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Mixture estimate in fractional sense and its application to the well-posedness of the Boltzmann equation with very soft potential

Yu-Chu Lin[1] **· Haitao Wang**[2] **· Kung-Chien Wu**[1,3]

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Abstract

In this paper, we consider the Boltzmann equation with angular-cutoff for very soft potential case $-3 < \gamma \leq -2$. We prove a regularization mechanism that transfers the microscopic velocity regularity to macroscopic space regularity in the *fractional sense*. The result extends the smoothing effect results of Liu–Yu (see "mixture lemma" in Comm Pure Appl Math 57:1543–1608, 2004), and of Gualdani–Mischler–Mouhot (see "iterated averaging lemma" in Mém Soc Math Fr 153, 2017), both established for the hard sphere case. A precise pointwise estimate of the fractional derivative of collision kernel, and a connection between velocity derivative and space derivative in the fractional sense are exploited to overcome the high singularity for very soft potential case. As an application of fractional regularization estimates, we prove the global well-posedness and large time behavior of the solution for non-smooth initial perturbation.

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✉ Kung-Chien Wu
kungchienwu@gmail.com

Yu-Chu Lin
yuchu@mail.ncku.edu.tw

Haitao Wang
haitallica@sjtu.edu.cn

1 Department of Mathematics, National Cheng Kung University, Tainan, Taiwan

2 School of Mathematical Sciences, Institute of Natural Sciences, MOE-LSC, CMA-Shanghai, Shanghai Jiao Tong University, Shanghai, China

3 National Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan

1 Introduction

1.1 The model

In this paper, we consider the following Boltzmann equation:

$$\begin{cases} \partial_t F + \xi \cdot \nabla_x F = Q(F, F), \\ F(0, x, \xi) = F_0(x, \xi), \end{cases} \quad (t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3, \tag{1}$$

where $F(t, x, \xi)$ is the velocity distribution function for the particles at time $t > 0$, position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and microscopic velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. The left-hand side of this equation models the transport of particles and the operator on the right-hand side models the effect of collisions during the transport,

$$Q(F, G) = \int_{\mathbb{R}^3 \times S^2} |\xi - \xi_*|^\gamma B(\vartheta) \left\{ F(\xi_*')G(\xi') - F(\xi_*)G(\xi) \right\} d\xi_* d\omega.$$

We consider the very soft potential ($-3 < \gamma \le -2$) case and $B(\vartheta)$ satisfies the Grad cutoff assumption

$$0 < B(\vartheta) \le C|\cos\vartheta|,$$

for some constant $C > 0$. Moreover, the post-collisional velocities satisfy

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi_*' = \xi + [(\xi - \xi_*) \cdot \omega]\omega,$$

and ϑ is defined by

$$\cos\vartheta = \frac{|(\xi - \xi_*) \cdot \omega|}{|\xi - \xi_*|}.$$

It is well known that the global Maxwellians are steady-state solutions to the Boltzmann equation (1). Therefore, it is natural to consider the Boltzmann equation (1) around a global Maxwellian

$$\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2} \right),$$

with the standard perturbation $f(t, x, \xi)$ to $\mathcal{M}$ as

$$F = \mathcal{M} + \mathcal{M}^{1/2} f, \quad F_0 = \mathcal{M} + \eta \mathcal{M}^{1/2} f_0,$$

where $\eta > 0$ is sufficiently small. After substituting F and F_0 into (1), the equation for the perturbation f is

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f), \\ f(0, x, \xi) = \eta f_0(x, \xi) = \dfrac{F_0 - \mathcal{M}}{\sqrt{\mathcal{M}}}, \end{cases} \tag{2}$$

where L is the linearized collision operator defined as

$$Lf = \mathcal{M}^{-1/2}\left[Q(\mathcal{M}, \mathcal{M}^{1/2}f) + Q(\mathcal{M}^{1/2}f, \mathcal{M})\right],$$

and Γ is the nonlinear operator defined as

$$\Gamma(f, f) = \mathcal{M}^{-1/2}Q(\mathcal{M}^{1/2}f, \mathcal{M}^{1/2}f).$$

It is well-known that the null space of L is a five-dimensional vector space with the orthonormal basis $\{\chi_i\}_{i=0}^4$, where

$$Ker(L) = \{\chi_0, \chi_i, \chi_4\} = \left\{\mathcal{M}^{1/2},\ \xi_i\mathcal{M}^{1/2},\ \frac{1}{\sqrt{6}}(|\xi|^2 - 3)\mathcal{M}^{1/2},\ i = 1, 2, 3\right\}.$$

Based on this property, we can introduce the macro-micro decomposition: let P_0 be the orthogonal projection with respect to the L^2_ξ inner product onto $\mathrm{Ker}(L)$, and $\mathrm{P}_1 \equiv \mathrm{Id} - \mathrm{P}_0$.

The collision operator L consists of a multiplicative operator $\nu(\xi)$ and an integral operator K:

$$Lf = -\nu(\xi)f + Kf,$$

where

$$\nu(\xi) = \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}(\xi_*)d\xi_* d\omega,$$

and

$$Kf = -K_1 f + K_2 f \tag{3}$$

is defined as [5, 8]:

$$\begin{aligned} K_1 f &:= \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi)\mathcal{M}^{1/2}(\xi_*)f(\xi_*)d\xi_* d\omega, \\ K_2 f &:= \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi_*)\mathcal{M}^{1/2}(\xi')f(\xi'_*)d\xi_* d\omega \\ &\quad + \int B(\vartheta)|\xi - \xi_*|^\gamma \mathcal{M}^{1/2}(\xi_*)\mathcal{M}^{1/2}(\xi'_*)f(\xi')d\xi_* d\omega. \end{aligned}$$

In the next section, we will present a number of properties and estimates of the operators L, $\nu(\xi)$ and K.

1.2 Notations

Before the presentation of the main theorem, let us define some notations used in this paper. We denote $\langle\xi\rangle^s = (1+|\xi|^2)^{s/2}$, $s\in\mathbb{R}$. For the microscopic variable ξ, we denote the Lebesgue spaces

$$|g|_{L^q_\xi} = \left(\int_{\mathbb{R}^3} |g|^q d\xi\right)^{1/q} \text{ if } 1\le q<\infty, \qquad |g|_{L^\infty_\xi} = \sup_{\xi\in\mathbb{R}^3} |g(\xi)|,$$

and the weighted norms can be defined by

$$|g|_{L^q_{\xi,\beta}} = \left(\int_{\mathbb{R}^3} \left|\langle\xi\rangle^\beta g\right|^q d\xi\right)^{1/q} \text{ if } 1\le q<\infty, \qquad |g|_{L^\infty_{\xi,\beta}} = \sup_{\xi\in\mathbb{R}^3} \left|\langle\xi\rangle^\beta g(\xi)\right|,$$

and

$$|g|_{L^\infty_\xi(m)} = \sup_{\xi\in\mathbb{R}^3} \{|g(\xi)|m(\xi)\},$$

where $\beta\in\mathbb{R}$ and m is a weight function. The L^2_ξ inner product in $\mathbb{R}^3$ will be denoted by $\langle\cdot,\cdot\rangle_\xi$, i.e.,

$$\langle f,g\rangle_\xi = \int f(\xi)\overline{g(\xi)}d\xi.$$

For the Boltzmann equation with cut-off potential, the natural norm in ξ is $|\cdot|_{L^2_\sigma}$, which is defined as

$$|g|^2_{L^2_\sigma} = \left|\langle\xi\rangle^{\frac{\gamma}{2}} g\right|^2_{L^2_\xi}.$$

For the space variable x, we have similar notations, namely,

$$|g|_{L^q_x} = \left(\int_{\mathbb{R}^3} |g|^q dx\right)^{1/q} \text{ if } 1\le q<\infty, \qquad |g|_{L^\infty_x} = \sup_{x\in\mathbb{R}^3} |g(x)|.$$

Furthermore, we define the high order Sobolev norm: let $s\in\mathbb{N}$ and define

$$|g|_{H^s_\xi} = \sum_{|\alpha|\le s} \left|\partial^\alpha_\xi g\right|_{L^2_\xi}, \quad |g|_{H^s_x} = \sum_{|\alpha|\le s} \left|\partial^\alpha_x g\right|_{L^2_x},$$

where α is any multi-index with $|\alpha|\le s$.

Next, we introduce two equivalent definitions of the fractional derivative $(-\Delta_y)^{\frac{s}{2}}$ for $0 < s < 2$, and the interested reader is referred to [16] for other equivalent definitions. Let $f(y) : \mathbb{R}^3 \to \mathbb{R}$ be a function in the Lebesgue space L^p $(1 \le p < \infty)$, then we have the following two equivalent definitions:

Definition 1 (*Singular integral definition*). Let $0 < s < 2$. The fractional derivative of f of order s is defined as

$$(-\Delta_y)^{\frac{s}{2}} f(y) = \text{p.v.} \int_{\mathbb{R}^3} \frac{f(y+z) - f(y)}{|z|^{3+s}} dz = \lim_{r \to 0^+} \int_{|z|>r} \frac{f(y+z) - f(y)}{|z|^{3+s}} dz,$$

provided that the limit exists.

Definition 2 (*Fourier transform definition*). Let $0 < s < 2$. The fractional derivative of f of order s is defined as

$$(-\Delta_y)^{\frac{s}{2}} f(y) = \mathcal{F}^{-1}\{|\hat{y}|^s \hat{f}(\hat{y})\},$$

where

$$\hat{f}(\hat{y}) = \int_{\mathbb{R}^3} e^{iy\cdot\hat{y}} f(y) dy$$

is the Fourier transform of $f(y)$ and $\mathcal{F}^{-1}$ is its corresponding inverse transform.

Finally, with $\mathcal{X}$ and $\mathcal{Y}$ being norm spaces, we define

$$\|g\|_{\mathcal{X}\mathcal{Y}} = \big|\, |g|_{\mathcal{Y}} \big|_{\mathcal{X}} .$$

We also denote

$$\|g\|_{L^2} = \|g\|_{L^2_\xi L^2_x} = \left(\int_{\mathbb{R}^3} |g|^2_{L^2_x} d\xi \right)^{1/2} .$$

For simplicity of notations, hereafter, we abbreviate "$\le C$ " to "$\lesssim$", where C is a positive constant depending only on fixed numbers.

1.3 Main result I: mixture estimate in fractional sense

Denote the solution operator of the damped transport equation

$$\begin{cases} \partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0, \\ h(0, x, \xi) = h_0, \end{cases} \tag{4}$$

by $\mathbb{S}^t_\gamma$, i.e., $h(t) = \mathbb{S}^t_\gamma h_0$. Moreover, if $\nu(\xi) = 0$, we denote the solution operator as $\mathbb{S}^t$. By the method of characteristics, the solutions $\mathbb{S}^t_\gamma h_0$ and $\mathbb{S}^t h_0$ can be written down explicitly,

$$\mathbb{S}^t_\gamma h_0(x, \xi) = e^{-\nu(\xi)t} h_0(x - \xi t, \xi) \tag{5}$$

and

$$\mathbb{S}^t h_0(x,\xi) = h_0(x-\xi t,\xi). \tag{6}$$

The Mixture Estimate reveals the mechanism that the mixture of two operators $\mathbb{S}^t_\gamma$ and K will transfer the regularity in microscopic velocity ξ coming from K to the regularity in space x. The precise statement of the Mixture Estimate is stated as follows.

Theorem 1 (Mixture Estimate) *Let* $-3 < \gamma \le -2$, $0 < s < 3+\gamma$. *If* $h_0 \in L^2$, *then*

$$\left\| (-\Delta_x)^{\frac{s}{2}} K \mathbb{S}^t_\gamma K h_0 \right\|_{L^2} \lesssim t^{-s} \|h_0\|_{L^2}. \tag{7}$$

Furthermore, let $\mathbb{M}_n$ be a multiple-mixture operator defined inductively as below

$$\mathbb{M}_1(h) = \left(\mathbb{S}^t_\gamma K\right) *_t h = \int_0^t (\mathbb{S}^{t-\tau}_\gamma K h)(\tau,x,\xi)d\tau,$$

and

$$\begin{aligned}\mathbb{M}_n(h) &= \mathbb{M}_1(\mathbb{M}_{n-1}(h)) = \left(\mathbb{S}^t_\gamma K\right) *_t \mathbb{M}_{n-1}(h)\\ &= \int_0^t (\mathbb{S}^{t-\tau}_\gamma K \mathbb{M}_{n-1}(h))(\tau,x,\xi)d\tau, \quad \text{for } n \ge 2.\end{aligned}$$

where $h(t,\cdot) \in L^2\left(\mathbb{R}^3 \times \mathbb{R}^3\right)$. Iterating the Mixture Estimate enough times, it is shown that the spatial regularity can be improved as many as one desires:

Corollary 2 *Let* $-3 < \gamma \le -2$, $0 < s < 3+\gamma$, $k \in \mathbb{N}$. *If* $h \in L^2$ *uniformly in time, then*

$$\|\mathbb{M}_{2k}(h)(t)\|_{L^2_\xi H^{ks}_x} \lesssim (1+t)^{k(2-s)} \sup_{\tau\in[0,t]} \|h(\tau)\|_{L^2}. \tag{8}$$

The Mixture Estimate plays a crucial role in Theorem 3, which enables us to obtain the well-posedness and large time behavior of the Boltzmann equation without any regularity assumption on the initial data.

Since the Mixture Estimate has its own independent mathematical interest, we present it as Theorem 1 separately. In the literature, there are several papers regarding " regularization effect" for the Boltzmann equation. Among them, two works which are most relevant to the current research are by Liu and Yu [21], and by Gualdani, Mischler and Mouhot [9]. The reader is also referred to other variants of "regularization effects" for the Boltzmann equation, such as the "Averaging Lemma" by Golse, Lions, Perthame and Sentis [7], the "A-smoothing Property" by Glassey and Strauss [6], and the "L^2-L^∞ approach" by Guo [12].

In [21], the authors introduced "Mixture Lemma" (Lemma 4.9) to extract the particle-like wave in construction of Green's function for Boltzmann equation with hard sphere. On the other hand, in [9], the authors obtained "Iterated Averaging Lemma" (Lemma 4.19) for factorization and enlargement theory of the Boltzmann equation with hard sphere on the torus. Roughly speaking, both "Mixture Lemma" and "Iterated Averaging Lemma" reveal the following mechanism: let $S_{\mathcal{B}}$ be a transport type semigroup and $\mathcal{A}$ be a smoothing integral operator in ξ, then mixing $S_{\mathcal{B}}$ and $\mathcal{A}$ will transfer the ξ regularity coming from $\mathcal{A}$ to the space regularity x. Interestingly, the proofs of these two lemmas are quite different. The proof of "Mixture Lemma" is based on the Fourier transform with respect to the space variable x, combing with H^1_ξ smoothing effect of the integral operator $\mathcal{A}$. While the key idea in the proof of "Iterated Averaging Lemma" is to build up a bridge between x derivative and ξ derivative by introducing a crucial differential operator $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, which commutes with the free transport operator. Later, Wu [28] gave an alternative proof of "Mixture Lemma" by employing the operator $\mathcal{D}_t$. This method is then adapted to prove variant versions of "Mixture Lemma" for the Boltzmann equation with $-2 < \gamma \leq 1$ [18, 19].

However, all the aforementioned proofs are not applicable to the very soft case, due to the weak smoothing effect of integral operator K when $-3 < \gamma \leq -2$. In fact, the kernel function of K has a singularity $|\xi - \eta|^{-1\wedge\gamma}$, so that one can have full derivative estimate $\nabla_\xi K$ only when $-2 < \gamma \leq 1$.

The above restriction gives rise to some fundamental and interesting questions for the higher singularity region $-3 < \gamma \leq -2$:

(i) Instead of the derivative estimate of $\nabla_\xi K$, can we still gain some fractional regularity in velocity $(-\Delta_\xi)^{s/2}K$ for appropriate $s > 0$?
(ii) Given the fractional derivative estimate for K, is it still possible to transfer the microscopic velocity regularity to macroscopic space regularity in the fractional case by mixture?
(iii) Once we establish the Mixture Estimate, can we apply it to get the well-posedness of Boltzmann equation with very soft potential for non-smooth initial perturbations?

We will answer the question (i) in Sect. 2, the question (ii) in Sect. 3 (see Theorem 1), and the question (iii) in Sect. 4 (see Theorem 3) sequentially.

As hinted by the singularity $|\xi-\eta|^\gamma$ in the kernel function of K when $-3 < \gamma \leq -2$, one may expect for a fractional regularity $(-\Delta_\xi)^{s/2}K$ for $0 < s < 3+\gamma$. However, it is nontrivial to achieve this goal. Firstly, the kernel function of K is given by an integral expression for very soft potential rather than a closed form for hard sphere. Secondly, the fractional derivative is a non-local operator, which brings more complexities when acting on an integral expression. Furthermore, a uniform upper bound of fractional derivative is insufficient for the Mixture Estimate, and what we need is a pointwise estimate. To this end, we adopt the singular integral definition of the fractional derivative (Definition 1) and obtain a precise pointwise estimate for $(-\Delta_\xi)^{s/2}K$. In the course of calculations, we need to control the singularity (for $|\xi - \eta|$ small) and maintain the decay estimates of K (for $|\xi|$ or $|\eta|$ large) simultaneously. The former one is important in the regularization estimate as it is. While the decay estimates are also indispensable to ensure the integrability. We decompose the integral domain into different regions by recognizing the dominant term, and the whole proof is finished based on refined estimates for each of them (see Sects. 2.2, 2.3, 2.4 and 5). To the best of our knowledge, this is the first result regarding the pointwise estimate of fractional derivative of the integral part of the linearized Boltzmann collision operator. As corollaries of the pointwise estimates, some function space inequalities for $(-\Delta_\xi)^{s/2}K$ are followed immediately (see Corollary 15).

With the fractional regularity of K in velocity ξ, we are able to develop the Mixture Estimate. Since we are dealing with the fractional derivative, it is natural to use the Fourier transform as a tool to clarify the mechanism of regularization effect, rather than using the differential operator $\mathcal{D}_t$ as in [9]. Taking the Fourier transform of the free transport equation with respect to both the x and ξ variables, together with the Fourier transform definition of the fractional derivative (Definition 2), we set up a connection (34) between $(-\Delta_x)^{s/2}$ and $(-\Delta_\xi)^{s/2}$ for the free transport equation in terms of Fourier variables. This actually can be viewed as a fractional version analogue of the connection given by the operator $\mathcal{D}_t$. To complete the proof of the Mixture Estimate, one also needs to bound $(-\Delta_\xi)^{s/2}[e^{-\nu(\xi)t}k(\xi,\eta)]$, which is induced by the extra damping term $\nu(\xi)$ in the damped transport equation, and the integration by parts to absorb the ξ derivative. We obtain its estimate mainly through the Kato-Ponce inequality or "fractional Leibniz rule" (see Proposition 16). It is exactly the Mixture Estimate that enables us to obtain the global well-posedness without imposing any regularity on the initial data. Worthy of mention is that at first glance we seemingly employ two different definitions of fractional derivative in the proof of the Mixture Estimate, but they are in fact equivalent in the Lebesgue space L^p for $1 \le p < \infty$.

1.4 Main result II: well-posedness of the Boltzmann equation

With the help of the Mixture Estimate, we are able to obtain the well-posedness and large time behavior of the Boltzmann equation for $-3 < \gamma \le -2$ with non-smooth initial perturbations, the result is stated as follows.

Theorem 3 *Let $-3 < \gamma \le -2$, $0 < p \le 2$, $\beta > 3/2$, $\alpha > 0$ sufficiently small, and $j > 0$ sufficiently large. Assume that the initial data ηf_0 satisfies $f_0 \in L^\infty_{\xi,\beta+3j}(e^{\alpha\langle\xi\rangle^p})(L^1_x \cap L^\infty_x)$ where $\eta > 0$ is sufficiently small. Then there is a unique solution f to (2) in $L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^2_x \cap L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^\infty_x$ with*

$$\|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^2_x} \le \eta C_1(1+t)^{-\frac{3}{4}}\|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{9}$$

$$\|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le \eta C_2(1+t)^{-\frac{3}{2}}\|f_0\|_{L^\infty_{\xi,\beta+3j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{10}$$

$$\|f(t)\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^2_x} \le \eta \bar{C}_1\|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{11}$$

$$\|f(t)\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le \eta \bar{C}_2\|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{12}$$

for some positive constants C_1, C_2, $\bar{C}_1$, $\bar{C}_2$ depending on γ, α, p, β, and j.

In this theorem, we generalize the Green function approach of Liu and Yu [21] from hard sphere case to very soft potential case ($-3 < \gamma \le -2$) and then establish the well-posedness and large time behavior for non-smooth initial perturbations. In the literature, there are several energy methods for the study of the Boltzmann equations near Maxwellian in the whole space, for instance [11, 14, 22, 26]. In these works, people are aware that the large time behavior is governed by the long wave part in terms of the Fourier variables of the linearized equation. Meanwhile, in order to close the nonlinear problem, some suitable Sobolev regularity assumptions on the initial condition are necessarily required. By contrast, in the current paper we further employ the wave-remainder decomposition to analyze the solution, which enables us to remove the regularity assumption of the initial condition. Indeed, there exist several works concerning the L^∞_x initial data. For whole space problem, the time decay result is proved in [27] for hard potential by investigating the $L^2 - L^\infty$ smoothing effect of Mixture operator. In [3], the global existence for some type large amplitude data is established, whereas no decay estimate in whole space. In the bounded domain, the initial-boundary value problem associated with non-smooth initial perturbations has been considered as well. An L^2-L^∞ theory was developed in [12] to obtain the global existence and the exponential decay rate of the solution around a global Maxwellian for hard potentials associated with appropriate boundary conditions. Then, its extension to soft potential in a bounded domain is proved by [20] in which a sub-exponential decay rate is obtained, and the reader is also referred to [4, 13, 15] for recent advancements of this theory. All the above-mentioned results are dealing with cut-off case. See [23] for the global smooth solution in torus for L^∞ data and non-cutoff hard potential.

In what follows, we discuss the strategy of the proof of Theorem 3. By using the long wave-short wave decomposition and the wave-remainder decomposition, we first obtain the large time behavior of the linearized equation in the normed spaces $L^2_\xi L^2_x$ and $L^2_\xi L^\infty_x$. Note that the combination of these two decompositions was initially investigated by Liu and Yu [21] for hard sphere case and then generalized to hard and soft potentials (i.e., $-2 < \gamma < 1$) in [18, 19]. The restriction of $-2 < \gamma < 1$ is due to the absence of regularization estimate for the remainder part when $\gamma \le -2$. This crucial difficulty can be resolved by the Mixture Estimate in this paper and then we can generalize the $L^2_\xi L^2_x$ and $L^2_\xi L^\infty_x$ estimate of the linearized problem to the case $-3 < \gamma \le -2$. To solve the nonlinear problem, we need the L^∞_ξ weighted estimate of the linearized problem. Inspired by Ukai's bootstrap argument to the integral equation, we can improve the L^2_ξ estimates to the L^∞_ξ weighted spaces. It worth mentioning that when $-3 < \gamma \le -2$, the singularity of the integral operator K is too high to bootstrap the solution from L^2_ξ to L^∞_ξ directly. Fortunately, it can be obtained by applying finite steps of bootstrap argument with the aid of the interpolation result of the integral operator K (see Sect. 4.1, Step 2). Furthermore, given a source term $\Gamma(h_1, h_2)$ with prescribed time decay (see (60)), we establish the large time behavior for the inhomogeneous linearized equation. The large time behavior of the nonlinear problem (2) then follows from an iteration scheme.

The rest of this paper is organized as follows: In Sect. 2, we first review some basic properties of the operators L, $\nu(\xi)$ and K, then provide the fractional derivative estimates of K and $\nu(\xi)$. The lengthy proof of the estimate $(-\Delta)^{s/2}k_2(\xi, \eta)$ is postponed to Sect. 5 for the sake of readability. Then, we prove the Mixture estimate (Theorem 1) in Sect. 3 and demonstrate the well-posedness and large time behavior of the solution (Theorem 3) in Sect. 4.

2 Revisit of the linearized collision operator

In this section we will present a number of properties and estimates of the operators L, $\nu(\xi)$ and K. To begin with, we list some fundamental properties of these operators, which can be found in [1, 8, 25]. The rest of this section is devoted to the estimate of the fractional derivative for $\nu(\xi)$ and K, here we use singular integral definition (Definition 1) to define the fractional derivative.

2.1 Basic estimates of *L*, *ν*(*ξ*) and *K*

Lemma 4 *For any $g \in L^2_\sigma$, we have the coercivity estimate of the linearized collision operator L:*

$$\langle g, Lg\rangle_\xi \lesssim -\left|\mathrm{P}_1 g\right|^2_{L^2_\sigma}.$$

Lemma 5 *For the multiplicative operator $\nu(\xi)$, there exist positive constants ν_0 and ν_1 such that*

$$\nu_0 \langle\xi\rangle^\gamma \le \nu(\xi) \le \nu_1 \langle\xi\rangle^\gamma . \tag{13}$$

Moreover, for each multi-index $\alpha \in \mathbb{N}^3$,

$$\left|\partial^\alpha_\xi \nu(\xi)\right| \lesssim \langle\xi\rangle^{\gamma-|\alpha|}. \tag{14}$$

For the integral operators K_1 and K_2, we have the following representations.

Lemma 6 *The integral operator K_1 can be represented as*

$$(K_1 f)(\xi) = \int_{\mathbb{R}^3} k_1(\xi,\eta) f(\eta) d\eta,$$

where the kernel $k_1(\xi,\eta)$ is given by

$$k_1(\xi,\eta) = \gamma_0 |\xi-\eta|^\gamma \exp\left\{-\frac{1}{4}\left(|\xi|^2+|\eta|^2\right)\right\}, \tag{15}$$

for some positive constant γ_0. The integral operator K_2 can be represented as

$$(K_2 f)(\xi) = \int_{\mathbb{R}^3} k_2(\xi,\eta) f(\eta) d\eta,$$

where the kernel $k_2(\xi,\eta)$ is given by

$$k_2(\xi,\eta) = \frac{|\xi-\eta|^{-1}}{\left(2\pi^3\right)^{1/2}} \int_{w\perp(\xi-\eta)} \exp\left(-\frac{|\eta+w|^2+|\xi+w|^2}{4}\right) \left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} \left(\frac{B(\theta)}{|\cos\theta|}+\frac{B(\frac{\pi}{2}-\theta)}{|\sin\theta|}\right) d^2w,$$

with $\arctan\theta := \frac{|w|}{|\xi-\eta|}$. *Moreover, the kernel* $k_2(\xi,\eta)$ *satisfies*

$$|k_2(\xi,\eta)| \lesssim |\xi-\eta|^{\gamma}(1+|\xi|+|\eta|)^{\gamma-1} \exp\left(-\frac{1}{8}\left[\frac{\left(|\xi|^2-|\eta|^2\right)^2}{|\xi-\eta|^2}+|\xi-\eta|^2\right]\right) \quad \text{for } \gamma\in(-3,-2]. \tag{16}$$

Immediately from Lemma 6 and [26], we have the following result.

Proposition 7 *Let* $-3<\gamma\le -2$, $\tau\in\mathbb{R}$, $0\le\beta<\frac{1}{4}$, *and* $0\le p<2$. *If* $k(\xi,\eta) = -k_1(\xi,\eta)+k_2(\xi,\eta)$, *then*

$$\begin{aligned}\int_{\mathbb{R}^3}|k(\xi,\eta)|^q\langle\eta\rangle^{\tau}e^{-\beta\langle\eta\rangle^p}d\eta &\lesssim \langle\xi\rangle^{\tau+q(\gamma-1)-1}e^{-\beta\langle\xi\rangle^p},\\ \int_{\mathbb{R}^3}|k(\xi,\eta)|^q\langle\xi\rangle^{\tau}e^{-\beta\langle\xi\rangle^p}d\xi &\lesssim \langle\eta\rangle^{\tau+q(\gamma-1)-1}e^{-\beta\langle\eta\rangle^p}\end{aligned} \tag{17}$$

provided that $1\le q<\frac{3}{-\gamma}$.

Consequently, we have

Proposition 8 *Let* $\tau\in\mathbb{R}$ *and* $-3<\gamma\le-2$. *Then*

$$|Kg|_{L^q_{\xi,\tau+2-\gamma}} \lesssim |g|_{L^q_{\xi,\tau}},\ 1\le q\le\infty \tag{18}$$

and

$$|Kg|_{L^{\infty}_{\xi,\tau+1-\gamma+\frac{1}{q}}} \le C|g|_{L^{q'}_{\xi,\tau}} \tag{19}$$

provided that $1/q+1/q'=1$ *and* $1\le q<\frac{3}{-\gamma}$ $\left(\textit{that is, } q'>\frac{3}{3+\gamma}\right)$.

Proposition 9 *Let* $-3<\gamma\le-2$, $\tau\in\mathbb{R}$, $0\le\beta<\frac{1}{4}$, *and* $0\le p<2$, *then*

$$\|Kg(\xi)\|_{L^{\infty}_{\xi,\tau+2-\gamma}\left(e^{\beta\langle\xi\rangle^p}\right)} \lesssim \|g(\xi)\|_{L^{\infty}_{\xi,\tau}\left(e^{\beta\langle\xi\rangle^p}\right)}. \tag{20}$$

Next, we will focus on the estimate of the fractional derivative for $e^{-\nu(\xi)t}$ and $|\xi-\eta|^{\gamma}$.

2.2 The fractional derivative of $e^{-\nu(\xi)t}$ and $|\xi-\eta|^{\gamma}$

Proposition 10 *Let* $-3<\gamma\le-2$. *For any* $t>0$ *and* $0<s<3+\gamma$, *we have*

$$\left|\left(-\Delta_{\xi}\right)^{\frac{s}{2}}e^{-\nu(\xi)t}\right| \lesssim \langle\xi\rangle^{-s}. \tag{21}$$

Proof By definition of the fractional derivative, we separate the following integral into two domains $|z| < (1 + |\xi|)/2$ and $|z| > (1 + |\xi|)/2$, i.e.,

$$\begin{aligned}
\left(-\Delta_\xi\right)^{\frac{s}{2}} e^{-\nu(\xi)t} &= \text{p.v.} \int_{\mathbb{R}^3} \frac{e^{-\nu(\xi+z)t} - e^{-\nu(\xi)t}}{|z|^{3+s}} dz \\
&= \text{p.v.} \left(\int_{|z|<\frac{1+|\xi|}{2}} + \int_{|z|>\frac{1+|\xi|}{2}} \right) \frac{e^{-\nu(\xi+z)t} - e^{-\nu(\xi)t}}{|z|^{3+s}} dz \\
&= T_1 + T_2.
\end{aligned}$$

By the Newton–Leibniz formula and Lemma 5, we have

$$\begin{aligned}
|T_1| &= \left| \int_{|z|<\frac{1+|\xi|}{2}} \int_0^1 \frac{-te^{-\nu(\xi+yz)t}}{|z|^{3+s}} \frac{d}{dy} \nu(\xi + yz) dy dz \right| \\
&\lesssim \int_{|z|<\frac{1+|\xi|}{2}} \int_0^1 \frac{t\nu(\xi + yz)e^{-\nu(\xi+yz)t}}{|z|^{2+s}} (1 + |\xi + yz|)^{-1} dy dz \\
&\lesssim \langle \xi \rangle^{-1} \int_{|z|<\frac{1+|\xi|}{2}} \frac{1}{|z|^{2+s}} dz \\
&\lesssim \langle \xi \rangle^{-s}.
\end{aligned}$$

Combining this with that

$$|T_2| \lesssim \int_{|z|>\frac{1+|\xi|}{2}} \frac{1}{|z|^{3+s}} dz \lesssim \langle \xi \rangle^{-s}.$$

the proof is completed. □

Before going to the estimate of the fractional derivative of Kf, we calculate the fractional derivative of $|\xi - \eta|^\gamma$ first.

Lemma 11 *Let* $-3 < \gamma \leq -2$ *and* $0 < s < 1$. *Then*

$$\left| \left(-\Delta_\xi\right)^{\frac{s}{2}} |\xi - \eta|^\gamma \right| \lesssim |\xi - \eta|^{\gamma - s}. \tag{22}$$

Proof Denote $\zeta = \xi - \eta$. By using spherical coordinates, the fractional derivative of $|\xi - \eta|^\gamma$ can be written as

$$\begin{aligned}
&(-\Delta_\xi)^{\frac{s}{2}} |\xi - \eta|^\gamma \\
&\quad = \text{p.v.} \int_{\mathbb{R}^3} \frac{|\xi - \eta + z|^\gamma - |\xi - \eta|^\gamma}{|z|^{3+s}} dz \\
&\quad = \text{p.v.} \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\left(|\zeta|^2 + r^2 + 2r|\zeta|\cos\theta\right)^{\frac{\gamma}{2}} - |\zeta|^\gamma}{r^{3+s}} r^2 \sin\theta d\varphi d\theta dr
\end{aligned}$$

$$= 2\pi \cdot \text{p.v.} \int_0^\infty \int_0^\pi \left[\left(|\zeta|^2 + r^2 + 2r|\zeta|\cos\theta \right)^{\frac{\gamma}{2}} - |\zeta|^\gamma \right] r^{-1-s} \sin\theta d\theta dr,$$

where θ is the angle between ζ and z. Direct computation gives

$$\int_0^\pi \left(|\zeta|^2 + r^2 + 2r|\zeta|\cos\theta \right)^{\frac{\gamma}{2}} \sin\theta d\theta = \begin{cases} \frac{\ln(|\zeta|+r) - \ln||\zeta|-r|}{r|\zeta|}, & \text{for } \gamma = -2, \\ \frac{||\zeta|-r|^{\gamma+2} - (|\zeta|+r)^{\gamma+2}}{(-2-\gamma)r|\zeta|}, & \text{for } -3 < \gamma < -2. \end{cases}$$

Hence, for $\gamma = -2$, we have

$$\begin{aligned} &(-\Delta_\xi)^{\frac{s}{2}} |\xi - \eta|^\gamma \\ &= 2\pi \cdot \text{p.v.} \int_0^\infty \frac{\ln(|\zeta| + r) - \ln||\zeta| - r| - 2r|\zeta|^{-1}}{r^{2+s}|\zeta|} dr \\ &= 2\pi \cdot \text{p.v.} \left(\int_0^{|\zeta|/2} + \int_{|\zeta|/2}^\infty \right) \frac{\ln(|\zeta| + r) - \ln||\zeta| - r| - 2r|\zeta|^{-1}}{r^{2+s}|\zeta|} dr \\ &=: 2\pi \cdot (T_{11} + T_{12}). \end{aligned}$$

By the Newton–Leibniz formula,

$$\begin{aligned} |T_{11}| &= \left| \int_0^{|\zeta|/2} \int_0^r \left(\frac{1}{|\zeta| + \rho} + \frac{1}{|\zeta| - \rho} - \frac{2}{|\zeta|} \right) \frac{1}{r^{2+s}|\zeta|} d\rho dr \right| \\ &= \int_0^{|\zeta|/2} \int_\rho^{|\zeta|/2} \frac{2\rho^2}{|\zeta|^2 (|\zeta| + \rho)(|\zeta| - \rho)} \frac{1}{r^{2+s}} dr d\rho \\ &= \frac{2}{1+s} \int_0^{|\zeta|/2} \frac{\rho^2}{|\zeta|^2 (|\zeta| + \rho)(|\zeta| - \rho)} \left(\frac{1}{\rho^{1+s}} - \frac{1}{(|\zeta|/2)^{1+s}} \right) d\rho \\ &\lesssim |\zeta|^{-2-s}. \end{aligned}$$

Letting $r = \lambda|\zeta|$ gives

$$|T_{12}| \le |\zeta|^{-2-s} \int_{1/2}^\infty \frac{\ln(1 + \lambda) + |\ln|1 - \lambda|| + 2\lambda}{\lambda^{2+s}} d\lambda \lesssim |\zeta|^{-2-s}.$$

Combining the estimates for T_{11} and T_{12}, we obtain the desired estimate when $\gamma = -2$.

Next, for $-3 < \gamma < -2$, we also split the integral into two domains, that is,

$$\begin{aligned} &(-\Delta_\xi)^{\frac{s}{2}} |\xi - \eta|^\gamma \\ &= 2\pi \cdot \text{p.v.} \int_0^\infty \frac{||\zeta| - r|^{\gamma+2} - (|\zeta| + r)^{\gamma+2} - 2r(-2-\gamma)|\zeta|^{\gamma+1}}{(-2-\gamma)r^{2+s}|\zeta|} dr \\ &= 2\pi \cdot \text{p.v.} \left(\int_0^{|\zeta|/2} + \int_{|\zeta|/2}^\infty \right) \frac{||\zeta| - r|^{\gamma+2} - (|\zeta| + r)^{\gamma+2} - 2r(-2-\gamma)|\zeta|^{\gamma+1}}{(-2-\gamma)r^{2+s}|\zeta|} dr \\ &=: 2\pi \cdot (T_{21} + T_{22}). \end{aligned}$$

Using similar argument as above, we have

$$
\begin{aligned}
|T_{21}| &= \left| \int_0^{|\zeta|/2} \int_0^r \left(\frac{1}{(|\zeta|+\rho)^{-\gamma-1}} + \frac{1}{(|\zeta|-\rho)^{-\gamma-1}} - \frac{2}{|\zeta|^{-\gamma-1}} \right) \frac{1}{r^{2+s}|\zeta|} d\rho dr \right| \\
&= \left| \int_0^{|\zeta|/2} \int_\rho^{|\zeta|/2} \left(\frac{1}{(|\zeta|+\rho)^{-\gamma-1}} + \frac{1}{(|\zeta|-\rho)^{-\gamma-1}} - \frac{2}{|\zeta|^{-\gamma-1}} \right) \frac{1}{r^{2+s}|\zeta|} dr d\rho \right| \\
&\le \frac{1}{s+1} \int_0^{|\zeta|/2} \left(\frac{1}{(|\zeta|+\rho)^{-\gamma-1}} + \frac{1}{(|\zeta|-\rho)^{-\gamma-1}} - \frac{2}{|\zeta|^{-\gamma-1}} \right) \frac{1}{|\zeta|} \\
&\quad \left(\frac{1}{\rho^{s+1}} - \frac{1}{(|\zeta|/2)^{s+1}} \right) d\rho \\
&\lesssim \int_0^{|\zeta|/2} \frac{\rho}{|\zeta|^{-\gamma}} \frac{1}{|\zeta|} \frac{1}{\rho^{s+1}} d\rho \lesssim |\zeta|^{\gamma-s}.
\end{aligned}
$$

Also, letting $r = \lambda|\zeta|$ gives

$$
\begin{aligned}
|T_{22}| &= \left| \int_{1/2}^\infty |\zeta|^{\gamma-s} \frac{|1-\lambda|^{\gamma+2} - (1+\lambda)^{\gamma+2} - 2\lambda(-2-\gamma)}{(-2-\gamma)\lambda^{2+s}} d\lambda \right| \\
&\lesssim |\zeta|^{\gamma-s}.
\end{aligned}
$$

Combining the estimates for T_{21} and T_{22}, we obtain the desired estimate when $-3 < \gamma < -2$. □

In what follows, we will compute $\left(-\Delta_\xi\right)^{\frac{s}{2}} k_1(\xi, \eta)$ and $\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi, \eta)$, respectively.

2.3 The fractional derivative of k_1

Proposition 12 *Let $-3 < \gamma \le -2$ and $0 < s < 3 + \gamma$. Then*

$$
\left| \left(-\Delta_\xi\right)^{\frac{s}{2}} k_1(\xi, \eta) \right| \lesssim |\xi - \eta|^{\gamma-s} \langle \xi \rangle^{\gamma-3-s} e^{-\frac{|\eta|^2}{6}} + \langle \xi \rangle^{-3-s} e^{-\frac{|\eta|^2}{4}}. \tag{23}
$$

Proof Up to a constant multiple, one has

$$
\begin{aligned}
&\left(-\Delta_\xi\right)^{\frac{s}{2}} k_1(\xi, \eta) \\
&= \text{p.v.} \int_{\mathbb{R}^3} \left[|\xi - \eta + z|^\gamma \exp\left(-\frac{|\xi+z|^2 + |\eta|^2}{4} \right) - |\xi - \eta|^\gamma \exp\left(-\frac{|\xi|^2 + |\eta|^2}{4} \right) \right] \\
&\quad \frac{1}{|z|^{3+s}} dz \\
&= \text{p.v.} \int_{\mathbb{R}^3} \frac{|\xi - \eta + z|^\gamma - |\xi - \eta|^\gamma}{|z|^{3+s}} e^{-\frac{|\xi|^2+|\eta|^2}{4}} dz + \text{p.v.} \\
&\quad \int_{\mathbb{R}^3} |\xi - \eta + z|^\gamma \frac{e^{-\frac{|\xi+z|^2+|\eta|^2}{4}} - e^{-\frac{|\xi|^2+|\eta|^2}{4}}}{|z|^{3+s}} dz =: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
$$

It immediately follows from (22) that

$$|\mathcal{I}_1| \lesssim |\xi-\eta|^{\gamma-s} e^{-\frac{|\xi|^2+|\eta|^2}{4}}.$$

By the Newton–Leibniz formula,

$$\begin{aligned}\mathcal{I}_2 &= \text{p.v.} \int_{\mathbb{R}^3} |\xi-\eta+z|^\gamma \, \frac{e^{-\frac{|\xi+z|^2+|\eta|^2}{4}} - e^{-\frac{|\xi|^2+|\eta|^2}{4}}}{|z|^{3+s}} dz \\ &= \text{p.v.} \int_{\mathbb{R}^3} \frac{|\xi-\eta+z|^\gamma}{|z|^{3+s}} e^{-\frac{|\eta|^2}{4}} \int_0^1 e^{-\frac{|\xi+tz|^2}{4}} (-1) \, \frac{(\xi+tz)\cdot z}{2} dt dz,\end{aligned}$$

which implies that

$$|\mathcal{I}_2| \lesssim \int_{\mathbb{R}^3} \int_0^1 |\xi-\eta+z|^\gamma \, \frac{e^{-\frac{|\eta|^2+|\xi+tz|^2}{5}}}{|z|^{2+s}} dt dz. \tag{24}$$

Now we discuss the estimate for $\mathcal{I}_2$ into three cases, respectively.

Case 1: $|\xi-\eta|<1$ and $|\xi|>10$. Then we have $|\xi|-1<|\eta|<|\xi|+1$, and

$$|\mathcal{I}_2| \lesssim e^{-\frac{|\eta|^2}{5}-\frac{|\xi|^2}{20}} \int_{|z|<\frac{|\xi|}{2}} \frac{|\xi-\eta+z|^\gamma}{|z|^{2+s}} dz + e^{-\frac{|\eta|^2}{5}} \int_{|z|>\frac{|\xi|}{2}} |z|^{\gamma-2-s} dz.$$

Since

$$\begin{aligned}&\int_{|z|<\frac{|\xi|}{2}} \frac{|\xi-\eta+z|^\gamma}{|z|^{2+s}} dz \\ &\lesssim \int_{|z|<\frac{|\xi-\eta|}{2}} \frac{|\xi-\eta|^\gamma}{|z|^{2+s}} dz + \int_{\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2}} \frac{|\xi-\eta+z|^\gamma}{|\xi-\eta|^{2+s}} dz + \int_{|z|>\frac{3|\xi-\eta|}{2}} |z|^{\gamma-2-s}\, dz \\ &\lesssim |\xi-\eta|^{\gamma+1-s} + \int_{|z+(\xi-\eta)|<\frac{5|\xi-\eta|}{2}} \frac{|\xi-\eta+z|^\gamma}{|\xi-\eta|^{2+s}} dz \lesssim |\xi-\eta|^{\gamma+1-s}\,,\end{aligned} \tag{25}$$

and

$$\int_{|z|>\frac{|\xi|}{2}} |z|^{\gamma-2-s} dz \lesssim |\xi|^{\gamma-s+1} \lesssim |\xi-\eta|^{\gamma+1-s}\,,$$

we get

$$|\mathcal{I}_2| \lesssim |\xi-\eta|^{\gamma-s}\, e^{-\frac{|\eta|^2}{5}-\frac{|\xi|^2}{20}}.$$

Case 2: $|\xi-\eta|>1$ and $|\xi|>10$. Let $z=w-\xi$ and thus

$$\mathcal{I}_2 = \text{p.v.} \int_{\mathbb{R}^3} \frac{e^{-\frac{|w|^2+|\eta|^2}{4}} - e^{-\frac{|\xi|^2+|\eta|^2}{4}}}{|w-\eta|^{-\gamma}|w-\xi|^{3+s}} dw$$

$$= \text{p.v.} \left(\int_{|w-\xi|<\frac{|\xi|}{2}} + \int_{|w-\xi|>\frac{|\xi|}{2}} \right) \frac{e^{-\frac{|w|^2+|\eta|^2}{4}} - e^{-\frac{|\xi|^2+|\eta|^2}{4}}}{|w-\eta|^{-\gamma}|w-\xi|^{3+s}} dw =: \mathcal{A}_1 + \mathcal{A}_2.$$

For $\mathcal{A}_1$, by the Newton–Leibniz formula, we have

$$\begin{aligned} |\mathcal{A}_1| &\leq e^{\frac{-|\eta|^2}{4}} \int_{|w-\xi|<\frac{|\xi|}{2}} \int_0^1 \frac{e^{-\frac{|\xi+t(w-\xi)|^2}{4}} |\xi + t(w-\xi)|}{|w-\eta|^{-\gamma}|w-\xi|^{2+s}} dt dw \\ &\lesssim e^{\frac{-|\eta|^2}{4} - \frac{|\xi|^2}{20}} \int_{|y|<\frac{|\xi|}{2}} \frac{1}{|y+(\xi-\eta)|^{-\gamma}|y|^{2+s}} dy. \end{aligned}$$

Similar to (25), it follows that

$$|\mathcal{A}_1| \lesssim e^{\frac{-|\eta|^2}{4} - \frac{|\xi|^2}{20}} |\xi-\eta|^{\gamma+1-s} \lesssim |\xi-\eta|^{\gamma-s} e^{\frac{-|\eta|^2}{5} - \frac{|\xi|^2}{25}}.$$

For $\mathcal{A}_2$,

$$|\mathcal{A}_2| \leq \int_{|w-\xi|>\frac{|\xi|}{2}} \frac{e^{-\frac{|w|^2+|\eta|^2}{4}} + e^{-\frac{|\xi|^2+|\eta|^2}{4}}}{|w-\eta|^{-\gamma}|w-\xi|^{3+s}} dw =: \mathcal{A}_{21} + \mathcal{A}_{22}.$$

In view of [10], we have

$$|\mathcal{A}_{21}| \lesssim \frac{e^{-\frac{|\eta|^2}{4}}}{|\xi|^{3+s}} \int_{|w-\xi|>\frac{|\xi|}{2}} \frac{e^{-\frac{|w|^2}{4}}}{|w-\eta|^{-\gamma}} dw \lesssim \frac{e^{-\frac{|\eta|^2}{4}}}{|\xi|^{3+s}} \langle \eta \rangle^{\gamma} \lesssim e^{-\frac{|\eta|^2}{4}} \langle \xi \rangle^{-3-s}.$$

Finally,

$$\begin{aligned} |\mathcal{A}_{22}| &\lesssim e^{-\frac{|\xi|^2+|\eta|^2}{4}} \int_{\left(|w-\xi|>\frac{|\xi|}{2}\right)\wedge(|w-\eta|>1)} \frac{1}{|w-\eta|^{-\gamma}|w-\xi|^{3+s}} dw \\ &\quad + e^{-\frac{|\xi|^2+|\eta|^2}{4}} \int_{\left(|w-\xi|>\frac{|\xi|}{2}\right)\wedge(|w-\eta|<1)} \frac{1}{|w-\eta|^{-\gamma}|w-\xi|^{3+s}} dw \\ &\lesssim |\xi-\eta|^{\gamma-s} \langle \xi \rangle^{\gamma-3-s} e^{-\frac{|\xi|^2+|\eta|^2}{5}}. \end{aligned}$$

Thus we have

$$|\mathcal{I}_2| \lesssim |\xi-\eta|^{\gamma-s} \langle \xi \rangle^{\gamma-3-s} e^{-\frac{|\xi|^2+|\eta|^2}{5}} + e^{-\frac{|\eta|^2}{4}} \langle \xi \rangle^{-3-s}.$$

Case 3: $|\xi| \leq 10$. In view of (24), it follows

$$|\mathcal{I}_2| \lesssim \int_{\mathbb{R}^3} \frac{|\xi-\eta+z|^{\gamma}}{|z|^{2+s}} e^{-\frac{|\eta|^2}{5}} dz.$$

Similar to (25), we have

$$|\mathcal{I}_2| \lesssim |\xi - \eta|^{\gamma+1-s} e^{-\frac{|\eta|^2}{5}} \lesssim |\xi - \eta|^{\gamma-s} \langle\xi\rangle^{\gamma-3-s} e^{-\frac{|\eta|^2}{6}}.$$

The last inequality is due to the fact that $|\xi| \leq 10$.

Gathering **Case 1–Case 3**, we conclude that

$$|\mathcal{I}_2| \lesssim |\xi - \eta|^{\gamma-s} \langle\xi\rangle^{\gamma-3-s} e^{-\frac{|\eta|^2}{6}} + \langle\xi\rangle^{-3-s} e^{-\frac{|\eta|^2}{4}}.$$

Combining this with the estimate for $\mathcal{I}_1$, we obtain

$$\left|\left(-\Delta_\xi\right)^{\frac{s}{2}} k_1(\xi,\eta)\right| \lesssim |\xi - \eta|^{\gamma-s} \langle\xi\rangle^{\gamma-3-s} e^{-\frac{|\eta|^2}{6}} + \langle\xi\rangle^{-3-s} e^{-\frac{|\eta|^2}{4}}.$$

□

2.4 The fractional derivative of k_2

Proposition 13 *Let* $-3 < \gamma \leq -2$ *and* $0 < s < 3 + \gamma$. *Then*

$$\begin{aligned}\left|\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi,\eta)\right| \lesssim\, & |\xi - \eta|^{\gamma-s}(1 + |\xi| + |\eta|)^{\gamma+1} e^{-\frac{|\xi-\eta|^2}{C}} \\ & + (1 + |\xi - \eta|)^{-3-s} (1 + |\xi| + |\eta|)^{\gamma-1} \\ & + (1 + |\eta|)^{\gamma-1} (1 + |\xi| + |\eta|)^{-3-s}, \end{aligned} \tag{26}$$

for some $C > 0$.

The estimate of $\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi,\eta)$ plays a central role in our technical preparations for fractional Mixture Estimate. Its proof is based on refined analysis simultaneously respecting singularity and decay of the kernel function. However, as the calculations are lengthy, to offer a panoramic view of the paper as soon as possible, we postpone the proof of Proposition 13 until Sect. 5.

According to Propositions 12 and 13, we have the following results for the fractional derivative of the operator K in Lebesgue spaces.

Proposition 14 *Let* $-3 < \gamma \leq -2$ *and* $0 < s < 3 + \gamma$. *If* $k(\xi,\eta) = -k_1(\xi,\eta) + k_2(\xi,\eta)$, *then*

$$\int_{\mathbb{R}^3} \left|\left(-\Delta_\xi\right)^{\frac{s}{2}} k(\xi,\eta)\right|^q d\eta \lesssim \langle\xi\rangle^{q(\gamma+1)}, \tag{27}$$

$$\int_{\mathbb{R}^3} \left|\left(-\Delta_\xi\right)^{\frac{s}{2}} k(\xi,\eta)\right|^q d\xi \lesssim \langle\eta\rangle^{q(\gamma+1)}, \tag{28}$$

provided that $1 \leq q < \frac{3}{-\gamma+s}$.

Consequently, we have

Corollary 15 *Let* $-3 < \gamma \leq -2$ *and* $0 < s < 3+\gamma$. *Then*

$$|\left(-\Delta_\xi\right)^{\frac{s}{2}} Kg|_{L^q_\xi} \lesssim |g|_{L^q_\xi},\ 1 \leq q \leq \infty. \tag{29}$$

3 Proof of the mixture estimate

In this section, we will prove theorem 1: the Mixture Estimate in fractional sense. As mentioned in the Introduction, the fractional derivative estimates of the kernel $k\,(\xi,\eta)\,e^{-\nu(\eta)t}$ should be taken into account firstly. We have the following two propositions:

Proposition 16 *Let* $-3 < \gamma \leq -2$ *and* $0 < s < 3+\gamma$. *If* $k\,(\xi,\eta) = -k_1\,(\xi,\eta) + k_2\,(\xi,\eta)$, *then*

$$\int_{\mathbb{R}^3} \left|\left(-\Delta_\eta\right)^{\frac{s}{2}} \left[k\,(\xi,\eta)\,e^{-\nu(\eta)t}\right]\right|^q d\eta \lesssim C, \tag{30}$$

for some constant $C > 0$, *provided that* $1 < q < \frac{3}{-\gamma+s}$.

Proof To prove this, let us recall the Kato–Ponce inequality or the so called "fractional Leibniz rule" ([2], Proposition 3.3): Let $1 < r, p_1, p_2, q_1, q_2 < \infty$ with $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. Given $0 < s < 1$, we have

$$\left|(-\Delta)^{\frac{s}{2}}(fg)\right|_{L^r} \lesssim \left|(-\Delta)^{\frac{s}{2}} f\right|_{L^{p_1}} |g|_{L^{q_1}} + |f|_{L^{p_2}} \left|(-\Delta)^{\frac{s}{2}} g\right|_{L^{q_2}}.$$

Now using (17), (21), (28) associated with the Kato-Ponce inequality, we have

$$\begin{aligned}
&\left|\left(-\Delta_\eta\right)^{\frac{s}{2}}\left(k(\xi,\eta)e^{-\nu(\eta)t}\right)\right|_{L^q_\eta} \\
&\lesssim \left|\left(-\Delta_\eta\right)^{\frac{s}{2}} e^{-\nu(\eta)t}\right|_{L^{\widehat{p}}_\eta} |k(\xi,\eta)|_{L^{\widehat{q}}_\eta} + \left|e^{-\nu(\eta)t}\right|_{L^{\widehat{p}}_\eta} \left|\left(-\Delta_\eta\right)^{\frac{s}{2}} k(\xi,\eta)\right|_{L^{\widehat{q}}_\eta} \\
&\lesssim C\,,
\end{aligned}$$

where $\frac{1}{q} = \frac{1}{\widehat{p}} + \frac{1}{\widehat{q}}$ with $1 < q < \widehat{q} < \frac{3}{-\gamma+s}$ and $s\widehat{p} > 3$. □

Proposition 17 *Let* $-3 < \gamma \leq -2$ *and* $0 < s < 3+\gamma$. *If* $k\,(\xi,\eta) = -k_1\,(\xi,\eta) + k_2\,(\xi,\eta)$, *then*

$$\left|\int_{\mathbb{R}^3} \left(-\Delta_\eta\right)^{\frac{s}{2}} \left[k(\xi,\eta)e^{-\nu(\eta)t}\right] g(\eta)d\eta\right|_{L^2_\xi} \lesssim |g|_{L^p_\xi},$$

where $\frac{1}{p} = \frac{1}{2}(3 - \frac{2}{q})$ *and* $1 < q < \frac{3}{-\gamma+s}$.

Proof In view of (30), we have by Minkowski integral inequality

$$\left|\int_{\mathbb{R}^3} (-\Delta_\eta)^{\frac{s}{2}} \left[k(\xi,\eta)e^{-\nu(\eta)t}\right] g(\eta)d\eta\right|_{L^q_\xi} \lesssim |g|_{L^1_\xi}, \tag{31}$$

and by Hölder's inequality

$$\left|\int_{\mathbb{R}^3} (-\Delta_\eta)^{\frac{s}{2}} \left[k(\xi,\eta)e^{-\nu(\eta)t}\right] g(\eta)d\eta\right|_{L^\infty_\xi} \lesssim |g|_{L^{q'}_\xi} \tag{32}$$

provided that $1/q + 1/q' = 1$ and $1 < q < \frac{3}{-\gamma+s}$. And then applying the Riesz–Thorin Interpolation Theorem to (31) and (32) yields

$$\left|\int_{\mathbb{R}^3} (-\Delta_\eta)^{\frac{s}{2}} \left[k(\xi,\eta)e^{-\nu(\eta)t}\right] g(\eta)d\eta\right|_{L^2_\xi} \lesssim |g|_{L^p_\xi},$$

where $\frac{1}{p} = \frac{1}{2}(3 - \frac{2}{q})$. □

In order to prove our main theorem, we consider the mixing operator $K\mathbb{S}^t_\gamma$ first. Note that Proposition 17 will be used in the proof of the following lemma.

Lemma 18 *Let* $-3 < \gamma \le -2$, $0 < s < 3+\gamma$. *If* $\frac{1}{p} = \frac{1}{2}(3 - \frac{2}{q})$ *with* $1 < q < \frac{3}{-\gamma+s}$, *we have*

$$\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}^t_\gamma h_0\right\|_{L^2} \lesssim t^{-s}\,\|h_0\|_{L^p_\xi L^2_x} + t^{-s}\|\left(-\Delta_\xi\right)^{\frac{s}{2}} h_0\|_{L^2}. \tag{33}$$

Proof Let

$$h(t,x,\xi) = \mathbb{S}^t h_0 = h_0(x-\xi t,\xi).$$

Taking the Fourier transform in both x and ξ variables, we have

$$\hat{h}(t,\hat{x},\hat{\xi}) = \hat{h}_0(\hat{x},\hat{\xi}+t\hat{x}),$$

where $\hat{x}$ and $\hat{\xi}$ are the Fourier dual variables of x and ξ, respectively. Notice that

$$|\hat{x}|^s\hat{h}(t,\hat{x},\hat{\xi}) = t^{-s}|\hat{\xi}|^s\hat{h}(t,\hat{x},\hat{\xi}) + t^{-s}\left(|t\hat{x}|^s - |\hat{\xi}|^s\right)\hat{h}(t,\hat{x},\hat{\xi}).$$

Then applying the inverse Fourier transform to both sides gives

$$(-\Delta_x)^{\frac{s}{2}} h = t^{-s}\left(-\Delta_\xi\right)^{\frac{s}{2}} h + t^{-s}\mathcal{F}^{-1}\left\{\left(|t\hat{x}|^s - |\hat{\xi}|^s\right)\hat{h}(t,\hat{x},\hat{\xi})\right\}. \tag{34}$$

Hence,

$$\begin{aligned}&\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}_\gamma^t h_0\right\|_{L^2}\\&\quad\le t^{-s}\left\|Ke^{-\nu(\xi)t}\left(-\Delta_\xi\right)^{\frac{s}{2}}\mathbb{S}^t h_0\right\|_{L^2}+t^{-s}\left\|Ke^{-\nu(\xi)t}\mathcal{F}^{-1}\left\{\left(|t\hat{x}|^s-|\hat{\xi}|^s\right)\hat{h}(t,\hat{x},\hat{\xi})\right\}\right\|_{L^2}\\&\quad=: t^{-s}\left(T_1+T_2\right).\end{aligned}$$

In view of (18),

$$\begin{aligned}T_2&\le\left\|\mathcal{F}^{-1}\left\{\left(|t\hat{x}|^s-|\hat{\xi}|^s\right)\hat{h}(t,\hat{x},\hat{\xi})\right\}\right\|_{L^2}\\&\le\left\||t\hat{x}+\hat{\xi}|^s\hat{h}_0(\hat{x},t\hat{x}+\hat{\xi})\right\|_{L^2_{\hat{x}}L^2_{\hat{\xi}}}\\&\le\left\|\left(-\Delta_\xi\right)^{\frac{s}{2}}h_0\right\|_{L^2}.\end{aligned}$$

For the estimate of T_1, note that

$$\begin{aligned}Ke^{-\nu(\xi)t}\left(-\Delta_\xi\right)^{\frac{s}{2}}h&=\int_{\mathbb{R}^3}k(\xi,\eta)e^{-\nu(\eta)t}\left(-\Delta_\eta\right)^{\frac{s}{2}}h(t,x,\eta)d\eta\\&=\int_{\mathbb{R}^3}\left(-\Delta_\eta\right)^{\frac{s}{2}}\left[k(\xi,\eta)e^{-\nu(\eta)t}\right]h(t,x,\eta)d\eta.\end{aligned}$$

By Proposition 17, we have

$$|T_1|\le\left|\int_{\mathbb{R}^3}\left(-\Delta_\eta\right)^{\frac{s}{2}}\left[k(\xi,\eta)e^{-\nu(\eta)t}\right]|h(t,\cdot,\eta)|_{L^2_x}\,d\eta\right|_{L^2_\xi}\lesssim\|h\|_{L^p_\xi L^2_x}\lesssim\|h_0\|_{L^p_\xi L^2_x}.$$

This completes the proof of the lemma. □

Proof of Theorem 1 Applying Lemma 18, (18) and (29), we have

$$\begin{aligned}\left\|(-\Delta_x)^{\frac{s}{2}} K\mathbb{S}_\gamma^t Kh_0\right\|_{L^2}&\lesssim t^{-s}\|Kh_0\|_{L^p_\xi L^2_x}+t^{-s}\|\left(-\Delta_\xi\right)^{\frac{s}{2}}Kh_0\|_{L^2}\\&\lesssim t^{-s}\left\|\langle\xi\rangle^{\gamma-2}h_0\right\|_{L^p_\xi L^2_x}+t^{-s}\|h_0\|_{L^2}\\&\lesssim t^{-s}\|h_0\|_{L^2}+t^{-s}\|h_0\|_{L^2}.\end{aligned}$$

This completes the proof of Theorem 1. □

4 Proof of theorem 3

In this section we go back to equation (2) and investigate the well-posedness and large time behavior of the Boltzmann equation for the very soft potential case $-3<\gamma\le-2$

(Theorem 3). First, we study the large time behavior of the linearized Boltzmann equation

$$\begin{cases} \partial_t g + \xi \cdot \nabla_x g = Lg, \\ g(0,x,\xi) = g_0(x,\xi), \end{cases} \quad (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3. \tag{35}$$

Then, we study the large time behavior of the linearized equation with extra source term $\Gamma(h_1, h_2)$, where h_1, h_2 are prescribed with time decay (60). Finally, based on the result of the inhomogeneous linearized equation, we design an iteration scheme for solving equation (2) and thus establish the well-posedness and large time behavior of the Boltzmann equation for the very soft potential case. In what follows, we elaborate our proof.

4.1 Large time behavior of solution to the linearized equation

In this subsection we will prove the large time behavior of the solution to (35) in L^2_x and L^∞_x with certain ξ-weight as below.

Proposition 19 *Let $-3 < \gamma \le -2$, $0 < p \le 2$, $\beta > 3/2$, $\alpha > 0$ sufficiently small, and let $j > 0$ be sufficiently large. Assume that $g_0 \in L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^\infty_x \cap L^1_x)$. Then there are positive constants C_i and $\bar{C}_i$, $i = 1, 2$, such that the solution g to (35) satisfies*

$$\|g(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^2_x} \le C_1(1+t)^{-\frac{3}{4}} \|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{36}$$

$$\|g(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le C_2(1+t)^{-\frac{3}{2}} \|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}. \tag{37}$$

Moreover,

$$\|g(t)\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})L^2_x} \le \bar{C}_1 \|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{38}$$

$$\|g(t)\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le \bar{C}_2 \|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}. \tag{39}$$

The main idea is to obtain the $L^2_\xi L^2_x$ and $L^2_\xi L^\infty_x$ estimates of the linearized equation followed by the bootstrap argument. The $L^2_\xi L^2_x$ and $L^2_\xi L^\infty_x$ estimates is based on the long wave-short wave and wave-remainder decomposition; and regularization estimate plays a significant role in the course of the proof. Furthermore, to obtain the ξ-weighted L^2_x and L^∞_x estimate, we apply finite steps of bootstrap argument with the aid of the interpolation result of the integral operator K.

• *Step 1.* $L^2_\xi L^2_x$ and $L^2_\xi L^\infty_x$ estimates of the linearized equation.
For $-3 < \gamma \le -2$, $0 < p \le 2$, $\beta > 3/2$, $\alpha > 0$ sufficiently small, we will show that if $g_0 \in L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})(L^\infty_x \cap L^1_x)$, then there are positive constants C_i, $i = 1, 2$, such that the solution g to (35) satisfies

$$\|g(t)\|_{L^2_\xi L^2_x} \le C_1(1+t)^{-\frac{3}{4}} \|g_0\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{40}$$

$$\|g(t)\|_{L^2_\xi L^\infty_x} \leq C_2(1+t)^{-\frac{3}{2}} \|g_0\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)} . \tag{41}$$

To this end, we first introduce the long wave-short wave decomposition. By the Fourier transform, the solution of the linearized Boltzmann equation (35) can be written as

$$\mathbb{G}^t g_0 = g(t,x,\xi) = \int_{\mathbb{R}^3} e^{i\hat{x}\cdot x+(-i\xi\cdot\hat{x}+L)t}\widehat{g}_0(\hat{x},\xi)d\hat{x}, \tag{42}$$

where $\widehat{f}$ means the Fourier transform of f in the space variable x and $\mathbb{G}^t$ is the solution operator (or Green's function) of the linearized Boltzmann equation. We can decompose the solution g into the long wave part g_L and the short wave part g_S given respectively by

$$\begin{aligned} g_L &= \int_{|\hat{x}|<\delta} e^{i\hat{x}\cdot x+(-i\xi\cdot\hat{x}+L)t}\widehat{g}_0(\hat{x},\xi)d\hat{x}, \\ g_S &= \int_{|\hat{x}|>\delta} e^{i\hat{x}\cdot x+(-i\xi\cdot\hat{x}+L)t}\widehat{g}_0(\hat{x},\xi)d\hat{x}, \end{aligned} \tag{43}$$

for $\delta > 0$ small. Using similar arguments as those in the papers of Kawashima [14], Strain [24] and Strain-Guo [25], we get time decay as follows:

$$\|g_L\|_{L^2_\xi L^\infty_x} \lesssim (1+t)^{-\frac{3}{2}} \|g_0\|_{L^2_\xi(e^{\alpha\langle\xi\rangle^p})L^1_x} , \tag{44}$$

$$\|g_L\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}} \|g_0\|_{L^2_\xi(e^{\alpha\langle\xi\rangle^p})L^1_x} , \tag{45}$$

$$\|g_S\|_{L^2} \lesssim e^{-c_{p,\gamma}\alpha^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}} \|g_0\|_{L^2(e^{\alpha\langle\xi\rangle^p})} , \tag{46}$$

if the initial data $g_0 \in L^2_\xi(e^{\alpha\langle\xi\rangle^p})\left(L^1_x \cap L^\infty_x\right)$, $0 < p \leq 2$ and $\alpha > 0$ small; therefore, (40) is obtained. All related estimates about the proof have been done in [25] and so we skip the proof.

To obtain $L^2_\xi L^\infty_x$ estimate for g, we further introduce the wave-remainder decomposition; the strategy is to design a Picard-type iteration, treating Kg as a source term. Specifically, we write

$$g = W^{(n)} + \mathcal{R}^{(n)}$$

where

$$W^{(n)} = \sum_{j=0}^{n} h^{(j)}, \quad \mathcal{R}^{(n)} = g - W^{(n)}, \tag{47}$$

are called the wave part and remainder part, respectively, which are defined as below:

$$h^{(0)} = \mathbb{S}^t_\gamma g_0, \quad h^{(j)} = \int_0^t \mathbb{S}^{t-s}_\gamma K h^{(j-1)}(s)ds , \tag{48}$$

for $1 \leq j \leq n$ and

$$\mathcal{R}^{(n)} = \int_0^t \mathbb{G}^{t-s} K h^{(n)}(s)ds, \tag{49}$$

where $\mathbb{G}^t$ is the Green's function of the full linearized Boltzmann equation. Note that $h^{(j)}$ can be represented in terms of the multiple-mixture operator $\mathbb{M}_j$, as

$$h^{(j)} = \mathbb{M}_j h^{(0)}, \;\; j \geq 1.$$

Because of the argument in [18, Lemmas 8 and 11] being also valid in the case $-3 < \gamma \leq 2$, we readily get the L^∞ and L^2 estimates of $h^{(j)}$ and so does the wave part $W^{(n)}$:

$$\left\| h^{(j)} \right\|_{L^\infty_\xi\left(\langle\xi\rangle^\beta\right)L^\infty_x} \lesssim t^j e^{-c_0 \overline{\alpha}^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}} \|g_0\|_{L^\infty_\xi\left(e^{(j+1)\overline{\alpha}\langle\xi\rangle^p}\langle\xi\rangle^\beta\right)L^\infty_x}, \tag{50}$$

$$\left\| h^{(j)} \right\|_{L^2} \lesssim t^j e^{-c \overline{\alpha}^{\frac{-\gamma}{p-\gamma}} t^{\frac{p}{p-\gamma}}} \|g_0\|_{L^2\left(e^{(j+1)\overline{\alpha}\langle\xi\rangle^p}\right)}, \tag{51}$$

for some constants c_0, $c > 0$, where $g_0 \in L^\infty_\xi\left(e^{(j+1)\overline{\alpha}\langle\xi\rangle^p}\langle\xi\rangle^\beta\right)(L^\infty_x \cap L^2_x)$, $0 < p \leq 2$, $\beta > 3/2$ and $\overline{\alpha} > 0$ is small with $(j+1)\,\overline{\alpha} < \alpha$.

Next, we obtain the regularization estimate of $h^{(2k)}$, which is a consequence of Corollary 2.

Lemma 20 (Regularization estimate on $h^{(2k)}$) *For* $-3 < \gamma \leq -2$*, choose* $0 < s < 3+\gamma$*,* $k \in \mathbb{N}$ *such that* $sk = 2$*, then we have*

$$\|h^{(2k)}\|_{L^2_\xi H^2_x} \lesssim (1+t)^{k(2-s)} \|g_0\|_{L^2}.$$

Therefore, in view of (49), taking $n = 2k$, we find

$$\|\mathcal{R}^{(n)}\|_{L^2_\xi H^2_x} \leq \int_0^t \|h^{(2k)}(\tau)\|_{L^2_\xi H^2_x} d\tau \lesssim (1+t)^{k(2-s)+1} \|g_0\|_{L^2}. \tag{52}$$

Combining above estimates, one can obtain (41) by following the same argument presented in [18, Section 5.1].

• *Step 2. Bootstrap*

Subsequently, we will prove (37) and the others can be proved in a similar way. In terms of the damped transport operator $\mathbb{S}^t_\gamma$, g can be written as

$$g(t) = \mathbb{S}^t_\gamma g_0 + \int_0^t \mathbb{S}^{t-\tau}_\gamma K g(\tau) d\tau. \tag{53}$$

Let $T > 0$. For any $0 \leq t \leq T$,

$$e^{\alpha\langle\xi\rangle^p} |g(t)|_{L^\infty_x} \leq e^{\alpha\langle\xi\rangle^p} \left|\mathbb{S}^t_\gamma g_0\right|_{L^\infty_x} + e^{\alpha\langle\xi\rangle^p} \int_0^t \left|\mathbb{S}^{t-\tau}_\gamma K g(\tau)\right|_{L^\infty_x} d\tau = I + II.$$

It is easy to see that

$$
\begin{aligned}
I &\leq \sup_{\xi}\left(e^{\alpha\langle\xi\rangle^p}\left|\mathbb{S}_{\gamma}^t g_0\right|_{L_x^\infty}\right) \leq \left(\sup_{\xi} e^{-\nu(\xi)t}\langle\xi\rangle^{-j}\right)\|g_0\|_{L^\infty_{\xi,j}(e^{\alpha\langle\xi\rangle^p})L_x^\infty} \\
&\lesssim (1+t)^{\frac{j}{\gamma}}\|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})L_x^\infty} \lesssim (1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})L_x^\infty}, \qquad (54)
\end{aligned}
$$

since j is sufficiently large. For II, let $1/q+1/q'=1$ and q in the region $1<q<\frac{3}{-\gamma}$ (that is, $q'>\frac{3}{3+\gamma}\geq 3$), it follows from (19) and (20) that

$$
\begin{aligned}
& e^{\alpha\langle\xi\rangle^p}e^{-(t-\tau)\nu(\xi)}|Kg(\tau)|_{L_x^\infty} \\
&\leq e^{-(t-\tau)\nu(\xi)}\langle\xi\rangle^{\gamma-1-\frac{1}{q}}\left[\sup_{|\xi|\leq\lambda}\left(e^{\alpha\langle\xi\rangle^p}\langle\xi\rangle^{-\gamma+1+\frac{1}{q}}|Kg(\tau)|_{L_x^\infty}\right)\right. \\
&\quad\left.+\sup_{|\xi|>\lambda}\left(\langle\xi\rangle^{-1+\frac{1}{q}}e^{\alpha\langle\xi\rangle^p}\langle\xi\rangle^{2-\gamma}|Kg(\tau)|_{L_x^\infty}\right)\right] \\
&\lesssim (1+t-\tau)^{\frac{-\gamma+1+\frac{1}{q}}{\gamma}}\left(e^{\alpha\langle\lambda\rangle^p}\|Kg(\tau)\|_{L^\infty_{\xi,-\gamma+1+\frac{1}{q}}L_x^\infty}\right. \\
&\quad\left.+(1+\lambda)^{-1+\frac{1}{q}}\left\|e^{\alpha\langle\xi\rangle^p}Kg(\tau)\right\|_{L^\infty_{\xi,2-\gamma}L_x^\infty}\right) \\
&\lesssim (1+t-\tau)^{\frac{-\gamma+1+\frac{1}{q}}{\gamma}}\left(e^{\alpha(1+\lambda)^p}\|g(\tau)\|_{L_\xi^{q'}L_x^\infty}\right. \\
&\quad\left.+(1+\lambda)^{-1+\frac{1}{q}}\|g(\tau)\|_{L_\xi^\infty(e^{\alpha\langle\xi\rangle^p})L_x^\infty}\right),
\end{aligned}
$$

for any $\lambda>0$. Here we restrict q such that $\frac{-\gamma+1+\frac{1}{q}}{\gamma}\leq-\frac{3}{2}$, that is,

$$
1<q<\min\left\{\frac{2}{-\gamma-2},\frac{3}{-\gamma}\right\}.
$$

However, $\|g(\tau)\|_{L_\xi^{q'}L_x^\infty}$ is unknown. We claim that

$$
\|g(t)\|_{L_\xi^{q'}L_x^\infty} \lesssim (1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(L_x^1\cap L_x^\infty)}, \qquad (55)
$$

for $0\leq t\leq T$. Suppose this is the case, we can deduce

$$
\begin{aligned}
& II \\
&\leq Ce^{\alpha\langle\lambda\rangle^p}\|g_0\|_{L^\infty_{\xi,\beta+j}(L_x^1\cap L_x^\infty)}\int_0^t(1+t-\tau)^{\frac{-\gamma+1+\frac{1}{q}}{\gamma}}(1+\tau)^{-\frac{3}{2}}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C(1+\lambda)^{-1+\frac{1}{q}} \sup_{0\le s\le T}\left[(1+\tau)^{\frac{3}{2}}\|g(\tau)\|_{L^\infty_\xi(e^{\alpha\langle\xi\rangle^p})L^\infty_x}\right]\\
&\cdot\int_0^t (1+t-\tau)^{\frac{-\gamma+1+\frac{1}{q}}{\gamma}}(1+\tau)^{-\frac{3}{2}}d\tau\\
\le\ & C' e^{\alpha\langle\lambda\rangle^p}(1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}\\
&+C'(1+\lambda)^{-1+\frac{1}{q}}(1+t)^{-\frac{3}{2}}\sup_{0\le\tau\le T}\left[(1+\tau)^{\frac{3}{2}}\|g(\tau)\|_{L^\infty_\xi(e^{\alpha\langle\xi\rangle^p})L^\infty_x}\right]. \quad (56)
\end{aligned}
$$

After selecting $\lambda > 0$ sufficiently large with $C'(1+\lambda)^{-1+\frac{1}{q}} < 1/2$, we obtain

$$
\sup_{0\le t\le T}\left[(1+t)^{\frac{3}{2}}\|g(t)\|_{L^\infty_\xi(e^{\alpha\langle\xi\rangle^p})L^\infty_x}\right]\le C_1\|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)},
$$

due to (54) and (56). It implies that

$$
\|g(t)\|_{L^\infty_\xi(e^{\alpha\langle\xi\rangle^p})L^\infty_x}\le C_1(1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)} \quad (57)
$$

for $0 \le t < \infty$, since $T > 0$ is arbitrary.

Now, to obtain (55), we need some interpolation result of the integral operator K. Applying the Riesz-Thorin interpolation theorem to the operator $\langle\xi\rangle^{1-\gamma+1/q}K$, $1 < q < \frac{3}{-\gamma}$, associated with (18) and (19), we have

$$
\|Kh\|_{L^{p_1}_{\xi,1-\gamma+1/q}L^\infty_x}\lesssim\|h\|_{L^2_\xi L^\infty_x}
$$

where $\frac{1}{p_1}=\left(\frac{1}{q}-\frac{1}{2}\right)/\left(1-\frac{1}{q}\right)=1/(2q)$ with $\frac{1}{q'}=\left(1-\frac{1}{q}\right)<\frac{1}{p_1}<\frac{1}{q}$. Continuing in this way up to $(m+1)$ times for which $1/p_{m+1} < 1/q' < 1/p_m$ where $\left(\frac{1}{q}-\frac{1}{p_\ell}\right)=\frac{1}{2-q}\left(\frac{1}{q}-\frac{1}{p_{\ell-1}}\right)$ for $1\le\ell\le m$ and $p_0=2$, we find

$$
\|Kh\|_{L^{p_\ell}_{\xi,1-\gamma+1/q}L^\infty_x}\lesssim\|h\|_{L^{p_{\ell-1}}_\xi L^\infty_x}.
$$

Thus, in view of (53),

$$
\begin{aligned}
\|g(t)\|_{L^{p_\ell}_\xi L^\infty_x} &\le \left\|S^t_\gamma g_0\right\|_{L^{p_\ell}_\xi L^\infty_x}+\int_0^t\left\|S^{t-z}_\gamma Kg(z)\right\|_{L^{p_\ell}_\xi L^\infty_x}dz\\
&\lesssim (1+t)^{\frac{j}{\gamma}}\|g_0\|_{L^\infty_{\xi,\beta+j}L^\infty_x}+\int_0^t(1+t-z)^{\frac{-\gamma+1+\frac{1}{q}}{\gamma}}\|Kg(z)\|_{L^{p_\ell}_{\xi,1-\gamma+1/q}L^\infty_x}dz\\
&\lesssim (1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(L^1_x\cap L^\infty_x)}+\int_0^t(1+t-z)^{-\frac{3}{2}}\|g(z)\|_{L^{p_{\ell-1}}_\xi L^\infty_x}dz,
\end{aligned}
$$

for $1 \le \ell \le m+1$. By the Bootstrap argument associated with (41), we have

$$\begin{aligned}\|g\|_{L_\xi^{p_{m+1}}L_x^\infty} &\lesssim (1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(L^1_x\cap L^\infty_x)}, \ \|g\|_{L_\xi^{p_m}L_x^\infty}\\ &\lesssim (1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(L^1_x\cap L^\infty_x)}.\end{aligned}$$

Therefore,

$$\|g(t)\|_{L_\xi^{q'}L_x^\infty} \le \|g\|^{\lambda}_{L_\xi^{p_{m+1}}L_x^\infty}\|g\|^{1-\lambda}_{L_\xi^{p_m}L_x^\infty} \lesssim (1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(L^1_x\cap L^\infty_x)}$$

where $\frac{1}{q'} = \frac{\theta}{p_{m+1}} + \frac{1-\theta}{p_m}$, for some $0<\theta<1$.

Finally, to complete the estimate of (37), applying the bootstrap argument again, together with (20) and (57), we get

$$\|g(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le C_1(1+t)^{-\frac{3}{2}}\|g_0\|_{L^\infty_{\xi,\beta+j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)},$$

as desired. The proof of Proposition 19 is completed.

4.2 Nonlinear estimate

We consider the following inhomogeneous Boltzmann equation

$$\begin{cases}\partial_t g + \xi\cdot\nabla_x g = Lg + \Gamma(h_1,h_2),\\ g(0,x,\xi) = g_0(x,\xi).\end{cases} \tag{58}$$

Let $0<p\le 2$, $\beta>3/2$, $\alpha\ge 0$ sufficiently small, and $j>0$ sufficiently large. We assume that g_0 satisfies

$$\|g_0\|_{L^\infty_\xi(\langle\xi\rangle^{\beta+2j}e^{\alpha\langle\xi\rangle^p})L^1_x} + \|g_0\|_{L^\infty_\xi(\langle\xi\rangle^{\beta+2j}e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le b_0, \tag{59}$$

and h_i $(i=1,2)$ satisfies

$$\begin{aligned}\sup_t\Big\{&(1+t)^{\frac{3}{4}}\|h_i(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta}e^{\alpha\langle\xi\rangle^p})L^2_x},\ (1+t)^{\frac{3}{4}}\|h_i(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta}e^{\alpha\langle\xi\rangle^p})L^\infty_x},\\ &\|h_i(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta+2j}e^{\alpha\langle\xi\rangle^p})L^2_x},\ \|h_i(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta+2j}e^{\alpha\langle\xi\rangle^p})L^\infty_x}\Big\} \le b_i,\end{aligned} \tag{60}$$

for some $b_0, b_1, b_2 > 0$. We can demonstrate that the solution g to (58) satisfies

Proposition 21 *Assume that g_0 satisfies (59) and that h_1 and h_2 satisfy (60). Then there exists a number $C>0$ such that the solution g to (58) satisfies*

$$\begin{aligned}\max\Big\{&(1+t)^{\frac{3}{4}}\|g(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta}e^{\alpha\langle\xi\rangle^p})L^2_x}, (1+t)^{\frac{3}{4}}\|g(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta}e^{\alpha\langle\xi\rangle^p})L^\infty_x},\\ &\|g(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta+2j}e^{\alpha\langle\xi\rangle^p})L^2_x}, \|g(t)\|_{L^\infty_\xi(\langle\xi\rangle^{\beta+2j}e^{\alpha\langle\xi\rangle^p})L^\infty_x}\Big\} \le C(b_0+b_1b_2).\end{aligned}$$

The proof of Proposition 21 is similar to those in the soft potential cases ($-2 < \gamma < 0$) and the reader is referred to [17] for more details.

Next, define a norm $|||\cdot|||$ as

$$|||h||| \equiv \sup_t \Big\{ (1+t)^{\frac{3}{4}} \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^\beta e^{\alpha\langle\xi\rangle^p}\right)L^2_x}, (1+t)^{\frac{3}{4}} \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^\beta e^{\alpha\langle\xi\rangle^p}\right)L^\infty_x}, \\ \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta+2j} e^{\alpha\langle\xi\rangle^p}\right)L^2_x}, \|h(t)\|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta+2j} e^{\alpha\langle\xi\rangle^p}\right)L^\infty_x} \Big\}.$$

We consider the iteration $\left\{f^{(i)}\right\}$ for which $f^{(0)}(t,x,\xi) \equiv 0$ and $f^{(i+1)}$, $i \in \mathbb{N} \cup \{0\}$, is a solution to the equation

$$\begin{cases} \partial_t f^{(i+1)} + \xi\cdot\nabla_x f^{(i+1)} = Lf^{(i+1)} + \Gamma(f^{(i)}, f^{(i)}), \\ f^{(i+1)}(0,x,\xi) = \eta f_0(x,\xi), \end{cases} \tag{61}$$

where $\eta > 0$ is sufficiently small. Denote

$$b_0 := \eta \left(\|f_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta+2j} e^{\alpha\langle\xi\rangle^p}\right)L^1_x} + \|f_0\|_{L^\infty_\xi\left(\langle\xi\rangle^{\beta+2j} e^{\alpha\langle\xi\rangle^p}\right)L^\infty_x} \right).$$

According to Proposition 21, we find that $\left\{f^{(i)}\right\}$ is a Cauchy sequence in the norm $|||\cdot|||$, and therefore it converges to the limit f satisfying

$$\|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^2_x} \le \eta C_1 (1+t)^{-\frac{3}{4}} \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{62}$$

$$\|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le \eta C_2 (1+t)^{-\frac{3}{4}} \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{63}$$

$$\|f(t)\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^2_x} \le \eta \bar{C}_1 \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}, \tag{64}$$

$$\|f(t)\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \le \eta \bar{C}_2 \|f_0\|_{L^\infty_{\xi,\beta+2j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}. \tag{65}$$

Finally, write f as

$$f = \eta\mathbb{G}^t f_0 + \int_0^t \mathbb{G}^{t-\tau}\Gamma(f,f)(\tau)d\tau,$$

we can use a bootstrap argument to improve the estimate (63) as

$$\|f(t)\|_{L^\infty_{\xi,\beta}(e^{\alpha\langle\xi\rangle^p})L^\infty_x} \lesssim \eta\,(1+t)^{-\frac{3}{2}} \|f_0\|_{L^\infty_{\xi,\beta+3j}(e^{\alpha\langle\xi\rangle^p})(L^1_x\cap L^\infty_x)}.$$

Therefore, the proof of Theorem 3 is completed.

5 Proof of Proposition 13

By definition, the fractional derivative of k_2 is given by

$$\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi,\eta) = \text{p.v.} \int_{\mathbb{R}^3} \frac{k_2(\xi+z,\eta)-k_2(\xi,\eta)}{|z|^{s+3}} dz.$$

To proceed, we simplify the expression of $k_2(\xi+z,\eta)$. By the representation formula in Lemma 6, up to a constant multiple,

$$\begin{aligned}
&k_2(\xi,\eta)\\
&= |\xi-\eta|^{-1} \int_0^\infty \int_0^{2\pi} e^{-\frac{|\xi|^2+|\eta|^2}{4}} e^{-\frac{r^2+r|(\xi+\eta)_\perp|\cos\varphi}{2}} \left(|\xi-\eta|^2+r^2\right)^{\frac{\gamma-1}{2}} B_*\left(\frac{r}{|\xi-\eta|}\right) r d\varphi dr\\
&= |\xi-\eta|^{\gamma} e^{-\frac{|\xi|^2+|\eta|^2}{4}} \int_0^\infty \int_0^{2\pi} e^{-\frac{|\xi-\eta|^2\rho^2+|\xi-\eta||(\xi+\eta)_\perp|\rho\cos\varphi}{2}} \left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho, \qquad (66)
\end{aligned}$$

where

$$B_*\left(\frac{r}{|\xi-\eta|}\right) = \frac{B\left(\tan^{-1}\left(\frac{r}{|\xi-\eta|}\right)\right)}{\left|\cos\left(\tan^{-1}\left(\frac{r}{|\xi-\eta|}\right)\right)\right|} + \frac{B\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{r}{|\xi-\eta|}\right)\right)}{\left|\sin\left(\tan^{-1}\left(\frac{r}{|\xi-\eta|}\right)\right)\right|},$$

and

$$(\xi+\eta)_\perp = (\xi+\eta) - \frac{(\xi+\eta)\cdot(\xi-\eta)}{|\xi-\eta|^2}(\xi-\eta) = \frac{\xi-\eta}{|\xi-\eta|} \wedge \left((\xi+\eta)\wedge\frac{\xi-\eta}{|\xi-\eta|}\right),$$

with

$$\left|(\xi+\eta)_\perp\right| = \frac{2\sqrt{|\xi|^2|\eta|^2-(\xi\cdot\eta)^2}}{|\xi-\eta|} = \frac{2\,|\xi\wedge\eta|}{|\xi-\eta|}.$$

Therefore,

$$\begin{aligned}
&\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi,\eta)\\
&= \text{p.v.}\left(\int_{D^c} \frac{k_2(\xi+z,\eta)-k_2(\xi,\eta)}{|z|^{3+s}} dz + \int_D \frac{k_2(\xi+z,\eta)-k_2(\xi,\eta)}{|z|^{3+s}} dz\right)\\
&= \text{p.v.}\left(\int_{D^c} + \int_D\right) \frac{|\xi+z-\eta|^\gamma - |\xi-\eta|^\gamma}{|z|^{3+s}} dz\, e^{-\frac{|\xi|^2+|\eta|^2}{4}}\\
&\quad \int_0^\infty \int_0^{2\pi} e^{-\frac{|\xi-\eta|^2\rho^2+2|\xi\wedge\eta|\rho\cos\varphi}{2}} \left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho\\
&\quad + \text{p.v.}\left(\int_{D^c} + \int_D\right) |\xi+z-\eta|^\gamma e^{-\frac{|\eta|^2}{4}}\\
&\quad \cdot \int_0^\infty \int_0^{2\pi} \frac{e^{-\frac{|\xi+z|^2+2|\xi+z-\eta|^2\rho^2+4|(\xi+z)\wedge\eta|\rho\cos\varphi}{4}} - e^{-\frac{|\xi|^2+2|\xi-\eta|^2\rho^2+4|\xi\wedge\eta|\rho\cos\varphi}{4}}}{|z|^{3+s}}
\end{aligned}$$

$$
\begin{aligned}
&\left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho dz \\
&=: \left(\mathcal{I}_{D^c,1}+\mathcal{I}_{D,1}\right)+\left(\mathcal{I}_{D^c,2}+\mathcal{I}_{D,2}\right),
\end{aligned}
$$

where $D=\{z : |z| \leq \ell\,|\xi|\}$ for some $\ell>0$ or $D=\{z : |z| \leq \frac{|\xi-\eta|}{2}\}$. By previous argument, it gives

$$
\begin{aligned}
&\left|\mathcal{I}_{D^c,1}\right|, \left|\mathcal{I}_{D,1}\right|, \left|\mathcal{I}_{D^c,1}+\mathcal{I}_{D,1}\right| \\
&\lesssim |\xi-\eta|^{\gamma-s} e^{-\frac{|\xi|^2+|\eta|^2}{4}} \int_0^\infty \int_0^{2\pi} e^{-\frac{|\xi-\eta|^2\rho^2+|\xi-\eta||(\xi+\eta)_\perp|\rho\cos\varphi}{2}} \left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho \\
&\lesssim |\xi-\eta|^{\gamma-s}\left(1+|\xi|+|\eta|\right)^{\gamma-1} e^{-\frac{|\xi-\eta|^2+\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}}, \qquad (67)
\end{aligned}
$$

for some positive constant C.

To estimate $\mathcal{I}_{D^c,2}$ and $\mathcal{I}_{D,2}$, we need some preparations. By the Newton–Leibniz formula,

$$
\begin{aligned}
&e^{-\frac{|\xi+z|^2+2|\xi+z-\eta|^2\rho^2+4|(\xi+z)\wedge\eta|\rho\cos\varphi}{4}} - e^{-\frac{|\xi|^2+2|\xi-\eta|^2\rho^2+4|\xi\wedge\eta|\rho\cos\varphi}{4}} \\
&= \int_0^1 e^{-\frac{|\xi+tz|^2+2|\xi+tz-\eta|^2\rho^2+4|(\xi+tz)\wedge\eta|\rho\cos\varphi}{4}} \left(-\frac{1}{4}\right) \\
&\cdot\left[2\left(\xi+tz\right)\cdot z+4\rho^2\left(\xi+tz-\eta\right)\cdot z+4\rho\cos\varphi\frac{(\xi+tz)\wedge\eta}{|(\xi+tz)\wedge\eta|}\cdot(z\wedge\eta)\right]dt,
\end{aligned}
$$

and thus the double integral can be written as

$$
\begin{aligned}
&e^{-\frac{|\eta|^2}{4}} \int_0^\infty \int_0^{2\pi} \frac{e^{-\frac{|\xi+z|^2+2|\xi+z-\eta|^2\rho^2+4|(\xi+z)\wedge\eta|\rho\cos\varphi}{4}} - e^{-\frac{|\xi|^2+2|\xi-\eta|^2\rho^2+4|\xi\wedge\eta|\rho\cos\varphi}{4}}}{|z|^{3+s}} \\
&\left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho \\
&= \int_0^1 \int_0^\infty \int_0^{2\pi} e^{-\frac{|\eta|^2+|\xi+tz|^2+2|\xi+tz-\eta|^2\rho^2+4|(\xi+tz)\wedge\eta|\rho\cos\varphi}{4}} \\
&\cdot\frac{\left[-\frac{(\xi+tz)\cdot z}{2}-\rho^2(\xi+tz-\eta)\cdot z-\rho\cos\varphi\frac{(\xi+tz)\wedge\eta}{|(\xi+tz)\wedge\eta|}\cdot(z\wedge\eta)\right]}{|z|^{3+s}} \\
&\left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho dt,
\end{aligned}
$$

whose magnitude is bounded by

$$
\begin{aligned}
&\int_0^1 \int_0^\infty \int_0^{2\pi} e^{-\frac{|\eta|^2+|\xi+tz|^2+2|\xi+tz-\eta|^2\rho^2+4|(\xi+tz)\wedge\eta|\rho\cos\varphi}{C}} \frac{1+\rho+\rho|\eta|}{|z|^{2+s}} \\
&\left(1+\rho^2\right)^{\frac{\gamma-1}{2}} B_*(\rho)\rho d\varphi d\rho dt
\end{aligned}
$$

$$\lesssim \int_0^1 \int_{w\perp(\xi+tz-\eta)} e^{-\frac{|\xi+tz+w|^2+|\eta+w|^2}{C}} \frac{1+\frac{|w|}{|\xi+tz-\eta|}+\frac{|w||\eta|}{|\xi+tz-\eta|}}{|z|^{2+s}}$$
$$\left(|\xi+tz-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} |\xi+tz-\eta|^{-\gamma-1}\, d^2w dt.$$

Now, we consider the integral

$$\int_{w\perp(\xi+tz-\eta)} e^{-\frac{|\xi+tz+w|^2+|\eta+w|^2}{C}} \frac{1+\frac{|w|}{|\xi+tz-\eta|}+\frac{|w||\eta|}{|\xi+tz-\eta|}}{|z|^{2+s}} \left(|\xi+tz-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} d^2w.$$

For simplicity, we replace $\xi+tz$ by ξ in the above integral, namely,

$$\int_{w\perp(\xi-\eta)} e^{-\frac{|\xi+w|^2+|\eta+w|^2}{4C}} \left(1+\frac{|w|}{|\xi-\eta|}+\frac{|w||\eta|}{|\xi-\eta|}\right)\left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} d^2w. \tag{68}$$

As w is perpendicular to $\xi-\eta$, we have the identity

$$\begin{aligned}|\xi+w|^2+|\eta+w|^2 &= |\xi|^2+|\eta|^2-\frac{|(\xi+\eta)_\perp|^2}{2}+2\left|w+\frac{(\xi+\eta)_\perp}{2}\right|^2\\ &= \frac{1}{2}|\xi-\eta|^2+2\left|\frac{(\xi+\eta)_\parallel}{2}\right|^2+2\left|w+\frac{(\xi+\eta)_\perp}{2}\right|^2,\end{aligned}$$

where $(\xi+\eta)_\parallel$ is the orthogonal projection of $\xi+\eta$ onto the vector $\xi-\eta$ and $(\xi+\eta)_\perp=(\xi-\eta)-(\xi+\eta)_\parallel$. Denote $\zeta=\frac{\xi+\eta}{2}$ and then (68) becomes

$$\begin{aligned}&\int_{w\perp(\xi-\eta)} e^{-\frac{|\xi+w|^2+|\eta+w|^2}{4C}} \left(1+\frac{|w|}{|\xi-\eta|}+\frac{|w||\eta|}{|\xi-\eta|}\right)\left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} d^2w\\
&= e^{-\frac{1}{8C}|\xi-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2}\int_{w\perp(\xi-\eta)} e^{-\frac{1}{2C}|w+\zeta_\perp|^2}\left(1+\frac{|w|}{|\xi-\eta|}+\frac{|w||\eta|}{|\xi-\eta|}\right)\\
&\quad\left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} d^2w\\
&= |\xi-\eta|^{-1}e^{-\frac{1}{8C}|\xi-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2}\\
&\quad\cdot\int_{w\perp(\xi-\eta)} e^{-\frac{1}{2C}|w|^2}\left(|\xi-\eta|+|w-\zeta_\perp|+|w-\zeta_\perp||\eta|\right)\\
&\quad\left(|\xi-\eta|^2+|w-\zeta_\perp|^2\right)^{\frac{\gamma-1}{2}} d^2w.\end{aligned}$$

Next, we split this integral into two regions: $\{|w|\le|\zeta_\perp|/2\}$ and $\{|w|>|\zeta_\perp|/2\}$. On the first region,

$$\int_{|w|\le|\zeta_\perp|/2} e^{-\frac{1}{2C}|w|^2}\left(|\xi-\eta|+|w-\zeta_\perp|+|w-\zeta_\perp||\eta|\right)\left(|\xi-\eta|^2+|w-\zeta_\perp|^2\right)^{\frac{\gamma-1}{2}} d^2w$$
$$\le\left(|\xi-\eta|+|\zeta_\perp|+|\zeta_\perp||\eta|\right)\left(|\xi-\eta|+|\zeta_\perp|\right)^{\gamma-1}\min\left(1,|\zeta_\perp|^2\right).$$

On the second region,

$$\begin{aligned}
&\int_{|w|>|\zeta_\perp|/2} e^{-\frac{1}{2C}|w|^2}\left(|\xi-\eta|+|w-\zeta_\perp|+|w-\zeta_\perp||\eta|\right)\left(|\xi-\eta|^2+|w-\zeta_\perp|^2\right)^{\frac{\gamma-1}{2}} d^2w\\
&\lesssim\int_{\frac{|\zeta_\perp|}{2}<|w|<\frac{3|\zeta_\perp|}{2}} e^{-\frac{1}{8C}|\zeta_\perp|^2}\left(|\xi-\eta|+|w-\zeta_\perp|+|w-\zeta_\perp||\eta|\right)\left(|\xi-\eta|^2+|w-\zeta_\perp|^2\right)^{\frac{\gamma-1}{2}} d^2w\\
&\quad+\int_{|w|\ge\frac{3|\zeta_\perp|}{2}} e^{-\frac{1}{2C}|w|^2}\left(|\xi-\eta|+|w|+|w||\eta|\right)\left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma-1}{2}} d^2w\\
&=: T_1+T_2.
\end{aligned}$$

For T_2, it is easy to see that for $-3<\gamma<-2$,

$$\begin{aligned}
T_2&\lesssim\int_{|w|\ge\frac{3|\zeta_\perp|}{2}} e^{-\frac{1}{2C}|w|^2}(1+|\eta|)\left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma}{2}} d^2w\\
&\lesssim(1+|\eta|)\int_{\frac{3|\zeta_\perp|}{2}}^{\infty} ye^{-\frac{1}{2C}y^2}\left(|\xi-\eta|^2+|y|^2\right)^{\frac{\gamma}{2}} dy\\
&\lesssim e^{-\frac{9}{8C}|\zeta_\perp|^2}(1+|\eta|)|\xi-\eta|^{\gamma+2},
\end{aligned}$$

and for $\gamma=-2$,

$$\begin{aligned}
T_2&\lesssim\int_{|w|\ge\frac{3|\zeta_\perp|}{2}} e^{-\frac{1}{2C}|w|^2}(1+|\eta|)\left(|\xi-\eta|^2+|w|^2\right)^{\frac{\gamma}{2}} d^2w\\
&\lesssim(1+|\eta|)\left(\int_{\frac{3|\zeta_\perp|}{2}}^{|\xi-\eta|+2|\zeta_\perp|}+\int_{|\xi-\eta|+2|\zeta_\perp|}^{\infty}\right) ye^{-\frac{1}{2C}y^2}\left(|\xi-\eta|^2+|y|^2\right)^{\frac{\gamma}{2}} dy\\
&\lesssim e^{-\frac{9}{8C}|\zeta_\perp|^2}(1+|\eta|)\ln\left[\frac{|\xi-\eta|^2+(|\xi-\eta|+2|\zeta_\perp|)^2}{|\xi-\eta|^2+\left(\frac{3|\zeta_\perp|}{2}\right)^2}\right]\\
&\quad+(1+|\eta|)\frac{1}{|\xi-\eta|^2+(|\xi-\eta|+2|\zeta_\perp|)^2}e^{-\frac{1}{2C}(|\xi-\eta|+2|\zeta_\perp|)^2}\\
&\lesssim e^{-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|).
\end{aligned}$$

As for T_1, we further split the region of integration into

$$D_1=\left\{w\,\middle|\,|\zeta_\perp|/2<|w|<3|\zeta_\perp|/2,\ \ |w-\zeta_\perp|>|\zeta_\perp|/2\right\}$$

and

$$D_2=\left\{w\,\middle|\,|\zeta_\perp|/2<|w|<3|\zeta_\perp|/2,\ \ |w-\zeta_\perp|<|\zeta_\perp|/2\right\},$$

and then compute

$$
\begin{aligned}
T_1 &\lesssim \left(\int_{D_1} + \int_{D_2}\right) e^{-\frac{1}{8C}|\zeta_\perp|^2} \left(|\xi-\eta| + |w-\zeta_\perp| + |w-\zeta_\perp||\eta|\right) \\
&\quad \left(|\xi-\eta|^2 + |w-\zeta_\perp|^2\right)^{\frac{\gamma-1}{2}} d^2w \\
&\lesssim e^{-\frac{1}{8C}|\zeta_\perp|^2} \left(|\xi-\eta| + |\zeta_\perp| + |\zeta_\perp||\eta|\right) \left(|\xi-\eta|^2 + |\zeta_\perp|^2\right)^{\frac{\gamma-1}{2}} |\zeta_\perp|^2 \\
&\quad + e^{-\frac{1}{8C}|\zeta_\perp|^2} \int_0^{\frac{|\zeta_\perp|}{2}} \left(|\xi-\eta| + r + r|\eta|\right) \left(|\xi-\eta|^2 + r^2\right)^{\frac{\gamma-1}{2}} r dr \\
&\lesssim e^{-\frac{1}{8C}|\zeta_\perp|^2} (1+|\eta|) |\xi-\eta|^{\gamma+2} \\
&\quad + e^{-\frac{1}{8C}|\zeta_\perp|^2} (1+|\eta|) \cdot \begin{cases} \ln\left(1 + \frac{|\zeta_\perp|}{2|\xi-\eta|}\right), & \text{for } \gamma = -2, \\ |\xi-\eta|^{\gamma+2}, & \text{for } -2 < \gamma < -3. \end{cases}
\end{aligned}
$$

Combining the above estimates, (68) satisfies

$$
\begin{aligned}
&\int_{w\perp(\xi-\eta)} e^{-\frac{|\xi+w|^2+|\eta+w|^2}{4C}} \left(1 + \frac{|w|}{|\xi-\eta|} + \frac{|w||\eta|}{|\xi-\eta|}\right) \left(|\xi-\eta|^2 + |w|^2\right)^{\frac{\gamma-1}{2}} d^2w \\
&\lesssim |\xi-\eta|^{-1} e^{-\frac{1}{8C}|\xi-\eta|^2 - \frac{1}{2C}|\zeta_\parallel|^2} \cdot \Big[e^{-\frac{1}{8C}|\zeta_\perp|^2} (1+|\eta|) |\xi-\eta|^{\gamma+2} \\
&\quad + \left(|\xi-\eta| + |\zeta_\perp| + |\zeta_\perp||\eta|\right) \left(|\xi-\eta| + |\zeta_\perp|\right)^{\gamma-1} \min\left(1, |\zeta_\perp|^2\right) \Big],
\end{aligned} \tag{69}
$$

if $-3 < \gamma < -2$, and

$$
\begin{aligned}
&\int_{w\perp(\xi-\eta)} e^{-\frac{|\xi+w|^2+|\eta+w|^2}{4C}} \left(1 + \frac{|w|}{|\xi-\eta|} + \frac{|w||\eta|}{|\xi-\eta|}\right) \left(|\xi-\eta|^2 + |w|^2\right)^{\frac{\gamma-1}{2}} d^2w \\
&\lesssim |\xi-\eta|^{-1} e^{-\frac{1}{8C}|\xi-\eta|^2 - \frac{1}{2C}|\zeta_\parallel|^2} \cdot \Big[e^{-\frac{1}{8C}|\zeta_\perp|^2} (1+|\eta|) \left(1 + \ln\left(1 + \frac{|\zeta_\perp|}{2|\xi-\eta|}\right)\right) \\
&\quad + \left(|\xi-\eta| + |\zeta_\perp| + |\zeta_\perp||\eta|\right) \left(|\xi-\eta| + |\zeta_\perp|\right)^{\gamma-1} \min\left(1, |\zeta_\perp|^2\right) \Big],
\end{aligned} \tag{70}
$$

if $\gamma = -2$.

To complete the estimate for $\mathcal{I}_{D^c,2}$ and $\mathcal{I}_{D,2}$, we consider the following four cases:

(i) $|\xi-\eta| < 1$ and $|\xi| < 10$: singularity;
(ii) $|\xi-\eta| < 1$ and $|\xi| > 10$: ξ-decay;
(iii) $|\xi-\eta| > 1$ and $|\xi-\eta| > \frac{|\xi|}{2}$;
(iv) $|\xi-\eta| > 1$ and $|\xi-\eta| < \frac{|\xi|}{2}$.

We will estimate each case one by one for $-3 < \gamma < -2$. For simplicity of notation, we denote

$$\mathbb{A}_\gamma(\xi,\eta,z,t,\zeta_\perp) = e^{-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\,|\xi+tz-\eta|^{\gamma+2} + \left(|\xi+tz-\eta|+|\zeta_\perp|+|\zeta_\perp||\eta|\right)(|\xi+tz-\eta|+|\zeta_\perp|)^{\gamma-1}\min\left(1,|\zeta_\perp|^2\right).$$

• *Case (i):* $|\xi-\eta|<1$ and $|\xi|<10$. In this case, $|\eta|<1+|\xi|<11$. Notice that when $|\zeta_\perp|<|\xi+tz-\eta|$, we have

$$\left(|\xi+tz-\eta|+|\zeta_\perp|+|\zeta_\perp||\eta|\right)(|\xi+tz-\eta|+|\zeta_\perp|)^{\gamma-1}\min\left(1,|\zeta_\perp|^2\right) \lesssim |\xi+tz-\eta|^{\gamma+2}(1+|\eta|);$$

and when $|\zeta_\perp|\geq|\xi+tz-\eta|$, we have

$$\left(|\xi+tz-\eta|+|\zeta_\perp|+|\zeta_\perp||\eta|\right)(|\xi+tz-\eta|+|\zeta_\perp|)^{\gamma-1}\min\left(1,|\zeta_\perp|^2\right) \lesssim |\zeta_\perp|^{\gamma+2}(1+|\eta|)\lesssim|\xi+tz-\eta|^{\gamma+2}(1+|\eta|).$$

Therefore, in view of (69), we obtain

$$\begin{aligned}
&|\mathcal{I}_{D^c,2}+\mathcal{I}_{D,2}|\\
&\lesssim \int_{\mathbb{R}^3}|\xi+z-\eta|^\gamma dz\int_0^1\frac{|\xi+tz-\eta|^{-\gamma-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2}\mathbb{A}_\gamma(\xi,\eta,z,t,\zeta_\perp)dt\\
&\lesssim \int_{\mathbb{R}^3}\frac{|\xi+z-\eta|^\gamma}{|z|^{2+s}}dz\\
&\lesssim \int_{|z|<\frac{|\xi-\eta|}{2}}|\xi-\eta|^\gamma|z|^{-2-s}dz+\int_{\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2}}|\xi-\eta|^{-2-s}|\xi+z-\eta|^\gamma dz\\
&\quad+\int_{|z|>\frac{3|\xi-\eta|}{2}}|z|^{\gamma-s-2}dz\\
&\lesssim |\xi-\eta|^{\gamma+1-s}\\
&\lesssim |\xi-\eta|^{\gamma+1-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}},
\end{aligned}$$

for some $C_1>0$ large enough. Combining this with (67), we complete the estimate of $\left(-\Delta_\xi\right)^{\frac{s}{2}}k_2(\xi,\eta)$ in this case; that is,

$$\begin{aligned}
\left|\left(-\Delta_\xi\right)^{\frac{s}{2}}k_2(\xi,\eta)\right| &\lesssim |\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1}e^{-\frac{|\xi-\eta|^2+\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}}\\
&\quad+|\xi-\eta|^{\gamma+1-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}\\
&\lesssim |\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1}e^{-\frac{|\xi-\eta|^2+\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}}\\
&\quad+|\xi-\eta|^{\gamma-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}.
\end{aligned}$$

• *Case (ii)* $|\xi-\eta|<1$ and $|\xi|>10$. In this case, $|\xi|-1<|\eta|<1+|\xi|$, and we choose $D=\{z:|z|\le|\xi|/4\}$. Then in view of (16),

$$\begin{aligned}
&\left|\mathcal{I}_{D^c,1}+\mathcal{I}_{D^c,2}\right| \\
&\quad\le\int_{|z|>\frac{|\xi|}{4}}\frac{k_2(\xi+z,\eta)+k_2(\xi,\eta)}{|z|^{3+s}}dz \\
&\quad\lesssim\int_{|z|>\frac{|\xi|}{4}}|\xi+z-\eta|^{\gamma}(1+|\xi+z|+|\eta|)^{\gamma-1}|z|^{-3-s}dz \\
&\qquad+|\xi-\eta|^{\gamma}(1+|\xi|+|\eta|)^{\gamma-1}\int_{|z|>\frac{|\xi|}{4}}|z|^{-3-s}dz \\
&\quad\lesssim(1+|\xi|+|\eta|)^{2\gamma-s-1}+|\xi-\eta|^{\gamma}\left(1+|\xi|+|\eta|\right)^{\gamma-1-s} \\
&\quad\lesssim|\xi-\eta|^{\gamma}\left(1+|\xi|+|\eta|\right)^{\gamma-1-s}.
\end{aligned}\tag{71}$$

Next, we split $\mathcal{I}_{D,2}$ into three regions as

$$\begin{aligned}
&\left|\mathcal{I}_{D,2}\right| \\
&\lesssim\left(\int_{|z|<\frac{|\xi-\eta|}{2}}+\int_{\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2}}+\int_{\frac{3|\xi-\eta|}{2}<|z|\le\frac{|\xi|}{4}}\right)\frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}} \\
&\quad\cdot\int_0^1|\xi+tz-\eta|^{-\gamma-2}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_{\parallel}|^2}\mathbb{A}_{\gamma}(\xi,\eta,z,t,\zeta_{\perp})dtdz \\
&=:\mathcal{I}_{D,21}+\mathcal{I}_{D,22}+\mathcal{I}_{D,23}.
\end{aligned}$$

Noticing that when $|\xi|>10$ and $|\xi-\eta|<1$, together with $|z|\le\frac{|\xi|}{4}$ and $0\le t\le 1$, we have

$$|\xi+tz+\eta|=|2\xi+tz+\eta-\xi|\ge 2|\xi|-|z|-|\xi-\eta|\ge\frac{33}{20}|\xi|\ge\frac{1}{\sqrt{2}}\left(|\xi|+|\eta|\right),$$

and thus

$$|\zeta_{\parallel}|^2+|\zeta_{\perp}|^2=|\zeta|^2=\frac{|\xi+tz+\eta|^2}{4}\ge\frac{(|\eta|+|\xi|)^2}{8}.$$

It implies that either $|\zeta_{\parallel}|$ or $|\zeta_{\perp}|$ must be greater than or equal to $\frac{|\xi|+|\eta|}{4}$ in this situation. Thus, we have

$$\begin{aligned}
\left|\mathcal{I}_{D,21}\right|&\lesssim\int_{|z|<\frac{|\xi-\eta|}{2}}\int_0^1\frac{|\xi-\eta|^{\gamma}}{|z|^{2+s}}\left(1+|\eta|\right)e^{-\frac{1}{8C}|\zeta|^2}dtdz \\
&\quad+\int_{|z|<\frac{|\xi-\eta|}{2}}\int_0^1\frac{|\xi-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_{\parallel}|^2} \\
&\qquad\cdot\left(|\xi+tz-\eta|+|\zeta_{\perp}|+|\zeta_{\perp}||\eta|\right)\left(|\xi+tz-\eta|+|\zeta_{\perp}|\right)^{\gamma-1}\min\left(1,|\zeta_{\perp}|^2\right)dtdz \\
&=:\mathcal{I}'_{D,21}+\mathcal{I}''_{D,21},
\end{aligned}$$

and it immediately follows that

$$\left|\mathcal{I}'_{D,21}\right| \lesssim |\xi-\eta|^{\gamma+1-s}\, e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}.$$

As for $\mathcal{I}''_{D,21}$, we refine the region of integration in the zt-space and consider two subregions: the region (a) satisfying $|\zeta_\perp| < \frac{1}{3}|\xi+tz-\eta|$ and the region (b) satisfying $|\zeta_\perp| > \frac{1}{3}|\xi+tz-\eta|$. Hence, we denote

$$\mathcal{I}''_{D,21} := \int_{(a)} + \int_{(b)}.$$

On the region (a), since

$$\frac{1}{8C}|\xi+tz-\eta|^2 + \frac{1}{2C}\left|\zeta_\parallel\right|^2 \geq \frac{1}{2C}\left(|\zeta_\perp|^2+|\zeta_\parallel|^2\right) \geq \frac{|\eta|^2+|\xi|^2}{16C},$$

we have

$$\int_{(a)} \lesssim \int_{|z|<\frac{|\xi-\eta|}{2}} \frac{|\xi-\eta|^\gamma}{|z|^{2+s}}(1+|\eta|)\, e^{-\frac{|\xi|^2+|\eta|^2}{16C}} dz \lesssim |\xi-\eta|^{\gamma+1-s}\, e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}.$$

On the region (b), we find

$$\begin{aligned}
\int_{(b)} &\lesssim \left(\int_{(b)\wedge\left(|\zeta_\parallel|>\frac{|\xi|+|\eta|}{4}\right)} + \int_{(b)\wedge\left(|\zeta_\perp|>\frac{|\xi|+|\eta|}{4}\right)}\right) \frac{|\xi-\eta|^{-2}}{|z|^{2+s}} e^{-\frac{1}{2C}|\zeta_\parallel|^2} \\
&\quad \left(|\xi+tz-\eta|+|\zeta_\perp|+|\zeta_\perp||\eta|\right) \\
&\quad \cdot (|\xi+tz-\eta|+|\zeta_\perp|)^{\gamma-1} \min\left(1, |\zeta_\perp|^2\right) dt dz \\
&\lesssim \int_{|z|<\frac{|\xi-\eta|}{2}} \frac{|\xi-\eta|^{-2}}{|z|^{2+s}} e^{-\frac{|\xi|^2+|\eta|^2}{32C}} |\xi-\eta|^{\gamma+2}(1+|\eta|)\, dz \\
&\quad + \int_{|z|<\frac{|\xi-\eta|}{2}} \frac{|\xi-\eta|^{-2}}{|z|^{2+s}} (1+|\xi|+|\eta|)^{\gamma+1} dz \\
&\lesssim |\xi-\eta|^{\gamma+1-s}\, e^{-\frac{|\xi|^2+|\eta|^2}{C_1}} + |\xi-\eta|^{-1-s}(1+|\xi|+|\eta|)^{\gamma+1}.
\end{aligned}$$

Therefore, we have

$$\left|\mathcal{I}_{D,21}\right| \lesssim |\xi-\eta|^{\gamma+1-s}\, e^{-\frac{|\xi|^2+|\eta|^2}{C_1}} + |\xi-\eta|^{-1-s}(1+|\xi|+|\eta|)^{\gamma+1}.$$

Following similar arguments as those for $\mathcal{I}_{D,21}$, we deduce

$$
\begin{aligned}
&\left|\mathcal{I}_{D,22}\right| \\
&\lesssim \int_{\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2}} \frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}} \int_0^1 |\xi+tz-\eta|^{-\gamma-2}\, e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_{\parallel}|^2} \\
&\quad \mathbb{A}_{\gamma}(\xi,\eta,z,t,\zeta_{\perp})dtdz \\
&\lesssim \int_{\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2}} \frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}} \int_0^1 e^{-\frac{|\zeta|^2}{8C}}\,(1+|\eta|)\,dtdz \\
&\quad + \int\int_{\left(\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2},\, 0\le t\le 1\right)\wedge\left(|\zeta_{\perp}|<\frac{1}{3}|\xi+tz-\eta|\right)} \frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}} e^{-\frac{|\zeta|^2}{2C}}\,(1+|\eta|)\,dtdz \\
&\quad + \int\int_{\left(\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2},\, 0\le t\le 1\right)\wedge\left(|\zeta_{\perp}|>\frac{1}{3}|\xi+tz-\eta|\right)\wedge\left(|\zeta_{\parallel}|>\frac{|\xi|+|\eta|}{4}\right)} \\
&\quad \frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}} e^{-\frac{|\xi|^2+|\eta|^2}{32C}}\,(1+|\eta|)\,dtdz \\
&+ \int\int_{\left(\frac{|\xi-\eta|}{2}<|z|<\frac{3|\xi-\eta|}{2},\, 0\le t\le 1\right)\wedge\left(|\zeta_{\perp}|>\frac{1}{3}|\xi+tz-\eta|\right)\wedge\left(|\zeta_{\perp}|>\frac{|\xi|+|\eta|}{4}\right)} \\
&\quad \frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}} |\xi+tz-\eta|^{-\gamma-2}|\zeta_{\perp}|^{\gamma}(1+|\eta|)dtdz \\
&\lesssim |\xi-\eta|^{\gamma+1-s} e^{-\frac{|\xi|^2+|\eta|^2}{C_1}} + |\xi-\eta|^{-1-s}(1+|\xi|+|\eta|)^{\gamma+1},
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\mathcal{I}_{D,23}\right| &\lesssim \int_{\frac{3|\xi-\eta|}{2}\le|z|\le\frac{|\xi|}{4}} |z|^{\gamma-2-s} \int_0^1 e^{-\frac{|\zeta|^2}{8C}}(1+|\eta|)dtdz \\
&\quad + \int\int_{\left(\frac{3|\xi-\eta|}{2}\le|z|\le\frac{|\xi|}{4},\, 0\le t\le 1\right)\wedge\left(|\zeta_{\perp}|<\frac{1}{3}|\xi+tz-\eta|\right)} |z|^{\gamma-2-s} e^{-\frac{|\zeta|^2}{2C}}\,(1+|\eta|)\,dtdz \\
&\quad + \int\int_{\left(\frac{3|\xi-\eta|}{2}\le|z|\le\frac{|\xi|}{4},\, 0\le t\le 1\right)\wedge\left(|\zeta_{\perp}|>\frac{1}{3}|\xi+tz-\eta|\right)\wedge\left(|\zeta_{\parallel}|>\frac{|\xi|+|\eta|}{4}\right)} \\
&\quad |z|^{\gamma-2-s} e^{-\frac{|\xi|^2+|\eta|^2}{32C}}(1+|\eta|)dtdz \\
&\quad + \int\int_{\left(\frac{3|\xi-\eta|}{2}\le|z|\le\frac{|\xi|}{4},\, 0\le t\le 1\right)\wedge\left(|\zeta_{\perp}|>\frac{1}{3}|\xi+tz-\eta|\right)\wedge\left(|\zeta_{\perp}|>\frac{|\xi|+|\eta|}{4}\right)} \\
&\quad |z|^{\gamma-2-s}|\zeta_{\perp}|^{\gamma}(1+|\eta|)dtdz \\
&\lesssim |\xi-\eta|^{\gamma+1-s} e^{-\frac{|\xi|^2+|\eta|^2}{C_1}} + |\xi-\eta|^{\gamma+1-s}(1+|\xi|+|\eta|)^{\gamma+1}.
\end{aligned}
$$

Combining the above estimates, we have

$$\begin{aligned}\left|\mathcal{I}_{D,2}\right| &\lesssim |\xi-\eta|^{\gamma+1-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}+|\xi-\eta|^{\gamma+1-s}(1+|\xi|+|\eta|)^{\gamma+1}\\ &\quad+|\xi-\eta|^{\gamma}\left(1+|\xi|+|\eta|\right)^{\gamma-1-s}\\ &\lesssim |\xi-\eta|^{\gamma+1-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}+|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C_1}}.\end{aligned}$$

Together with (67) and (71), we obtain

$$\begin{aligned}\left|\left(-\Delta_{\xi}\right)^{\frac{s}{2}}k_2(\xi,\eta)\right| &= \left|\mathcal{I}_{D,1}+\mathcal{I}_{D^c,1}+\mathcal{I}_{D,2}+\mathcal{I}_{D^c,2}\right|\\ &\le \left|\mathcal{I}_{D,1}\right|+\left|\mathcal{I}_{D^c,1}+\mathcal{I}_{D^c,2}\right|+\left|\mathcal{I}_{D,2}\right|\\ &\lesssim |\xi-\eta|^{\gamma-s}\left(1+|\xi|+|\eta|\right)^{\gamma-1}e^{-\frac{|\xi-\eta|^2+\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}}\\ &\quad+|\xi-\eta|^{\gamma-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}\\ &\quad+|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C_1}}.\end{aligned}$$

• *Case (iii)* $|\xi-\eta|>1$ and $|\xi-\eta|>\frac{|\xi|}{2}$. In this case,

$$|\xi|<2|\xi-\eta| \text{ and } |\eta|\le|\xi-\eta|+|\xi|<3|\xi-\eta|,$$

so that

$$|\xi-\eta|\ge\frac{1}{6}\left(1+|\xi|+|\eta|\right).$$

Here we choose $D=\{z:|z|\le\frac{|\xi-\eta|}{2}\}$. In view of (16),

$$\begin{aligned}&\left|\left(\mathcal{I}_{D^c,1}+\mathcal{I}_{D^c,2}\right)\right|\\ &\quad=\left|\int_{|z|>\frac{|\xi-\eta|}{2}}\frac{k_2(\xi+z,\eta)-k_2(\xi,\eta)}{|z|^{3+s}}dz\right|\\ &\quad\le\int_{|z|>\frac{|\xi-\eta|}{2}}\frac{k_2(\xi+z,\eta)+k_2(\xi,\eta)}{|z|^{3+s}}dz\\ &\quad\lesssim\int_{|z|>\frac{|\xi-\eta|}{2}}e^{-\frac{|\xi+z-\eta|^2}{C}}|\xi+z-\eta|^{\gamma}(1+|\xi+z|+|\eta|)^{\gamma-1}|z|^{-3-s}dz\\ &\qquad+e^{-\frac{|\xi-\eta|^2}{C}}|\xi-\eta|^{\gamma}(1+|\xi|+|\eta|)^{\gamma-1}\int_{|z|>\frac{|\xi-\eta|}{2}}|z|^{-3-s}dz\\ &\lesssim(1+|\eta|)^{\gamma-1}\left(1+|\xi|+|\eta|\right)^{-3-s}+e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}|\xi-\eta|^{\gamma}(1+|\xi|+|\eta|)^{\gamma-1-s}.\end{aligned} \tag{72}$$

On the other hand, if $|z| \le \frac{|\xi-\eta|}{2}$, then

$$|\xi + tz + \eta| \ge \frac{1}{2}|\xi - \eta| \ge \frac{1}{10}(|\xi| + |\eta|),$$

so that

$$\left|\zeta_{\parallel}\right|^2 + |\zeta_{\perp}|^2 = |\zeta|^2 = \frac{|\xi + tz + \eta|^2}{4} \ge \frac{|\xi|^2 + |\eta|^2}{400}.$$

Hence, we get

$$\begin{aligned}
&\left|\mathcal{I}_{D,2}\right| \\
&\lesssim \int_{|z|\le\frac{|\xi-\eta|}{2}} \frac{|\xi+z-\eta|^\gamma}{|z|^{2+s}} \int_0^1 |\xi + tz - \eta|^{-\gamma-2} e^{-\frac{1}{8C}|\xi+tz-\eta|^2 - \frac{1}{2C}|\zeta_{\parallel}|^2} \\
&\quad \mathbb{A}_\gamma(\xi, \eta, z, t, \zeta_\perp) dt dz \\
&\lesssim \int_{|z|\le\frac{|\xi-\eta|}{2}} \frac{|\xi-\eta|^\gamma}{|z|^{2+s}} (1+|\eta|) e^{-\frac{|\zeta|^2}{8C}} dz + \int_{|z|\le\frac{|\xi-\eta|}{2}} \frac{|\xi-\eta|^\gamma}{|z|^{2+s}} e^{-\frac{|\xi-\eta|^2}{32C}} (1+|\eta|) dz \\
&\lesssim |\xi-\eta|^{\gamma+1-s} e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}.
\end{aligned}$$

Together with (67) and (72), we complete the estimate for $\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi,\eta)$ with

$$\begin{aligned}
\left|\left(-\Delta_\xi\right)^{\frac{s}{2}} k_2(\xi,\eta)\right| &\lesssim |\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1} e^{-\frac{|\xi-\eta|^2 + \frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}} \\
&\quad + (1+|\eta|)^{\gamma-1}(1+|\xi|+|\eta|)^{-3-s} \\
&\quad + e^{-\frac{|\xi|^2+|\eta|^2}{C_1}} |\xi-\eta|^\gamma (1+|\xi|+|\eta|)^{\gamma-1-s} \\
&\quad + |\xi-\eta|^{\gamma+1-s} e^{-\frac{|\xi|^2+|\eta|^2}{C_1}} \\
&\lesssim |\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1} e^{-\frac{|\xi-\eta|^2 + \frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}} \\
&\quad + (1+|\eta|)^{\gamma-1}(1+|\xi|+|\eta|)^{-3-s} \\
&\quad + |\xi-\eta|^{\gamma-s} e^{-\frac{|\xi|^2+|\eta|^2}{C_1'}}.
\end{aligned}$$

• *Case (iv)* $|\xi-\eta| > 1$ and $|\xi-\eta| < \frac{|\xi|}{2}$. In this case, we have $1 < \frac{|\xi|}{2} < |\eta| < \frac{3}{2}|\xi|$, and $\frac{3|\xi-\eta|}{2} < \frac{3|\xi|}{4}$. Now select $D = \{z : |z| \le \frac{1}{2}|\xi-\eta|\}$. In view of (16),

$$\begin{aligned}
\left|\mathcal{I}_{D^c,1} + \mathcal{I}_{D^c,2}\right| &\le \int_{|z|>\frac{|\xi-\eta|}{2}} \frac{k_2(\xi+z,\eta)+k_2(\xi,\eta)}{|z|^{3+s}} dz \\
&\lesssim \int_{|z|>\frac{|\xi-\eta|}{2}} e^{-\frac{|\xi+z-\eta|^2}{C}} |\xi+z-\eta|^\gamma (1+|\xi+z|+|\eta|)^{\gamma-1} |z|^{-3-s} dz \\
&\quad + e^{-\frac{|\xi-\eta|^2}{C}} |\xi-\eta|^\gamma (1+|\xi|+|\eta|)^{\gamma-1} \int_{|z|>\frac{|\xi-\eta|}{2}} |z|^{-3-s} dz
\end{aligned}$$

$$\begin{aligned}&\lesssim |\xi-\eta|^{-3-s}(1+|\eta|)^{\gamma-1}+e^{-\frac{|\xi-\eta|^2}{C}}\\&\qquad |\xi-\eta|^{\gamma}(1+|\xi|+|\eta|)^{\gamma-1}|\xi-\eta|^{-s}\\&\lesssim (1+|\xi-\eta|)^{-3-s}(1+|\xi|+|\eta|)^{\gamma-1}\\&\quad +e^{-\frac{|\xi-\eta|^2}{C}}|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1}.\end{aligned}$$

For $\mathcal{I}_{D,2}$, we have

$$\begin{aligned}&\left|\mathcal{I}_{D,2}\right|\\&\leq \int_{|z|<\frac{|\xi-\eta|}{2}}\frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}}\int_0^1|\xi+tz-\eta|^{-\gamma-2}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_{\|}|^2}\\&\mathbb{A}_{\gamma}(\xi,\eta,z,t,\zeta_{\perp})dtdz\\&\leq \int_{|z|<\frac{|\xi-\eta|}{2}}\frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}}\int_0^1(1+|\eta|)e^{-\frac{|\xi+tz-\eta|^2+|\zeta|^2}{8C}}dtdz\\&\quad+\int_{|z|<\frac{|\xi-\eta|}{2}}\int_0^1\frac{|\xi+z-\eta|^{\gamma}}{|z|^{2+s}}|\xi+tz-\eta|^{-\gamma-2}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_{\|}|^2}\\&\cdot\left(|\xi+tz-\eta|+|\zeta_{\perp}|+|\zeta_{\perp}||\eta|\right)(|\xi+tz-\eta|+|\zeta_{\perp}|)^{\gamma-1}\min\left(1,|\zeta_{\perp}|^2\right)dtdz\\&=:\mathcal{A}_1+\mathcal{A}_2.\end{aligned}$$

As $|z|\leq\frac{|\xi-\eta|}{2}<\frac{|\xi|}{4}$, $0\leq t\leq 1$, it gives

$$\begin{aligned}|\xi+tz-\eta|^2+4|\zeta|^2&=|\xi+tz-\eta|^2+|\xi+tz+\eta|^2\\&=2|\xi+tz|^2+2|\eta|^2\geq|\xi|^2+2|\eta|^2.\end{aligned}$$

Hence,

$$\mathcal{A}_1\lesssim|\xi-\eta|^{\gamma+1-s}e^{-\frac{|\xi|^2+|\eta|^2}{32C}}\lesssim e^{-\frac{|\xi|^2+|\eta|^2}{32C}}.$$

Moreover, as $|z|\leq\frac{|\xi-\eta|}{2}<\frac{|\xi|}{4}$, $0\leq t\leq 1$, we have

$$2\sqrt{|\zeta_{\|}|^2+|\zeta_{\perp}|^2}=2|\zeta|=|\xi+tz+\eta|\geq 2|\xi|-|z|-|\xi-\eta|\geq\frac{5}{4}|\xi|,$$

which leads to that either $|\zeta_{\|}|$ or $|\zeta_{\perp}|$ must be greater than or equal to $\frac{5\sqrt{2}}{16}|\xi|$ when $|z|\leq\frac{|\xi-\eta|}{2}<\frac{|\xi|}{4}$, $0\leq t\leq 1$. Therefore,

$$\begin{aligned}\mathcal{A}_2&\lesssim\int_{|z|<\frac{|\xi-\eta|}{2}}\int_0^1\frac{|\xi-\eta|^{\gamma}}{|z|^{2+s}}|\xi-\eta|^{-\gamma-2}e^{-\frac{1}{32C}|\xi-\eta|^2-\frac{1}{2C}|\zeta_{\|}|^2}\\&\quad\cdot(|\xi-\eta|+|\zeta_{\perp}|(1+|\eta|))(|\xi-\eta|+|\zeta_{\perp}|)^{\gamma-1}\min\left(1,|\zeta_{\perp}|^2\right)dtdz\\&\lesssim\iint_{\left(|z|<\frac{|\xi-\eta|}{2},0\leq t\leq 1\right)\wedge(|\zeta_{\perp}|<|\xi-\eta|)}\frac{|\xi-\eta|^{\gamma}}{|z|^{2+s}}e^{-\frac{|\zeta|^2}{32C}}(1+|\eta|)dtdz\end{aligned}$$

$$+\iint_{\left(|z|<\frac{|\xi-\eta|}{2},0\le t\le 1\right)\wedge(|\zeta_\perp|>|\xi-\eta|)\wedge\left(|\zeta_\parallel|>\frac{5\sqrt{2}}{16}|\xi|\right)}\frac{|\xi-\eta|^\gamma}{|z|^{2+s}}e^{-\frac{25|\xi|^2}{256C}}(1+|\eta|)\,dtdz$$
$$+\iint_{\left(|z|<\frac{|\xi-\eta|}{2},0\le t\le 1\right)\wedge(|\zeta_\perp|>|\xi-\eta|)\wedge\left(|\zeta_\perp|>\frac{5\sqrt{2}}{16}|\xi|\right)}\frac{|\xi-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{|\xi-\eta|^2}{32C}}(1+|\xi|+|\eta|)^{\gamma+1}\,dtdz$$
$$\lesssim|\xi-\eta|^{\gamma-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}+|\xi-\eta|^{-1-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C_1}}.$$

Combining the above estimates with (67), we conclude that for $|\xi-\eta|>1$ and $|\xi-\eta|<\frac{|\xi|}{2}$,

$$\left|(-\Delta_\xi)^{\frac{s}{2}}k_2(\xi,\eta)\right|$$
$$\lesssim|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1}e^{-\frac{|\xi-\eta|^2+\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}}$$
$$+(1+|\xi-\eta|)^{-3-s}(1+|\xi|+|\eta|)^{\gamma-1}+e^{-\frac{|\xi-\eta|^2}{C}}|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1}$$
$$+|\xi-\eta|^{\gamma-s}e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}+|\xi-\eta|^{-1-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C_1}}$$
$$\lesssim|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma-1}e^{-\frac{|\xi-\eta|^2+\frac{(|\xi|^2-|\eta|^2)^2}{|\xi-\eta|^2}}{C}}+(1+|\xi-\eta|)^{-3-s}(1+|\xi|+|\eta|)^{\gamma-1}$$
$$+|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C'}}.$$

In summary,

$$\left|(-\Delta_\xi)^{\frac{s}{2}}k_2(\xi,\eta)\right|\lesssim|\xi-\eta|^{\gamma-s}(1+|\xi|+|\eta|)^{\gamma+1}e^{-\frac{|\xi-\eta|^2}{C}}$$
$$+(1+|\xi-\eta|)^{-3-s}(1+|\xi|+|\eta|)^{\gamma-1}$$
$$+(1+|\eta|)^{\gamma-1}(1+|\xi|+|\eta|)^{-3-s},$$

for $C>0$ whenever $-3<\gamma\le-2$.

Last, we deal with the estimate of $\mathcal{I}_{2,D}$ and $\mathcal{I}_{2,D^c}$ for the case $\gamma=-2$. In view of (69) and (70), it remains to deal with the integral

$$\int_{\mathbb{R}^3}\int_0^1\frac{|z+\xi-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\ln\left(1+\frac{|\zeta_\perp|}{2|\xi+tz-\eta|}\right)dtdz$$
$$\lesssim\int_{\mathbb{R}^3}\int_0^1\frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\left(\frac{|\zeta_\perp|}{|\xi+tz-\eta|}\right)^\epsilon dtdz$$

for any $\epsilon>0$ small whenever $\gamma=-2$; the other terms have been done by replacing $\gamma=-2$ during the procedure of the proof in which $-3<\gamma\le-2$. By Hölder's inequality,

$$\int_{\mathbb{R}^3}\int_0^1\frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\left(\frac{|\zeta_\perp|}{|\xi+tz-\eta|}\right)^\epsilon dtdz$$
$$\lesssim\int_{\mathbb{R}^3}\int_0^1\frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{16C}|\zeta_\perp|^2}(1+|\eta|)\,|\xi+tz-\eta|^{-\epsilon}\,dtdz$$

$$\lesssim \left(\int_{\mathbb{R}^3}\int_0^1 \frac{|\xi+z-\eta|^{-2p}}{|z|^{2+s}} e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{16C}|\zeta_\perp|^2}(1+|\eta|)^p\,dtdz\right)^{1/p}$$
$$\cdot\left(\int_{\mathbb{R}^3}\int_0^1 \frac{|\xi+tz-\eta|^{-q\epsilon}}{|z|^{2+s}} e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{16C}|\zeta_\perp|^2}dtdz\right)^{1/q},$$

where $1 < p < 3/2$ with $1/p + 1/q = 1$.

$$\begin{aligned}
&\left(\int_{\mathbb{R}^3}\int_0^1 \frac{|\xi+tz-\eta|^{-q\epsilon}}{|z|^{2+s}} e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{16C}|\zeta_\perp|^2}dtdz\right)^{1/q}\\
&\lesssim \left(\int_0^1 t^{-1+s}\int_{\mathbb{R}^3}\frac{|\xi+w-\eta|^{-q\epsilon}}{|w|^{2+s}}e^{-\frac{1}{8C}|\xi+w-\eta|^2}dwdt\right)^{1/q}\\
&\lesssim \left(\int_{\mathbb{R}^3}\frac{|\xi+w-\eta|^{-q\epsilon}}{|w|^{2+s}}e^{-\frac{1}{8C}|\xi+w-\eta|^2}dw\right)^{1/q}\\
&\lesssim \left[\left(\int_{|w|<\frac{|\xi-\eta|}{2}}+\int_{|w|>\frac{|\xi-\eta|}{2}}\right)\frac{|\xi+w-\eta|^{-q\epsilon}}{|w|^{2+s}}e^{-\frac{1}{8C}|\xi+w-\eta|^2}dw\right]^{1/q}\\
&\lesssim |\xi-\eta|^{(1-s)/q-\epsilon}+|\xi-\eta|^{-(2+s)/q}\left(\int_{|w|>\frac{|\xi-\eta|}{2}}|\xi+w-\eta|^{-q\epsilon}\,e^{-\frac{1}{8C}|\xi+w-\eta|^2}dw\right)^{1/q}\\
&\lesssim |\xi-\eta|^{(1-s)/q-\epsilon}+|\xi-\eta|^{-(2+s)/q}.
\end{aligned}$$

In the Case (i) $|\xi-\eta|<1$ and $|\xi|<10$,

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_0^1 \frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\left(\frac{|\zeta_\perp|}{|\xi+tz-\eta|}\right)^\epsilon dtdz\\
&\lesssim |\xi-\eta|^{(-2p+1-s)/p}\left(|\xi-\eta|^{(1-s)/q-\epsilon}+|\xi-\eta|^{-(2+s)/q}\right)\\
&\lesssim |\xi-\eta|^{-2+1-s-\epsilon}+|\xi-\eta|^{-2+1-s-3/q}\\
&\lesssim \left(|\xi-\eta|^{-2+1-s-\epsilon}+|\xi-\eta|^{-2+1-s-3/q}\right)e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}\\
&\lesssim \left(|\xi-\eta|^{-2-s}+|\xi-\eta|^{-2-s}\right)e^{-\frac{|\xi|^2+|\eta|^2}{C_1}}.
\end{aligned}$$

In the Case (ii) $|\xi-\eta|<1$ and $|\xi|>10$,

$$\begin{aligned}
&\int_{|z|<\frac{|\xi|}{4}}\int_0^1 \frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}}e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\left(\frac{|\zeta_\perp|}{|\xi+tz-\eta|}\right)^\epsilon dtdz\\
&\lesssim \left(|\xi-\eta|^{(-2p+1-s)/p}\,e^{-\frac{|\xi|^2+|\eta|^2}{pC_1}}\right)\left(|\xi-\eta|^{(1-s)/q-\epsilon}+|\xi-\eta|^{-(2+s)/q}\right)\\
&\lesssim \left(|\xi-\eta|^{-2-s+1-\epsilon}+|\xi-\eta|^{-2-s+1-3/q}\right)e^{-\frac{|\xi|^2+|\eta|^2}{pC_1}}\\
&\lesssim \left(|\xi-\eta|^{-2-s}+|\xi-\eta|^{-2-s}\right)e^{-\frac{|\xi|^2+|\eta|^2}{pC_1}}.
\end{aligned}$$

In the Case (iii) $|\xi-\eta|>1$ and $|\xi-\eta|>\frac{|\xi|}{2}$,

$$\int_{|z|\le\frac{|\xi-\eta|}{2}}\int_0^1 \frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}} e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\left(\frac{|\zeta_\perp|}{|\xi+tz-\eta|}\right)^\epsilon dtdz$$
$$\lesssim\left(|\xi-\eta|^{(-2p+1-s)/p}e^{-\frac{|\xi|^2+|\eta|^2}{pC_1}}\right)\left(|\xi-\eta|^{(1-s)/q-\epsilon}+|\xi-\eta|^{-(2+s)/q}\right)$$
$$\lesssim\left(|\xi-\eta|^{-2-s+1-\epsilon}+|\xi-\eta|^{-2-s+1-3/q}\right)e^{-\frac{|\xi|^2+|\eta|^2}{pC_1}}$$
$$\lesssim\left(|\xi-\eta|^{-2-s}+|\xi-\eta|^{-2-s}\right)e^{-\frac{|\xi|^2+|\eta|^2}{pC_1'}}.$$

In the Case (iv) $|\xi-\eta|>1$ and $|\xi-\eta|<\frac{|\xi|}{2}$,

$$\int_{||z|\le\frac{3}{4}|\xi|}\int_0^1 \frac{|\xi+z-\eta|^{-2}}{|z|^{2+s}} e^{-\frac{1}{8C}|\xi+tz-\eta|^2-\frac{1}{2C}|\zeta_\parallel|^2-\frac{1}{8C}|\zeta_\perp|^2}(1+|\eta|)\left(\frac{|\zeta_\perp|}{|\xi+tz-\eta|}\right)^\epsilon dtdz$$
$$\lesssim\left(e^{-\frac{|\xi|^2+|\eta|^2}{256pC}}\right)\left(|\xi-\eta|^{(1-s)/q-\epsilon}+|\xi-\eta|^{-(2+s)/q}\right)$$
$$\lesssim\left(|\xi-\eta|^{-2-s+1-\epsilon}+|\xi-\eta|^{-2-s+1-3/q}\right)e^{-\frac{|\xi|^2+|\eta|^2}{256pC}}.$$
$$\lesssim\left(|\xi-\eta|^{-2-s}+|\xi-\eta|^{-2-s}\right)e^{-\frac{|\xi|^2+|\eta|^2}{C'}}.$$

Therefore, the proof of Proposition 13 is completed.

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Data Availability My manuscript has no associated data.

Declaration

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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