' \int_0^t (1+t)^{-(5+k)/2} ds \|g_{in}\|_{L_x^1 L_p^2} \\
&\le C''(1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2},
\end{aligned}$$

here $\overline{a} = 1/2 \min\{a_2, a_{2,2}\} > 0$ whenever $\delta > 0$ is small.

Furthermore, owing to the conservation of momentum, the operator

$$L_{AB} J + \frac{\sqrt{M_B}}{\sqrt{M_A}} L_{BA} J$$

is orthogonal to all collision invariants $p_1\sqrt{M_A}$, $p_2\sqrt{M_A}$ and $p_3\sqrt{M_A}$. With this applied to $J = E_{D,1} = L_{AB}^{-1}(p\cdot\omega/m_A)E_D$, it follows

$$\int_{\mathbb{R}^3} \left(L_{AB}\left[L_{AB}^{-1}(p\cdot\omega/m_A)E_D\right] + \frac{\sqrt{M_B}}{\sqrt{M_A}} L_{BA}\left[L_{AB}^{-1}(p\cdot\omega/m_A)E_D\right]\right)\Phi_2\sqrt{M_A}dp = 0,$$

for any linear combination Φ_2 of p_1, p_2 and p_3, so that

$$\int_{\mathbb{R}^3} \Phi_2\sqrt{M_B} L_{BA}\left[L_{AB}^{-1}(p\cdot\omega/m_A)E_D\right]dp = -\int_{\mathbb{R}^3} \Phi_2 (p\cdot\omega/m_A) M_A dp.$$

Choosing $\Phi_2 = E_3/\sqrt{M_B} = \omega_1^\perp \cdot p$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^3} E_3 L_{BA}\left[L_{AB}^{-1}(p\cdot\omega/m_A)E_D\right]dp \\
&= -\int_{\mathbb{R}^3} (p\cdot\omega_1^\perp)(p\cdot\omega)\frac{M_A}{m_A}dp = -\sum_{i,j=1}^3 \left(\int p_i p_j \frac{M_A}{m_A} dp\right)(\omega_1^\perp)_i \omega_j = 0,
\end{aligned}$$

due to the fact that $\int p_i p_j \frac{M_A}{m_A} dp = \frac{1}{3m_B}\delta_{ij}$ and $\omega_1^\perp \perp \omega$. It turns out that $\left\langle e_3(-\eta), L_{BA} e_D(\eta)\right\rangle_p = O(|\eta|^2)$. Likewise, $\left\langle e_4(-\eta), L_{BA} e_D(\eta)\right\rangle_p = O(|\eta|^2)$. Therefore,

$$\|\nabla_x^k I_3\|_{L_x^\infty L_p^2}, \|\nabla_x^k I_4\|_{L_x^\infty L_p^2} \lesssim (1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2}.$$

This completes the estimate of h_{00}^L.

For $h_{0\perp}^L$, direct computation gives

$$\begin{aligned}
\|\nabla_x^k h_{0\perp}^L\|_{L_x^\infty L_p^2} &\le C\sum_{j=1}^3 \int_0^t \int_{|\eta|<\delta} |\eta|^k e^{-a(\tau_2)s} \left| e^{\sigma_j(\eta)(t-s)} \right| |\hat{g}_{in}|_{L_p^2} d\eta ds \\
&\le C' \left(\int_0^{t/2} e^{-a(\tau_2)s}(1+t)^{-(3+k)/2} ds + \int_{t/2}^t e^{-a(\tau_2)s} ds \right) \|g_{in}\|_{L_x^1 L_p^2}, \\
&\lesssim (1+t)^{-(3+k)/2} \|g_{in}\|_{L_x^1 L_p^2}.
\end{aligned}$$

From the last statements of Lemmas 2.5 and 2.6, one can improve p regularity freely; that is, for any $\ell,\ k\in\mathbb{N}\cup\{0\}$,

$$\|\nabla_p^\ell\nabla_x^k(h_{00}^L+h_{0\perp}^L)\|_{L_x^\infty L_p^2}\lesssim(1+t)^{-(3+k)/2}\|g_{in}\|_{L_x^1L_p^2}.$$

For $h_{\perp 0}^L$, we have

$$\begin{aligned}\|\nabla_x^k h_{\perp 0}^L\|_{L_x^\infty L_p^2}&\lesssim\int_0^t\int_{|\eta|<\delta}|\eta|^{k+1}e^{-a(\tau_1)(t-s)}\left|e^{\lambda(\eta)s}\right||\hat{g}_{in}|_{L_p^2}d\eta ds\\&\lesssim\left(\int_{t/2}^t e^{-a(\tau_1)(t-s)}(1+s)^{-(k+4)/2}ds+\int_0^{t/2}e^{-a(\tau_1)(t-s)}ds\right)\|g_{in}\|_{L_x^1L_p^2}\\&\lesssim(1+t)^{-(4+k)/2}\|g_{in}\|_{L_x^1L_p^2}.\end{aligned}$$

Finally,

$$\|\nabla_x^k h_{\perp\perp}^L\|_{L_x^\infty L_p^2}\lesssim\int_0^t\int_{|\eta|<\delta}|\eta|^k e^{-a(\tau_1)(t-s)-a(\tau_2)s}d\eta ds\|g_{in}\|_{L_x^1L_p^2}\lesssim te^{-a(\tau)t}\|g_{in}\|_{L_x^1L_p^2},$$

here $a(\tau)=\min\{a(\tau_1),a(\tau_2)\}$. This completes the proof of the proposition. ∎

4.2. Improvement of the x and p regularities (general discussion). Let f be a solution of the linearized Landau equation

$$\begin{cases}\partial_t f+\frac{1}{m_B}p\cdot\nabla_x f=L_{BB}f+L_{BA}g,\\ f(0,x,p)=f_{in}(x,p),\end{cases}\tag{4.8}$$

where g is known and satisfies the equation $\partial_t g+\frac{1}{m_A}p\cdot\nabla_x g=L_{AB}g$. In the following, we first apply similar arguments as those in Section 3 to establish the regularization estimates for f in short time and in large time. After that, based on these regularization estimates, one can readily obtain the decay rate of h_S through the interpolation trick as before. Contrary to h_S, the $L_x^\infty L_p^2$ norm of $\nabla_x h_{\perp 0}^L$ decays only algebraically, rather than exponentially. To maintain the decay rate of $\nabla_p^\ell\nabla_x^k h_{\perp 0}^L$ the same as that of the homogeneous problem, we interpolate the $L_x^\infty L_p^2$ norm of it from higher p-derivatives. However, the method used before for improving p-regularity results in the weights imposed on initial data growing drastically (in fact, exponentially growing with respect to ℓ). We refine the p-regularization estimate by making use of Equation (4.15) and the p-regularity of source term $g_{L;0}$. In such a way, the p-regularity of $h_{\perp 0}^L$ has been improved without imposing any weight on initial data.

Let the operator $\mathcal{L}_{BB}=-\frac{1}{m_B}p\cdot\nabla_x-\Lambda^{BB}$. As shown in [6], the operator $e^{t\mathcal{L}_{BB}}$ has the same energy estimate and regularization estimate as $e^{t\mathcal{L}_{AB}}$ in Lemma 3.1. Based on this estimate, we deduce the x-regularity of f in small time as below.

LEMMA 4.1. *Let f be a solution of the linearized Landau Equation (4.8). Then for $k\in\mathbb{N}\cup\{0\}$,*

(i) there is $C_k>0$ such that for $0<t\le 1$

$$\left\|\nabla_x^k f\right\|_{L^2(m_\theta)}\le C_k t^{-\frac{3}{2}k}\left(\|f_{in}\|_{L^2(m_k)}+\|g_{in}\|_{L^2(m_k)}\right).$$

(ii) For $t\ge 1$,

$$\left\|\nabla_x^k f\right\|_{L^2(m_\theta)}\lesssim\left(\|f_{in}\|_{L^2(m_k)}+t^{1/2}\|g_{in}\|_{L^2(m_k)}\right).$$

Proof. Firstly, similar to (3.3) and (3.4), there exists a universal constant $C>0$ such that

$$\left\|\mathbb{G_{BB}}^{t} f_{in}\right\|_{L^2(m_\theta)} \leq C\left\|f_{in}\right\|_{L^2(m_\theta)}, \tag{4.9}$$

and

$$\|\partial_x^\alpha \mathbb{G_{BB}}^{s_2} f_{in}\|_{L^2(m_\theta)} \leq C\|\partial_x^\alpha \mathbb{G_{BB}}^{s_1} f_{in}\|_{L^2(m_\theta)}, \tag{4.10}$$

for any $0<s_1<s_2$. By the Duhamel principle,

$$f=\mathbb{G_{BB}}^{t} f_{in}+\int_0^t \mathbb{G_{BB}}^{t-s} L_{BA} g(s)\,ds.$$

Notice that L_{BA} is an integral operator like K^{BB}. Hence, for $k=0$, we find

$$\begin{aligned}
\|f\|_{L^2(m_\theta)} &\lesssim \left(\left\|\mathbb{G_{BB}}^{t} f_{in}\right\|_{L^2(m_\theta)}^2+\int_0^t \left\|\mathbb{G_{BB}}^{t-s} L_{BA} g(s)\right\|_{L^2(m_\theta)}^2 ds\right)^{1/2} \\
&\lesssim \left(\|f_{in}\|_{L^2(m_\theta)}^2+\int_0^t \|g(s)\|_{L^2(m_\theta)}^2 ds\right)^{1/2} \\
&\lesssim \|f_{in}\|_{L^2(m_\theta)}+t^{1/2}\|g_{in}\|_{L^2(m_\theta)},
\end{aligned} \tag{4.11}$$

from (3.3) and (4.9). In addition, since ∂_x^α is commutative with Equation (4.8), we as well have

$$\|\partial_x^\alpha f(s_2)\|_{L^2(m_\theta)} \lesssim \|\partial_x^\alpha f(s_1)\|_{L^2(m_\theta)}+(s_2-s_1)^{1/2}\|\partial_x^\alpha g(s_1)\|_{L^2(m_\theta)}, \tag{4.12}$$

for any $0<s_1<s_2$.

For $k\geq 1$, using a Picard-type iteration introduced in Lemma 3.2, we rewrite

$$f=f^{(0)}+f^{(1)}+\cdots+f^{(2k)}+\mathcal{R}^{(k)},$$

where $f^{(0)}$ satisfies

$$\begin{cases}
\partial_t f^{(0)}=\mathcal{L}_{BB} f^{(0)}, \\
f^{(0)}(0,x,p)=f_{in}(x,p),
\end{cases}$$

$f^{(j)}$, $1\leq j\leq 2k$, satisfies

$$\begin{cases}
\partial_t f^{(j)}=\mathcal{L}_{BB} f^{(j)}+K^{BB} f^{(j-1)}, \\
f^{(j)}(0,x,p)=0,
\end{cases}$$

and $\mathcal{R}^{(k)}$ solves the equation

$$\begin{cases}
\partial_t \mathcal{R}^{(k)}=L_{BB}\mathcal{R}^{(k)}+K^{BB} f^{(2k)},+L_{BA} g \\
\mathcal{R}^{(k)}(0,x,p)=0.
\end{cases}$$

Under this decomposition, following the same procedure as in Lemma 3.2, we finish the proof. ∎

PROPOSITION 4.2. *Let f be a solution to Equation (4.8) and let $k,\ \ell\in\mathbb{N}\cup\{0\}$. Then for $0<t\leq 1$*

$$\left\|\nabla_p^\ell\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq Ct^{-\frac{\ell}{2}-\frac{3}{2}(k+\ell)}\left(\|f_{in}\|_{L^2(m_{k+\ell})}+\|g_{in}\|_{L^2(m_{k+\ell})}\right).$$

Proof. Let u be a solution of equation

$$\begin{cases}\partial_t u=\mathcal{L}_{BB}u,\\ u(0,x,p)=u_0(x,p).\end{cases}\tag{4.13}$$

Then for any $\ell\in\mathbb{N}$

$$t\|\nabla_p^\ell u\|_{L^2(m_\theta)}^2\lesssim\mathcal{F}_\ell(0,u),\ 0<t\leq 1\tag{4.14}$$

where $\mathcal{F}_\ell(t,u)$ is defined as in the proof of Proposition 3.2.

Let $0<t_0\leq 1$. By Duhamel's principle,

$$f(t)=e^{(t-t_0/2)\mathcal{L}_{BB}}f(t_0/2)+\int_{t_0/2}^t e^{(t-s)\mathcal{L}_{BB}}K^{BB}f(s)\,ds+\int_{t_0/2}^t e^{(t-s)\mathcal{L}_{BB}}L_{AB}g(s)\,ds,$$

for $0<t_0/2\leq t\leq 1$, so that

$$\begin{aligned}\|\nabla_p f(t)\|_{L^2(m_\theta)}\leq{}& C(t-t_0/2)^{-\frac{1}{2}}\left(\|f(t_0/2)\|_{L^2(m_1)}+\|\nabla_x f(t_0/2)\|_{L^2(m_\theta)}\right)\\ &+C\int_{t_0/2}^t(t-s)^{-\frac{1}{2}}\Bigg(\|f(s)\|_{L^2(m_1)}+\|\nabla_x f(s)\|_{L^2(m_\theta)}\\ &\qquad+\|g(s)\|_{L^2(m_1)}+\|\nabla_x g(s)\|_{L^2(m_\theta)}\Bigg)ds\\ \leq{}& C(t-t_0/2)^{-\frac{1}{2}}t_0^{-\frac{3}{2}}\left(\|f_{in}\|_{L^2(m_1)}+\|g_{in}\|_{L^2(m_1)}\right),\end{aligned}$$

due to Lemma 4.1, (4.12) and (4.14). It follows that

$$\|\nabla_p f(t)\|_{L^2(m_\theta)}\lesssim t^{-\frac{1}{2}-\frac{3}{2}}\left(\|f_{in}\|_{L^2(m_1)}+\|g_{in}\|_{L^2(m_1)}\right),\qquad 0<t\leq 1.$$

Since ∂_x^α commutes with the Equation (4.8), we also have

$$\left\|\nabla_p\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\lesssim t^{-\frac{1}{2}-\frac{3}{2}(k+1)}\left(\|f_{in}\|_{L^2(m_{k+1})}+\|g_{in}\|_{L^2(m_{k+1})}\right),\qquad 0<t\leq 1.$$

For general k,ℓ, we apply the induction argument combined with (4.14) to conclude that

$$\left\|\nabla_p^\ell\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\lesssim t^{-\frac{\ell}{2}-\frac{3}{2}(k+\ell)}\left(\|f_{in}\|_{L^2(m_{k+\ell})}+\|g_{in}\|_{L^2(m_{k+\ell})}\right)$$

for $0<t\leq 1$. ∎

PROPOSITION 4.3. *Let f be a solution to Equation (4.8) and let $k,\ \ell\in\mathbb{N}\cup\{0\}$. Then for $t\geq 1$,*

$$\left\|\nabla_p^\ell\nabla_x^k f(t)\right\|_{L^2(m_\theta)}\leq C\left(\|f_{in}\|_{L^2(m_{k+\ell})}+t^{1/2}\|g_{in}\|_{L^2(m_{k+\ell})}\right),$$

the constant C depending only upon k and ℓ.

Proof. Define

$$H^n[f](t)=\sum_{j=0}^{n}\left\|\nabla_p^j f(t)\right\|_{L^2(m_\theta)}^2 \quad \text{and} \quad H_x^n[f](t)=\sum_{j=0}^{n}\left\|\nabla_p^j\nabla_x f(t)\right\|_{L^2(m_\theta)}^2.$$

In view of Lemmas 2.3 and 2.4,

$$\frac{d}{dt}\|\nabla_p^\ell f\|_{L^2(m_\theta)}^2 \lesssim -\|\nabla_p^\ell f\|_{H^1_\sigma(m_\theta)}^2 + C\left(\sum_{j=0}^{\ell-1}\|\nabla_p^j f\|_{H^1_\sigma(m_\theta)}^2 + \sum_{j=0}^{\ell-1}\|\nabla_p^j\nabla_x f\|_{L^2(m_\theta)}^2\right)$$
$$+C\left(\|f\|_{L^2}^2+\sum_{j=0}^{\ell}\|\nabla_p^j g\|_{L^2(m_\theta)}^2\right).$$

Hence, following a similar argument as in Proposition 3.3, we deduce

$$\frac{d}{dt}H^n[f](t)\le -cH^n[f](t)+C\left(H_x^{n-1}[f](t)+\|f(t)\|_{L^2}^2+\sum_{j=0}^{n}\|\nabla_p^j g(t)\|_{L^2(m_\theta)}^2\right)$$

for some constants $c>0$ small and $C>0$ large. It implies that for $t\ge 1$

$$H^n[f](t)\le e^{-c(t-1)}H^n[f](1)+Ce^{-ct}\int_1^t e^{cs}\left(\|f(s)\|_{L^2}^2+\sum_{j=0}^{n}\|\nabla_p^j g(s)\|_{L^2(m_\theta)}^2\right)ds,$$
$$+Ce^{-ct}\int_1^t e^{cs}H_x^{n-1}[f](s)\,ds,$$
$$\lesssim\left(\|f_{in}\|_{L^2(m_n)}^2+t\|g_{in}\|_{L^2(m_n)}^2\right)+e^{-ct}\int_1^t e^{cs}H_x^{n-1}[f](s)\,ds,$$

due to (4.11) and Proposition 4.2. Furthermore, applying a similar argument to the same functional $\mathcal{A}_N(t,f)$ defined in Proposition 3.3 gives

$$\frac{d}{dt}\mathcal{A}_N(t,f)\le -c\mathcal{A}_N(t,f)+C\left(\sum_{j=1}^{N+1}\left\|\nabla_x^j f(t)\right\|_{L^2}^2+\sum_{j=1}^{N+1}\left\|\nabla_x^j g(t)\right\|_{L^2(m_\theta)}^2\right.$$
$$\left.+\sum_{j=1}^{N}\sum_{q=1}^{N+1-j}\|\nabla_p^j\nabla_x^q g(t)\|_{L^2(m_\theta)}^2\right),$$

so that

$$\mathcal{A}_N(t,f)\le e^{-c(t-1)}\mathcal{A}_N(1,f)$$
$$+C\int_1^t e^{-c(t-s)}\left(\sum_{j=1}^{N+1}\left\|\nabla_x^j f(s)\right\|_{L^2}^2+\sum_{j=1}^{N+1}\left\|\nabla_x^j g(t)\right\|_{L^2(m_\theta)}^2\right.$$
$$\left.+\sum_{j=1}^{N}\sum_{q=1}^{N+1-j}\|\nabla_p^j\nabla_x^q g(t)\|_{L^2(m_\theta)}^2\right)ds$$

$$\lesssim \left(\|f_{in}\|^2_{L^2(m_{N+1})} + t\|g_{in}\|^2_{L^2(m_{N+1})} \right).$$

It implies that for $t \geq 1$

$$H^{n-1}_x[f](t) \lesssim \mathcal{A}_{n-1}(t,f) \lesssim \left(\|f_{in}\|^2_{L^2(m_n)} + t\|g_{in}\|^2_{L^2(m_n)} \right).$$

As a result, we have

$$H^n[f](t) \lesssim \left(\|f_{in}\|^2_{L^2(m_n)} + t\|g_{in}\|^2_{L^2(m_n)} \right),$$

and

$$H^n_x[f](t) \lesssim \left(\|f_{in}\|^2_{L^2(m_{n+1})} + t\|g_{in}\|^2_{L^2(m_{n+1})} \right).$$

Therefore, for $\ell \in \mathbb{N}$,

$$\left\|\nabla^\ell_p f(t)\right\|_{L^2(m_\theta)} \lesssim \left(\|f_{in}\|_{L^2(m_\ell)} + t^{1/2}\|g_{in}\|_{L^2(m_\ell)} \right), \qquad t \geq 1.$$

On the other hand, for any k, $\ell \in \mathbb{N}$ we also have

$$\left\|\nabla^\ell_p \nabla^k_x f(t)\right\|_{L^2(m_\theta)} \lesssim \sqrt{\mathcal{A}_{k+\ell-1}(t,f)} \lesssim \left(\|f_{in}\|_{L^2(m_{k+\ell})} + t^{1/2}\|g_{in}\|_{L^2(m_{k+\ell})} \right), \qquad t \geq 1.$$

Together with Lemma 4.1 (ii), the proof is completed. □

4.3. Proof of Theorem 4.1. Let k, $\ell \in \mathbb{N} \cup \{0\}$. It follows from Lemma 4.1, Propositions 4.2 and 4.3 that for $0 < t \leq 1$

$$\left\|\nabla^\ell_p \nabla^k_x h_L(t)\right\|_{L^2}, \ \left\|\nabla^\ell_p \nabla^k_x h_S(t)\right\|_{L^2} \lesssim t^{-\frac{\ell}{2} - \frac{3}{2}(k+\ell)} \|g_{in}\|_{L^2(w_{k+\ell})},$$

and for $t \geq 1$

$$\left\|\nabla^\ell_p \nabla^k_x h_L(t)\right\|_{L^2}, \ \left\|\nabla^\ell_p \nabla^k_x h_S(t)\right\|_{L^2} \lesssim \|g_{in}\|_{L^2(w_{k+\ell})}.$$

Owing to the fact that

$$\|h_S\|_{L^2} \lesssim e^{-Ct} \|g_{in}\|_{L^2},$$

it is easy to derive the regularity estimate on h_S via the Sobolev inequality and the interpolation inequality, i.e., for any $k, \ell \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \left\|\nabla^\ell_p \nabla^k_x h_S\right\|_{L^\infty_x L^2_p} \leq \left\|\nabla^\ell_p \nabla^k_x h_S\right\|_{L^2_p L^\infty_x} &\lesssim \left\|\nabla^\ell_p \nabla^k_x h_S\right\|^{3/4}_{H^2_x L^2_p} \left\|\nabla^\ell_p \nabla^k_x h_S\right\|^{1/4}_{L^2} \\ &\lesssim \|h_S\|^{2^{-(k+\ell+2)}}_{L^2} \left(\|g_{in}\|_{L^2(w_{k+\ell+2})} \right)^{1-2^{-(k+\ell+2)}} \\ &\lesssim e^{-2^{-(k+\ell+2)}Ct} \|g_{in}\|_{L^2(w_{k+\ell+2})}, \end{aligned}$$

for all $t \geq 1$. We complete the estimate for h_S.

Recall the regularization estimate of the homogeneous part of h. To maintain the decay rate of h_L without imposing higher weights on the initial data, we further refine the regularization estimate on $h^L_{\perp 0}$ and $h^L_{\perp\perp}$ before employing the Sobolev inequality and interpolation trick. Specifically, we prove that for any $k, \ell \in \mathbb{N} \cup \{0\}$,

$$\|\nabla^\ell_p \nabla^k_x h^L_{\perp 0}\|_{L^2} \lesssim t^{1/2} \|g_{in}\|_{L^2},$$

and

$$\left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L(t)\right\|_{L^2} \leq C t^{1/2} \|g_{in}\|_{L^2(w_\ell)},$$

for $t \geq 1$. Now, let $h_{*,0}^L = h_{00}^L + h_{\perp 0}^L$ and then it satisfies the equation:

$$\begin{cases} \partial_t h_{*,0}^L + \dfrac{1}{m_B} p \cdot \nabla_x h_{*,0}^L = L_{BB} h_{*,0}^L + L_{BA} g_{L,0}, \quad (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ h_{*,0}^L(0,x,p) = 0. \end{cases} \tag{4.15}$$

Following the energy estimate in Proposition 4.3, together with $m_\theta = 1$, gives

$$\frac{d}{dt}\mathcal{A}_N[h_{*,0}^L](t) \leq -c\mathcal{A}_N[h_{*,0}^L](t) + C\left(\sum_{j=1}^{N+1} \left\|\nabla_x^j h_{*,0}^L(t)\right\|_{L^2}^2 + \sum_{j=1}^{N+1} \left\|\nabla_x^j g_{L,0}(t)\right\|_{L^2}^2 + \sum_{j=1}^{N} \sum_{q=1}^{N+1-j} \|\nabla_p^j \nabla_x^q g_{L,0}(t)\|_{L^2}^2\right),$$

which implies that

$$\mathcal{A}_N[h_{*,0}^L](t) \lesssim t\|(g_{L,0})_{in}\|_{L^2(w_{N+1})}^2 + \int_1^t e^{-c(t-s)} \left(\sum_{j=1}^{N+1} \left\|\partial_x^j h_{*,0}^L(s)\right\|_{L^2}^2 + \sum_{j=1}^{N} \sum_{q=1}^{N+1-j} \|\nabla_p^j \nabla_x^q g_{L,0}(s)\|_{L^2}^2\right) ds.$$

Combined

$$\|(g_{L,0})_{in}\|_{L^2(w_n)}^2 \lesssim \|g_{in}\|_{L^2}^2$$

with

$$\left\|\nabla_x^k h_{*,0}^L\right\|_{L^2}, \quad \|\nabla_p^\ell \nabla_x^k g_{L,0}\|_{L^2} \lesssim \|g_{in}\|_{L^2}, \quad t \geq 1,$$

it follows that

$$\mathcal{A}_N[h_{*,0}^L](t) \lesssim t\|g_{in}\|_{L^2}^2, \quad t \geq 1,$$

which implies that

$$\|\nabla_p^\ell h_{*,0}^L\|_{L^2}^2 \lesssim H^\ell\left[h_{*,0}^L\right](t) \lesssim t\|(g_{L,0})_{in}\|_{L^2(w_\ell)}^2 \lesssim t\|g_{in}\|_{L^2}^2,$$

and

$$\|\nabla_p^\ell \nabla_x^k h_{*,0}^L\|_{L^2} \lesssim \sqrt{\mathcal{A}_{\ell+k-1}[h_{*,0}^L](t)} \lesssim t^{1/2}\|g_{in}\|_{L^2},$$

for all $t \geq 1$. Hence,

$$\|\nabla_p^\ell \nabla_x^k h_{\perp 0}^L\|_{L^2} \leq \|\nabla_p^\ell \nabla_x^k h_{*,0}^L\|_{L^2} + \|\nabla_p^\ell \nabla_x^k h_{00}^L\|_{L^2} \lesssim t^{1/2}\|g_{in}\|_{L^2}.$$

By the Sobolev inequality, we obtain

$$\|\nabla_p^\ell \nabla_x^k h_{\perp 0}^L\|_{L_x^\infty L_p^2} \lesssim \|\nabla_p^\ell \nabla_x^k h_{\perp 0}^L\|_{H_x^2 L_p^2}^{3/4} \|\nabla_p^\ell \nabla_x^k h_{\perp 0}^L\|_{L^2}^{1/4} \lesssim t^{1/2}\|g_{in}\|_{L^2}.$$

Consequently, employing the interpolation inequality in p (**NOT** in x) yields

$$\begin{aligned}\left\|\nabla_p^\ell \nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2} &\lesssim \left\|\nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2}^{1/2} \left\|\nabla_p^{2\ell} \nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2}^{1/2} \\ &\lesssim \left\|\nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2}^{(1/2+1/4)} \left\|\nabla_p^{4\ell} \nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2}^{1/4} \\ &\lesssim \left\|\nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2}^{(1/2+1/4+\cdots+1/2^q)} \left\|\nabla_p^{2^q\ell} \nabla_x^k h_{\perp 0}^L\right\|_{L_x^\infty L_p^2}^{2^{-q}} \\ &\lesssim (1+t)^{-\frac{4+k}{2}\left(1-2^{-q}\right)} \cdot t^{2^{-q-1}} \left(\|g_{in}\|_{L^2} + \|g_{in}\|_{L_x^1 L_p^2}\right) \\ &= (1+t)^{-\frac{k+3}{2}+2^{-q-1}(5+k-2^q)} \cdot \left(\|g_{in}\|_{L^2} + \|g_{in}\|_{L_x^1 L_p^2}\right) \\ &\le (1+t)^{-\frac{k+3}{2}} \cdot \left(\|g_{in}\|_{L^2} + \|g_{in}\|_{L_x^1 L_p^2}\right), \end{aligned} \tag{4.16}$$

whenever $q \ge \lceil \log_2(5+k) \rceil$.

Finally, let $h_{*,\perp}^L = h_{0\perp}^L + h_{\perp\perp}^L$ and then it satisfies the equation:

$$\begin{cases} \partial_t h_{*,\perp}^L + \dfrac{1}{m_B} p \cdot \nabla_x h_{*,\perp} = L_{BB} h_{*,\perp}^L + L_{BA} g_{L,\perp}, & (t,x,p) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \\ h_{*,\perp}^L(0,x,p) = 0. \end{cases} \tag{4.17}$$

Going through the proofs of Propositions 4.2 and 4.3 with the fact that $\left\|\nabla_p^\ell \nabla_x^k g_{L,\perp}(t)\right\|_{L^2} \le C \|g_{in}\|_{L^2(w_\ell)}$, we find

$$\left\|\nabla_p^\ell \nabla_x^k h_{*,\perp}^L(t)\right\|_{L^2} \le C t^{1/2} \|g_{in}\|_{L^2(w_\ell)}, \quad t \ge 1.$$

Since

$$\left\|\nabla_p^\ell \nabla_x^k h_{0\perp}^L(t)\right\|_{L^2} \le C \|g_{in}\|_{L^2},$$

we improve

$$\left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L(t)\right\|_{L^2} \le C t^{1/2} \|g_{in}\|_{L^2(w_\ell)}, \quad t \ge 1.$$

Therefore,

$$\begin{aligned}\left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L\right\|_{L_x^\infty L_p^2} &\lesssim \left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L\right\|_{H_x^2 L_p^2}^{3/4} \left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L\right\|_{L^2}^{1/4} \\ &\lesssim \left\|\nabla_p^\ell \nabla_x^k h_{\perp\perp}^L\right\|_{H_x^2 L_p^2}^{3/4} \Bigg[\left\|\nabla_x^{k+1} h_{\perp\perp}^L\right\|_{L^2}^{2^{-\ell}} \\ &\qquad \left(\prod_{j=2}^{\ell} \left\|\nabla_p^j \nabla_x^{k+1} h_{\perp\perp}^L\right\|_{L^2}^{2^{-(\ell-j+2)}} \right) \left\|\nabla_p^{\ell+1} \nabla_x^{k+1} h_{\perp\perp}^L\right\|_{L^2}^{1/2} \Bigg]^{1/4} \\ &\lesssim e^{-2^{-(\ell+2)} a(\tau) t} \|g_{in}\|_{L^2(w_{\ell+1})}. \end{aligned}$$

This completes the proof of Theorem 4.1.

Acknowledgment. The authors want to thank Professor Tai-Ping Liu for helpful comments. Y.-C. Lin was supported by the Ministry of Science and Technology under the grant MOST 107-2115-M-006-006 -. H.T. Wang was sponsored by Shanghai Sailing

Program (Grant No. 18YF1411800) and Shanghai Jiao Tong University (Grant No. WF220441907). K.-C. Wu was supported by the Ministry of Science and Technology under the grant MOST 108-2636-M-006-005-MY4 and National Center for Theoretical Sciences.

REFERENCES

[1] R. Adams and J. Fournier, *Sobolev Spaces*, Second Edition, Academic Press, 2003. 3.1, 3.3

[2] R. Alexandre and C. Villani, *On the Landau approximation in plasma physics*, Ann. Inst. H. Poincaré C, Analyse Non Linéaire, 21:61–95, 2004. 1.3

[3] K. Aoki, C. Bardos, and S. Takata, *Knudsen layer for gas mixtures*, J. Stat. Phys., 112:629–655, 2003. 1.3

[4] M. Briant, *Stability of global equilibrium for the multi-species Boltzmann equation in L^1 settings*, Discrete Contin. Dyn. Syst., 36:6669–6688, 2016. 1.3, 1.5

[5] M. Briant and E. Daus, *The Boltzmann equation for multi-species mixture close to global equilibrium*, Arch. Rational Mech. Anal., 222:1367–1443, 2016. 1.3, 1.5

[6] K. Carrapatoso, I. Tristani, and K.-C. Wu, *Cauchy problem and exponential stability for the inhomogeneous Landau equation*, Arch. Rational Mech. Anal., 221:363–418, 2016. 1.3, 1.5, 3.1, 4.2

[7] K. Carrapatoso and S. Mischler, *Landau equation for very soft and Coulomb potentials near Maxwellians*, Ann. PDE, 3:1, 2017. 1.3

[8] E. S. Daus, A. Jungel, C. Mouhot, and N. Zamponi, *Hypocoercivity for a linearized multi-species Boltzmann system*, SIAM J. Math. Anal., 48:538–568, 2016. 1.3, 1.5

[9] P. Degond and M. Lemou, *Dispersion relations for the linearized Fokker-Planck equation*, Arch. Rational Mech. Anal., 138:137–167, 1997. 2

[10] L. Desvillettes and C. Villani, *On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation*, Invent. Math., 159:245–316, 2005. 1.3

[11] M. Gualdani and N. Zamponi, *Spectral gap and exponential convergence to equilibrium for a multi-species Landau system*, Bull. Sci. Math., 141(6):509–538, 2017. 1.1, 1.3, 1.5

[12] Y. Guo, *The Landau equation in a periodic box*, Comm. Math. Phys., 231:391–434, 2002. 1.3, 2, 2.4

[13] S.-Y. Ha, S.-E. Noh, and S.-B. Yun, *Global existence and stability of mild solutions to the Boltzmann system for gas mixtures*, Q. Appl. Math., 65:757–779, 2007. 1.3

[14] E. M. Lifschitz and L. P. Pitajewski, *Physical Kinetics*, Textbook of Theoretical Physics, Oxford: Pergamon Press, 1983. 1.3

[15] T.-P. Liu and S.-H. Yu, *The Green function and large time behavior of solutions for the one-dimensional Boltzmann equation*, Commun. Pure App. Math., 57:1543–1608, 2004. 1.5

[16] C. Mouhot and L. Neumann, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus*, Nonlinearity, 19:969–998, 2006. 3.1

[17] Y. Sone, *Kinetic Theory and Fluid Dynamics*, Birkhauser, Boston, 2002. 1.3

[18] A. Sotirov and S. H. Yu, *On the solution of a Boltzmann system for gas mixtures*, Arch. Rational Mech. Anal., 195:675–700, 2010. 1.2, 1.3, 1.5

[19] R. M. Strain and Y. Guo, *Almost exponential decay near Maxwellian*, Comm. Partial Diff. Eqs., 31:417–429, 2006. 1.3, 1.5

[20] R. M. Strain and Y. Guo, *Exponential decay for soft potentials near Maxwellian*, Arch. Rational Mech. Anal., 187:287–339, 2008. 1.3, 1.5

[21] S. Takata and K. Aoki, *The ghost effect in the continuum limit for a vapor-gas mixture around condensed phases: asymptotic analysis of the Boltzmann equation*, Transp. Theory Stat. Phys., 30:205–237, 2001. 1.3

[22] K.-C. Wu, *Boltzmann systems of gas mixtures in 1D torus*, Quart. Appl. Math., 75:415-431, 2017. 1.3

[23] T. Yang and H. J. Yu, *Spectrum analysis of some kinetic equations*, Arch. Rational Mech. Anal., 222:731–768, 2016.

1.5, 2, 2.5, 2, 2.6