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Quantitative Pointwise Estimate of the Solution of the Linearized Boltzmann Equation

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Abstract We study the quantitative pointwise behavior of the solutions of the linearized Boltzmann equation for hard potentials, Maxwellian molecules and soft potentials, with Grad's angular cutoff assumption. More precisely, for solutions inside the finite Mach number region (time like region), we obtain the pointwise fluid structure for hard potentials and Maxwellian molecules, and optimal time decay in the fluid part and sub-exponential time decay in the non-fluid part for soft potentials. For solutions outside the finite Mach number region (space like region), we obtain sub-exponential decay in the space variable. The singular wave estimate, regularization estimate and refined weighted energy estimate play important roles in this paper. Our results extend the classical results of Liu and Yu (Commun Pure Appl Math 57:1543–1608, 2004), (Bull Inst Math Acad Sin 1:1–78, 2006), (Bull Inst Math Acad Sin 6:151–243, 2011) and Lee et al. (Commun Math Phys 269:17–37, 2007) to hard and soft potentials by imposing suitable exponential velocity weight on the initial condition.

Keywords Boltzmann equation · Fluid-like wave · Kinetic-like wave · Maxwellian states · Mixture Lemma · Singular wave · Pointwise estimate

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1 Introduction

1.1 The Models

In this paper, we consider the following Boltzmann equation:

$$\begin{cases} \partial_t F + \xi \cdot \nabla_x F = Q(F, F), \\ F(0, x, \xi) = F_0(x, \xi), \end{cases} \quad (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{1}$$

where $F(t, x, \xi)$ is the distribution function for the particles at time $t > 0$, position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and microscopic velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. The left-hand side of this equation models the transport of particles and the operator on the right-hand side models the effect of collisions on the transport

$$Q(F, G) = \int_{\mathbb{R}^3 \times S^2} |\xi - \xi_*|^\gamma B(\vartheta) \left\{ F(\xi_*')G(\xi') - F(\xi_*)G(\xi) \right\} d\xi_* d\omega.$$

In this paper, we consider the Maxwellian molecules ($\gamma = 0$), hard potentials ($0 < \gamma < 1$) and soft potentials ($-2 < \gamma < 0$); and $B(\vartheta)$ satisfies the Grad cutoff assumption

$$0 < B(\vartheta) \le C|\cos\vartheta|,$$

for some constant $C > 0$. Moreover, the post-collisional velocities satisfy

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi_*' = \xi + [(\xi - \xi_*) \cdot \omega]\omega,$$

and ϑ is defined by

$$\cos\vartheta = \frac{|(\xi - \xi_*) \cdot \omega|}{|\xi - \xi_*|}.$$

It is well known that the Maxwellians are steady state solutions to the Boltzmann equation. Thus, it is natural to linearize the Boltzmann equation (1) around a global Maxwellian

$$\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2} \right),$$

with the standard perturbation $f(t, x, \xi)$ to $\mathcal{M}$ as

$$F = \mathcal{M} + \mathcal{M}^{1/2} f.$$

After substituting the above ansatz into (1) and dropping the nonlinear term, we then have the linearized Boltzmann equation:

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x = Lf, \\ f(0, x, \xi) = f_0(x, \xi). \end{cases} \quad (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{2}$$

where

$$Lf := \mathcal{M}^{-1/2} \left[Q(\mathcal{M}, \mathcal{M}^{1/2} f) + Q(\mathcal{M}^{1/2} f, \mathcal{M}) \right].$$

The existence theorem of the linearized Boltzmann equation (2) can be found in [17]. It is well-known that the null space of L is a five-dimensional vector space with the orthonormal basis $\{\chi_i\}_{i=0}^4$, where

$$Ker(L) = \{\chi_0, \chi_i, \chi_4\} = \left\{ \mathcal{M}^{1/2}, \ \xi_i \mathcal{M}^{1/2}, \ \frac{1}{\sqrt{6}}(|\xi|^2 - 3)\mathcal{M}^{1/2} \right\}, \quad i = 1, 2, 3.$$

Based on this property, we can introduce the macro-micro decomposition: let P_0 be the orthogonal projection with respect to the L^2_ξ inner product onto $\mathrm{Ker}(L)$, and $\mathrm{P}_1 \equiv \mathrm{Id} - \mathrm{P}_0$.

1.2 Main Results

Before the presentation of the main theorem, let us define some notation in this paper. We denote $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$, $s \in \mathbb{R}$. For the microscopic variable ξ, we denote

$$|g|_{L^2_\xi} = \Big(\int_{\mathbb{R}^3} |g|^2 d\xi\Big)^{1/2}, \quad |g|_{L^\infty_\xi} = \sup_{\xi\in\mathbb{R}^3} |g(\xi)|,$$

and the weighted norms can be defined by

$$|g|_{L^2_\xi(m)} = \Big(\int_{\mathbb{R}^3} |g|^2 m d\xi\Big)^{1/2}, \quad |g|_{L^\infty_\xi(m)} = \sup_{\xi\in\mathbb{R}^3} \{|g(\xi)|m\},$$

here $m = m(t,x,\xi)$ is a weight function. The L^2_ξ inner product in $\mathbb{R}^3$ will be denoted by $\langle\cdot,\cdot\rangle_\xi$:

$$\langle f, g\rangle_\xi = \int f(\xi)\overline{g(\xi)}d\xi.$$

For the space variable x, we have similar notation. In fact,

$$|g|_{L^2_x} = \Big(\int_{\mathbb{R}^3} |g|^2 dx\Big)^{1/2}, \quad |g|_{L^\infty_x} = \sup_{x\in\mathbb{R}^3} |g(x)|.$$

The standard inner product will be denoted by (a,b) or $a\cdot b$ for any vectors $a, b \in \mathbb{R}^3$. For the Boltzmann equation, the natural norm in ξ is $|\cdot|_{L^2_\sigma}$, which is defined by

$$|g|^2_{L^2_\sigma} = |\,\langle\xi\rangle^{\frac{\gamma}{2}}\, g|^2_{L^2_\xi}.$$

Moreover, we define

$$\|g\|^2_{L^2} = \int_{\mathbb{R}^3} |g|^2_{L^2_\xi} dx, \quad \|g\|^2_{L^2(m)} = \int_{\mathbb{R}^3} |g|^2_{L^2_\xi(m)} dx,$$

and

$$\|g\|_{L^\infty_x L^\infty_\xi(m)} = \sup_{(x,\xi)\in\mathbb{R}^6} \{|g(x,\xi)|m\}, \quad \|g\|_{L^1_x L^2_\xi(m)} = \int_{\mathbb{R}^3} |g|_{L^2_\xi(m)} dx.$$

Finally, we define the high order Sobolev norms: let $s_1, s_2 \in \mathbb{N}$ and let α_1, α_2 be any multi-indexes with $|\alpha_1| \le s_1$ and $|\alpha_2| \le s_2$,

$$\|g\|_{H^{s_1}_x L^2_\xi(m)} = \sum_{|\alpha_1|\le s_1} \big\|\partial^{\alpha_1}_x g\big\|_{L^2(m)}, \quad \|g\|_{L^2_x H^{s_2}_\xi(m)} = \sum_{|\alpha_2|\le s_2} \|\partial^{\alpha_2}_\xi g\|_{L^2(m)}.$$

The domain decomposition plays an important role in our analysis, hence we need to define a cut-off function $\chi : \mathbb{R} \to \mathbb{R}$, which is a smooth non-increasing function, $\chi(s) = 1$ for $s \le 1$, $\chi(s) = 0$ for $s \ge 2$ and $0 \le \chi \le 1$. Moreover, we define $\chi_R(s) = \chi(s/R)$.

For simplicity of notation, hereafter, we abbreviate "$\le C$" to "$\lesssim$", where C is a positive constant depending only on fixed numbers.

The precise description of our main result is as follows.

Theorem 1 *Let f be a solution to (2) with initial data f_0 compactly supported in x-variable and bounded in the weighted ξ -space:*

$$f_0(x,\xi) \equiv 0, \quad \text{for } |x| \geq 1.$$

There exists a positive constant M such that the following hold:

(1) *As $0 \leq \gamma < 1$, for any given positive integer N, any given $0 < p \leq 2$ and any sufficiently small α, $\epsilon > 0$, there exist positive constants C, C_N, c_0 and c_ϵ such that f satisfies*

(a) *For $\langle x \rangle \leq 2Mt$,*

$$\begin{aligned} |f(t,x,\cdot)|_{L^2_\xi} \leq C_N \Bigg[& (1+t)^{-2}\left(1+\frac{(|x|-\mathrm{v}t)^2}{1+t}\right)^{-N} + (1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ & + \mathbf{1}_{\{|x|\leq \mathrm{v}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2} \\ & + e^{-c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)} + e^{-t/C}\Bigg] |||f_0|||. \end{aligned}$$

(b) *For $\langle x \rangle \geq 2Mt$,*

$$|f(t,x,\cdot)|_{L^2_\xi} \leq C\left(e^{-c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)} + t^5 e^{-c_\epsilon(\langle x\rangle+t)^{\frac{p}{p+1-\gamma}}}\right)|||f_0|||.$$

(2) *As $-2 < \gamma < 0$, for any given $0 < p \leq 2$ and any sufficiently small α, $\epsilon > 0$, there exist positive constants C, c, c_0 and c_ϵ such that f satisfies*

(a) *For $\langle x \rangle \leq 2Mt$,*

$$\begin{aligned} |f(t,x,\cdot)|_{L^2_\xi} \leq C\Bigg[& (1+t)^{-3/2} + e^{-c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}} \\ & + e^{-c_0\left(\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)}\Bigg]|||f_0|||. \end{aligned}$$

(b) *For $\langle x \rangle \geq 2Mt$,*

$$|f(t,x,\cdot)|_{L^2_\xi} \leq C\left[e^{-c_0\left(\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)} + t^5 e^{-c_\epsilon(\langle x\rangle+t)^{\frac{p}{p+1-\gamma}}}\right]|||f_0|||.$$

Here $\mathbf{1}_{\{\cdot\}}$ is the indicator function and

$$|||f_0||| \equiv \max\left\{\|f_0\|_{L^2(e^{7\alpha|\xi|^p})}, \|f_0\|_{L^1_x L^2_\xi}, \|f_0\|_{L^2(e^{\epsilon|\xi|^p})}, \|f_0\|_{L^\infty_x L^\infty_\xi(e^{8\alpha|\xi|^p})}\right\}.$$

The constant $\mathrm{v} = \sqrt{5/3}$ is the sound speed associated with the normalized global Maxwellian.

1.3 Method of Proof

The pointwise behavior of the solutions of the linearized Boltzmann equation has been investigated in [11–13] for the hard sphere case and [10] for the hard potential case. On the other hand, we are aware that a stronger velocity weight yields not only faster time decay (see Caflisch [2] and Strain-Guo [15]), but also space decay (see Golse-Poupand [6]). In this regard, we are interested in the pointwise behavior of the solution of the linearized Boltzmann equation with hard, Maxwellian and soft potentials, under an exponential velocity weight assumption on the initial condition.

In this paper, as in [10–13], we assume the initial condition is compactly supported in the space variable. This means that we want to understand the detailed propagation of localized perturbation. Furthermore, we assume an extra exponential velocity weight ($e^{\alpha|\xi|^p}$, $\alpha > 0$ small and $0 < p \leq 2$) on the initial data. Under this assumption, we get an accurate relationship between decay rates and weight functions.

The main idea of this paper is to combine the long wave-short wave decomposition, the wave-remainder decomposition, the weighted energy estimate and the regularization estimate together to analyze the solution. The long wave-short wave decomposition, which is based on the Fourier transform, gives the fluid structure or time decay estimate of the solution. The wave-remainder decomposition is used for extracting the singular waves. The weighted energy estimate is used for the pointwise estimate of the remainder term, in which the regularization estimate is used. We explain the idea in more details as below.

Inside the finite Mach number region (time like region), the solution is dominated by the long wave part. In order to obtain its decay rate, we devise different methods for $0 \leq \gamma \leq 1$ and $-2 < \gamma < 0$ respectively. For $0 \leq \gamma \leq 1$, it is well known that taking advantage of the spectrum information of the Boltzmann collision operator [4], the complex analysis (for $\gamma = 1$) and Fourier multiplier (for $0 \leq \gamma < 1$) techniques can be applied to obtain the pointwise structure of the fluid part. However, for $-2 < \gamma < 0$, the spectrum information is missing due to the weak damping for large velocity. Instead, we use similar arguments as those in the papers by Kawashima [9], Strain [14] and Strain-Guo [15] to get optimal decay in time. It is shown that the L^2 norm of the short wave exponentially decays in time for $0 \leq \gamma < 1$ essentially due to the spectrum gap, while it decays only sub-exponentially for $-2 < \gamma < 0$ if imposing an exponential velocity weight on the initial data.

As mentioned before, we use the wave-remainder decomposition to extract the singular waves in the short wave. This decomposition is based on a Picard-type iteration. Such an iteration is manipulated to construct the increasingly regular particle-like waves; in other words, the first several terms in the iteration (indeed, the first seven terms of the iteration) contain the most singular part of the solution, the so-called wave part. In virtue of the pointwise estimate of the damped transport equation, we have a rather accurate pointwise estimate for the wave part. On the other hand, the regularization estimate enables us to show the remainder becomes regular, and together with the L^2 decay of the short wave yields the L^∞ decay of the short wave. Combining this with the long wave, we establish the pointwise structure inside the finite Mach number region.

As for the structure outside the finite Mach number region (space like region), it remains to estimate the remainder part since we have already gained an explicit estimate for the wave part. The weighted energy estimate plays an essential role here. The weight functions not only are chosen delicately for different γ and p, but also takes the domain decomposition into account. It is noted that the sufficient understanding of the structure of the wave part, which has been obtained previously, is needed in the estimate. The regularization estimate makes it possible to do the higher order weighted energy estimate. Then the desired pointwise estimate follows from the Sobolev inequality.

1.4 Significant Points of the Paper

We point out some significant points of this paper as follows:

- Singular waves: The wave-remainder decomposition plays an important role in this paper. The significant points of the remainder part will be discussed later. To comprehend the wave part (singular waves), we have to establish a quantitative estimate of the damped transport operator $\mathbb{S}^t$ (see Lemma 8) first, since the singular waves can be represented by the combination of operators $\mathbb{S}^t$ and integral operator K (the definition of K can be found in (3)). For the hard sphere case ($\gamma = 1$), one can get the space and time decay of the wave part precisely without assuming any velocity weight on the initial data (see [11]). However, for the hard potential case ($0 < \gamma < 1$, see [10]), the Gaussian velocity weight is required to get the decay estimate. How the decay rate depends on the weight function is not well understood. In this regard, we reinvestigate the hard potential case, as well as Maxwellian molecules and soft potentials, assuming the initial condition is compactly supported in x and has a $L^\infty_\xi(e^{\alpha|\xi|^p})$ bound, here $0 < p \le 2$. Under this assumption, we get exponential time decay for $0 \le \gamma < 1$ and sub-exponential time decay for $-2 < \gamma < 0$. Simultaneously, we get sub-exponential space decay for $-2 < \gamma < 1$. This wave structure reveals accurate dependence of decay rates on initial weights, as opposed to the classical hard sphere case ([11]). There are some interesting observations:
  - (a) For the soft potential case, we get sub-exponential time decay with rate $e^{-\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}}$, which coincides with the results of Caflisch [2] and Strain-Guo [15]. These rates should be consistent since they studied the torus case with zero moment, and our wave part excludes the fluid part of the solution.
  - (b) We give a very precise relation between initial velocity weights and the asymptotic behavior of the solution ($|x|$ large), i.e., if the initial condition is with weight $e^{\alpha|\xi|^p}$, then we have sub-exponential decay $e^{-\alpha^{\frac{1-\gamma}{p+1-\gamma}} |x|^{\frac{p}{p+1-\gamma}}}$. Moreover, the asymptotic behavior of the wave part (Lemma 8) and the remainder part (Equation (64)) are matched with our estimate.
- Regularization estimate: The regularization estimate plays a crucial role in this paper (see Lemma 14), which enables us to obtain the pointwise estimate without regularity assumption on the initial data. We here emphasize that there are two types of regularization estimates: in the standard L^2 norm and L^2 norm with weight (see Lemma 14). For the hard sphere case [11,12,13], the regularization estimate in the standard L^2 norm is enough to control the solution both inside and outside the finite Mach number regions. However, for other cases ($-2 < \gamma < 1$), in addition to the standard L^2 regularization estimate to control the solution inside the finite Mach number region, one also needs the regularization in the weighted (in both velocity and space) L^2 space to control the solution outside the finite Mach number region.
  In the proof of Lemma 14, it reveals that the mixture of the two operators $\mathbb{S}^t$ and K transports the regularity in the microscopic velocity ξ from K to the regularity in the space x. Note that K is an integral operator from L^2_ξ to H^1_ξ only when $\gamma > -2$, that is why we restrict ourselves to the case $\gamma > -2$ in this paper. This notion (in the standard

L^2 space) was firstly introduced by Liu and Yu for the hard sphere case [11–13] and they call it as Mixture Lemma. However, their machinery is specifically designed for the hard sphere case. When naively applying to hard potentials, it will result in $e^{\infty\cdot\alpha|\xi|^p}$ weight imposed on the initial data. To resolve this difficulty, we introduce the differential operator $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, which commutes with the free transport operator. This operator is crucial since it is a bridge between the x derivative and the ξ derivative. We remark that the crucial operator $\mathcal{D}_t$ was firstly introduced in the paper by Gualdani, Mischler and Mouhot [8], and Wu [19] applied it to give an alternative proof of the Mixture Lemma used in [10–13]. Through this operator $\mathcal{D}_t$, mixing the operator $\mathbb{S}^t$ with K enough times will help the ξ regularity transfer to the x regularity (here "enough times" depends on how many ξ regularities we want to transfer) without any regularity assumption in ξ on the initial data. In other words, mixing operators $\mathbb{S}^t$ and K enough times will lead to x regularity **automatically.**

- Weighted energy estimate: The pointwise estimate of the solution outside the finite Mach number region is constructed by the weighted energy estimate. The time dependent weight functions are chosen accordingly to different γ (interactions between particles) and p (initial velocity weight). For the hard sphere case $\gamma = 1$ (see Liu-Yu [11–13]), the weight function depends only on the time and the space variables, and exponentially grows in space (it takes the form $\exp\{\frac{|x|-Mt}{D}\}$). Since it commutes with the integral operator K, the estimate is relatively simple. However, for this paper $-2 < \gamma < 1$, the weight function is much more complicated. Indeed, it depends on the velocity variable as well and thus does not commute with the integral operator K. This results in that the coercivity of linearized collision operator cannot be applied straightforwardly. The difficulty is eventually overcome by fine tuning the weight functions, introducing refined space-velocity domain decomposition and analyzing the integral operator K with weight accordingly (see Lemma 16).

The rest of this paper is organized as follows: We first prepare some basic properties of the collision operator in Sect. 2. After that, we construct the long wave-short wave decomposition in Sect. 3 and the wave-remainder decomposition in Sect. 4. Finally, we establish the global wave structures in Sect. 5.

2 Basic Properties of the Collision Operator

For the linearized Boltzmann equation (2) , the collision operator L consists of a multiplicative operator $\nu(\xi)$ and an integral operator K:

$$Lf = -\nu(\xi)f + Kf,$$

where

$$\nu(\xi) = \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}(\xi_*)d\xi_* d\omega,$$

and

$$Kf = -K_1 f + K_2 f \tag{3}$$

is defined as [5,7]:

$$K_1 f := \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi)\mathcal{M}^{1/2}(\xi_*) f(\xi_*) d\xi_* d\omega,$$

$$K_2 f := \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi_*)\mathcal{M}^{1/2}(\xi') f(\xi_*') d\xi_* d\omega$$
$$+ \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi_*)\mathcal{M}^{1/2}(\xi_*') f(\xi') d\xi_* d\omega.$$

In this section we will present a number of properties and estimates of the operators L, $\nu(\xi)$ and K. To begin with, we list some fundamental properties of these operators, which can be found in [2,6,7,15].

Lemma 2 *For any $g \in L^2_\sigma$, we have the coercivity estimate of the linearized collision operator L:*

$$\langle g, Lg\rangle_\xi \lesssim -|\mathrm{P}_1 g|^2_{L^2_\sigma}.$$

For the multiplicative operator $\nu(\xi)$, there exist positive constants ν_0 and ν_1 such that

$$\nu_0(1 + |\xi|)^\gamma \le \nu(\xi) \le \nu_1(1 + |\xi|)^\gamma. \tag{4}$$

Moreover, for each multi-index α,

$$|\partial^\alpha_\xi \nu(\xi)| \lesssim \langle\xi\rangle^{\gamma - |\alpha|}. \tag{5}$$

For the integral operator K,

$$Kf = -K_1 f + K_2 f = \int_{\mathbb{R}^3} -k_1(\xi, \xi_*) f(\xi_*) d\xi_* + \int_{\mathbb{R}^3} k_2(\xi, \xi_*) f(\xi_*) d\xi_*,$$

the kernels $k_1(\xi, \xi_*)$ and $k_2(\xi, \xi_*)$ satisfy

$$k_1(\xi, \xi_*) \lesssim |\xi - \xi_*|^\gamma \exp\left\{-\frac{1}{4}\left(|\xi|^2 + |\xi_*|^2\right)\right\},$$

and

$$k_2(\xi, \xi_*) = a(\xi, \xi_*, \kappa) \exp\left(-\frac{(1-\kappa)}{8}\left[\frac{\left(|\xi|^2 - |\xi_*|^2\right)^2}{|\xi - \xi_*|^2} + |\xi - \xi_*|^2\right]\right),$$

for any $0 < \kappa < 1$, together with

$$a(\xi, \xi_*, \kappa) \le C_\kappa |\xi - \xi_*|^{-1}(1 + |\xi| + |\xi_*|)^{\gamma - 1}.$$

In addition, their derivatives as well have similar estimates, i.e.,

$$|\nabla_\xi k_1(\xi, \xi_*)|, |\nabla_{\xi_*} k_1(\xi, \xi_*)| \lesssim |\xi - \xi_*|^{\gamma - 1} \exp\left\{-\frac{1}{4}\left(|\xi|^2 + |\xi_*|^2\right)\right\},$$

$$|\nabla_\xi k_2(\xi, \xi_*)|, |\nabla_{\xi_*} k_2(\xi, \xi_*)| \lesssim |\nabla_\xi a(\xi, \xi_*)| \exp\left(-\frac{(1-\kappa)}{8}\left[\frac{\left(|\xi|^2 - |\xi_*|^2\right)^2}{|\xi - \xi_*|^2} + |\xi - \xi_*|^2\right]\right).$$

According to the above estimates of the integral operator K, it follows that for any $g_1, g_2 \in L^2_\sigma \cap L^2_\xi$,

$$\left|\langle g_1, K g_2\rangle_\xi\right| \lesssim \begin{cases} |g_1|_{L^2_\xi} |g_2|_{L^2_\xi} & \text{for} \quad 0 \le \gamma < 1, \\ |g_1|_{L^2_\sigma} |g_2|_{L^2_\sigma} & \text{for} \quad -2 < \gamma < 0, \end{cases} \tag{6}$$

and for any $g_0 \in L^2$,

$$\|K g_0\|_{H^1_\xi L^2_x} \lesssim \|g_0\|_{L^2}. \tag{7}$$

Next, we will provide the sup norm estimate for the integral operator K, which extends (6.2) in Proposition 6.1 of Caflisch [2] to the case $-2 < \gamma < 1$.

Lemma 3 *Let* $0 < p \leq 2$. *For any* $\beta_1 \geq 0$ *and* $0 \leq \beta_2 < \frac{1}{4}$, *the operator* K *satisfies*

$$|Kg(\xi)| = \left| \int k(\xi, \xi_*) g(\xi_*) d\xi_* \right| \lesssim \langle \xi \rangle^{-\beta_1+\gamma-2} e^{-\beta_2|\xi|^p} |g|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta_1} e^{\beta_2|\xi|^p}\right)}. \tag{8}$$

Proof We first give an estimate on the kernel k, which extends Proposition 5.1 in [2] to the case $-2 < \gamma < 1$. For any $0 < \kappa < 1$, we have

$$\begin{aligned} k_1(\xi, \xi_*) &\lesssim |\xi - \xi_*|^\gamma \exp\left\{-\frac{1}{4}\left(|\xi|^2 + |\xi_*|^2\right)\right\} \\ &\leq C_\kappa |\xi - \xi_*|^{-2} (1 + |\xi| + |\xi_*|)^{\gamma-1} \exp\left\{-\frac{1}{4}(1-\kappa)\left(|\xi|^2 + |\xi_*|^2\right)\right\}, \end{aligned}$$

for some constant $C_\kappa > 0$. Since

$$\frac{1}{4}\left(|\xi|^2 + |\xi_*|^2\right) \geq \frac{1}{8}\left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2}\right),$$

we deduce

$$\begin{aligned} k_1(\xi, \xi_*) \lesssim\, & |\xi - \xi_*|^{-2} (1 + |\xi| + |\xi_*|)^{\gamma-1} \\ & \times \exp\left\{-\frac{(1-\kappa)}{8}\left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2}\right)\right\}. \end{aligned}$$

Together with $k_2(\xi, \xi_*)$, it follows that for any $0 < \kappa < 1$,

$$\begin{aligned} |k(\xi, \xi_*)| \lesssim\, & |\xi - \xi_*|^{-2} (1 + |\xi| + |\xi_*|)^{\gamma-1} \\ & \times \exp\left\{-\frac{(1-\kappa)}{8}\left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2}\right)\right\}. \end{aligned} \tag{9}$$

Now, in view of (9),

$$\begin{aligned} &\left| \int k(\xi, \xi_*) g(\xi_*) d\xi_* \right| \\ &\lesssim \int_{\mathbb{R}^3} \frac{1}{|\xi - \xi_*|^2} (1 + |\xi| + |\xi_*|)^{\gamma-1} \\ &\quad \times \exp\left(-\frac{(1-\kappa)}{8}\left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2}\right)\right) |g(\xi_*)| d\xi_* \\ &\lesssim e^{-\beta_2|\xi|^p} (1 + |\xi|)^{\gamma-1} |g|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta_1} e^{\beta_2|\xi|^p}\right)} \cdot \mathbb{A}(\xi), \end{aligned}$$

where

$$\mathbb{A}(\xi) = \int_{\mathbb{R}^3} \frac{1}{|\xi - \xi_*|^2} \exp\left\{ -\frac{(1-\kappa)}{8} \left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2} \right) \right.$$
$$\left. + \beta_2 |\xi|^p - \beta_2 |\xi_*|^p \right\} (1 + |\xi_*|)^{-\beta_1} \, d\xi_*.$$

Notice that

$$|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2} \geq 2 \left||\xi|^2 - |\xi_*|^2\right|,$$

and

$$\left||\xi|^p - |\xi_*|^p\right| \leq \left||\xi|^2 - |\xi_*|^2\right|^{\frac{p}{2}}.$$

Picking $\kappa = \frac{1-4\beta_2}{2}$ and $\varpi = \frac{1-4\beta_2}{16}$ yields

$$\begin{aligned}
\mathbb{A}(\xi) &\lesssim \int_{\mathbb{R}^3} \frac{(1+|\xi_*|)^{-\beta_1}}{|\xi - \xi_*|^2} \exp\left\{ -\varpi \left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2} \right) \right. \\
&\qquad \left. - \beta_2 \left||\xi|^2 - |\xi_*|^2\right| + \beta_2 \left||\xi|^2 - |\xi_*|^2\right|^{\frac{p}{2}} \right\} d\xi_* \\
&\lesssim \int_{\mathbb{R}^3} \frac{1}{|\xi - \xi_*|^2} (1+|\xi_*|)^{-\beta_1} \exp\left\{ -\varpi \left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2} \right) \right\} d\xi_* \\
&\equiv I,
\end{aligned}$$

since

$$\exp\left(-\beta_2 \left||\xi|^2 - |\xi_*|^2\right| + \beta_2 \left||\xi|^2 - |\xi_*|^2\right|^{\frac{p}{2}} \right) < e^{\beta_2},$$

uniformly in ξ, ξ_* and p. We split I into two parts: I_1, with $|\xi_*| < \frac{1}{3}|\xi|$, and I_2, with $|\xi_*| > \frac{1}{3}|\xi|$. Then

$$I_1 \lesssim e^{-\frac{\varpi}{4}|\xi|^2} \tag{10}$$

since $|\xi - \xi_*|^2 \geq \frac{4}{9}|\xi|^2$ in that domain. In the domain integration for I_2, we have $(1 + |\xi_*|) > \frac{1}{3}(1 + |\xi|)$, so that

$$\begin{aligned}
I_2 &\lesssim (1+|\xi|)^{-\beta_1} \int_{|\xi_*| > \frac{1}{3}|\xi|} \frac{1}{|\xi - \xi_*|^2} \exp\left\{ -\varpi \left(|\xi - \xi_*|^2 + \frac{\left||\xi|^2 - |\xi_*|^2\right|^2}{|\xi - \xi_*|^2} \right) \right\} d\xi_* \\
&\lesssim (1+|\xi|)^{-\beta_1 - 1},
\end{aligned} \tag{11}$$

due to Proposition 5.3 in [2]. Combining (10) and (11), we find

$$I \lesssim (1+|\xi|)^{-\beta_1 - 1},$$

and hence

$$|Kg(\xi)| \lesssim \langle\xi\rangle^{-\beta_1 + \gamma - 2} e^{-\beta_2 |\xi|^p} |g|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta_1} e^{\beta_2|\xi|^p}\right)}.$$

□

In fact, in the course of the proof of this lemma, one can also infer that $k(\xi, \xi_*)$ is integrable in ξ_* with

$$\int_{\mathbb{R}^3} |k(\xi, \xi_*)| \, d\xi_* \lesssim (1 + |\xi|)^{\gamma - 2}, \tag{12}$$

for $-2 < \gamma < 1$.

Regarding the weighted energy estimate, the following weight functions $\mu(x, \xi)$ will be taken into account:

$$\mu(x, \xi) = 1, \quad \text{or} \quad \exp(\epsilon\theta(x, \xi)), \tag{13}$$

where

$$\begin{aligned}\theta(x,\xi) = {} & 5\Big(\delta\langle x\rangle\Big)^{\frac{p}{p+1-\gamma}}\left(1-\chi\left(\delta\langle x\rangle\langle\xi\rangle^{\gamma-p-1}\right)\right) \\ & + \left[\left(1-\chi\Big(\delta\langle x\rangle\langle\xi\rangle^{\gamma-p-1}\Big)\right)[\delta\langle x\rangle]\langle\xi\rangle^{\gamma-1} + 3\langle\xi\rangle^{p}\right]\chi\left(\delta\langle x\rangle\langle\xi\rangle^{\gamma-p-1}\right),\end{aligned}$$

with $0 < p \leq 2$; the constants ϵ and $\delta > 0$ will be chosen sufficiently small later on. Among them, the choices of the functions θ are motivated by [3]. Under these considerations, we need the estimates of K as below.

Lemma 4 *Let* $0 < p \leq 2$ *and* $g_1, g_2 \in L^2_\sigma \cap L^2_\xi$. *Then for any* $\epsilon \geq 0$ *sufficiently small,*

$$\left|\left\langle g_1, e^{\epsilon\theta(x,\xi)} K e^{-\epsilon\theta(x,\xi)} g_2\right\rangle_\xi - \langle g_1, K g_2\rangle_\xi\right| \lesssim \begin{cases} \epsilon\, |g_1|_{L^2_\xi} |g_2|_{L^2_\xi} & \textit{for} \quad 0 \leq \gamma < 1, \\ \epsilon\, |g_1|_{L^2_\sigma} |g_2|_{L^2_\sigma} & \textit{for} \quad -2 < \gamma < 0. \end{cases} \tag{14}$$

In particular,

$$\left|\left\langle g_1, e^{\epsilon\theta(x,\xi)} K e^{-\epsilon\theta(x,\xi)} g_2\right\rangle_\xi\right| \lesssim \begin{cases} |g_1|_{L^2_\xi} |g_2|_{L^2_\xi} & \textit{for} \quad 0 \leq \gamma < 1, \\ |g_1|_{L^2_\sigma} |g_2|_{L^2_\sigma} & \textit{for} \quad -2 < \gamma < 0. \end{cases} \tag{15}$$

Consequently, for $g_0 \in L^2(\mu)$,

$$\|K g_0\|_{L^2(\mu)} \lesssim \|g_0\|_{L^2(\mu)}. \tag{16}$$

Proof It suffices to show that for $j = 1$ and 2,

$$\left| e^{\epsilon\theta(x,\xi)} K_j e^{-\epsilon\theta(x,\xi)} - K_j \right|_{L^2_\xi} \lesssim \epsilon. \tag{17}$$

By the Cauchy-Schwartz inequality,

$$\frac{\left(|\xi|^2 - |\xi_*|^2\right)^2}{|\xi - \xi_*|^2} + |\xi - \xi_*|^2 \geq 2\left||\xi|^2 - |\xi_*|^2\right|. \tag{18}$$

Further, we rewrite

$$
\begin{aligned}
&e^{\epsilon\theta(x,\xi)}k_2(\xi,\xi_*)\,e^{-\epsilon\theta(x,\xi_*)}-k_2(\xi,\xi_*)\\
&\quad=\left\{\widetilde{a}(\xi,\xi_*)\exp\left(-\frac{1}{32}\left[\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2}+|\xi-\xi_*|^2\right]\right)\right\}\\
&\qquad\times\left\{\exp\left(-\frac{1}{32}\left[\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2}+|\xi-\xi_*|^2\right]\right)\times\left(\exp\{\epsilon(\theta(x,\xi)-\theta(x,\xi_*))\}-1\right)\right\}\\
&\quad\equiv p(\xi,\xi_*)\,s(\epsilon,\xi,\xi_*),
\end{aligned}
$$

where $\widetilde{a}(\xi,\xi_*)=a\left(\xi,\xi_*,\frac{1}{2}\right)$. We claim that

$$\sup_{\xi,\xi_*}|s(\epsilon,\xi,\xi_*)|\to 0\ \text{ as } \epsilon\to 0.$$

Since $\left|\frac{\partial}{\partial|\xi|}\theta(x,\xi)\right|\lesssim\langle\xi\rangle^{p-2}|\xi|$ uniformly in x for $\xi\neq 0$ and $p\in(0,2]$, we obtain

$$
\begin{aligned}
|\theta(x,\xi)-\theta(x,\xi_*)|=|\theta(x,|\xi|)-\theta(x,|\xi_*|)|&\lesssim\left\langle\xi^*\right\rangle^{p-2}\left|\xi^*\right|\,||\xi|-|\xi_*||\\
&\le c_1\left||\xi|^2-|\xi_*|^2\right|,
\end{aligned}\tag{19}
$$

for some $|\xi^*|$ between $|\xi|$ and $|\xi_*|$, and some constant $c_1>0$ depending only upon γ and p. Together with (18), whenever $\epsilon>0$ is sufficiently small with $0\le\epsilon c_1<\frac{1}{32}$,

$$\sup_{\xi,\xi_*}|s(\epsilon,\xi,\xi_*)|\le\epsilon c_1\sup_{\xi,\xi_*}\left(\left||\xi|^2-|\xi_*|^2\right|\exp\left[-\frac{1}{32}\left||\xi|^2-|\xi_*|^2\right|\right]\right).$$

In other words,

$$\sup_{\xi,\xi_*}|s(\epsilon,\xi,\xi_*)|\to 0\ \text{ as } \epsilon\to 0.$$

Since $p(\xi,\xi_*)$ is also a kernel of a bounded operator on $L^2_\xi\,(L^2_\sigma)$ for $0\le\gamma<1$ $(-2<\gamma<0)$, this completes the estimate for K_2. As to the case K_1, it is easy and we omit the details.

According to the above discussion, we readily obtain that for $g_0\in L^2(\mu)$,

$$\|Kg_0\|_{L^2(\mu)}\lesssim\|g_0\|_{L^2(\mu)}.$$

Precisely,

$$
\begin{aligned}
\|Kg_0\|_{L^2(\mu)}&=\sup_{g_1\in L^2(\mu),\|g_1\|_{L^2(\mu)}\le 1}\int(Kg_0)\,g_1\mu\,dx\,d\xi\\
&=\sup_{g_1\in L^2(\mu),\|g_1\|_{L^2(\mu)}\le 1}\int\left\langle\mu^{1/2}K\mu^{-1/2}\left(\mu^{1/2}g_0\right),\mu^{1/2}g_1\right\rangle_{L^2_\xi}dx\\
&\le C\|\mu^{1/2}g_0\|_{L^2}=C\|g_0\|_{L^2(\mu)}.
\end{aligned}
$$

□

We here remark that this lemma also includes the following weighted estimate: for any $g_0\in L^2\left(e^{\alpha|\xi|^p}\right)$ with $\alpha>0$ small and $0<p\le 2$,

$$\|Kg_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}\lesssim\|g_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}.\tag{20}$$

Before we end this section, we recall the spectrum $\mathrm{Spec}(\eta)$, $\eta \in \mathbb{R}$, of the operator $-i\xi\cdot\eta+L$, in preparation for estimating the Green function of the linearized Boltzmann equation in the next section.

Lemma 5 [4] *Set $\eta = |\eta|\omega$. For any $0 \le \gamma < 1$, there exist $\delta > 0$ and $\tau = \tau(\delta) > 0$ such that*

(1) *For any $|\eta| > \delta$,*

$$Spec(\eta) \subset \{z \in \mathbb{C} : Re(z) < -\tau\}.$$

(2) *For any $|\eta| < \delta$, the spectrum within the region $\{z \in \mathbb{C} : Re(z) > -\tau\}$ consists of exactly five eigenvalues $\{\varrho_j(\eta)\}_{j=0}^4$,*

$$Spec(\eta) \cap \{z \in \mathbb{C} : Re(z) > -\tau\} = \{\varrho_j(\eta)\}_{j=0}^4,$$

associated with corresponding eigenvectors $\{e_j(\eta)\}_{j=0}^4$. They have the expansions

$$\varrho_j(\eta) = -i\, a_j|\eta| - A_j|\eta|^2 + O(|\eta|^3),$$

$$e_j(\eta) = E_j + O(|\eta|),$$

with $A_j > 0$ and

$$\begin{cases} a_0 = \sqrt{\frac{5}{3}}, \quad a_1 = -\sqrt{\frac{5}{3}}, \quad a_2 = a_3 = a_4 = 0, \\ E_0 = \sqrt{\frac{3}{10}}\chi_0 + \sqrt{\frac{1}{2}}\omega\cdot\overline{\chi} + \sqrt{\frac{1}{5}}\chi_4, \\ E_1 = \sqrt{\frac{3}{10}}\chi_0 - \sqrt{\frac{1}{2}}\omega\cdot\overline{\chi} + \sqrt{\frac{1}{5}}\chi_4, \\ E_2 = -\sqrt{\frac{2}{5}}\chi_0 + \sqrt{\frac{3}{5}}\chi_4, \\ E_3 = \omega_1\cdot\overline{\chi}, \\ E_4 = \omega_2\cdot\overline{\chi}, \end{cases}$$

where $\overline{\chi} = (\chi_1, \chi_2, \chi_3)$, and $\{\omega_1, \omega_2, \omega\}$ is an orthonormal basis of $\mathbb{R}^3$. Here $\{e_j(\eta)\}_{j=0}^4$ can be normalized by $\langle e_j(-\eta), e_l(\eta)\rangle_\xi = \delta_{jl}$, $0 \le j, l \le 4$.

Moreover, the semigroup $e^{(-i\xi\cdot\eta+L)t}$ can be decomposed as

$$e^{(-i\xi\cdot\eta+L)t}g = e^{(-i\xi\cdot\eta+L)t}\Pi_\eta^\perp g + \mathbf{1}_{\{|\eta|<\delta\}}\sum_{j=0}^{4} e^{\varrho_j(\eta)t}\langle e_j(-\eta), g\rangle_\xi e_j(\eta),$$

where $\mathbf{1}_{\{\cdot\}}$ is the indicator function and there exists $C > 0$ such that

$$\left|e^{(-i\xi\cdot\eta+L)t}\Pi_\eta^\perp g\right|_{L^2_\xi} \le e^{-Ct}|g|_{L^2_\xi}.$$

3 Long Wave-Short Wave Decomposition

In order to study the large time behavior, we introduce the long wave-short wave decomposition. By the Fourier transform, the solution of the linearized Boltzmann equation can be written as

$$\mathbb{G}^t f_0 = f(t, x, \xi) = \int_{\mathbb{R}^3} e^{i\eta x + (-i\xi\cdot\eta+L)t}\widehat{f}_0(\eta, \xi)d\eta, \tag{21}$$

where $\widehat{f}$ means the Fourier transform in the space variable and $\mathbb{G}^t$ is the solution operator (or Green function) of the linearized Boltzmann equation. We can decompose the solution f into the long wave part f_L and the short wave part f_S given respectively by

$$\begin{aligned} f_L &= \int_{|\eta|<\delta} e^{i\eta x+(-i\xi\cdot\eta+L)t}\widehat{f_0}(\eta,\xi)d\eta, \\ f_S &= \int_{|\eta|>\delta} e^{i\eta x+(-i\xi\cdot\eta+L)t}\widehat{f_0}(\eta,\xi)d\eta, \end{aligned} \tag{22}$$

here the positive number δ is defined as in Lemma 5.

For the case $0 \le \gamma < 1$, we further decompose the long wave part as the fluid part and non-fluid part, i.e., $f_L = f_{L;0} + f_{L;\perp}$, where

$$\begin{aligned} f_{L;0} &= \int_{|\eta|<\delta} \sum_{j=0}^{4} e^{\varrho_j(\eta)t} e^{i\eta x} \big\langle e_j(-\eta), \hat{f}_0\big\rangle_\xi e_j(\eta)d\eta, \\ f_{L;\perp} &= \int_{|\eta|<\delta} e^{i\eta x} e^{(-i\xi\cdot\eta+L)t}\Pi_\eta^\perp \hat{f}_0 d\eta. \end{aligned} \tag{23}$$

Taking advantage of the spectrum information of the Boltzmann collision operator (Lemma 5), we will obtain the L^2 estimates of the non-fluid long wave part and short wave part directly. On the other hand, the Fourier multiplier techniques can be applied to obtain the pointwise structure of the fluid part. The estimate of this part is exactly the same as in the Landau case [18] and hence we omit the details.

Proposition 6 *Let $0 \le \gamma < 1$ and let f be the solution of the linearized Boltzmann equation.*

(a) (Fluid wave $f_{L;0}$) Let $\mathrm{v} = \sqrt{5/3}$ be the sound speed associated with the normalized global Maxwellian. For any given positive integer N and any given Mach number $\mathbb{M} > 1$, there exists $C_N > 0$ such that if $|x| \le (\mathbb{M}+1)\,\mathrm{v}t$, then

$$\begin{aligned} \left|f_{L;0}\right|_{L^2_\xi} \le C_N \Bigg[&(1+t)^{-2}\left(1+\frac{(|x|-\mathrm{v}t)^2}{1+t}\right)^{-N} + (1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ &+\mathbf{1}_{\{|x|\le \mathrm{v}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2}\Bigg]\|f_0\|_{L^1_x L^2_\xi}. \end{aligned} \tag{24}$$

(b) (Non-fluid long wave $f_{L;\perp}$) There exists a constant $c > 0$ such that

$$\|f_{L;\perp}\|_{H^s_x L^2_\xi} \lesssim e^{-ct}\|f_0\|_{L^2} \tag{25}$$

for any $s > 0$.

(c) (Short wave f_S) There exists a constant $c > 0$ such that

$$\|f_S\|_{L^2} \lesssim e^{-ct}\|f_0\|_{L^2}. \tag{26}$$

Alternatively, for $-2 < \gamma < 0$, the spectrum information is missing due to the weak damping for large velocity. Instead, we use similar arguments as those in the papers by Kawashima [9], Strain [14] and Strain-Guo [15] to get optimal time decay. All related estimates have been done in [15] and thereby we simply sketch the proof.

Proposition 7 *Let $-2 < \gamma < 0$ and let f be the solution of the linearized Boltzmann equation. For $0 < p \leq 2$ and $\alpha > 0$ small, we have*

(a) (Long wave f_L)

$$\|f_L\|_{L_x^\infty L_\xi^2} \lesssim (1+t)^{-\frac{3}{2}} \|f_0\|_{L_x^1 L_\xi^2(e^{\alpha|\xi|^p})}. \tag{27}$$

(b) (Short wave f_S) There exists $c_{p,\gamma} > 0$ such that

$$\|f_S\|_{L^2} \lesssim e^{-c_{p,\gamma}\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}} \|f_0\|_{L^2(e^{\alpha|\xi|^p})}. \tag{28}$$

Proof Following the same argument as in [14], we find that there exists a time-frequency functional $\mathcal{E}(t,\eta)$ such that

$$\mathcal{E}(t,\eta) \approx \left|\widehat{f}(t,\eta)\right|_{L_\xi^2}^2, \tag{29}$$

where for any $t > 0$ and $\eta \in \mathbb{R}^3$, we have

$$\partial_t \mathcal{E}(t,\eta) + \sigma\widehat{\rho}(\eta)\left|\widehat{f}(t,\eta)\right|_{L_\sigma^2}^2 \leq 0, \tag{30}$$

for some constant $\sigma > 0$. Here we use the notation $\widehat{\rho}(\eta) = \min\{1, |\eta|^2\}$. Moreover, there exists a weighted time-frequency functional $\mathcal{E}_{\alpha,p}(t,\eta)$ such that

$$\mathcal{E}_{\alpha,p}(t,\eta) \approx \left|e^{\frac{\alpha|\xi|^p}{2}}\widehat{f}(t,\eta)\right|_{L_\xi^2}^2, \tag{31}$$

where for any $t > 0$ and $\eta \in \mathbb{R}^3$, we have

$$\partial_t \mathcal{E}_{\alpha,p}(t,\eta) \leq 0. \tag{32}$$

For the long wave part, the argument basically follows the paper [14]. In fact, by (30) and (32), we have

$$\|f_L\|_{H_x^k L_\xi^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}} \|f_0\|_{L_x^1 L_\xi^2(e^{\alpha|\xi|^p})}. \tag{33}$$

With the aid of the Sobolev inequality, (27) holds. For the short wave f_S, applying the same argument as in [15, Sect. 5], together with (30) and (32), we get (28). □

4 Wave-Remainder Decomposition

In this section we introduce the wave-remainder decomposition, which is the key decomposition in our paper. The strategy is to design a Picard-type iteration, treating Kf as a source term. Specifically, the zero order approximation $h^{(0)}$ of the linearized Boltzmann equation (2) is defined as

$$\begin{cases} \partial_t h^{(0)} + \xi\cdot\nabla_x h^{(0)} + \nu(\xi)h^{(0)} = 0, \\ h^{(0)}(0,x,\xi) = f_0(x,\xi), \end{cases} \tag{34}$$

and thus the difference $f - h^{(0)}$ satisfies

$$\begin{cases} \partial_t(f-h^{(0)}) + \xi\cdot\nabla_x(f-h^{(0)}) + \nu(\xi)(f-h^{(0)}) = K(f-h^{(0)}) + Kh^{(0)}, \\ (f-h^{(0)})(0,x,\xi) = 0. \end{cases}$$

We can define the j^{th} order approximation $h^{(j)}$, $j \geq 1$, as

$$\begin{cases} \partial_t h^{(j)} + \xi\cdot\nabla_x h^{(j)} + \nu(\xi)h^{(j)} = Kh^{(j-1)}, \\ h^{(j)}(0,x,\xi) = 0. \end{cases} \tag{35}$$

Now, the wave part and the remainder part can be defined as follows:

$$W^{(6)} = \sum_{j=0}^{6} h^{(j)}, \quad \mathcal{R}^{(6)} = f - W^{(6)}, \tag{36}$$

$\mathcal{R}^{(6)}$ solving the equation:

$$\begin{cases} \partial_t \mathcal{R}^{(6)} + \xi \cdot \nabla_x \mathcal{R}^{(6)} = L\mathcal{R}^{(6)} + Kh^{(6)}, \\ \mathcal{R}^{(6)}(0, x, \xi) = 0. \end{cases} \tag{37}$$

In fact, $\mathcal{R}^{(6)}$ can be solved by using Green function $\mathbb{G}^t$ for the full linearized Boltzmann equation, namely

$$\mathcal{R}^{(6)} = \int_0^t \mathbb{G}^{t-s} K h^{(6)}(s) ds. \tag{38}$$

4.1 Estimates on the Wave Part

We denote the solution operator of the damped transport equation

$$\begin{cases} \partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0, \\ h(0, x, \xi) = h_0, \end{cases} \tag{39}$$

by $\mathbb{S}^t$, i.e., $h(t) = \mathbb{S}^t h_0$. By the method of characteristics, the solution $\mathbb{S}^t h_0$ can be written down explicitly; that is,

$$\mathbb{S}^t h_0(x, \xi) = h(t, x, \xi) = e^{-\nu(\xi)t} h_0(x - \xi t, \xi). \tag{40}$$

In addition, it is easy to see that $h^{(j)}$ can be represented by the combination of operators $\mathbb{S}^t$ and K.

In the sequel, we will find the pointwise decay of the solution $\mathbb{S}^t h_0$ in both time variable t and space variable x upon imposing some weights on the velocity variable ξ. Through the pointwise decay of the solution $\mathbb{S}^t h_0$ and Duhamel's principle, we thereby obtain the pointwise estimate of the wave part $W^{(6)}$. Moreover, we will provide the L^2 estimate for $\mathbb{S}^t h_0$ with an exponential weight as well, which leads us to obtain the L^2 estimates for $h^{(j)}$ $(0 \le j \le 6)$ and $\mathcal{R}^{(6)}$.

Lemma 8 *Let $\alpha > 0$, $0 < p \le 2$ and $\beta > 3/2$. Then for $0 \le \gamma < 1$,*

$$\left|\mathbb{S}^t h_0(x, \cdot)\right|_{L^\infty_\xi(\langle\xi\rangle^\beta)} \le \sup_y e^{-c_0\left(t + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right)} \left|h_0(y, \cdot)\right|_{L^\infty_\xi\left(e^{\alpha|\xi|^p}\langle\xi\rangle^\beta\right)}, \tag{41}$$

and for $-2 < \gamma < 0$,

$$\left|\mathbb{S}^t h_0(x, \cdot)\right|_{L^\infty_\xi(\langle\xi\rangle^\beta)} \le C(\alpha, \gamma) \sup_y e^{-c_0\left(\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}} + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right)} \left|h_0(y, \cdot)\right|_{L^\infty_\xi\left(e^{\alpha|\xi|^p}\langle\xi\rangle^\beta\right)}, \tag{42}$$

where $c_0 = c(\gamma) > 0$ and $C(\alpha, \gamma) > 0$ are constants.

Proof In view of (40), let $x - \xi t = y$ and then it suffices to find the lower bound of

$$\nu_0 (t + |x - y|)^\gamma t^{1-\gamma} + \alpha |x - y|^p t^{-p}.$$

Case 1. Hard potentials $0 \le \gamma < 1$:

Case 1a. As $p > \gamma$. We discuss the lower bound separately in the two regions

$$|x-y| \le \alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \text{ and } |x-y| > \alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}}.$$

If $|x-y| \le \alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \left(\Leftrightarrow t \ge \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right)$, then

$$(t+|x-y|)^{\gamma} t^{1-\gamma} \ge \begin{cases} t, \\ |x-y|^{\gamma} t^{1-\gamma} \ge \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}, \end{cases}$$

which implies that

$$(t+|x-y|)^{\gamma} t^{1-\gamma} \ge \frac{1}{2}\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right).$$

If $|x-y| > \alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \left(\Leftrightarrow t < \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right)$, then we have

$$(t+|x-y|)^{\gamma} t^{1-\gamma} \ge t,$$

and

$$\alpha\,|x-y|^{p}\, t^{-p} \ge \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}.$$

As a consequence,

$$\nu_0 (t+|x-y|)^{\gamma} t^{1-\gamma} + \alpha\,|x-y|^{p}\, t^{-p} \ge c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}\,|x-y|^{\frac{p}{p+1-\gamma}}\right),$$

for some $c_0 = c(\gamma) > 0$, so that

$$|h(t,x,\cdot)|_{L^{\infty}_{\xi}(\langle\xi\rangle^{\beta})} \le \sup_{y} e^{-c\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}\,|x-y|^{\frac{p}{p+1-\gamma}}\right)} |h_0(y,\cdot)|_{L^{\infty}_{\xi}(e^{\alpha|\xi|^{p}}\langle\xi\rangle^{\beta})}.$$

Case 1b. As $0 < p < \gamma$. We can apply a similar argument in Case 1a to obtain (41) as well.
Case 1c. As $0 < p = \gamma$, it is easy to see that

$$\begin{aligned} &\nu_0 (t+|x-y|)^{\gamma} t^{1-\gamma} + \alpha\,|x-y|^{p}\, t^{-p} \\ &\ge \left(\nu_0\,|x-y|^{\gamma}\, t^{1-\gamma}\right)^{\frac{p}{p}} + \left(\alpha\,|x-y|^{p}\, t^{-p}\right)^{\frac{1-\gamma}{1-\gamma}} \\ &\ge \nu_0^{\gamma} \alpha^{1-\gamma}\,|x-y|^{p}, \end{aligned}$$

due to Young's inequality. Therefore,

$$\nu_0 (t+|x-y|)^{\gamma} t^{1-\gamma} + \alpha\,|x-y|^{p}\, t^{-p} \ge \begin{cases} \nu_0 t, \\ \nu_0^{\gamma} \alpha^{1-\gamma}\,|x-y|^{p}, \end{cases}$$

which follows that

$$\nu_0 (t+|x-y|)^{\gamma} t^{1-\gamma} + \alpha\,|x-y|^{p}\, t^{-p} \ge c_0\left(t+\alpha^{1-\gamma}\,|x-y|^{p}\right),$$

for some $c_0 = c(\gamma) > 0$, as desired.
Case 2. Soft potentials $-2 < \gamma < 0$:

Case 2a. $|x-y| \le \alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \left(\Leftrightarrow t \ge \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right)$ and $|x-y| \ge t$. In this case we have $t \ge \alpha$ and

$$(t+|x-y|)^\gamma t^{1-\gamma} \ge \begin{cases} \left(t+\alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}}\right)^\gamma t^{1-\gamma} \ge 2^\gamma \alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}, \\ 2^\gamma |x-y|^\gamma t^{1-\gamma} \ge 2^\gamma \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}, \end{cases}$$

so that

$$(t+|x-y|)^\gamma t^{1-\gamma} \ge 2^{\gamma-1}\left(\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}} + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right).$$

Thus, (42) holds.

Case 2b. $|x-y| \le \alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \left(\Leftrightarrow t \ge \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right)$ and $|x-y| \le t$. In this case we have

$$|x-y| \le \min\{\alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}}, t\}.$$

If $\alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \ge t$, then $t \ge \alpha$ and

$$(t+|x-y|)^\gamma t^{1-\gamma} \ge \begin{cases} \left(t+\alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}}\right)^\gamma t^{1-\gamma} \ge 2^\gamma \alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}, \\ 2^\gamma t = 2^\gamma t^{\frac{1-\gamma}{p+1-\gamma}} t^{\frac{p}{p+1-\gamma}} \ge 2^\gamma \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}, \end{cases}$$

which implies that

$$(t+|x-y|)^\gamma t^{1-\gamma} \ge 2^{\gamma-1}\left(\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}} + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right).$$

Hence, (42) holds.

If $\alpha^{\frac{1}{\gamma-p}} t^{\frac{p+1-\gamma}{p-\gamma}} \le t$, then we deduce $t \le \alpha$ and thus $|x-y| \le \alpha$. Since

$$(t+|x-y|)^\gamma t^{1-\gamma} \ge 2^\gamma t \ge 2^\gamma \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}},$$

we have

$$(t+|x-y|)^\gamma t^{1-\gamma} \ge 2^{\gamma-1}\left(t+\alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right).$$

Together with the fact that $t \le \alpha$ and $|x-y| \le \alpha$, we deduce

$$\begin{aligned} |h(t,x,\xi)| &\le e^{-\nu_0 (t+|x-y|)^\gamma t^{1-\gamma}} |h_0(y,\xi)| \\ &\le e^{-2^{\gamma-1}\nu_0 t} \cdot e^{-2^{\gamma-1}\nu_0 \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}} |h_0(y,\xi)| \\ &\le \left[C_1 e^{-2^{\gamma-1}\nu_0 \alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}}\right]\left[C_1 e^{-2^{\gamma-1}\nu_0 \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}}\right] |h_0(y,\xi)| \\ &= C e^{-2^{\gamma-1}\nu_0\left(\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}} + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}}\right)} |h_0(y,\xi)|, \quad C = C_1^2, \end{aligned}$$

where

$$\exp\left(2^{\gamma-1}\nu_0\left(\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}} - t\right)\right) \le \exp\left[2^\gamma \nu_0 \alpha\right] = C_1(\alpha,\gamma),$$

and

$$\exp\left[2^{\gamma-1}\nu_0\left(\alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}} - \alpha^{\frac{1}{p+1-\gamma}} |x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right)\right] \le \exp\left(2^\gamma \nu_0 \alpha\right) = C_1(\alpha,\gamma).$$

Thus, (42) holds.

Case 2c. $|x-y|>\alpha^{\frac{1}{\gamma-p}}t^{\frac{p+1-\gamma}{p-\gamma}}\left(\Leftrightarrow t<\alpha^{\frac{1}{p+1-\gamma}}|x-y|^{\frac{p-\gamma}{p+1-\gamma}}\right)$. In this case we have

$$\alpha|x-y|^p t^{-p}\geq\begin{cases}\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}},\\ \alpha^{\frac{1-\gamma}{p+1-\gamma}}|x-y|^{\frac{p}{p+1-\gamma}},\end{cases}$$

so that

$$|h(t,x,\cdot)|_{L^\infty_\xi(\langle\xi\rangle^\beta)}\leq\sup_y e^{-\frac{1}{2}\left(\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x-y|^{\frac{p}{p+1-\gamma}}\right)}|h_0(y,\cdot)|_{L^\infty_\xi\left(e^{\alpha|\xi|^p}\langle\xi\rangle^\beta\right)}.$$

□

Immediately from Lemmas 3 and 8, we get the pointwise estimate of $h^{(j)}$, $0\leq j\leq 6$, as below.

Lemma 9 (Pointwise estimate of $h^{(j)}$, $0\leq j\leq 6$) *Let* $f_0(x,\cdot)\in L^\infty_\xi\left(e^{7\alpha|\xi|^p}\langle\xi\rangle^\beta\right)$ *with compact support in variable* x*, where* $0<p\leq 2$, $\beta>3/2$ *and* $\alpha>0$ *is small. Then there exists* $c_0=c(\gamma)>0$ *such that for* $0\leq\gamma<1$,

$$\left|h^{(j)}(t,x,\cdot)\right|_{L^\infty_\xi(\langle\xi\rangle^\beta)}\lesssim t^j e^{-c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)}\|f_0\|_{L^\infty_x L^\infty_\xi\left(e^{(j+1)\alpha|\xi|^p}\langle\xi\rangle^\beta\right)},$$

and for $-2<\gamma<0$,

$$\left|h^{(j)}(t,x,\cdot)\right|_{L^\infty_\xi(\langle\xi\rangle^\beta)}\lesssim t^j e^{-c_0\left(\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)}\|f_0\|_{L^\infty_x L^\infty_\xi\left(e^{(j+1)\alpha|\xi|^p}\langle\xi\rangle^\beta\right)}.$$

Remark 10 It is noted that the pointwise structure of singular waves for hard potentials [10] implicitly assumed the Gaussian velocity weight $e^{\alpha|\xi|^2}$ on the initial condition.

In order to get the L^2 estimate of $h^{(j)}$, we need the L^2 estimate of the damped transport operator $\mathbb{S}^t$.

Lemma 11 *Let* $\alpha>0$ *and* $0<p\leq 2$*. Then for* $0\leq\gamma<1$,

$$\left\|\mathbb{S}^t h_0\right\|_{L^2}\lesssim e^{-\nu_0 t}\|h_0\|_{L^2},$$

and for $-2<\gamma<0$,

$$\left\|\mathbb{S}^t h_0\right\|_{L^2}\lesssim e^{-c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}}\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)},$$

where the constant $c>0$ *depends only upon* γ *and* p.

Proof We only prove the case $-2<\gamma<0$ since the case $0\leq\gamma<1$ is obvious. In view of (4) and (40),

$$\left\|\mathbb{S}^t h_0\right\|_{L^2}\leq\left(\sup_\xi e^{-\nu_0 t(1+|\xi|)^\gamma-\alpha|\xi|^p}\right)\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)},$$

since h_0 has compact support in x. As for $-2<\gamma<0$,

$$\begin{aligned}\left\|\mathbb{S}^t h_0\right\|_{L^2} &\le \left(\sup_\xi e^{-\nu_0 t(1+|\xi|)^\gamma-\alpha|\xi|^p}\right)\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}\\
&\le \left(\sup_{|\xi|\le 1} e^{-\nu_0 t(1+|\xi|)^\gamma-\alpha|\xi|^p}+\sup_{|\xi|>1} e^{-\nu_0 t(1+|\xi|)^\gamma-\alpha|\xi|^p}\right)\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}\\
&\le \left(e^{-2^\gamma\nu_0 t}+\sup_{|\xi|>1} e^{-2^\gamma\nu_0 t|\xi|^\gamma-\alpha|\xi|^p}\right)\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}\\
&\le \left(e^{-2^\gamma\nu_0 t}+e^{-\left(\frac{p-\gamma}{-\gamma}\left(\frac{-\gamma 2^\gamma\nu_0}{p}\right)^{\frac{p}{p-\gamma}}\alpha^{\frac{-\gamma}{p-\gamma}}\right)t^{\frac{p}{p-\gamma}}}\right)\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}\\
&\lesssim e^{-c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}}\|h_0\|_{L^2\left(e^{\alpha|\xi|^p}\right)}\end{aligned}$$

for some constant $c=c(\gamma,p)>0$, since $2^\gamma\nu_0 t|\xi|^\gamma+\alpha|\xi|^p$ attains a minimum at $|\xi|=\left(\frac{-\gamma 2^\gamma\nu_0 t}{\alpha p}\right)^{\frac{1}{p-\gamma}}$. □

Combining Lemma 11 and (20), we thereby get the L^2 estimates of $h^{(j)}$, $0\le j\le 6$.

Lemma 12 (L^2 estimate of $h^{(j)}$, $0\le j\le 6$) *Let $f_0(x,\cdot)\in L^2\left(e^{7\alpha|\xi|^p}\right)$, where $0<p\le 2$ and $\alpha>0$ is small. Then there exists a constant $c>0$ such that for $0\le\gamma<1$,*

$$\left\|h^{(j)}\right\|_{L^2}\lesssim t^j e^{-\nu_0 t}\|f_0\|_{L^2},$$

and for $-2<\gamma<0$,

$$\left\|h^{(j)}\right\|_{L^2}\lesssim t^j e^{-c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}}\|f_0\|_{L^2\left(e^{(j+1)\alpha|\xi|^p}\right)}.$$

4.2 Regularization Estimate

In the previous subsection, we have carried out the pointwise estimate of the wave part and the L^2 estimates of $\mathcal{R}^{(6)}$. To obtain the pointwise estimate on $\mathcal{R}^{(6)}$, we still need the regularization estimate for $\mathcal{R}^{(6)}$. In light of (38), we turn to the regularization estimate for $h^{(6)}$ in advance. To proceed, we introduce a differential operator:

$$\mathcal{D}_t=t\nabla_x+\nabla_\xi.$$

This operator $\mathcal{D}_t$ is important since it commutes with the free transport operator, i.e.,

$$[\mathcal{D}_t,\partial_t+\xi\cdot\nabla_x]=0,$$

where $[A,B]=AB-BA$ is the commutator.

Lemma 13 *For any $g_0\in L^2_x H^1_\xi(\mu)$, we have*

$$\|Kg_0\|_{L^2_xH^1_\xi(\mu)}\lesssim\|g_0\|_{L^2(\mu)},\ \ \|K(\nabla_\xi g)\|_{L^2(\mu)}\lesssim\|g_0\|_{L^2(\mu)},\tag{43}$$

$$\|\mathbb{S}^t g_0\|_{L^2(\mu)}\lesssim e^{-c_\gamma t}\|g_0\|_{L^2(\mu)},\tag{44}$$

$$\|\mathcal{D}_t\mathbb{S}^t g_0\|_{L^2(\mu)}\lesssim(1+t)\,e^{-c_\gamma t}\|g_0\|_{L^2_xH^1_\xi(\mu)},\tag{45}$$

here $c_\gamma > 0$ *for* $0 \le \gamma < 1$ *and* $c_\gamma = 0$ *for* $-2 < \gamma < 0$.

Consequently,

$$\|\mathcal{D}_t \mathbb{S}^t K g_0\|_{L^2(\mu)} \lesssim (1+t)\, e^{-c_\gamma t} \|g_0\|_{L^2(\mu)}. \tag{46}$$

Proof The estimate of (43) can follow the same procedure as in Lemma 4 and hence we omit the details.

Denote $g(t) = \mathbb{S}^t g_0$. Direct Computation shows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2(\mu)} &= \int \left(-\xi\cdot\nabla_x g - \nu(\xi)\, g\right) g\mu dx d\xi \\
&= \frac{1}{2}\int (\xi\cdot\nabla_x\mu)\, |g|^2\, dx d\xi - \int \nu(\xi)\, |g|^2\, \mu dx d\xi \\
&\le \epsilon\delta c \int \langle\xi\rangle^\gamma |g|^2 \mu dx d\xi - \int \nu(\xi)\, |g|^2\, \mu dx d\xi,
\end{aligned}$$

since $|\nabla_x\mu| \lesssim \epsilon\delta\, \langle\xi\rangle^{\gamma-1}$. After choosing $\epsilon,\ \delta > 0$ sufficiently small with $c\epsilon\delta < \nu_0$, we have

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2(\mu)} \le -c' \int \langle\xi\rangle^\gamma |g|^2 \mu dx d\xi = -c'\|g_0\|^2_{L^2_\sigma(\mu)},$$

for some constant $c' = c(\gamma) > 0$. As a result,

$$\|g\|_{L^2(\mu)} \le e^{-c_\gamma t}\|g_0\|_{L^2(\mu)}, \tag{47}$$

here the constant $c_\gamma > 0$ for $0 \le \gamma < 1$ and $c_\gamma = 0$ for $-2 < \gamma < 0$.

Furthermore, set $y = \mathcal{D}_t \mathbb{S}^t g_0 = \mathcal{D}_t g$ and then y satisfies the equation

$$\begin{cases} \partial_t y + \xi\cdot\nabla_x y = -\nu(\xi)\, y - \left(\nabla_\xi \nu(\xi)\right) g, \\ y(0,x,\xi) = \left(\nabla_\xi g_0\right)(x,\xi). \end{cases}$$

Immediately, by Duhamel's principle and (47)

$$\begin{aligned}
\left\|\mathcal{D}_t \mathbb{S}^t g_0\right\|_{L^2(\mu)} = \|y\|_{L^2(\mu)} &\lesssim e^{-c_\gamma t}\|\nabla_\xi g_0\|_{L^2(\mu)} + \int_0^t e^{-c_\gamma(t-s)}\|g(s)\|_{L^2(\mu)} ds \\
&\lesssim e^{-c_\gamma t}\left(\|\nabla_\xi g_0\|_{L^2(\mu)} + t\|g_0\|_{L^2(\mu)}\right) \lesssim (1+t)\, e^{-c_\gamma t}\|g_0\|_{L^2_x H^1_\xi(\mu)}.
\end{aligned}$$

□

We are now in the position to get the regularization estimate of $h^{(6)}$. We find that without any regularity assumption on the initial condition, $h^{(6)}$ has H^2_x regularity automatically.

Lemma 14 (Regularization estimate on $h^{(6)}$)

$$\|h^{(6)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^4 (1+t)^2 e^{-c_\gamma t}\|f_0\|_{L^2(\mu)},$$

here $c_\gamma > 0$ *for* $0 \le \gamma < 1$ *and* $c_\gamma = 0$ *for* $-2 < \gamma < 0$.

Proof It follows immediately from Lemma 13 that

$$\|h^{(6)}\|_{L^2(\mu)} \lesssim t^6 e^{-c_\gamma t}\|f_0\|_{L^2(\mu)}.$$

Next, we prove the estimate for the first x-derivative of $h^{(6)}$. Notice that

$$
\begin{aligned}
&\nabla_x h^{(6)}(t)\\
&= \nabla_x \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \int_0^{s_5} \frac{s_1 - s_2}{s_1 - s_3} \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0\, ds\\
&\quad + \nabla_x \int_0^t \int_0^{s_1} \int_0^{s_2} \frac{s_2 - s_3}{s_1 - s_3} \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0\, ds\\
&= \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \int_0^{s_5} \frac{1}{s_1 - s_3} A\left(s_1, s_2, \ldots, s_6, x, \xi, t\right)\, ds,
\end{aligned}
$$

where $ds = ds_6 ds_5 ds_4 ds_3 ds_2 ds_1$ and

$$
\begin{aligned}
&A\left(s_1, s_2, \ldots, s_6, x, \xi, t\right)\\
&\quad = \mathbb{S}^{t-s_1} K \left(\mathcal{D}_{s_1-s_2} - \nabla_\xi\right) \mathbb{S}^{s_1-s_2} K \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0\\
&\qquad + \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \left(\mathcal{D}_{s_2-s_3} - \nabla_\xi\right) \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0.
\end{aligned}
$$

From Lemma 13, it follows that

$$
\begin{aligned}
\left\| \nabla_x h^{(6)}(t) \right\|_{L^2(\mu)} &\lesssim e^{-c_\gamma t} \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \int_0^{s_5} \left(\frac{1}{s_1 - s_3} + 1\right) \| f_0 \|_{L^2(\mu)}\, ds\\
&\lesssim e^{-c_\gamma t} \left[\int_0^t \int_0^{s_1} \int_{s_3}^{s_1} \frac{s_3^3}{s_1 - s_3} ds_2 ds_3 ds_1 + t^6\right] \| f_0 \|_{L^2(\mu)}\\
&\lesssim \left(t^6 + t^5\right) e^{-c_\gamma t} \| f_0 \|_{L^2(\mu)}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\nabla_x^2 h^{(6)}(t)\\
&\quad = \nabla_x^2 \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \int_0^{s_5}\\
&\qquad \left[\frac{(s_1 - s_2 + s_2 - s_3)}{s_1 - s_3} \frac{(s_4 - s_5 + s_5 - s_6)}{s_4 - s_6}\right.\\
&\qquad \left. \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0 \right] ds\\
&\quad = \int_0^t \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \int_0^{s_4} \int_0^{s_5} \frac{1}{(s_1 - s_3)(s_4 - s_6)} B\left(s_1, s_2, \ldots, s_6, x, \xi, t\right)\, ds
\end{aligned}
$$

where $ds = ds_6 ds_5 ds_4 ds_3 ds_2 ds_1$ and

$$
\begin{aligned}
&B\left(s_1, s_2, \ldots, s_6, x, \xi, t\right)\\
&\quad = \mathbb{S}^{t-s_1} K \left(\mathcal{D}_{s_1-s_2} - \nabla_\xi\right) \mathbb{S}^{s_1-s_2} K \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \left(\mathcal{D}_{s_4-s_5} - \nabla_\xi\right) \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0\\
&\qquad + \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \left(\mathcal{D}_{s_1-s_2} - \nabla_\xi\right) \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \left(\mathcal{D}_{s_5-s_6} - \nabla_\xi\right) \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0\\
&\qquad + \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \left(\mathcal{D}_{s_2-s_3} - \nabla_\xi\right) \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \left(\mathcal{D}_{s_4-s_5} - \nabla_\xi\right) \mathbb{S}^{s_4-s_5} K \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0\\
&\qquad + \mathbb{S}^{t-s_1} K \mathbb{S}^{s_1-s_2} K \left(\mathcal{D}_{s_2-s_3} - \nabla_\xi\right) \mathbb{S}^{s_2-s_3} K \mathbb{S}^{s_3-s_4}\, K \mathbb{S}^{s_4-s_5} K \left(\mathcal{D}_{s_5-s_6} - \nabla_\xi\right) \mathbb{S}^{s_5-s_6} K \mathbb{S}^{s_6} f_0.
\end{aligned}
$$

By Lemma 13 again, we deduce

$$\begin{aligned}&\left\|\nabla_x^2 h^{(6)}(t)\right\|_{L^2(\mu)}\\&\lesssim e^{-c_\gamma t}\,\|f_0\|_{L^2(\mu)}\int_0^t\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\int_0^{s_4}\int_0^{s_5}\left(1+\frac{1}{s_1-s_3}\right)\left(1+\frac{1}{s_4-s_6}\right)ds\\&\lesssim e^{-c_\gamma t}\left(t^6+t^5+t^4\right)\|f_0\|_{L^2(\mu)}.\end{aligned}$$

Therefore, $\|h^{(6)}\|_{H_x^2L_\xi^2(\mu)}\lesssim t^4\,(1+t)^2\,e^{-c_\gamma t}\|f_0\|_{L^2(\mu)}$. □

As a consequence, owing to (38) and Lemma 14, we find

$$\|\mathcal{R}^{(6)}\|_{H_x^2L_\xi^2(\mu)}\le\int_0^t\|h^{(6)}(s)\|_{H_x^2L_\xi^2(\mu)}ds\lesssim\begin{cases}\{t^5\wedge 1\}\|f_0\|_{L^2(\mu)}, & 0\le\gamma<1,\\ t^5(1+t)^2\,\|f_0\|_{L^2(\mu)}, & -2<\gamma<0.\end{cases}\tag{48}$$

Here $\{a\wedge b\}=\min\{a,b\}$.

5 Global Wave Structures

In this section we will complete the proof of Theorem 1 by discussing the global wave structures inside the finite Mach number region and outside the finite Mach number region separately.

5.1 Inside the Finite Mach Number Region

By the long wave-short wave decomposition and wave-remainder decomposition, we have

$$f=f_L+f_S=W^{(6)}+\mathcal{R}^{(6)}.$$

We now define the tail part as $f_R=\mathcal{R}^{(6)}-f_L=f_S-W^{(6)}$. Therefore f can be rewritten as $f=f_L+W^{(6)}+f_R$.

From Propositions 6, 7 and Lemma 9, the pointwise estimate of the long wave part f_L and the wave part $W^{(6)}$ are completed. It remains to study the tail part f_R. It is easy to see that

$$\|f_R\|_{H_x^2L_\xi^2}=\|(\mathcal{R}^{(6)}-f_L)\|_{H_x^2L_\xi^2}\lesssim\begin{cases}\|f_0\|_{L^2}, & \text{for }\ 0\le\gamma<1,\\ (1+t)^7\|f_0\|_{L^2}, & \text{for }\ -2<\gamma<0,\end{cases}$$

due to (48), and using Propositions 6–7 and Lemma 12 gives

$$\|f_R\|_{L^2}=\|f_S-W^{(6)}\|_{L^2}\lesssim\begin{cases}e^{-Ct}\|f_0\|_{L^2}, & \text{for }\ 0\le\gamma<1,\\ e^{-c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}}\|f_0\|_{L^2\left(e^{7\alpha|\xi|^p}\right)} & \text{for }\ -2<\gamma<0,\end{cases}$$

for some constants $C,c>0$. The Sobolev inequality [1, Theorem 5.8] implies

$$\begin{aligned}|f_R|_{L_\xi^2}\le\|f_R\|_{L_\xi^2L_x^\infty}&\lesssim\|f_R\|_{H_x^2L_\xi^2}^{3/4}\|f_R\|_{L^2}^{1/4}\\&\lesssim\begin{cases}e^{-\frac14 Ct}\|f_0\|_{L^2}, & \text{for }\ 0\le\gamma<1,\\ e^{-\frac18 c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}}\|f_0\|_{L^2\left(e^{7\alpha|\xi|^p}\right)}, & \text{for }\ -2<\gamma<0.\end{cases}\end{aligned}\tag{49}$$

Combining Propositions 6–7, Lemma 9 and (49), we obtain the pointwise estimate for the solution inside the finite Mach number region.

Proposition 15 *Let f be the solution to the linearized Boltzmann equation (2) and let $\mathrm{v} = \sqrt{5/3}$ be the sound speed associated with the normalized global Maxwellian. Then*

(1) *As $0 \le \gamma < 1$, for any given positive integer N, and any given $0 < p \le 2$, $\beta > 3/2$, sufficiently small $\alpha > 0$, there exist positive constants C_N, C and c_0 such that*

$$|f(t,x,\cdot)|_{L^2_\xi} \le C_N \Bigg[(1+t)^{-2}\left(1+\frac{(|x|-\mathrm{v}t)^2}{1+t}\right)^{-N} + (1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N}$$
$$+\mathbf{1}_{\{|x|\le \mathrm{v}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2}$$
$$+e^{-c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)} + e^{-t/C}\Bigg] ||f_0||_I.$$

(2) *As $-2 < \gamma < 0$, for any given $0 < p \le 2$, $\beta > 3/2$ and sufficiently small $\alpha > 0$, there exist positive constants C, c and c_0 such that*

$$|f(t,x,\cdot)|_{L^2_\xi} \le C \Bigg[(1+t)^{-3/2} + e^{-c\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}}$$
$$+e^{-c_0\left(\alpha^{\frac{-\gamma}{p-\gamma}}t^{\frac{p}{p-\gamma}}+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x|^{\frac{p}{p+1-\gamma}}\right)}\Bigg] ||f_0||_I.$$

Here $\mathbf{1}_{\{\cdot\}}$ is the indicator function and

$$||f_0||_I \equiv \max\left\{\|f_0\|_{L^2(e^{7\alpha|\xi|^p})}, \|f_0\|_{L^1_x L^2_\xi}, \|f_0\|_{L^\infty_x L^\infty_\xi(e^{7\alpha|\xi|^p}\langle\xi\rangle^\beta)}\right\}.$$

5.2 Outside the Finite Mach Number Region

In the previous section we have well investigated the pointwise behavior for the wave part $W^{(6)}$ (see Lemma 9). To clarify the wave structure outside the finite Mach number region, we still need to estimate the remainder part $\mathcal{R}^{(6)}$. Here, the weighted energy estimate plays a decisive role. We remark that the estimate for $-2 < \gamma < 1$ is nontrivial in the sense that a subtle space-velocity domain decomposition and delicate estimates of the integral operator K with weights are needed. In this subsection, the analysis has been carried out in detail.

Consider the weight

$$w(t,x,\xi) = \exp\left(\epsilon\rho(t,x,\xi)/2\right), \tag{50}$$

with

$$\rho(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{p}{p+1-\gamma}}\left(1-\chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right)\right)$$
$$+\left[\left(1-\chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right)\right)\left[\delta\left(\langle x\rangle - Mt\right)\right]\langle\xi\rangle^{\gamma-1} + 3\langle\xi\rangle^{p}\right]$$
$$\times\chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right).$$

where $\epsilon, \delta > 0$ will be chosen sufficiently small and $M > 0$ large enough later on. We define

$$H_+ = \{(x,\xi) : \delta\left(\langle x\rangle - Mt\right) > 2\langle\xi\rangle^{p+1-\gamma}\},$$
$$H_0 = \{(x,\xi) : \langle\xi\rangle^{p+1-\gamma} \le \delta\left(\langle x\rangle - Mt\right) \le 2\langle\xi\rangle^{p+1-\gamma}\},$$

and

$$H_- = \{(x,\xi) : \delta\left(\langle x\rangle - Mt\right) < \langle\xi\rangle^{p+1-\gamma}\}.$$

To go further, we need to estimate $\left|\int \langle g, (K_\epsilon - K)\,g\rangle_\xi\, dx\right|$, where $K_\epsilon = e^{\epsilon\rho(t,x,\xi)}Ke^{-\epsilon\rho(t,x,\xi)}$. This estimate will be used in the weighted energy estimate of $\mathcal{R}^{(6)}$ (Proposition 17). For simplicity of notation, let $\mathrm{P}_0 g = \sum_{j=0}^{4} b_j\chi_j$, $b_j = \left\langle g, \chi_j\right\rangle_\xi$.

Lemma 16 *Let* $0 < p \le 2$. *There exists a constant* $C = C(\gamma, p) > 0$ *such that for any* $0 < \epsilon \ll 1$,

$$\left|\int \langle g, (K_\epsilon - K)\,g\rangle_\xi\, dx\right| \le C\epsilon\int \langle\xi\rangle^{\gamma}\left|\mathrm{P}_1 g\right|^2 d\xi dx$$
$$+ C\epsilon\left[\int_{H_+}\left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0 g\right|^2 d\xi dx\right.$$
$$\left. + \int_{H_0\cup H_-}\left|\mathrm{P}_0 g\right|^2 d\xi dx\right]. \tag{51}$$

Consequently,

$$\int \langle g, L_\epsilon g\rangle_\xi\, dx \le \int \langle g, Lg\rangle_\xi\, dx + C\epsilon\int \langle\xi\rangle^{\gamma}\left|\mathrm{P}_1 g\right|^2 d\xi dx$$
$$+ C\epsilon\left[\int_{H_+}\left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0 g\right|^2 d\xi dx\right.$$
$$\left. + \int_{H_0\cup H_-}\left|\mathrm{P}_0 g\right|^2 d\xi dx\right], \tag{52}$$

where $L_\epsilon = e^{\epsilon\rho(t,x,\xi)}Le^{-\epsilon\rho(t,x,\xi)}$.

Proof We split the integral into several pieces:

$$\int \langle g, (K_\epsilon - K)\,g\rangle_\xi\, dx = \int \langle \mathrm{P}_1 g, (K_\epsilon - K)\,\mathrm{P}_1 g\rangle_\xi\, dx + \int \langle \mathrm{P}_0 g, (K_\epsilon - K)\,\mathrm{P}_0 g\rangle_\xi\, dx$$
$$+ \int \langle \mathrm{P}_1 g, (K_\epsilon - K)\,\mathrm{P}_0 g\rangle_\xi\, dx + \int \langle \mathrm{P}_1 g, (K_{-\epsilon} - K)\,\mathrm{P}_0 g\rangle_\xi\, dx. \tag{53}$$

Firstly, following the same procedure as in Lemma 4, we have that for any $\epsilon > 0$ sufficiently small,

$$\left|\int \langle \mathrm{P}_1 g, (K_\epsilon - K)\, \mathrm{P}_1 g\rangle_\xi \, dx\right| \lesssim \epsilon \, \|\mathrm{P}_1 g\|^2_{L^2_\sigma} . \tag{54}$$

Next, we estimate $\int \langle \mathrm{P}_0 g, (K_\epsilon - K)\, \mathrm{P}_0 g\rangle_\xi \, dx$.

Estimate on $\int \langle \mathrm{P}_0 g, (K_\epsilon - K)\, \mathrm{P}_0 g\rangle_\xi \, dx$. We split the integral

$$\begin{aligned}
&\int \langle \mathrm{P}_0 g, (K_\epsilon - K)\, \mathrm{P}_0 g\rangle_\xi \, dx \\
&= \begin{bmatrix} \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} \\ + \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\ge 2\langle\xi_*\rangle^{p+1-\gamma}} \\ + \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} \\ + \int_{\delta(\langle x\rangle - Mt)\le 0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} \end{bmatrix} \\
&\quad \mathrm{P}_0 g(\xi)\, k(\xi, \xi_*)\, A_\epsilon(t, x, \xi, \xi_*)\, \mathrm{P}_0 g(\xi_*)\, d\xi_* d\xi dx \\
&= I_a + I_b + I_c + II,
\end{aligned}$$

where $A_\epsilon(t, x, \xi, \xi_*) = \left[e^{\epsilon(\rho(t,x,\xi) - \rho(t,x,\xi_*))} - 1\right]$. We remark here that $A_\epsilon(t, x, \xi, \xi_*) = 0$ whenever $\delta(\langle x\rangle - Mt) > 2\langle\xi\rangle^{p+1-\gamma}$ and $\delta(\langle x\rangle - Mt) > 2\langle\xi_*\rangle^{p+1-\gamma}$; in other words, there is no contribution to the integral in this region. Note that $\rho(t, x, \xi)$ also satisfies

$$|\rho(t, x, \xi) - \rho(t, x, \xi_*)| \le c_1 \left||\xi|^2 - |\xi_*|^2\right|,$$

whose proof is similar to Lemma 4, hence

$$|A_\epsilon(t, x, \xi, \xi_*)| \lesssim \epsilon \left||\xi|^2 - |\xi_*|^2\right| e^{c_1 \epsilon \left||\xi|^2 - |\xi_*|^2\right|}.$$

Now, for I_a, we have

$$|A_\epsilon(t, x, \xi, \xi_*)\, \mathrm{P}_0 g(\xi_*)| \lesssim \epsilon \left||\xi|^2 - |\xi_*|^2\right| e^{c_1 \epsilon \left||\xi|^2 - |\xi_*|^2\right|} \left(\sum_{j=0}^4 \left|\chi_j(\xi_*)\right|^2\right)^{1/2} \left(\sum_{j=0}^4 b_j^2\right)^{1/2},$$

and thus

$$\begin{aligned}
&\int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} |\mathrm{P}_0 g(\xi)\, k(\xi, \xi_*)\, A_\epsilon(t, x, \xi, \xi_*)\, \mathrm{P}_0 g(\xi_*)|\, d\xi_* d\xi \\
&\lesssim \epsilon \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} |\mathrm{P}_0 g(\xi)| \left(\sum_{j=0}^4 b_j^2\right)^{1/2} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} |k(\xi, \xi_*)|\, e^{-c|\xi_*|^2} d\xi_* d\xi \\
&\lesssim \epsilon \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \left(\sum_{j=0}^4 \left|\chi_j(\xi)\right|^2\right)^{1/2} \left(\sum_{j=0}^4 b_j^2\right) \exp\left(-c' \left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right) d\xi \\
&\lesssim \epsilon \exp\left(-c' \left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right) \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} |\mathrm{P}_0 g|^2 d\xi \\
&\quad + \epsilon \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} |\mathrm{P}_0 g|^2 d\xi.
\end{aligned}$$

The first inequality is valid since $|\xi| < |\xi_*|$ and $\left(\sum_{j=0}^4 \left|\chi_j(\xi)\right|^2\right)^{1/2}$ decays exponentially; the second inequality holds due to the fact that $e^{-c|\xi_*|^2} \lesssim \exp\left(-c' \left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right)$ for some constant $c' > 0$ whenever $\delta(\langle x\rangle - Mt) \le 2\langle \xi_*\rangle^{p+1-\gamma}$ and that $|k(\xi,\cdot)|$ is integrable. Hence,

$$|I_a| \lesssim \epsilon \left[\int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 g|^2 \, d\xi dx \right].$$

Similarly for I_b, it follows

$$|I_b| \lesssim \epsilon \left[\int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 g|^2 \, d\xi dx \right].$$

On the other hand, by symmetry

$$\begin{aligned} |I_c| \le 2c_1\epsilon \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}, |\xi|\le|\xi_*|} \\ |\mathrm{P}_0 g(\xi)\, k(\xi,\xi_*)\, \mathrm{P}_0 g(\xi_*)| \left| |\xi|^2 - |\xi_*|^2 \right| e^{c_1\epsilon \left||\xi|^2 - |\xi_*|^2\right|} d\xi_* d\xi dx, \end{aligned}$$

applying a similar argument for I_a gives

$$|I_c| \lesssim \epsilon \left[\int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 g|^2 \, d\xi dx \right],$$

as well.

Following the same argument as the proof of (14) in Lemma 4, it is easy to see that

$$\begin{aligned} |II| &\lesssim \epsilon \int_{\delta(\langle x\rangle - Mt)\le 0} \int |\mathrm{P}_0 g|^2 \, d\xi dx \\ &= \epsilon \int_{\delta(\langle x\rangle - Mt)\le 0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx \\ &\le \epsilon \int_{H_-} |\mathrm{P}_0 g|^2 \, d\xi dx. \end{aligned}$$

Gathering the estimates for I_a, I_b, I_c and II yields

$$\begin{aligned} \left| \int \langle \mathrm{P}_0 g, (K_\epsilon - K)\, \mathrm{P}_0 g\rangle_\xi \, dx \right| \lesssim \epsilon \Bigg[\int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx \\ + \int_{H_0 \cup H_-} |\mathrm{P}_0 g|^2 \, d\xi dx \Bigg]. \end{aligned} \tag{55}$$

Estimate on $\int \langle \mathrm{P}_1 g, (K_\epsilon - K)\, \mathrm{P}_0 g\rangle_\xi \, dx + \int \langle \mathrm{P}_1 g, (K_{-\epsilon} - K)\, \mathrm{P}_0 g\rangle_\xi \, dx$. We split the integral

$$\begin{aligned} &\int \langle \mathrm{P}_1 g, (K_\epsilon - K)\, \mathrm{P}_0 g\rangle_\xi \, dx + \int \langle \mathrm{P}_1 g, (K_{-\epsilon} - K)\, \mathrm{P}_0 g\rangle_\xi \, dx \\ &= \begin{bmatrix} \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} \\ + \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\ge 2\langle\xi_*\rangle^{p+1-\gamma}} \\ + \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} \\ + \int_{\delta(\langle x\rangle - Mt)\le 0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi_*\rangle^{p+1-\gamma}} \end{bmatrix} \\ &\quad \mathrm{P}_1 g(\xi)\, k(\xi,\xi_*)\, B_\epsilon(t,x,\xi,\xi_*)\, \mathrm{P}_0 g(\xi_*)\, d\xi_* d\xi dx \\ &= I^a + I^b + I^c + II', \end{aligned}$$

where $B_\epsilon(t,x,\xi,\xi_*) = A_\epsilon(t,x,\xi,\xi_*) + A_\epsilon(t,x,\xi_*,\xi)$. It readily follows from the definition of B_ϵ that

$$|B_\epsilon(t,x,\xi,\xi_*)| \lesssim \epsilon \left||\xi|^2 - |\xi_*|^2\right| e^{c_1\epsilon\left||\xi|^2-|\xi_*|^2\right|}. \tag{56}$$

According to the above discussion, we obtain

$$\begin{aligned}\left|I^a\right| &\lesssim \epsilon \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} |\mathrm{P}_1 g(\xi)| \exp\left(-c'(\delta(\langle x\rangle - Mt))^{\frac{2}{p+1-\gamma}}\right) \\ &\quad \times \left(\sum_{j=0}^4 b_j^2\right)^{1/2} d\xi dx \\ &\lesssim \epsilon \int \langle\xi\rangle^\gamma |\mathrm{P}_1 g|^2 d\xi dx \\ &\quad + \epsilon \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \langle\xi\rangle^{-\gamma} \exp\left(-2c'(\delta(\langle x\rangle - Mt))^{\frac{2}{p+1-\gamma}}\right) \\ &\quad \times \left(\sum_{j=0}^4 b_j^2\right) d\xi dx.\end{aligned}$$

For $0 \le \gamma < 1$,

$$\begin{aligned}&\int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \langle\xi\rangle^{-\gamma} \exp\left(-2c'(\delta(\langle x\rangle - Mt))^{\frac{2}{p+1-\gamma}}\right) \left(\sum_{j=0}^4 b_j^2\right) d\xi dx \\ &\lesssim \int_{\delta(\langle x\rangle - Mt)>0} [(\delta(\langle x\rangle - Mt))]^{\frac{3}{p+1-\gamma}} \exp\left(-2c'(\delta(\langle x\rangle - Mt))^{\frac{2}{p+1-\gamma}}\right) \left(\sum_{j=0}^4 b_j^2\right) dx \\ &\lesssim \int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 d\xi dx + \int_{H_0\cup H_-} |\mathrm{P}_0 g|^2 d\xi dx,\end{aligned}$$

and for $-2 < \gamma < 0$,

$$\begin{aligned}&\int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)>2\langle\xi\rangle^{p+1-\gamma}} \langle\xi\rangle^{-\gamma} \exp\left(-2c'(\delta(\langle x\rangle - Mt))^{\frac{2}{p+1-\gamma}}\right) \left(\sum_{j=0}^4 b_j^2\right) d\xi dx \\ &\lesssim \int_{\delta(\langle x\rangle - Mt)>0} [(\delta(\langle x\rangle - Mt))]^{\frac{3-\gamma}{p+1-\gamma}} \exp\left(-2c'(\delta(\langle x\rangle - Mt))^{\frac{2}{p+1-\gamma}}\right) \left(\sum_{j=0}^4 b_j^2\right) dx \\ &\lesssim \int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 d\xi dx + \int_{H_0\cup H_-} |\mathrm{P}_0 g|^2 d\xi dx.\end{aligned}$$

Hence, we conclude

$$\begin{aligned}\left|I^a\right| \lesssim \epsilon \Bigg(&\int \langle\xi\rangle^\gamma |\mathrm{P}_1 g|^2 d\xi dx + \int_{H_+} [(\delta(\langle x\rangle - Mt))]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 d\xi dx \\ &+ \int_{H_0\cup H_-} |\mathrm{P}_0 g|^2 d\xi dx\Bigg).\end{aligned}$$

For II', similar to Lemma 4,

$$
\begin{aligned}
\left|II'\right| &\lesssim \epsilon \int_{\delta(\langle x\rangle - Mt)\le 0} \left|\langle\xi\rangle^{\gamma/2}\,\mathrm{P}_1 g\right|_{L^2_\xi} \left|\langle\xi\rangle^{\gamma/2}\,\mathrm{P}_0 g\right|_{L^2_\xi} dx \\
&\lesssim \epsilon \int \langle\xi\rangle^{\gamma} \left|\mathrm{P}_1 g\right|^2 d\xi dx + \epsilon \int_{\delta(\langle x\rangle - Mt)\le 0} \int_{\delta(\langle x\rangle - Mt)\le 2\langle\xi\rangle^{p+1-\gamma}} \langle\xi\rangle^{\gamma} \left|\mathrm{P}_0 g\right|^2 d\xi dx \\
&\lesssim \epsilon \int \langle\xi\rangle^{\gamma} \left|\mathrm{P}_1 g\right|^2 d\xi dx + \epsilon \int_{H_-} \left|\mathrm{P}_0 g\right|^2 d\xi dx.
\end{aligned}
$$

For I^b, we observe that $|\xi_*| \le |\xi|$ in this region. Further, note that

$$
\begin{aligned}
& k_1(\xi,\xi_*)\left|B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*)\right| \\
&\quad \lesssim |\xi-\xi_*|^{\gamma} \exp\left(-\frac{1}{4}|\xi|^2 - \frac{1}{4}|\xi_*|^2\right)\left|B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*)\right| \\
&\quad = \widetilde{p}(\xi,\xi_*) \times \left[\exp\left(-\frac{1}{8}|\xi|^2 - \frac{1}{8}|\xi_*|^2\right)\left|B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*)\right|\right]
\end{aligned}
$$

satisfies

$$
\begin{aligned}
& \left|\exp\left(-\frac{1}{8}|\xi|^2 - \frac{1}{8}|\xi_*|^2\right) B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*)\right| \\
&\quad \lesssim \left[\epsilon \exp\left(-\frac{1}{16}|\xi|^2\right)\left(\sum_{j=0}^{4} b_j^2\right)^{1/2}\right] \exp\left(-c_2|\xi_*|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
& k_2(\xi,\xi_*)\,B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*) \\
&\quad = p(\xi,\xi_*)\left\{\exp\left(-\frac{1}{32}\left[\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2} + |\xi-\xi_*|^2\right]\right) \times B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*)\right\}
\end{aligned}
$$

satisfies

$$
\begin{aligned}
& \left|\exp\left(-\frac{1}{32}\left[\frac{\left(|\xi|^2-|\xi_*|^2\right)^2}{|\xi-\xi_*|^2} + |\xi-\xi_*|^2\right]\right) B_\epsilon(t,x,\xi,\xi_*)\,\mathrm{P}_0 g(\xi_*)\right| \\
&\quad \lesssim \epsilon \exp\left(-\frac{1}{32}\left||\xi|^2-|\xi_*|^2\right|\right)\left(\sum_{j=0}^{4}\left|\chi_j(\xi_*)\right|^2\right)^{1/2}\left(\sum_{j=0}^{4} b_j^2\right)^{1/2} \\
&\quad \lesssim \left[\epsilon \exp\left(-\frac{1}{32}|\xi|^2\right)\left(\sum_{j=0}^{4} b_j^2\right)^{1/2}\right] \exp\left(-c_2|\xi_*|^2\right),
\end{aligned}
$$

for some $c_2 > 0$, where $\widetilde{p}(\xi,\xi_*)$ and $p(\xi,\xi_*)$ are kernels of bounded operators on L^2_ξ. Since $e^{-c_2|\xi_*|^2} \in L^2_{\xi_*}$ and

$$
\langle\xi\rangle^{-\frac{\gamma}{2}} e^{-\frac{1}{32}|\xi|^2} \lesssim \exp\left(-c_3\left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right)
$$

for some constant $c_3 > 0$ as $\delta(\langle x\rangle - Mt) \leq 2\langle\xi\rangle^{p+1-\gamma}$, we obtain

$$
\begin{aligned}
\left|I^b\right| &\lesssim \epsilon \int \langle\xi\rangle^{\gamma} |\mathrm{P}_1 g|^2 \, d\xi dx + \epsilon \int_{\delta(\langle x\rangle - Mt)>0} \exp\left(-2c_3 \left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right) \left(\sum_{j=0}^{4} b_j^2\right) dx \\
&\lesssim \epsilon \int \langle\xi\rangle^{\gamma} |\mathrm{P}_1 g|^2 \, d\xi dx \\
&\quad + \epsilon \left[\int_{H_+} \left[\delta(\langle x\rangle - Mt)\right]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 g|^2 \, d\xi dx\right].
\end{aligned}
$$

Finally, we split the integral

$$
\begin{aligned}
I^c =& \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi_*\rangle^{p+1-\gamma}, |\xi_*|\leq|\xi|} \\
&+ \int_{\delta(\langle x\rangle - Mt)>0} \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi\rangle^{p+1-\gamma} \delta(\langle x\rangle - Mt)\leq 2\langle\xi\rangle^{p+1-\gamma}} \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi_*\rangle^{p+1-\gamma}, |\xi_*|>|\xi|} \\
\equiv& I_1^c + I_2^c.
\end{aligned}
$$

Similar to I^b,

$$
\begin{aligned}
\left|I_1^c\right| \lesssim & \epsilon \int \langle\xi\rangle^{\gamma} |\mathrm{P}_1 g|^2 \, d\xi dx \\
& + \epsilon \left[\int_{H_+} \left[\delta(\langle x\rangle - Mt)\right]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 g|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 g|^2 \, d\xi dx\right].
\end{aligned}
$$

In view of (56),

$$
\begin{aligned}
&\int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi_*\rangle^{p+1-\gamma},\ |\xi|<|\xi_*|} |k(\xi,\xi_*)\, B_\epsilon(t,x,\xi,\xi_*)\, \mathrm{P}_0 g(\xi_*)| \, d\xi_* \\
&\lesssim \epsilon \left(\sum_{j=0}^{4} b_j^2\right)^{1/2} \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi_*\rangle^{p+1-\gamma}} |k(\xi,\xi_*)|\, |\xi_*|^2 e^{c_1\epsilon|\xi_*|^2} \left(\sum_{j=0}^{4} \left|\chi_j(\xi_*)\right|^2\right)^{1/2} d\xi_* \\
&\lesssim \epsilon \left(\sum_{j=0}^{4} b_j^2\right)^{1/2} \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi_*\rangle^{p+1-\gamma}} |k(\xi,\xi_*)|\, e^{-c'|\xi_*|^2} d\xi_* \\
&\lesssim \epsilon \left(\sum_{j=0}^{4} b_j^2\right)^{1/2} \exp\left(-\frac{c''}{2}\left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right) \\
&\quad \times \int_{\delta(\langle x\rangle - Mt)\leq 2\langle\xi_*\rangle^{p+1-\gamma}} |k(\xi,\xi_*)|\, e^{-\frac{c'}{2}|\xi_*|^2} d\xi_* \\
&\lesssim \epsilon \left(\sum_{j=0}^{4} b_j^2\right)^{1/2} \exp\left(-\frac{c''}{2}\left[\delta(\langle x\rangle - Mt)\right]^{\frac{2}{p+1-\gamma}}\right) e^{-\frac{c'}{2}|\xi|^2} \qquad \text{(by (8))}
\end{aligned}
$$

for some constants $0 < c' < 1/2$ and $c'' > 0$, and then by the Cauchy inequality,

$$\begin{aligned}
\left|I_2^c\right| &\lesssim \epsilon \int \langle\xi\rangle^{\gamma} \left|\mathrm{P}_1 g\right|^2 d\xi dx \\
&\quad + \epsilon \int \langle\xi\rangle^{-\gamma} e^{-\frac{c'}{2}|\xi|^2} d\xi \cdot \int_{\delta(\langle x\rangle - Mt)>0} \exp\left(-c'' \left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{2}{p+1-\gamma}}\right) \sum_{j=0}^{4} b_j^2 dx \\
&\lesssim \epsilon \int \langle\xi\rangle^{\gamma} \left|\mathrm{P}_1 g\right|^2 d\xi dx \\
&\quad + \epsilon \left[\int_{H_+} \left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}} \left|\mathrm{P}_0 g\right|^2 d\xi dx + \int_{H_0 \cup H_-} \left|\mathrm{P}_0 g\right|^2 d\xi dx\right].
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\left|\int \langle \mathrm{P}_1 g, \left(K_\epsilon - K\right) \mathrm{P}_0 g\rangle_\xi \, dx + \int \langle \mathrm{P}_1 g, \left(K_{-\epsilon} - K\right) \mathrm{P}_0 g\rangle_\xi \, dx\right| \\
&\quad \lesssim \epsilon \left[\int \langle\xi\rangle^{\gamma} \left|\mathrm{P}_1 g\right|^2 d\xi dx + \int_{H_+} \left[\delta\left(\langle x\rangle - Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}} \left|\mathrm{P}_0 g\right|^2 d\xi dx \right. \\
&\qquad \left. + \int_{H_0 \cup H_-} \left|\mathrm{P}_0 g\right|^2 d\xi dx\right]. \qquad (57)
\end{aligned}$$

Combining (54), (55) and (57), we get our result. □

Now, we are ready to get the weighted energy estimate of $\mathcal{R}^{(6)}$.

Proposition 17 (Weighted energy for $\mathcal{R}^{(6)}$) *Consider the weight*

$$w\left(t, x, \xi\right) = \exp\left(\epsilon\rho\left(t, x, \xi\right)/2\right), \qquad (58)$$

with

$$\begin{aligned}
\rho\left(t, x, \xi\right) &= 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{p}{p+1-\gamma}} \left(1 - \chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right)\right) \\
&\quad + \left[\left(1 - \chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right)\right)\left[\delta\left(\langle x\rangle - Mt\right)\right]\langle\xi\rangle^{\gamma-1} + 3\langle\xi\rangle^{p}\right] \\
&\quad \times \chi\left(\frac{\delta\left(\langle x\rangle - Mt\right)}{\langle\xi\rangle^{p+1-\gamma}}\right),
\end{aligned}$$

where $\epsilon, \ \delta > 0$ are sufficiently small, $M > 0$ sufficiently large, and $0 < p \le 2$. Then we have

$$\left\|w\mathcal{R}^{(6)}\right\|_{H_x^2 L_\xi^2} \lesssim \left\{t^5 \wedge t\right\} \left\|f_0\right\|_{L^2(\mu)}, \qquad 0 \le \gamma < 1,$$

and

$$\left\|w\mathcal{R}^{(6)}\right\|_{H_x^2 L_\xi^2} \lesssim t^5 \left(1+t\right)^3 \left\|f_0\right\|_{L^2(\mu)}, \quad -2 < \gamma < 0.$$

Proof Let $u = w\mathcal{R}^{(6)} = e^{\frac{\epsilon\rho}{2}}\mathcal{R}^{(6)}$, and then u solves the equation

$$\partial_t u + \xi \cdot \nabla_x u - \frac{\epsilon}{2}\left(\partial_t \rho + \xi \cdot \nabla_x \rho\right) u - e^{\frac{\epsilon\rho}{2}} L\left(e^{-\frac{\epsilon\rho}{2}} u\right) = e^{\frac{\epsilon\rho}{2}} K h^{(6)}.$$

The energy estimate gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\langle u,u\rangle_\xi\,dx-\int_{\mathbb{R}^3}\left\langle u,e^{\frac{\epsilon\rho}{2}}Kh^{(6)}\right\rangle_\xi dx\\&\quad=\int_{\mathbb{R}^3}\frac{\epsilon}{2}\langle u,(\partial_t\rho+\xi\cdot\nabla_x\rho)u\rangle_\xi\,dx+\int_{\mathbb{R}^3}\left\langle u,e^{\frac{\epsilon\rho}{2}}L\left(e^{-\frac{\epsilon\rho}{2}}u\right)\right\rangle_\xi dx.\end{aligned}$$

In view of Lemma 16,

$$\begin{aligned}\int_{\mathbb{R}^3}\left\langle u,e^{\frac{\epsilon\rho}{2}}L\left(e^{-\frac{\epsilon\rho}{2}}u\right)\right\rangle_\xi dx\le&-\mu\int_{\mathbb{R}^3}|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1u|^2_{L^2_\xi}dx\\&+C_1\epsilon\left[\int_{H_+}[\delta(\langle x\rangle-Mt)]^{\frac{\gamma-1}{p+1-\gamma}}|\mathrm{P}_0u|^2\,d\xi dx\right.\\&\left.+\int_{H_0\cup H_-}|\mathrm{P}_0u|^2\,d\xi dx\right],\end{aligned}$$

for some constants $\mu>0$ and $C_1>0$. One can easily check that

$$\begin{aligned}\partial_t\rho=&-\delta M\langle\xi\rangle^{\gamma-1}\left(\frac{5p}{p+1-\gamma}\left[(\delta(\langle x\rangle-Mt))\langle\xi\rangle^{\gamma-p-1}\right]^{\frac{\gamma-1}{p+1-\gamma}}(1-\chi)+\chi(1-\chi)\right)\\&+\delta M\left(5\left[(\delta(\langle x\rangle-Mt))\langle\xi\rangle^{\gamma-p-1}\right]^{\frac{p}{p+1-\gamma}}-(1-2\chi)\left[(\delta(\langle x\rangle-Mt))\langle\xi\rangle^{\gamma-p-1}\right]\right.\\&\left.-3\right)\langle\xi\rangle^{\gamma-1}\chi'\le 0\end{aligned}$$

(the constants 5 and 3 are chosen intentionally such that the quantity in the latter bracket is nonnegative on H_0), and

$$\begin{aligned}\nabla_x\rho=&\,\delta(\nabla_x\langle x\rangle)\langle\xi\rangle^{\gamma-1}\left(\frac{5p}{p+1-\gamma}\left[\delta(\langle x\rangle-Mt)\langle\xi\rangle^{\gamma-p-1}\right]^{\frac{\gamma-1}{p+1-\gamma}}(1-\chi)+\chi(1-\chi)\right)\\&-\delta(\nabla_x\langle x\rangle)\left(5\left[\delta(\langle x\rangle-Mt)\langle\xi\rangle^{\gamma-p-1}\right]^{\frac{p}{p+1-\gamma}}\right.\\&\left.-(1-2\chi)\left[(\delta(\langle x\rangle-Mt))\langle\xi\rangle^{\gamma-p-1}\right]-3\right)\langle\xi\rangle^{\gamma-1}\chi'.\end{aligned}$$

Hence,

$$\begin{aligned}&\partial_t\rho=\xi\cdot\nabla_x\rho=0\quad\text{on}\quad H_-,\\&|\partial_t\rho|\lesssim\delta M\langle\xi\rangle^{\gamma-1}\quad\text{and}\quad|\xi\cdot\nabla_x\rho|\lesssim\delta\langle\xi\rangle^{\gamma}\quad\text{on}\quad H_0,\end{aligned}$$

and we have

$$\begin{aligned}\partial_t\rho&=-\frac{5p\delta M}{p+1-\gamma}[\delta(\langle x\rangle-Mt)]^{\frac{\gamma-1}{p+1-\gamma}},\\\xi\cdot\nabla_x\rho&=\frac{5p\delta}{p+1-\gamma}\frac{\xi\cdot x}{\langle x\rangle}[\delta(\langle x\rangle-Mt)]^{\frac{\gamma-1}{p+1-\gamma}},\end{aligned}$$

on H_+. Direct calculation together with the Cauchy inequality show that

$$\epsilon \left| \int_{\mathbb{R}^3} \langle u, (\xi \cdot \nabla_x \rho)\, u \rangle_\xi \, dx \right|$$
$$\leq C_2 \epsilon \delta \left[\begin{array}{l} \int_{\mathbb{R}^3} |\langle \xi \rangle^{\frac{\gamma}{2}} \mathrm{P}_1 u|^2_{L^2_\xi} dx \\ + \int_{H_+} [\delta (\langle x \rangle - Mt)]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 u|^2 d\xi dx + \int_{H_0} |\mathrm{P}_0 u|^2 d\xi dx \end{array} \right],$$

and

$$\epsilon \int_{\mathbb{R}^3} \langle u, (\partial_t \rho)\, u \rangle_\xi \, dx \leq \epsilon \delta M C_3 \int_{\mathbb{R}^3} |\langle \xi \rangle^{\frac{\gamma}{2}} \mathrm{P}_1 u|^2_{L^2_\xi} dx$$
$$- \epsilon \delta M C_4 \int_{H_+} [\delta (\langle x \rangle - Mt)]^{\frac{\gamma-1}{p+1-\gamma}} |\mathrm{P}_0 u|^2 d\xi dx$$
$$+ \epsilon \delta M C_5 \int_{H_0} |\mathrm{P}_0 u|^2 d\xi dx.$$

In conclusion, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} \langle u, u \rangle_\xi \, dx - \int_{\mathbb{R}^3} \left\langle u, wKh^{(6)} \right\rangle_\xi dx$$
$$\leq -(\mu - \epsilon \delta C_2 - \epsilon \delta M C_3) \int_{\mathbb{R}^3} |\langle \xi \rangle^{\frac{\gamma}{2}} \mathrm{P}_1 u|^2_{L^2_\xi} dx$$
$$- \epsilon\, (\delta M C_4 - \delta C_2 - C_1) \int_{H_+} [\delta(\langle x \rangle - Mt)]^{\frac{r-1}{p+1-\gamma}} |\mathrm{P}_0 u|^2 d\xi dx$$
$$+ \epsilon (\delta C_2 + \delta M C_5 + C_1) \int_{H_0} |\mathrm{P}_0 u|^2 d\xi dx + \epsilon C_1 \int_{H_-} |\mathrm{P}_0 u|^2 d\xi dx.$$

Choosing $\delta, \epsilon > 0$ small and $M > 0$ large enough, we have

$$\frac{d}{dt} \|u\|^2_{L^2} \lesssim \|u\|_{L^2} \|wKh^{(6)}\|_{L^2} + \int_{H_0 \cup H_-} |\mathrm{P}_0 u|^2 d\xi dx$$
$$\lesssim \|u\|_{L^2} \|wKh^{(6)}\|_{L^2} + \|u\|_{L^2} \, \|\mathcal{R}^{(6)}\|_{L^2}.$$

Moreover, since $\partial_t \rho \leq 0$, the weight w is decreasing in t, so that

$$\|wKh^{(6)}\|_{L^2} \leq \|\mu^{1/2} Kh^{(6)}\|_{L^2} = \|Kh^{(6)}\|_{L^2(\mu)} \lesssim \|h^{(6)}\|_{L^2(\mu)},$$

due to (16). It implies

$$\frac{d}{dt} \|u\|_{L^2} \lesssim \|h^{(6)}\|_{L^2(\mu)} + \|\mathcal{R}^{(6)}\|_{L^2}.$$

For the x-derivative estimate, we only need to control the commutator terms:

$$\epsilon \int_{\mathbb{R}^3} \left\langle \partial_{x_i} u, \partial_{x_i} (\partial_t \rho + \xi \cdot \nabla_x \rho)\, u \right\rangle_\xi dx, \tag{59}$$

$$\epsilon \int_{\mathbb{R}^3} \left\langle \partial_{x_i} u, e^{\frac{\epsilon \rho}{2}} K \left(e^{-\frac{\epsilon \rho}{2}} \partial_{x_i} \rho u \right) \right\rangle_\xi dx = \epsilon \int_{\mathbb{R}^3} \left\langle \left(\partial_{x_i} \rho \right) u, e^{-\frac{\epsilon \rho}{2}} K e^{\frac{\epsilon \rho}{2}} \left(\partial_{x_i} u \right) \right\rangle_\xi dx, \tag{60}$$

$$\epsilon \int_{\mathbb{R}^3} \left\langle \partial_{x_i} u, \partial_{x_i} \rho e^{\frac{\epsilon \rho}{2}} K e^{-\frac{\epsilon \rho}{2}} u \right\rangle_\xi dx = \epsilon \int_{\mathbb{R}^3} \left\langle \partial_{x_i} \rho \partial_{x_i} u, e^{\frac{\epsilon \rho}{2}} K e^{-\frac{\epsilon \rho}{2}} u \right\rangle_\xi dx, \tag{61}$$

and

$$\int_{\mathbb{R}^3} \left\langle \partial_{x_i} u, \partial_{x_i} \left(wKh^{(6)} \right) \right\rangle_\xi dx. \tag{62}$$

It is obvious that the decay of $\partial_{x_i}(\partial_t\rho+\xi\cdot\nabla_x\rho)$ is faster than $(\partial_t\rho+\xi\cdot\nabla_x\rho)$, hence the first term (59) is easy to control. Since $\partial_{x_i}\rho=0$ on H_-, $\left|\partial_{x_i}\rho\right|\lesssim\delta\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}$ on H_+ and $\left|\partial_{x_i}\rho\right|\lesssim\delta\langle\xi\rangle^{r-1}$ on H_0, we have

$$\begin{aligned}&\epsilon\left|\int_{\mathbb{R}^3}\left\langle\left(\partial_{x_i}\rho\right)u,e^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right),\right\rangle_\xi dx\right|\\&\lesssim\epsilon\delta\left(\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|ue^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right)\right|d\xi dx\right.\\&\quad\left.+\int_{H_0}\langle\xi\rangle^{\gamma-1}\left|ue^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right)\right|d\xi dx\right).\end{aligned}$$

Similar to (15), $e^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}$ and $e^{\frac{\epsilon\rho}{2}}Ke^{-\frac{\epsilon\rho}{2}}$ are bounded operators on L^2_ξ. Notice that $\delta\left(\langle x\rangle-Mt\right)\geq1$ and $\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\leq\langle\xi\rangle^{r-1}$ on $H_+\cup H_0$, hence direct computation shows that

$$\begin{aligned}&\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|ue^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right)\right|d\xi dx\\&\lesssim\int_{\delta(\langle x\rangle-Mt)\geq1}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left(\int_{\delta(\langle x\rangle-Mt)>2\langle\xi\rangle^{p+1-\gamma}}|u|^2d\xi\right)^{1/2}\\&\quad\times\left(\int_{\mathbb{R}^3}\left|\partial_{x_i}u\right|^2d\xi\right)^{1/2}dx\\&\lesssim\int_{\delta(\langle x\rangle-Mt)\geq1}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left(\int_{\delta(\langle x\rangle-Mt)>2\langle\xi\rangle^{p+1-\gamma}}|u|^2d\xi\right)^{1/2}\\&\quad\times\left(\int_{\delta(\langle x\rangle-Mt)>2\langle\xi\rangle^{p+1-\gamma}}\left|\partial_{x_i}u\right|^2d\xi\right)^{1/2}dx\\&\quad+\int_{\delta(\langle x\rangle-Mt)\geq1}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left(\int_{\delta(\langle x\rangle-Mt)>2\langle\xi\rangle^{p+1-\gamma}}|u|^2d\xi\right)^{1/2}\\&\quad\times\left(\int_{\langle\xi\rangle^{p+1-\gamma}\leq\delta(\langle x\rangle-Mt)\leq2\langle\xi\rangle^{p+1-\gamma}}\left|\partial_{x_i}u\right|^2d\xi\right)^{1/2}dx\\&\quad+\int_{\delta(\langle x\rangle-Mt)\geq1}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left(\int_{\delta(\langle x\rangle-Mt)>2\langle\xi\rangle^{p+1-\gamma}}|u|^2d\xi\right)^{1/2}\\&\quad\times\left(\int_{\delta(\langle x\rangle-Mt)<\langle\xi\rangle^{p+1-\gamma}}\left|\partial_{x_i}u\right|^2d\xi\right)^{1/2}dx\\&\lesssim\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}|u|^2d\xi dx+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\partial_{x_i}u\right|^2d\xi dx\\&\quad+\int_{H_0\cup H_-}\left|\partial_{x_i}u\right|^2d\xi dx\\&\lesssim\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1\partial_{x_i}u\right\|^2_{L^2}+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0\partial_{x_i}u\right|^2d\xi dx\end{aligned}$$

$$
\begin{aligned}
&+\int_{H_0\cup H_-}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\\
&+\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1 u\right\|_{L^2}^2+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}|\mathrm{P}_0u|^2 d\xi dx,
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\int_{H_0}\langle\xi\rangle^{\gamma-1}\left|ue^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right)\right|d\xi dx\\
&\lesssim\int_{H_0}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|ue^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right)\right|d\xi dx\\
&\lesssim\int_{H_0}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}|u|^2 d\xi dx+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\partial_{x_i}u\right|^2 d\xi dx\\
&\quad+\int_{H_0\cup H_-}\left|\partial_{x_i}u\right|^2 d\xi dx\\
&\lesssim\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1\partial_{x_i}u\right\|_{L^2}^2+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\\
&\quad+\int_{H_0\cup H_-}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\\
&\quad+\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1 u\right\|_{L^2}^2+\int_{H_0}|\mathrm{P}_0u|^2 d\xi dx.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\epsilon\left|\int_{\mathbb{R}^3}\left\langle\left(\partial_{x_i}\rho\right)u,e^{-\frac{\epsilon\rho}{2}}Ke^{\frac{\epsilon\rho}{2}}\left(\partial_{x_i}u\right),\right\rangle_\xi dx\right|\\
&\lesssim\epsilon\delta\left(\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1\partial_{x_i}u\right\|_{L^2}^2+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\right.\\
&\quad\left.+\int_{H_0\cup H_-}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\right)\\
&\quad+\epsilon\delta\left(\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1 u\right\|_{L^2}^2+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}|\mathrm{P}_0u|^2 d\xi dx+\int_{H_0}|\mathrm{P}_0u|^2 d\xi dx\right).
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
&\epsilon\int_{\mathbb{R}^3}\left\langle\partial_{x_i}\rho\partial_{x_i}u,e^{\frac{\epsilon\rho}{2}}Ke^{-\frac{\epsilon\rho}{2}}u\right\rangle_\xi dx\\
&\lesssim\epsilon\delta\left(\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1\partial_{x_i}u\right\|_{L^2}^2+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\right.\\
&\quad\left.+\int_{H_0}\left|\mathrm{P}_0\partial_{x_i}u\right|^2 d\xi dx\right)\\
&\quad+\epsilon\delta\left(\left\|\langle\xi\rangle^{\frac{\gamma}{2}}\mathrm{P}_1 u\right\|_{L^2}^2+\int_{H_+}\left[\delta\left(\langle x\rangle-Mt\right)\right]^{\frac{\gamma-1}{p+1-\gamma}}|\mathrm{P}_0u|^2 d\xi dx\right.\\
&\quad\left.+\int_{H_0\cup H_-}|\mathrm{P}_0u|^2 d\xi dx\right).
\end{aligned}
$$

On the other hand,

$$\left|\int_{\mathbb{R}^3}\left\langle \partial_{x_i}u,\partial_{x_i}\left(wKh^{(6)}\right)\right\rangle_\xi dx\right| \lesssim \left\|\partial_{x_i}u\right\|_{L^2}\left\|\partial_{x_i}\left(wKh^{(6)}\right)\right\|_{L^2}.$$

The second derivative estimate is similar and hence we omit the details. We then deduce that

$$\begin{aligned}\frac{d}{dt}\|u\|^2_{H^2_xL^2_\xi} &\lesssim \|u\|_{H^2_xL^2_\xi}\|wKh^{(6)}\|_{H^2_xL^2_\xi} + \int_{H_0\cup H_-}|\mathrm{P}_0u|^2+|\mathrm{P}_0\nabla_xu|^2+\left|\mathrm{P}_0\nabla_x^2u\right|d\xi dx\\ &\lesssim \|u\|_{H^2_xL^2_\xi}\|wKh^{(6)}\|_{H^2_xL^2_\xi}+\|u\|_{H^2_xL^2_\xi}\|\mathcal{R}^{(6)}\|_{H^2_xL^2_\xi}\\ &\lesssim \|u\|_{H^2_xL^2_\xi}\left(\|h^{(6)}\|_{H^2_xL^2_\xi(\mu)}+\|\mathcal{R}^{(6)}\|_{H^2_xL^2_\xi}\right),\end{aligned}$$

the last inequality holds since $|\nabla_x\rho|$, $\left|\nabla_x^2\rho\right|\lesssim\langle\xi\rangle^{\gamma-1}$ and the weight w is decreasing in t. It follows that

$$\frac{d}{dt}\|u\|_{H^2_xL^2_\xi}\lesssim\|h^{(6)}\|_{H^2_xL^2_\xi(\mu)}+\|\mathcal{R}^{(6)}\|_{H^2_xL^2_\xi}.$$

In view of Lemma 14 and (48),

$$\frac{d}{dt}\|u\|_{H^2_xL^2_\xi}\lesssim\begin{cases}\{t^4\wedge 1\}\left(\|f_0\|_{L^2(\mu)}+\|f_0\|_{L^2}\right), & 0\le\gamma<1,\\ t^4(1+t)^3\left(\|f_0\|_{L^2(\mu)}+\|f_0\|_{L^2}\right), & -2<\gamma<0.\end{cases}$$

This completes the proof of the proposition. □

Through Proposition 17 and the Sobolev inequality, we will establish the pointwise estimate for $\mathcal{R}^{(6)}$ in the following. Combining this with the wave part $W^{(6)}$ (see Lemma 9), we complete the wave structure of the solution outside the finite Mach number region.

Proposition 18 *Let $\mathcal{R}^{(6)}$ be the remainder part of the linearized Boltzmann equation* (2) *with $-2<\gamma<1$, and $0<p\le 2$. There exists a positive constant M such that for $\langle x\rangle>2Mt$, we have*

$$\left|\mathcal{R}^{(6)}(t,x,\cdot)\right|_{L^2_\xi}\le Ct^5e^{-c_\epsilon(\langle x\rangle+t)^{\frac{p}{p+1-\gamma}}}\|f_0\|_{L^2\left(e^{\epsilon|\xi|^p}\right)},\tag{63}$$

where the constant $\epsilon>0$ is sufficiently small and C, c_ϵ are some positive constants.

Proof Let w be the weight function defined as (50). Observe that for $\langle x\rangle>2Mt$,

$$\rho(t,x,\xi)\gtrsim(\delta(\langle x\rangle-Mt))^{\frac{p}{p+1-\gamma}}.$$

Applying Proposition 17, it follows from the Sobolev inequality [16, Proposition 3.8] that

$$\begin{aligned}e^{\epsilon(\delta(\langle x\rangle-Mt))^{\frac{p}{p+1-\gamma}}}\left|\mathcal{R}^{(6)}(t,x,\cdot)\right|_{L^2_\xi}&\le\left|w\mathcal{R}^{(6)}\right|_{L^2_\xi}\le\left\|w\mathcal{R}^{(6)}\right\|_{L^2_\xi L^\infty_x}\\ &\lesssim\left\|\nabla_x^2\left(w\mathcal{R}^{(6)}\right)\right\|^{1/2}_{L^2}\left\|\nabla_x\left(w\mathcal{R}^{(6)}\right)\right\|^{1/2}_{L^2}\lesssim\left\|w\mathcal{R}^{(6)}\right\|_{H^2_xL^2_\xi}\\ &\lesssim\begin{cases}\{t^5\wedge t\}\|f_0\|_{L^2(\mu)}, & 0\le\gamma<1,\\ t^5(1+t)^3\|f_0\|_{L^2(\mu)}, & -2<\gamma<0.\end{cases}\end{aligned}$$

Here $\epsilon>0$ can be chosen as small as we want. Note that for $\langle x\rangle>2Mt$,

$$\langle x\rangle-Mt>\frac{\langle x\rangle}{3}+\frac{Mt}{3},$$

and

$$\|f_0\|_{L^2(\mu)} \lesssim \|f_0\|_{L^2(e^{\epsilon|\xi|^p})},$$

due to the fact that f_0 has compact support in variable x. Therefore there exist positive constants C and c_ϵ such that

$$\left|\mathcal{R}^{(6)}(t,x,\cdot)\right|_{L^2_\xi} \le Ct^5 e^{-c_\epsilon(\langle x\rangle+t)^{\frac{p}{p+1-\gamma}}} \|f_0\|_{L^2(e^{\epsilon|\xi|^p})}. \tag{64}$$

□

6 Conclusion

In this paper, we obtain the quantitative pointwise behavior of the solutions of the linearized Boltzmann equation for hard potentials ($0 < \gamma < 1$), Maxwellian molecules ($\gamma = 0$) and soft potentials ($-2 < \gamma < 0$), with Grad's angular cutoff assumption, by assuming the exponential velocity weight $e^{\alpha|\xi|^p}$ on the initial data. Here α is a small positive number and $0 < p \le 2$. For hard potentials, we extend the result [10] with the Gaussian velocity weight $e^{\alpha|\xi|^2}$ to more general exponential velocity weights $e^{\alpha|\xi|^p}$, $0 < p \le 2$. For Maxwellian molecules and soft potentials, our result is the first attempt aiming at the pointwise structure of the solution.

It would also be interesting to consider the quantitative pointwise behavior for other kinetic equations. In fact, our approach is applicable to the Landau kinetic equation [18]. Furthermore, it has potential to be adapted to the Boltzmann equation with non cut-off hard potentials, where the regularization mechanism is analogous to Landau type equations rather than cut-off cases. The study of the non cut-off Boltzmann equation is in progress.

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