

EXPLICIT STRUCTURE OF THE FOKKER-PLANCK EQUATION WITH POTENTIAL

BY

YU-CHU LIN (*Department of Mathematics, National Cheng Kung University, Tainan, Taiwan*),

HAITAO WANG (*Institute of Natural Sciences and School of Mathematical Sciences, MOE-LSC, Shanghai Jiao Tong University, Shanghai, People's Republic of China*),

AND

KUNG-CHIEN WU (*Department of Mathematics, National Cheng Kung University, Tainan, Taiwan; and National Center for Theoretical Sciences, National Taiwan University, Taipei, Taiwan*)

Abstract. We study the pointwise (in the space and time variables) behavior of the Fokker-Planck equation with potential. An explicit description of the solution is given, including the large time behavior, initial layer, and spatially asymptotic behavior. Moreover, it is shown that the structure of the solution sensitively depends on the potential function.

1. Introduction.

1.1. *The models.* In this paper, we study the kinetic Fokker-Planck equation with potential in \mathbb{R}^3 . It reads

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \nabla_v \cdot [\nabla_v F + (\nabla_v \Phi)F], & x, v \in \mathbb{R}^3, t > 0, \\ F(0, x, v) = F_0(x, v), \end{cases} \quad (1.1)$$

where the potential $\Phi(v)$ is of the form

$$\Phi = \frac{1}{\gamma} \langle v \rangle^\gamma + \Phi_0, \quad \gamma > 0$$

Received October 27, 2018, and, in revised form, January 24, 2019.

2010 *Mathematics Subject Classification.* Primary 35Q84, 82C40.

Key words and phrases. Fokker-Planck, fluid-like waves, kinetic-like waves, pointwise estimate, regularization estimate.

The first author was supported by the Ministry of Science and Technology under the grant MOST 107-2115-M-006-006-.

The second author was sponsored by Shanghai Sailing Program(18YF1411800) and Shanghai Jiao Tong University(WF220441907).

The third author was supported by the Ministry of Science and Technology under the grant 108-2636-M-006-005- and National Center for Theoretical Sciences.

Email address: yuchu@mail.ncku.edu.tw

Email address: haitallica@sjtu.edu.cn

Email address: kungchienwu@gmail.com

for some constant Φ_0 , where $\langle v \rangle = (1 + |v|^2)^{1/2}$. We define

$$\mathcal{M}(v) = e^{-\Phi(v)}$$

with $\Phi_0 \in \mathbb{R}$ such that \mathcal{M} is a probability measure. It is easy to see that \mathcal{M} is a steady state to the Fokker-Planck equation (1.1). Thus it is natural to study the fluctuation of the Fokker-Planck equation (1.1) around $\mathcal{M}(v)$ with the standard perturbation $f(t, x, v)$ to \mathcal{M} as

$$F = \mathcal{M} + \mathcal{M}^{1/2} f.$$

The Fokker-Planck equation for $f(t, x, v) = \mathbb{G}^t f_0$ now takes the form

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \Delta_v f - \frac{1}{4} |v|^2 \langle v \rangle^{2\gamma-4} f + \left(\frac{3}{2} \langle v \rangle^{\gamma-2} + \frac{\gamma-2}{2} |v|^2 \langle v \rangle^{\gamma-4} \right) f = Lf, \\ f(0, x, v) = f_0(x, v), \quad (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{cases} \tag{1.2}$$

Here \mathbb{G}^t is the solution operator of the Fokker-Planck equation (1.2).

The goal of this paper is to study the pointwise (in the space and time variables) behavior of (1.2).

1.2. Review of previous works. The Fokker-Planck equations arise in many areas of sciences, including probability, statistical physics, plasma physics, and gas and stellar dynamics. In particular, when $\Phi = \langle v \rangle^2 / 2$, it is closely related to the Langevin equation, which is used to describe the Brownian motion [23]. Some general Φ has also been investigated by physical literatures in order to approximate the Boltzmann equation [9, 10]. So in this paper, we would like to study the generalized potential Φ .

The study of the Fokker-Planck equation can be traced back to the 1930s. When the potential $\Phi = 0$, the equation (1.1) is known as the Kolmogorov-Fokker-Planck equation. In 1934 Kolmogorov [21] derived the Green function for the whole space problem:

$$\mathbb{G}_{FP}(t, x, v; \tau, y, u) = \frac{1}{(t - \tau)^6} \exp \left(-\frac{3|(x - y) - [(t - \tau)/2](v + u)|^2}{(t - \tau)^3} - \frac{|v - u|^2}{4(t - \tau)} \right).$$

This explicit formula surprisingly showed that the solution becomes smooth in the t, x, v variables when $t > 0$ immediately. Moreover, it can be applied to boundary value problems [16–18] and the Vlasov-Poisson-Fokker-Planck system [2, 12, 32].

Later the regularization effect was investigated further and recovered by some more general and robust methods. For example, it is known that the Fokker-Planck operator $-v \cdot \nabla_x + \Delta_v$ is a hypoelliptic operator. So one can apply Hörmander’s commutator [15] to the linear Fokker-Planck operator to obtain that diffusion in v together with the transport term $v \cdot \nabla_x$ has a regularizing effect on solutions not only in v but also in t and x . It can also be obtained through the functional method; see [13, 36]. On the other hand, the Fokker-Planck operator is also known as a hypocoercive operator, which concerns the rate of convergence to equilibrium. Indeed, the trend to equilibria with a certain rate has been investigated in many papers (cf. [7, 8, 13, 14, 30, 31]) for the close to Maxwellian regime in the whole space or in the periodic box.

Let us point out the recent important results constructed by Mischler and Mouhot [30]. They developed an abstract method for deriving decay estimates of the semigroup associated to non-symmetric operators in Banach spaces. Applying this method to the

kinetic Fokker-Planck equation in the torus with potential in the close to equilibrium setting, they obtained spectral gap estimates for the associated semigroup in various norms, including Lebesgue norms, negative Sobolev norms, and the Monge-Kantorovich distance (or 1-Wasserstein distance).

1.3. *Main theorem.* Before the presentation of the main theorem, let us define some notation in this paper. We denote $\langle v \rangle^s = (1 + |v|^2)^{s/2}$, $s \in \mathbb{R}$. For the microscopic variable v , we denote

$$|f|_{L^2_v} = \left(\int_{\mathbb{R}^3} |f|^2 dv \right)^{1/2},$$

and the weighted norms $|\cdot|_{L^2_v(m)}$ and $|\cdot|_{L^2_\theta}$ can be defined by

$$|f|_{L^2(m)} = \left(\int_{\mathbb{R}^3} |f|^2 m dv \right)^{1/2}, \quad |f|_{L^2_\theta} = \left(\int_{\mathbb{R}^3} \langle v \rangle^{2\theta} |f|^2 dv \right)^{1/2},$$

respectively, where $m = m(t, x, v)$ is a weight function. The L^2_v inner product in \mathbb{R}^3 will be denoted by $\langle \cdot, \cdot \rangle_v$,

$$\langle f, g \rangle_v = \int_{\mathbb{R}^3} f(v) \overline{g(v)} dv.$$

For the space variable x , we have the similar notation. In fact, L^2_x is the classical Hilbert space with norm

$$|f|_{L^2_x} = \left(\int_{\mathbb{R}^3} |f|^2 dx \right)^{1/2}.$$

We denote the supremum norm as

$$|f|_{L^\infty_x} = \sup_{x \in \mathbb{R}^3} |f(x)|.$$

The standard inner product in \mathbb{R}^3 will be denoted by (\cdot, \cdot) . For the Fokker-Planck equation, the natural space in the v variable is equipped with the norm $|\cdot|_{L^2_\sigma}$, which is defined as

$$|f|_{L^2_\sigma} = |\langle v \rangle^{\gamma-1} f|_{L^2_v} + |\nabla_v f|_{L^2_v},$$

and the corresponding weighted norms are defined as

$$|f|_{L^2_\sigma(m)} = |\langle v \rangle^{\gamma-1} f|_{L^2_v(m)} + |\nabla_v f|_{L^2_v(m)}, \quad |f|_{L^2_{\sigma,\theta}} = |\langle v \rangle^{\gamma-1} f|_{L^2_\theta} + |\nabla_v f|_{L^2_\theta}.$$

Moreover, we define

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^3} |f|_{L^2_v}^2 dx, \quad \|f\|_{L^2_\sigma}^2 = \int_{\mathbb{R}^3} |f|_{L^2_\sigma}^2 dx,$$

and

$$\|f\|_{L^\infty_x L^2_v} = \sup_{x \in \mathbb{R}^3} |f|_{L^2_v}, \quad \|f\|_{L^1_x L^2_v} = \int_{\mathbb{R}^3} |f|_{L^2_v} dx.$$

Finally, we define the high order Sobolev norm in the x variable: let $k \in \mathbb{N}$ and let α be any multi-index,

$$\|f\|_{H^k_x L^2_v} := \sum_{|\alpha| \leq k} \|\partial_x^\alpha f\|_{L^2}.$$

The weighted spaces in the (x, v) -variable can be defined in a similar way.

For multi-indices $\alpha, \beta_j (j = 1, \dots, s) \in \mathbb{N}_0^3$ with $\alpha = \sum_{j=1}^s \beta_j$, we denote the multinomial coefficients by

$$\binom{\alpha}{\beta_1 \beta_2 \dots \beta_s} = \prod_{k=1}^3 \frac{\alpha_k!}{\prod_{j=1}^s (\beta_j)_k!}.$$

The domain decomposition plays an essential role in our analysis, hence we define a cut-off function $\chi : \mathbb{R} \rightarrow \mathbb{R}$, which is a smooth non-increasing function, $\chi(s) = 1$ for $s \leq 1$, $\chi(s) = 0$ for $s \geq 2$ and $0 \leq \chi \leq 1$. Moreover, we define $\chi_R(s) = \chi(s/R)$.

For simplicity of notation, hereafter, we abbreviate “ $\leq C$ ” to “ \lesssim ”, where C is a positive constant depending only upon fixed numbers.

Here is the precise description of our main results (combining Theorems 3.7, 3.12, 4.2, and 4.4).

THEOREM 1.1. Let f be a solution to the Fokker-Planck equation (1.2) with initial data compactly supported in the x variable and bounded in L_v^2 (we need some exponential weight for $0 < \gamma < 3/2$) space

$$f_0(x, v) \equiv 0 \text{ for } |x| \geq 1.$$

There exists a positive constant M such that the following hold:

- (1) As $\gamma \geq 3/2$, there exists a positive constant C such that the solution f satisfies
 - (a) For $\langle x \rangle \leq 2Mt$,

$$|f(t, x)|_{L_v^2} \lesssim \left[(1 + t^{-9/4})e^{-Ct} + (1 + t)^{-3/2}e^{-C\frac{|x|^2}{t+1}} \right] \|f_0\|_{L_x^\infty L_v^2}.$$

- (b) For $\langle x \rangle \geq 2Mt$,

$$|f(t, x)|_{L_v^2} \lesssim (1 + t^{-9/4})e^{-C(\langle x \rangle + t)} \|f_0\|_{L_x^\infty L_v^2}.$$

- (2) As $1 \leq \gamma < 3/2$, for any given positive integer N and any sufficiently small $\alpha > 0$, there exists a positive constant C such that the solution f satisfies
 - (a) For $\langle x \rangle \leq 2Mt$,

$$|f(t, x)|_{L_v^2} \lesssim \left[(1 + t^{-9/4})e^{-Ct} + (1 + t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-N} \right] \|f_0\|_{L_x^\infty L_v^2}.$$

- (b) For $\langle x \rangle \geq 2Mt$,

$$|f(t, x)|_{L_v^2} (1 + t^{-9/4})e^{-C(\langle x \rangle + t)^{\frac{\gamma}{3-\gamma}}} \|f_0\|_{L^2(e^{4\alpha(v)^\gamma})}.$$

- (3) As $0 < \gamma < 1$, for any sufficiently small $\alpha > 0$, there exists a positive constant C such that the solution f satisfies
 - (a) For $\langle x \rangle \leq 2Mt$,

$$|f(t, x)|_{L_v^2} \lesssim \left[(1 + t^{-9/4})e^{-Ct^{\frac{\gamma}{2-\gamma}}} + (1 + t)^{-3/2} \right] \|f_0\|_{L^2(e^{4\alpha(v)^\gamma})}.$$

- (b) For $\langle x \rangle \geq 2Mt$,

$$|f(t, x)|_{L_v^2} \lesssim (1 + t^{-9/4})e^{-C(\langle x \rangle + t)^{\frac{\gamma}{3-\gamma}}} \|f_0\|_{L^2(e^{4\alpha(v)^\gamma})}.$$

1.4. *Significant points of the paper.* In this paper, we study the Fokker-Planck equation with potential in the close to equilibrium setting. In the literature, this kind of problem basically focuses on the rate of convergence to equilibrium (see the reference listed above). Instead, in this paper we supply an explicit description of the solution in the sense of pointwise estimate. It turns out the structure of the solution sensitively depends on the potential function. Let us illustrate the novelties of the paper:

- We obtain the global picture of the solution, which consists of three parts: the time-like region (large time behavior), the space-like region (spatially asymptotic behavior), and the small time region (the evolution of initial singularity).
 - (1) In the time-like region, we have distinctly different descriptions according to potential functions. For $\gamma \geq 1$, thanks to the spectrum analysis, we have a pointwise fluid structure, which is more precise than previous results. The leading term of the wave propagation has been recognized. More specifically, for $\gamma \geq 3/2$ the leading term is a diffusion wave with heat kernel-type, while for $1 \leq \gamma < 3/2$ the diffusion wave is of algebraic-type. By contrast, the spectral information is missing for $0 < \gamma < 1$ due to the weak damping for large velocity, which leads to the unavailability of pointwise structure. Nevertheless, we can apply Kawashima's argument [20, 34] to get a uniform time decay rate.
 - (2) Concerning the space-like region, we have exponential decay for $\gamma \geq 3/2$ and sub-exponential decay for $0 < \gamma < 3/2$. The results are consistent with the wave behaviors inside the time-like region for different γ 's, respectively. To our knowledge, this is the first result for the asymptotic behavior of the Fokker-Planck equation with potential.
 - (3) Owing to the regularization effect, the initial singularity is eliminated instantaneously.
- The regularization estimate is a key ingredient of this paper (see Lemmas 2.5 and 2.7), which enables us to obtain the pointwise estimate without regularity assumptions on the initial condition. In the literature, the regularization estimates for the kinetic Fokker-Planck equation and Landau equation have been proved for various purposes; see for instance [13], [30], and [36, Appendix A.21.2] for the Fokker-Planck case and [5] for the Landau case. The above mentioned regularization estimates are sufficient for studying the time decay of the solution. However, to gain understanding of the spatially asymptotic behavior, one needs to analyze the solution in some appropriate weighted spaces. Taking this into account, we construct the regularization estimates in suitable weighted spaces. The calculation of the estimates is interesting and more sophisticated than before. Moreover, this type of regularization estimate is itself new.
- The pointwise estimate of the solution in the space-like region is constructed by the weighted energy estimate. The time-dependent weight functions are chosen according to different confinement potentials. For $\gamma \geq 3/2$, from an estimate in the time-like region, the solution decays exponentially along the wave cone, i.e., $|x| = Mt$, suggesting the exponential decay at the spatial infinity. It turns out that a simple weight function is satisfactory (see Proposition 4.1). However,

when $0 < \gamma < 3/2$, we notice that in (1.2) the exponent of damping coefficient ($\sim \langle v \rangle^{2(\gamma-1)}$) is less than 1. From the scaling of the transport equation, we cannot expect exponential decay in the spatial variable. In fact, motivated by the transport equation with weak damping, we devise appropriate weight functions, introduce a refined space-velocity domain decomposition, and eventually show the sub-exponential decay for $0 < \gamma < 3/2$ (Proposition 4.3).

- We believe that our idea in this paper can have potential applications in other important kinetic equations, such as the Landau equation or Boltzmann equation without angular cutoff. In fact, these projects are in progress.

To the best of our knowledge, the first pointwise result of the kinetic-type equation is the Boltzmann equation for hard sphere [26–28]; the authors have established important results regarding the pointwise behavior of the Green function and completed the non-linear problem. Later, the result was generalized to the Boltzmann equation with cutoff hard potentials [22]. Very recently, the authors of the current paper extended the pointwise result to more general potentials, the range $-2 < \gamma < 1$, and obtained an explicit relation between the decay rate and velocity weight assumption [24]. Let us point out some similarities and differences between the Fokker-Planck equation with potential and the Boltzmann equation with hard sphere or cutoff hard potentials.

- The solutions of both equations in large time are dominated by the fluid part. For the Fokker-Planck with $\gamma \geq 1$ and for the Boltzmann with hard sphere or hard potentials with cutoff, the fluid parts are characterized by diffusion waves. To extract them, both need the long wave-short wave decomposition. However, the wave structures of them are quite different. For the Boltzmann equation, there are diffusion waves propagating with different speeds: one with the background speed of the global Maxwellian while the other with the superposed speed of the background speed and the sound speed. In comparison, there is only one diffusion wave for the Fokker-Planck equation. The fluid behavior can be seen formally from the Chapman-Enskog expansion, which indicates that the macroscopic part (the fluid part) of the solution satisfies the viscous system of conservation laws. For the Boltzmann equation there are conservation laws of mass, momentum, and energy, while the Fokker-Planck equation only preserves the mass, explaining the difference of their wave structures.
- Since the leading term of the solution in large time is the fluid part and it essentially has finite propagation speed, the solution in the space-like region, compared to the leading part, should be much smaller. In fact it is shown that the asymptotic behaviors exponentially or sub-exponentially decay. This is similar to the solution of the Boltzmann equation outside the finite Mach number region.
- The regularization mechanism of the Fokker-Planck equation is distinct from that of the Boltzmann equation. For the Boltzmann equation, the initial singularity will be preserved (although decays in time very fast), one has to single them out. Since the singular waves satisfy a damped transport equation, there is an explicit solution formula, from which the pointwise structure can be deduced. Then the regularity of the resulting remainder part comes from the compact part of the collision operator (see the Mixture Lemma in [22], [26], and [27]). By contrast,

for the Fokker-Planck equation the regularity comes from the combined effect of ellipticity in the velocity variable v and the transport term (see Lemmas 2.5 and 2.7). The initial singularities have been identified. However, there is no explicit formula for singular waves. Instead, they are accurately estimated by suitable weighted energy estimates.

1.5. *Method of proof and plan of the paper.* The main idea of this paper is to combine the long wave-short wave decomposition, the wave-remainder decomposition, the weighted energy estimate, and the regularization estimate to analyze the solution. The long wave-short wave decomposition, based on the Fourier transform, gives the fluid structure of the solution. The wave-remainder decomposition is used for extracting the initial singularity. The weighted energy estimate is used for the pointwise estimate of solution inside the space-like region, where the regularization estimate is also used. We explain the idea in more detail below.

In the time-like region (inside the region $|x| \leq Mt$ for some M), the solution is dominated by the fluid part, which is contained in the long wave part. In order to obtain its estimate, we devise different methods for $\gamma \geq 1$ and $0 < \gamma < 1$, respectively. For $\gamma \geq 1$, taking advantage of the spectrum information of the Fokker-Planck operator [29] (in fact, the paper [29] only studies the case $\gamma = 2$ and we can extend it to the case $\gamma \geq 1$), the complex analytic or Fourier multiplier techniques can be applied to obtain pointwise structure of the fluid part. However, for $0 < \gamma < 1$, the spectrum information is missing due to the weak damping for large velocity. Instead, we use Kawashima's argument [20] to get the optimal decay only in time. It is shown that the L^2 norm of the short wave exponentially decays in time for $\gamma \geq 1$ essentially due to the spectrum gap, while it decays only algebraically for $0 < \gamma < 1$ if imposing certain velocity weight on initial data.

We use the wave-remainder decomposition to extract the possible initial singularity in the short wave. This decomposition is based on a Picard-type iteration. The first several terms in the iteration contain the most singular part of the solution, and they are the so-called wave part. By functional methods, we prove the iteration equation has a regularization effect, which enables us to show the remainder becomes more regular. Noticing the singularity will disappear after initial time, the regularization estimate together with L^2 decay of the short wave yields the L^∞ decay of the short wave. Combining this with the long wave, we finish the pointwise structure inside the wave cone.

To get the global structure of the solution, we need the estimate outside the wave cone, i.e., inside the space-like region. The weighted energy estimates play a decisive role here. The weight functions are carefully chosen for different γ 's. It is noted that the sufficient understanding of the structure of the wave part obtained previously is essential in the estimate. Moreover, the regularization effect makes it possible to do the higher order weighted energy estimate. Then the desired pointwise estimate follows from the Sobolev inequality.

The rest of this paper is organized as follows: We first prepare some important properties in Section 2 for the long wave-short wave decomposition, the wave-remainder decomposition, and regularization estimates. Then we study the large time behavior in Section 3. Finally, we study the initial layer and the asymptotic behavior in Section 4.

2. Preliminary.

2.1. *The operator L.* It is obvious that L is a non-positive self-adjoint operator on L^2_v . More precisely, its Dirichlet form is given by

$$\langle Lf, f \rangle_v = - \int_{\mathbb{R}^3} \left| \nabla_v f + \frac{\nabla \Phi}{2} f \right|^2 dv = - \int_{\mathbb{R}^3} \left| \nabla_v \left(\frac{f}{\sqrt{\mathcal{M}}} \right) \right|^2 \mathcal{M} dv.$$

Therefore, the null space of L is given by

$$\text{Ker}(L) = \text{span} \{ E_D \} ,$$

where $E_D = \sqrt{\mathcal{M}}$. Based on this property, we can introduce the macro-micro decomposition as follows: the macro projection P_0 is the orthogonal projection with respect to the L^2_v inner product onto $\text{Ker}(L)$, and the micro projection $P_1 \equiv \text{Id} - P_0$.

Now, we introduce a new norm $|\cdot|_{L^2_{\sigma,\theta}}$:

$$|g|_{L^2_{\sigma,\theta}} := \int \langle v \rangle^{2\theta} |\nabla g|^2 dv + \int \langle v \rangle^{2\theta} \frac{|v|^2 \langle v \rangle^{2\gamma-4}}{2} |g|^2 dv, \quad \theta \in \mathbb{R}, \gamma > 0,$$

which is equivalent to the natural norm $|\cdot|_{L_{\sigma,\theta}}$. Through this equivalent norm, we can derive the coercivity of the operator L for all $\gamma > 0$, as below. The proof is analogous to the Landau case [11].

LEMMA 2.1 (Coercivity). Let $\theta \in \mathbb{R}, \gamma > 0$. For any $m > 1$, there is $0 < C(m) < \infty$, such that

$$\begin{aligned} & \left| \left\langle \langle v \rangle^{2\theta} \frac{\Delta_v \Phi}{2} g_1, g_2 \right\rangle_v \right| \tag{2.1} \\ & \leq \frac{C}{m^\gamma} |g_1|_{L^2_{\sigma,\theta}} |g_2|_{L^2_{\sigma,\theta}} + C(m) \left(\int_{|v| \leq m} |\langle v \rangle^\theta g_1|^2 dv \right)^{1/2} \left(\int_{|v| \leq m} |\langle v \rangle^\theta g_2|^2 dv \right)^{1/2}. \end{aligned}$$

Moreover, there exists $\nu_0 > 0$ such that

$$\langle -Lg, g \rangle_v \geq \nu_0 |P_1 g|_{L^2_v}^2. \tag{2.2}$$

Now, let us decompose the collision operator $L = -\Lambda + K$, where

$$\Lambda = -L + \varpi \chi_R(|v|), \quad K = \varpi \chi_R(|v|);$$

here $\varpi > 0$ and $R > 0$ are as large as desired.

Regarding the behavior of solutions to equation (1.2) in the space-like region, the following weight functions $\mu(x, v)$ will be taken into account:

$$\mu(x, v) = 1 \quad \text{or} \quad \exp(\langle x \rangle / D) \quad \text{if} \quad \gamma \geq 3/2,$$

for D large, and

$$\mu(x, v) = 1 \quad \text{or} \quad \exp(\alpha c(x, v)) \quad \text{if} \quad 0 < \gamma < 3/2,$$

where

$$c(x, v) = 5 \left(\delta \langle x \rangle \right)^{\frac{\gamma}{3-\gamma}} \left(1 - \chi \left(\delta \langle x \rangle \langle v \rangle^{\gamma-3} \right) \right) + \left[\left(1 - \chi \left(\delta \langle x \rangle \langle v \rangle^{\gamma-3} \right) \right) \delta \langle x \rangle \langle v \rangle^{2\gamma-3} + 3 \langle v \rangle^\gamma \right] \chi \left(\delta \langle x \rangle \langle v \rangle^{\gamma-3} \right),$$

and the positive constants δ and α will be determined later.

LEMMA 2.2. Assuming that $\gamma > 0$, we have the following properties of the operators Λ and K :

(i) There exists $c > 0$ such that

$$\int (\Lambda g) g \mu dx dv \geq c \|g\|_{L^2_\sigma(\mu)}^2.$$

(ii)

$$\int (Kg) g \mu dx dv \leq \varpi \|g\|_{L^2(\mu)}^2.$$

Proof. We only prove part (i) when $\mu(x, v) = e^{\alpha c(x, v)}$, since the other cases of part (i) and part (ii) are trivial. Notice that there is a constant $c_1 > 0$ such that

$$\begin{aligned} \|\nabla_v g\|_{L^2(\mu)}^2 + \int \left[\frac{|v|^2 \langle v \rangle^{2\gamma-4}}{4} - \left(\frac{3}{2} \langle v \rangle^{\gamma-2} + \frac{(\gamma-2)}{2} |v|^2 \langle v \rangle^{\gamma-4} \right) + \varpi \chi_R \right] g^2 \mu dx dv \\ \geq c_1 \|g\|_{L^2_\sigma(\mu)}^2 \end{aligned}$$

whenever $\varpi, R > 0$ are sufficiently large. On the other hand, it follows from

$$|\nabla_v c(x, v)| \leq C(\gamma) \langle v \rangle^{\gamma-1} \left(1 + \left| \chi' \left([\delta \langle x \rangle] \langle v \rangle^{\gamma-3} \right) \right| \right) \tag{2.3}$$

that

$$\left| \int \nabla_v g \cdot \nabla_v (\mu) g dx dv \right| \leq \alpha C(\gamma) \sup (1 + |\chi'|) \int \langle v \rangle^{\gamma-1} |g| |\nabla_v g| \mu dx dv \leq \frac{\alpha Q}{2} \|g\|_{L^2_\sigma(\mu)}^2,$$

where $Q = [C(\gamma) \sup (1 + |\chi'|)]$. Therefore, we choose $\alpha > 0$ sufficiently small with $\alpha Q < c_1$ and thus deduce that

$$\begin{aligned} \int (\Lambda g) g \mu dx dv &= \|\nabla_v g\|_{L^2(\mu)}^2 + \int \nabla_v g \cdot \nabla_v (\mu) g dx dv \\ &+ \int \left[\frac{|v|^2 \langle v \rangle^{2\gamma-4}}{4} - \left(\frac{3}{2} \langle v \rangle^{\gamma-2} + \frac{(\gamma-2)}{2} |v|^2 \langle v \rangle^{\gamma-4} \right) + \varpi \chi_R \right] g^2 \mu dx dv \\ &\geq \frac{c_1}{2} \|g\|_{L^2_\sigma(\mu)}^2 = c \|g\|_{L^2_\sigma(\mu)}^2, \end{aligned}$$

which completes the proof of the lemma. □

On the other hand, in preparation for studying the time-like region, we provide the spectrum $\text{Spec}(\eta)$, $\eta \in \mathbb{R}^3$, of the operator $L_\eta = -iv \cdot \eta + L$. In fact, we extend the results for the case $\gamma = 2$ in [29] to the case $\gamma \geq 1$.

LEMMA 2.3 (Spectrum of L_η). Assuming that $\gamma \geq 1$, given $0 < \delta \ll 1$:

(i) There exists $\tau = \tau(\delta) > 0$ such that if $|\eta| > \delta$,

$$\text{Spec}(\eta) \subset \{z \in \mathbb{C} : \text{Re}(z) < -\tau\}. \tag{2.4}$$

(ii) If $|\eta| < \delta$,

$$\text{Spec}(\eta) \cap \{z \in \mathbb{C} : \text{Re}(z) > -\tau\} = \{\lambda(\eta)\}, \tag{2.5}$$

where $\lambda(\eta)$ is the eigenvalue of L_η which is real and smooth in η only through $|\eta|^2$, i.e., $\lambda(\eta) = \mathcal{A}(|\eta|^2)$ for some real smooth function \mathcal{A} ; the eigenfunction $e_D(v, \eta)$ is smooth in η as well. In addition, they are analytic in η if $\gamma \geq 3/2$. Their asymptotic expansions are given as below:

$$\lambda(\eta) = -a_\gamma |\eta|^2 + O(|\eta|^4), \tag{2.6}$$

$$e_D(\eta) = E_D + iE_{D,1}|\eta| + O(|\eta|^2),$$

with $a_\gamma > 0$, $E_{D,1} = L^{-1}(v \cdot \omega E_D)$, $\omega = \eta/|\eta|$. Here $\{e_D(\eta)\}$ can be normalized by

$$\langle e_D(-\eta), e_D(\eta) \rangle_v = 1.$$

(iii) Moreover, the semigroup $e^{(-i\eta \cdot v + L)t}$ can be decomposed as

$$e^{(-i\eta \cdot v + L)t} f = e^{(-i\eta \cdot v + L)t} \Pi_\eta^{D\perp} f + \mathbf{1}_{\{|\eta| < \delta\}} e^{\lambda(\eta)t} \langle e_D(-\eta), f \rangle_v e_D(\eta), \tag{2.7}$$

where $\mathbf{1}_D$ is the characteristic function of the domain D , and there exist $a(\tau) > 0$ and $\bar{a} > 0$ such that

$$|e^{(-i\eta \cdot v + L)t} \Pi_\eta^{D\perp}|_{L_v^2} \lesssim e^{-a(\tau)t} \quad \text{and} \quad |e^{\lambda(\eta)t}| \leq e^{-\bar{a}|\eta|^2 t}. \tag{2.8}$$

Proof. Let $L = -\Lambda + K$ with

$$\Lambda_\eta f = (-\Lambda - i\eta \cdot v) f, \quad L_\eta = (L - i\eta \cdot v) f.$$

Here $f \in D(\Lambda_\eta) = \{f \in L_v^2; \Lambda_\eta f \in L_v^2\}$ and $D(\Lambda_\eta) = D(L_\eta)$. Since K is a bounded operator in L_v^2 , L_η is regarded as a bounded perturbation of Λ_η . We shall verify that such a decomposition satisfies the four hypotheses **H1-H4** stated in [38]. Under the assumptions **H1-H4**, using semigroup theory and linear operator perturbation theory, Theorem 1.1 in [38] asserts that the spectrum of L_η has the similar structure of the Boltzmann equation with cutoff hard potential. Since the null space of the linear Fokker-Planck operator is one-dimensional, for $|\eta|$ small enough, we only obtain one smooth eigenvalue of L_η while there are five smooth eigenvalues for the Boltzmann equation with cutoff hard potentials. As to the verification of **H1-H4**, the proof is a slight modification of the paper [29] and hence we omit the details. The hypothesis **H1** is worthy of being mentioned. For ϖ sufficiently large, there exists a constant $c > 0$ such that

$$\langle \Lambda f, f \rangle_v \geq c |f|_{L_v^2}^2 \geq c |f|_{L_v^2}^2$$

for all $\gamma \geq 1$. The last inequality holds since $|f|_{L_v^2}$ is stronger than $|f|_{L_v^2}$ as $\gamma \geq 1$. This is why we miss the spectrum structure for the case $0 < \gamma < 1$.

To prove (ii), we need to explore the symmetric properties of $\lambda(\eta)$ and $e_D(\eta)$. Here we follow the framework of Section 7.3 in [28]. First we notice there is a natural three-dimensional orthogonal group $O(3)$ -action on L_v^2 : Let $a \in O(3)$, $f \in L_v^2$,

$$(a \circ f)(v) \equiv f(a^{-1}v).$$

Then it is easy to check the $O(3)$ -action commutes with operators L , P_0 , and P_1 . Consider the eigenvalue problem

$$L_\eta e_D(\eta) = (-iv \cdot \eta + L) e_D(\eta) = \lambda(\eta) e_D(\eta). \tag{2.9}$$

Apply $a \in O(3)$ to (2.9). By commutative properties and the fact that a preserves the vector inner product in \mathbb{R}^3 ,

$$(-iv \cdot (a\eta) + L)(a \circ e_D(\eta)) = \lambda(\eta)(a \circ e_D(\eta)).$$

Then $\lambda(a\eta) = \lambda(\eta)$, $e_D(a\eta) = a \circ e(\eta)$, which implies that $\lambda(\eta)$ is dependent only upon $|\eta|$. Now let $a \in O(3)$ be an orthogonal transformation that sends $\frac{\eta}{|\eta|}$ to $(1, 0, 0)^T$. Thus the original eigenvalue problem (2.9) is reduced to

$$(-iv_1|\eta| + L)e(|\eta|) = \lambda(|\eta|)e(|\eta|) \tag{2.10}$$

with $\lambda(\eta) = \lambda(|\eta|)$, $e_D(\eta) = a^{-1} \circ e_D(|\eta|)$. Apply the macro-micro decomposition to (2.10) to yield

$$-i|\eta|P_0v_1(P_0e + P_1e) = \lambda P_0e, \tag{2.11a}$$

$$-i|\eta|P_1v_1P_0e - i|\eta|P_1v_1P_1e + LP_1e = \lambda P_1e. \tag{2.11b}$$

Set $\lambda(|\eta|) = i|\eta|\zeta(|\eta|)$. We can solve P_1e in terms of P_0e from (2.11b),

$$P_1e = i|\eta|[L - i|\eta|P_1v_1 - i|\eta|\zeta(|\eta|)]^{-1}P_1v_1P_0e; \tag{2.12}$$

then substitute this back to (2.11a) to get

$$(P_0v_1 + i|\eta|P_0[L - i|\eta|P_1v_1 - i|\eta|\zeta(|\eta|)]^{-1}P_1v_1)P_0e = -\zeta P_0e. \tag{2.13}$$

We notice that this is actually a finite dimensional eigenvalue problem. The solvability of it and the asymptotic expansions of eigenvalue and eigenfunction for $|\eta| \ll 1$ are essentially due to the implicit function theorem. The procedure is basically the same as the case $\gamma = 2$; we refer the readers to Theorem 3.2 in [29] for details. We obtain $\lambda(|\eta|)$ and $P_0e(|\eta|) = \beta(|\eta|)E_D$ with λ and β being smooth functions. Furthermore, $\lambda(|\eta|)$ and $\beta(|\eta|)$ are not merely smooth but analytic for $\gamma \geq 3/2$. To prove this, it suffices to check that the perturbation ivf is L -bounded, i.e.,

$$|vf|_{L_v^2}^2 \leq C_1|Lf|_{L_v^2}^2 + C_2|f|_{L_v^2}^2.$$

Then the Kato-Rellich theorem guarantees the operator $B(z) = -iv_1z + L$ is in the analytic family of Type (A), see [19], which in turn implies the eigenvalue and eigenfunction associated with (2.10) are analytic in $|\eta|$, cf. [6]. Now, let us calculate $\langle \Lambda f, \Lambda f \rangle_v$ first. For simplicity of notation, let

$$\psi(v) = \frac{1}{4}|v|^2 \langle v \rangle^{2\gamma-4} - \left(\frac{3}{2} \langle v \rangle^{\gamma-2} + \frac{\gamma-2}{2}|v|^2 \langle v \rangle^{\gamma-4} \right) + \varpi \chi_R(|v|);$$

then

$$\begin{aligned} \langle \Lambda f, \Lambda f \rangle_v &= |\Delta_v f|_{L_v^2}^2 + |\psi(v)f|_{L_v^2}^2 \\ &\quad + 2\langle \psi(v), (\nabla_v f, \nabla_v f) \rangle_v + 2\langle f, (\nabla_v \psi(v), \nabla_v f) \rangle_v. \end{aligned}$$

By the Cauchy inequality, we have

$$|\langle f, (\nabla_v \psi(v), \nabla_v f) \rangle_v| \leq \langle \psi(v), (\nabla_v f, \nabla_v f) \rangle_v + \frac{1}{4} \left\langle \frac{f^2}{\psi(v)}, (\nabla_v \psi(v), \nabla_v \psi(v)) \right\rangle_v.$$

Let us compare $(\nabla_v \psi(v), \nabla_v \psi(v))$ and $\psi^3(v)$. For $|v|$ large, we have

$$(\nabla_v \psi(v), \nabla_v \psi(v)) \approx |v|^{4\gamma-6}$$

and

$$\psi^3(v) \approx |v|^{6\gamma-6}.$$

For $|v|$ small, one can choose ϖ large enough such that

$$(\nabla_v \psi(v), \nabla_v \psi(v)) \ll \psi^3(v).$$

This means

$$\langle \Lambda f, \Lambda f \rangle_v \geq |\Delta_v f|_{L^2_v}^2 + \frac{1}{2} |\psi(v) f|_{L^2_v}^2 \gtrsim |\langle v \rangle^{2\gamma-2} f|_{L^2_v}^2.$$

Hence if $\gamma \geq 3/2$,

$$\begin{aligned} |vf|_{L^2_v}^2 &\leq C \langle \Lambda f, \Lambda f \rangle_v = C \langle Lf - Kf, Lf - Kf \rangle_v \\ &\leq C_1 |Lf|_{L^2_v}^2 + C_2 |f|_{L^2_v}^2. \end{aligned}$$

However, we cannot simply deduce smoothness (analyticity) in η from smoothness (analyticity) in $|\eta|$. Our goal is to show $\lambda(z)$ and $\beta(z)$ are in fact even functions in z . If so, due to a classical theorem of Whitney [37], we have

$$\lambda(|\eta|) = \mathcal{A}(|\eta|^2), \quad \beta(|\eta|) = \mathcal{B}(|\eta|^2)$$

for some smooth or analytic functions \mathcal{A} and \mathcal{B} provided $\lambda(|\eta|)$ and $\beta(|\eta|)$ are smooth or analytic, respectively. To show they are even, let us define a map $\mathcal{R} : (v_1, v_2, v_3) \mapsto (-v_1, v_2, v_3)$; then obviously $\mathcal{R} \in O(3)$. We apply \mathcal{R} to (2.10),

$$(-iv_1(-|\eta|) + L)(\mathcal{R} \circ e(|\eta|)) = \lambda(|\eta|)(\mathcal{R} \circ e(|\eta|)),$$

which is an eigenvalue problem with $|\eta| \rightarrow -|\eta|$. This follows that the eigenpair $\{\lambda(|\eta|), \mathcal{R} \circ e(|\eta|)\}$ coincides with $\{\lambda(-|\eta|), e(-|\eta|)\}$. Hence

$$\lambda(|\eta|) = \lambda(-|\eta|), \quad \mathcal{R} \circ e(|\eta|) = e(-|\eta|). \tag{2.14}$$

In addition, use $\mathcal{R} \circ P_0 e(|\eta|) = P_0 \mathcal{R} \circ e(|\eta|) = P_0 e(-|\eta|)$ and $\mathcal{R} \circ E_D = E_D$ to find $\beta(|\eta|) = \beta(-|\eta|)$, namely β is also an even function. We can show

$$\overline{\lambda(|\eta|)} = \lambda(-|\eta|), \quad \overline{e(|\eta|)} = e(-|\eta|), \tag{2.15}$$

by taking the complex conjugate of (2.10). This together with (2.14) shows $\lambda(|\eta|)$ and $\beta(|\eta|)$ are real functions. By (2.12), we can construct $e(|\eta|)$ from $P_0 e(|\eta|)$,

$$e(|\eta|) = P_0 e(|\eta|) + P_1 e(|\eta|) = \left(1 + i|\eta| [L - i|\eta| P_1 v_1 - \lambda(|\eta|)]^{-1} P_1 v_1\right) \beta(|\eta|) E_D.$$

The eigenfunction $e_D(\eta)$ to the original eigenvalue-problem (2.9) can be recovered by applying a^{-1} , namely

$$e_D(\eta) = a^{-1} \circ e(|\eta|) = \left(1 + [L - P_1 i\eta \cdot v - \mathcal{A}(|\eta|^2)]^{-1} P_1 i\eta \cdot v\right) \mathcal{B}(|\eta|^2) E_D.$$

Therefore the proof is complete. □

2.2. *The semigroup operator $e^{t\mathcal{L}}$.* Now, let h be the solution of the equation

$$\begin{cases} \partial_t h = \mathcal{L}h, & \text{where } \mathcal{L}h = -v \cdot \nabla_x h - \Lambda h, \\ h(0, x, v) = h_0(x, v). \end{cases} \tag{2.16}$$

In this subsection, we will study some properties of the semigroup operator $e^{t\mathcal{L}}$.

LEMMA 2.4. For any $k \in \mathbb{N} \cup \{0\}$:

(i) If $\gamma \geq 1$, there exists $C > 0$ such that

$$\|e^{t\mathcal{L}}h_0\|_{H_x^k L_v^2(\mu)} \lesssim e^{-Ct} \|h_0\|_{H_x^k L_v^2(\mu)}. \tag{2.17}$$

(ii) If $0 < \gamma < 1$, we have

$$\|e^{t\mathcal{L}}h_0\|_{H_x^k L_v^2(\mu)} \lesssim \|h_0\|_{H_x^k L_v^2(\mu)}. \tag{2.18}$$

Proof. It suffices to show that there exists $c_0 > 0$ such that for any multi-index β ,

$$\frac{d}{dt} \|\partial_x^\beta h\|_{L^2(\mu)}^2 \leq -c_0 \|\partial_x^\beta h\|_{L^2(\mu)}^2. \tag{2.19}$$

In view of Lemma 2.2, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_x^\beta h\|_{L^2(\mu)}^2 &= \int -v \cdot \nabla_x (\partial_x^\beta h) \partial_x^\beta h \mu dx dv - \int \Lambda (\partial_x^\beta h) \partial_x^\beta h \mu dx dv \\ &\leq \int \frac{1}{2} (\partial_x^\beta h)^2 v \cdot \nabla_x \mu dx dv - c_0 \|\partial_x^\beta h\|_{L^2(\mu)}^2. \end{aligned}$$

If $\mu(x, v) \equiv 1$, (2.19) is obvious since $\int \frac{1}{2} (\partial_x^\beta h)^2 v \cdot \nabla_x \mu dx dv = 0$.

If $\mu(x, v) = \exp(\langle x \rangle / D)$ and $\gamma \geq 3/2$, we choose D sufficiently large such that $1/D < \min\{c_0, 1\}$ and thus obtain

$$\begin{aligned} \left| \int \frac{1}{2} (\partial_x^\beta h)^2 v \cdot \nabla_x \mu dx dv \right| &= \left| \int \frac{1}{2} (\partial_x^\beta h)^2 v \cdot \frac{x}{D \langle x \rangle} \mu dx dv \right| \\ &\leq \frac{1}{2D} \int \langle v \rangle^{2\gamma-2} (\partial_x^\beta h)^2 \mu dx dv \leq \frac{c_0}{2} \|\partial_x^\beta h\|_{L^2(\mu)}^2. \end{aligned}$$

If $\mu(x, v) = e^{\alpha c(x, v)}$ and $0 < \gamma < 3/2$, since

$$|\nabla_x c(x, v)| \leq \delta C \langle v \rangle^{2\gamma-3},$$

for some constant $C > 0$ depending only upon γ , we have

$$\left| \int \frac{1}{2} (\partial_x^\beta h)^2 v \cdot \nabla_x \mu dx dv \right| \leq \frac{\delta C \alpha}{2} \int \langle v \rangle^{2\gamma-2} (\partial_x^\beta h)^2 \mu dx dv \leq \frac{c_0}{2} \|\partial_x^\beta h\|_{L^2(\mu)}^2$$

by choosing $\alpha, \delta > 0$ small enough with $0 < \delta C \alpha < \min\{c_0, 1\}$.

Grouping the above discussions, we obtain (2.19) and thus deduce that for $\gamma > 0$, $k \in \mathbb{N} \cup \{0\}$,

$$\|e^{t\mathcal{L}}h_0\|_{H_x^k L_v^2(\mu)} \leq \|h_0\|_{H_x^k L_v^2(\mu)}.$$

Moreover, if $\gamma \geq 1$, then (2.19) becomes

$$\frac{1}{2} \frac{d}{dt} \|\partial_x^\beta h\|_{L^2(\mu)}^2 \leq -\frac{c_0}{2} \|\partial_x^\beta h\|_{L^2(\mu)}^2 \leq -\frac{c_0}{2} \|\partial_x^\beta h\|_{L^2(\mu)}^2,$$

which leads to the exponential time decay of all x -derivatives of the solution $e^{t\mathcal{L}}h_0$ in the weighted L^2 norm. □

The following is the regularization estimate of the semigroup operator $e^{t\mathcal{L}}$ in small time.

LEMMA 2.5 (Regularization estimate). For $\gamma > 0$ and $0 < t \leq 1$, we have

$$\int |\nabla_v e^{t\mathcal{L}}h_0|^2 \mu dx dv = O(t^{-1}) \int |h_0|^2 \mu dx dv$$

and

$$\int |\nabla_x e^{t\mathcal{L}}h_0|^2 \mu dx dv = O(t^{-3}) \int |h_0|^2 \mu dx dv .$$

Proof. Recall that $\Lambda = -L + K$ with

$$\Lambda f = -\Delta_v f + \left[\frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right] f + \varpi \chi_R (|v|) f, \quad K = \varpi \chi_R (|v|) f.$$

Here we choose $R > 0$ and $\varpi > 0$ sufficiently large such that

$$\begin{cases} \left| \frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right| \leq \frac{1}{3} \langle v \rangle^{2\gamma-2}, & \left| \nabla_v \left(\frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right) \right| \lesssim \langle v \rangle^{2\gamma-3} \text{ for } |v| > R, \\ \left| \frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right|, & \left| \nabla_v \left(\frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right) \right| < \frac{\varpi}{2} \text{ for } |v| \leq R. \end{cases} \tag{2.20}$$

Now, define the energy functional

$$\mathcal{F}(t, h_t) := A \|h\|_{L^2(\mu)}^2 + at \|\nabla_v h\|_{L^2(\mu)}^2 + 2ct^2 \langle \nabla_x h, \nabla_v h \rangle_{L^2(\mu)} + bt^3 \|\nabla_x h\|_{L^2(\mu)}^2$$

with $a, b, c > 0$ and $c < \sqrt{ab}$ (positive definite) and $A > 0$ sufficiently large. We shall show that $d\mathcal{F}/dt \leq 0$, $t \in (0, 1)$, via choosing suitable positive constants A, a, b , and c .

In (2.19), it has been shown that

$$\frac{d}{dt} \|h\|_{L^2(\mu)}^2 \leq -c_0 \|h\|_{L^2(\mu)}^2, \tag{2.21}$$

$$\frac{d}{dt} \|\partial_{x_i} h\|_{L^2(\mu)}^2 \leq -c_0 \|\partial_{x_i} h\|_{L^2(\mu)}^2. \tag{2.22}$$

Next, we show that

$$\frac{1}{2} \frac{d}{dt} \|\partial_{v_i} h\|_{L^2(\mu)}^2 \leq - \int \partial_{x_i} h \partial_{v_i} h \mu dx dv - \frac{c_0}{2} \|\partial_{v_i} h\|_{L^2(\mu)}^2 + C_\varepsilon \|h\|_{L^2(\mu)}^2 + \varepsilon \|\partial_{v_i} h\|_{L^2(\mu)}^2, \tag{2.23}$$

where $\varepsilon > 0$ is arbitrarily small and $C_\varepsilon = O(1/\varepsilon)$. Compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_{v_i} h\|_{L^2(\mu)}^2 &= - \int \partial_{x_i} h \partial_{v_i} h \mu dx dv + \int \left[\frac{v}{2} (\partial_{v_i} h)^2 \right] \cdot \nabla_x \mu dx dv \\ &\quad - \int (\Lambda \partial_{v_i} h) \partial_{v_i} h \mu dx dv - \int ([\partial_{v_i}, \Lambda] h) \partial_{v_i} h \mu dx dv, \end{aligned}$$

where

$$[\partial_{v_i}, \Lambda] h = \left[\partial_{v_i} \left(\frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right) \right] h + \varpi \partial_{v_i} (\chi_R) h.$$

In the course of the proof of Lemma 2.4, one can see that

$$|\nabla_x \mu| \leq \min\{c_0, 1\} \cdot \langle v \rangle^{2\gamma-3} \mu \quad \text{and} \quad |\nabla_x \mu| \leq \min\{c_0, 1\} \mu, \tag{2.24}$$

so

$$\left| \int \left[\frac{v}{2} (\partial_{v_i} h)^2 \right] \cdot \nabla_x \mu dx dv \right| \leq \frac{c_0}{2} \int \langle v \rangle^{2\gamma-2} (\partial_{v_i} h)^2 \mu dx dv \leq \frac{c_0}{2} \|\partial_{v_i} h\|_{L^2_\sigma(\mu)}^2.$$

Furthermore, by (2.20) we obtain

$$\begin{aligned} & \left| \int ([\partial_{v_i}, \Lambda] h) \partial_{v_i} h \mu dx dv \right| \\ & \leq C \int \langle v \rangle^{2\gamma-2} |h \partial_{v_i} h| \mu dx dv + \frac{\varpi}{2} \int \chi_R |h \partial_{v_i} h| \mu dx dv + \frac{\varpi}{R} \int |\chi'_R| |h \partial_{v_i} h| \mu dx dv \\ & \leq C' \int \langle v \rangle^{2\gamma-2} |h \partial_{v_i} h| \mu dx dv \leq C_\varepsilon \|h\|_{L^2_\sigma(\mu)}^2 + \varepsilon \|\partial_{v_i} h\|_{L^2_\sigma(\mu)}^2, \end{aligned}$$

where $\varepsilon > 0$ is arbitrarily small and $C_\varepsilon = O(1/\varepsilon)$. It turns out that

$$\frac{1}{2} \frac{d}{dt} \|\partial_{v_i} h\|_{L^2(\mu)}^2 \leq - \int \partial_{x_i} h \partial_{v_i} h \mu dx dv - \frac{c_0}{2} \|\partial_{v_i} h\|_{L^2_\sigma(\mu)}^2 + C_\varepsilon \|h\|_{L^2_\sigma(\mu)}^2 + \varepsilon \|\partial_{v_i} h\|_{L^2_\sigma(\mu)}^2.$$

Finally, direct computation gives

$$\begin{aligned} & \frac{d}{dt} \int \partial_{x_i} h \partial_{v_i} h \mu dx dv \\ & = - \int |\partial_{x_i} h|^2 \mu dx dv - 2 \int \nabla_v (\partial_{x_i} h) \cdot \nabla_v (\partial_{v_i} h) \mu dx dv \\ & \quad - 2 \int \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) \partial_{x_i} h \partial_{v_i} h \mu dx dv \\ & \quad + \int (v \cdot \nabla_x \mu) \partial_{v_i} h \partial_{x_i} h dx dv + \frac{1}{2} \int \partial_{v_i} \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \\ & \quad - \int \nabla_v (\partial_{x_i} h) \cdot \nabla_v (\mu) \partial_{v_i} h dx dv - \int \nabla_v (\partial_{v_i} h) \cdot \nabla_v (\mu) \partial_{x_i} h dx dv. \end{aligned}$$

From (2.3), it follows that

$$\begin{aligned} & \left| \int \nabla_v (\partial_{x_i} h) \cdot \nabla_v (\mu) \partial_{v_i} h dx dv + \int \nabla_v (\partial_{v_i} h) \cdot \nabla_v (\mu) \partial_{x_i} h dx dv \right| \\ & \leq \alpha C(\gamma) \sup(1 + |\chi'|) \int \langle v \rangle^{\gamma-1} (|\nabla_v (\partial_{x_i} h)| |\partial_{v_i} h| + |\nabla_v (\partial_{v_i} h)| |\partial_{x_i} h|) \mu dx dv \\ & \leq \int \left(\langle v \rangle^{\gamma-1} |\nabla_v (\partial_{x_i} h)| |\partial_{v_i} h| + \langle v \rangle^{\gamma-1} |\nabla_v (\partial_{v_i} h)| |\partial_{x_i} h| \right) \mu dx dv \end{aligned}$$

from $\alpha C(\gamma) \sup(1 + |\chi'|) = \alpha Q < 1$. Note that this inequality is valid in the cases $\mu(x, v) = 1$ and $\mu(x, v) = \exp(\langle x \rangle / D)$ as well, since $\nabla_v \mu = 0$ in both cases. Therefore,

$$\begin{aligned} & \frac{d}{dt} \int \partial_{x_i} h \partial_{v_i} h \mu dx dv \\ \leq & - \int |\partial_{x_i} h|^2 \mu dx dv - 2 \int \nabla_v (\partial_{x_i} h) \cdot \nabla_v (\partial_{v_i} h) \mu dx dv \\ & - 2 \int \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) \partial_{x_i} h \partial_{v_i} h \mu dx dv \\ & + \int (v \cdot \nabla_x \mu) \partial_{v_i} h \partial_{x_i} h dx dv + \frac{1}{2} \int \partial_{v_i} \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \\ & + \int (\langle v \rangle^{\gamma-1} |\nabla_v (\partial_{x_i} h)| |\partial_{v_i} h| + \langle v \rangle^{\gamma-1} |\nabla_v (\partial_{v_i} h)| |\partial_{x_i} h|) \mu dx dv. \end{aligned}$$

Collecting terms gives

$$\begin{aligned} & \frac{d}{dt} \mathcal{F}(t, h_t) \\ \leq & -c_0 A \|h\|_{L^2_\sigma(\mu)}^2 + a \|\nabla_v h\|_{L^2(\mu)}^2 + 4ct \langle \nabla_x h, \nabla_v h \rangle_{L^2(\mu)} + 3bt^2 \|\nabla_x h\|_{L^2(\mu)}^2 \\ & + 2at \left[- \sum_{i=1}^3 \int \partial_{x_i} h \partial_{v_i} h \mu dx dv - \frac{c_0}{2} \|\nabla_v h\|_{L^2_\sigma(\mu)}^2 + 3C_\varepsilon \|h\|_{L^2_\sigma(\mu)}^2 + \varepsilon \|\nabla_v h\|_{L^2_\sigma(\mu)}^2 \right] \\ & + 2ct^2 \left[- \int |\nabla_x h|^2 \mu dx dv - 2 \sum_{i=1}^3 \int \nabla_v (\partial_{x_i} h) \cdot \nabla_v (\partial_{v_i} h) \mu dx dv \right. \\ & - 2 \sum_{i=1}^3 \int \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) \partial_{x_i} h \partial_{v_i} h \mu dx dv - \sum_{i=1}^3 \int (v \cdot \nabla_x \mu) \partial_{v_i} h \partial_{x_i} h dx dv \\ & + \frac{1}{2} \sum_{i=1}^3 \int \partial_{v_i} \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \\ & \left. + \sum_{i=1}^3 \int (\langle v \rangle^{\gamma-1} |\nabla_v (\partial_{x_i} h)| |\partial_{v_i} h| + \langle v \rangle^{\gamma-1} |\nabla_v (\partial_{v_i} h)| |\partial_{x_i} h|) \mu dx dv \right] - bc_0 t^3 \|\nabla_x h\|_{L^2_\sigma(\mu)}^2. \end{aligned}$$

By (2.20) and (2.24),

$$\begin{aligned} & \left| -2 \sum_{i=1}^3 \int \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) \partial_{x_i} h \partial_{v_i} h \mu dx dv + \sum_{i=1}^3 \int (v \cdot \nabla_x \mu) \partial_{v_i} h \partial_{x_i} h dx dv \right| \\ & \leq \sum_{i=1}^3 2 \int \langle v \rangle^{2\gamma-2} |\partial_{x_i} h \partial_{v_i} h| \mu dx dv + 3\varpi \int \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv \\ & \leq \sum_{i=1}^3 \left(\frac{bc_0}{8c} t \left\| \langle v \rangle^{\gamma-1} (\partial_{x_i} h) \right\|_{L^2(\mu)}^2 + \frac{8c}{bc_0 t} \left\| \langle v \rangle^{\gamma-1} (\partial_{v_i} h) \right\|_{L^2(\mu)}^2 \right) \\ & \quad + 3\varpi \int \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv \\ & = \frac{bc_0}{8c} t \|\nabla_x h\|_{L^2_\sigma(\mu)}^2 + \frac{8c}{bc_0 t} \|\nabla_v h\|_{L^2_\sigma(\mu)}^2 + 3\varpi \int \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv. \end{aligned}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned} & \left| 2 \sum_{i=1}^3 \int \nabla_v (\partial_{x_i} h) \cdot \nabla_v (\partial_{v_i} h) \mu dx dv \right| \\ & \leq \sum_{i=1}^3 \left(\frac{bc_0}{8c} t \|\nabla_v (\partial_{x_i} h)\|_{L^2(\mu)}^2 + \frac{8c}{bc_0 t} \|\nabla_v (\partial_{v_i} h)\|_{L^2(\mu)}^2 \right) \\ & \leq \frac{bc_0}{8c} t \|\nabla_x h\|_{L^2_\sigma(\mu)}^2 + \frac{8c}{bc_0 t} \|\nabla_v h\|_{L^2_\sigma(\mu)}^2, \end{aligned}$$

$$\begin{aligned} & \left| (4ct - 2at) \int \nabla_x h \cdot \nabla_v h \mu dx dv \right| + \int 6c\varpi t^2 \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv \\ & \leq (4c + 2a + 6c\varpi t) t \left[\varepsilon t \|\nabla_x h\|_{L^2(\mu)}^2 + \frac{C_\varepsilon}{t} \|\nabla_v h\|_{L^2(\mu)}^2 \right], \quad C_\varepsilon = O\left(\frac{1}{\varepsilon}\right), \end{aligned}$$

and

$$\begin{aligned} & \left| \sum_{i=1}^3 \int \left(\langle v \rangle^{\gamma-1} |\nabla_v (\partial_{x_i} h)| |\partial_{v_i} h| + \langle v \rangle^{\gamma-1} |\nabla_v (\partial_{v_i} h)| |\partial_{x_i} h| \right) \mu dx dv \right| \\ & \leq \sum_{i=1}^3 \left(\frac{2c}{bc_0 t} \int \langle v \rangle^{2\gamma-2} |\partial_{v_i} h|^2 \mu dx dv + \frac{bc_0}{8c} t \int |\nabla_v (\partial_{x_i} h)|^2 \mu dx dv \right) \\ & \quad + \sum_{i=1}^3 \left(\frac{bc_0}{8c} t \int \langle v \rangle^{2\gamma-2} |\partial_{x_i} h|^2 \mu dx dv + \frac{2c}{bc_0 t} \int |\nabla_v (\partial_{v_i} h)|^2 \mu dx dv \right) \\ & \leq \frac{2c}{bc_0 t} \|\nabla_v h\|_{L^2_\sigma(\mu)}^2 + \frac{bc_0}{8c} t \|\nabla_x h\|_{L^2_\sigma(\mu)}^2. \end{aligned}$$

In view of (2.20) and (2.24),

$$\begin{aligned} & \left| \frac{1}{2} \sum_{i=1}^3 \int \partial_{v_i} \left(\frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \right| \\ & \leq \frac{C}{2} \int \langle v \rangle^{2\gamma-3} h^2 \mu dx dv + \frac{\varpi}{4} \int \langle v \rangle^{2\gamma-3} \chi_R h^2 \mu dx dv + \frac{\varpi}{2R} \int \langle v \rangle^{2\gamma-3} |\chi'_R| h^2 \mu dx dv \\ & \leq M' \int \langle v \rangle^{2\gamma-2} h^2 \mu dx dv \leq M' \|h\|_{L^2_\sigma(\mu)}^2, \end{aligned}$$

where $M' > 0$ is dependent only upon R and ϖ . Gathering the above estimates, we therefore deduce

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(t, h_t) & \leq \|h\|_{L^2_\sigma(\mu)}^2 [-c_0 A + a + 6atC_\varepsilon + (4c + 2a + 6c\varpi t) C_\varepsilon + 2cM't^2] \\ & \quad + \|\nabla_x h\|_{L^2(\mu)}^2 (-2c + 3b + (4c + 2a + 6c\varpi t) \varepsilon) t^2 \\ & \quad + \|\nabla_x h\|_{L^2_\sigma(\mu)}^2 \left(-\frac{bc_0}{4} \right) t^3 \\ & \quad + \|\nabla_v h\|_{L^2_\sigma(\mu)}^2 \left(-ac_0 + 2a\varepsilon + \frac{36c^2}{bc_0} \right) t. \end{aligned}$$

Set $a = \varepsilon$, $4b = c = \varepsilon^{3/2}$. After choosing $A > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small, we obtain

$$\frac{d}{dt} \mathcal{F}(t, h_t) \leq 0, \quad t \in (0, 1),$$

which implies that

$$\mathcal{F}(t, h_t) \leq \mathcal{F}(0, h_0) = A \|h_0\|_{L^2}^2, \quad t \in [0, 1].$$

This completes the proof of the lemma. □

Before the end of this section, we introduce the wave-remainder decomposition, which is the key decomposition in our paper. The strategy is to design a Picard-type iteration, treating Kf as a source term. The zeroth order approximation of the Fokker-Planck equation (1.2) is

$$\begin{cases} \partial_t h^{(0)} + v \cdot \nabla_x h^{(0)} + \Lambda h^{(0)} = 0, \\ h^{(0)}(0, x, v) = f_0(x, v). \end{cases} \tag{2.25}$$

Thus the difference $f - h^{(0)}$ satisfies

$$\begin{cases} \partial_t (f - h^{(0)}) + v \cdot \nabla_x (f - h^{(0)}) + \Lambda (f - h^{(0)}) = K(f - h^{(0)}) + Kh^{(0)}, \\ (f - h^{(0)})(0, x, v) = 0. \end{cases}$$

Therefore the first order approximation $h^{(1)}$ can be defined by

$$\begin{cases} \partial_t h^{(1)} + v \cdot \nabla_x h^{(1)} + \Lambda h^{(1)} = Kh^{(0)}, \\ h^{(1)}(0, x, v) = 0. \end{cases} \tag{2.26}$$

In general, we can define the j th order approximation $h^{(j)}$, $j \geq 1$, as

$$\begin{cases} \partial_t h^{(j)} + v \cdot \nabla_x h^{(j)} + \Lambda h^{(j)} = K h^{(j-1)}, \\ h^{(j)}(0, x, v) = 0. \end{cases} \tag{2.27}$$

The wave part and the remainder part can be defined as follows:

$$W^{(3)} = \sum_{j=0}^3 h^{(j)}, \quad \mathcal{R}^{(3)} = f - W^{(3)},$$

$\mathcal{R}^{(3)}$ solving the equation:

$$\begin{cases} \partial_t \mathcal{R}^{(3)} + v \cdot \nabla_x \mathcal{R}^{(3)} = L \mathcal{R}^{(3)} + K h^{(3)}, \\ \mathcal{R}^{(3)}(0, x, v) = 0. \end{cases} \tag{2.28}$$

The next two lemmas are the fundamental properties of $h^{(j)}$, $0 \leq j \leq 3$.

LEMMA 2.6 (L^2 estimate of $h^{(j)}$, $0 \leq j \leq 3$). For all $0 \leq j \leq 3$ and $t > 0$:

(i) If $\gamma \geq 1$, there exists $C > 0$ such that

$$\|h^{(j)}\|_{L^2(\mu)} \lesssim t^j e^{-Ct} \|f_0\|_{L^2(\mu)}.$$

(ii) If $0 < \gamma < 1$, we have

$$\|h^{(j)}\|_{L^2(\mu)} \lesssim t^j \|f_0\|_{L^2(\mu)}.$$

This lemma is immediate from Lemma 2.4 and hence we omit the details.

LEMMA 2.7 (x -derivative estimate of $h^{(j)}$, $0 \leq j \leq 3$). Let $\gamma > 0$ and $k = 1, 2$. Then:

(i) For $0 < t \leq 1$, we have

$$\|\nabla_x^k h^{(j)}\|_{L^2(\mu)} \lesssim t^{j-\frac{3}{2}k} \|f_0\|_{L^2(\mu)}.$$

(ii) For $t > 1$, we have that if $\gamma \geq 1$,

$$\|\nabla_x^k h^{(j)}\|_{L^2(\mu)} \lesssim t^j e^{-Ct} \|f_0\|_{L^2(\mu)},$$

and if $0 < \gamma < 1$,

$$\|\nabla_x^k h^{(j)}\|_{L^2(\mu)} \lesssim t^j \|f_0\|_{L^2(\mu)}.$$

Proof. We divide our proof into several steps.

STEP 1. First x -derivative of $h^{(j)}$, $0 \leq j \leq 3$ in small time. We want to show that for $0 < t \leq 1$,

$$\|\nabla_x h^{(j)}\|_{L^2(\mu)} \lesssim t^{(-3+2j)/2} \|f_0\|_{L^2(\mu)}.$$

The estimate of $h^{(0)}$ is immediate from Lemma 2.5. Note that

$$h^{(1)} = \int_0^t e^{(t-s)\mathcal{L}} K e^{s\mathcal{L}} f_0 ds,$$

hence

$$\begin{aligned} \nabla_x h^{(1)} &= \int_0^t \frac{(t-s) + s}{t} \nabla_x e^{(t-s)\mathcal{L}} K e^{s\mathcal{L}} f_0 ds \\ &= \int_0^t \frac{1}{t} \left[(t-s) \nabla_x e^{(t-s)\mathcal{L}} K e^{s\mathcal{L}} f_0 + e^{(t-s)\mathcal{L}} K (s \nabla_x e^{s\mathcal{L}} f_0) \right] ds. \end{aligned}$$

From Lemmas 2.4 and 2.5, it follows

$$\begin{aligned} \left\| \nabla_x h^{(1)} \right\|_{L^2(\mu)} &\lesssim \int_0^t t^{-1} \left[(t-s)^{-1/2} + s^{-1/2} \right] ds \|f_0\|_{L^2(\mu)} \\ &\lesssim t^{-1/2} \|f_0\|_{L^2(\mu)}. \end{aligned}$$

Likewise, note that

$$h^{(2)} = \int_0^t \int_0^{s_1} e^{(t-s_1)\mathcal{L}} K e^{(s_1-s_2)\mathcal{L}} K e^{s_2\mathcal{L}} f_0 ds_2 ds_1,$$

and

$$\nabla_x h^{(2)} = \int_0^t \int_0^{s_1} \frac{(s_1-s_2) + s_2}{s_1} \nabla_x e^{(t-s_1)\mathcal{L}} K e^{(s_1-s_2)\mathcal{L}} K e^{s_2\mathcal{L}} f_0 ds_2 ds_1,$$

hence we have

$$\begin{aligned} \left\| \nabla_x h^{(2)} \right\|_{L^2(\mu)} &\lesssim \int_0^t \int_0^{s_1} s_1^{-1} \left[(s_1-s_2)^{-1/2} + s_2^{-1/2} \right] ds_2 ds_1 \|f_0\|_{L^2(\mu)} \\ &\lesssim t^{1/2} \|f_0\|_{L^2(\mu)}. \end{aligned}$$

The estimate of $h^{(3)}$ is analogous and hence we omit the details.

STEP 2. Second x -derivatives of $h^{(j)}$, $0 \leq j \leq 3$ in small time. We want to show that for any $0 < t \leq 1$,

$$\left\| \nabla_x^2 h^{(j)} \right\|_{L^2(\mu)} \leq C_j t^{-3+j} \|f_0\|_{L^2(\mu)}.$$

We only give the estimates for $h^{(0)}$ and $h^{(1)}$, and the others are similar. For any $0 < t_0 \leq 1$ and $t_0/2 < t \leq t_0$, we have

$$\nabla_x h^{(0)}(t, x, v) = e^{(t-t_0/2)\mathcal{L}} \left[\nabla_x h^{(0)}(t_0/2, x, v) \right].$$

By Lemma 2.5,

$$\left\| \nabla_x^2 h^{(0)} \right\|_{L^2(\mu)}(t) \lesssim (t-t_0/2)^{-3/2} (t_0/2)^{-3/2} \|f_0\|_{L^2(\mu)}. \tag{2.29}$$

Taking $t = t_0$ yields

$$\left\| \nabla_x^2 h^{(0)} \right\|_{L^2(\mu)}(t_0) \lesssim t_0^{-3} \|f_0\|_{L^2(\mu)}. \tag{2.30}$$

Since $t_0 \in (0, 1]$ is arbitrary, this completes the estimate for $h^{(0)}$. For $0 < t_1 \leq 1$ and $t_1/2 < t \leq t_1$, we have

$$\nabla_x h^{(1)}(t, x, v) = e^{(t-t_1/2)\mathcal{L}} \left[\nabla_x h^{(1)}(t_1/2, x, v) \right] + \int_{t_1/2}^t e^{(t-s)\mathcal{L}} \left[K \nabla_x h^{(0)}(s, x, v) \right] ds.$$

Using Lemma 2.5 and (2.30) gives

$$\left\| \nabla_x^2 h^{(1)} \right\|_{L^2(\mu)}(t) \lesssim (t-t_1/2)^{-3/2} (t_1/2)^{-1/2} \|f_0\|_{L^2(\mu)} + \int_{t_1/2}^t s^{-3} \|f_0\|_{L^2(\mu)} ds.$$

Taking $t = t_1$, we get

$$\left\| \nabla_x^2 h^{(1)} \right\|_{L^2(\mu)}(t_1) \lesssim t_1^{-2} \|f_0\|_{L^2(\mu)}.$$

Since $t_1 \in (0, 1]$ is arbitrary, this completes the estimate for $h^{(1)}$.

Next, we shall prove the large time behavior for $\gamma \geq 1$; the estimate for the case $0 < \gamma < 1$ can be obtained by employing the same argument.

STEP 3. First x -derivative of $h^{(j)}$, $0 \leq j \leq 3$, in large time for $\gamma \geq 1$. We want to show that for $t > 1$,

$$\|\nabla_x h^{(j)}\|_{L^2(\mu)} \lesssim t^j e^{-Ct} \|f_0\|_{L^2(\mu)}, \quad 0 \leq j \leq 3.$$

In view of Lemma 2.4, we have

$$\left\| \nabla_x h^{(0)} \right\|_{L^2(\mu)}(t) \leq e^{-C(t-1)} \left\| \nabla_x h^{(0)} \right\|_{L^2(\mu)}(1) \lesssim e^{-Ct} \|f_0\|_{L^2(\mu)}, \quad t > 1. \tag{2.31}$$

For $h^{(1)}$, we have

$$h^{(1)}(t, x, v) = e^{(t-1)\mathcal{L}} h^{(1)}(1, x, v) + \int_1^t e^{(t-s)\mathcal{L}} \left[Kh^{(0)}(s, x, v) \right] ds, \quad t > 1.$$

Using Lemma 2.4 and (2.31) gives

$$\begin{aligned} \left\| \nabla_x h^{(1)} \right\|_{L^2(\mu)}(t) &\leq e^{-C(t-1)} \left\| \nabla_x h^{(1)} \right\|_{L^2(\mu)}(1) + \int_1^t e^{-C(t-s)} \left\| \nabla_x h^{(0)} \right\|_{L^2(\mu)}(s) ds \\ &\lesssim t e^{-Ct} \|f_0\|_{L^2(\mu)}, \quad t > 1, \end{aligned}$$

and similarly for $\nabla_x h^{(2)}$ and $\nabla_x h^{(3)}$.

STEP 4. Second x -derivatives of $h^{(j)}$, $0 \leq j \leq 3$ in large time for $\gamma \geq 1$. We demonstrate that for $t > 1$,

$$\|\nabla_x^2 h^{(j)}\|_{L^2(\mu)} \lesssim t^j e^{-Ct} \|f_0\|_{L^2(\mu)},$$

whose proof is similar to Step 3. □

3. In the time-like region. In this section, we will see the large time behavior of solutions to equation (1.2). In what follows, we separate our discussion for the case $\gamma \geq 1$ and the case $0 < \gamma < 1$.

3.1. *The case $\gamma \geq 1$.* According to Lemma 2.7, together with the Sobolev inequality [1, Theorem 5.8]:

$$\|f\|_{L_x^\infty L_v^2} \lesssim \|f\|_{H_x^2 L^2}^{3/4} \|f\|_{L^2}^{1/4},$$

we immediately obtain the behavior of the wave part as follows.

PROPOSITION 3.1. Assume that $\gamma \geq 1$. Then for $0 \leq j \leq 3$ and $t > 0$, there exists $C > 0$ such that

$$|h^{(j)}|_{L_v^2} \lesssim e^{-Ct} t^{j-\frac{3}{4}} \|f_0\|_{L^2}.$$

Based on the wave-remainder decomposition, it remains to study the large time behavior of the remainder part. By the Fourier transform with respect to the x variable, the solution of the Fokker-Planck equation (1.2) can be represented as

$$f(t, x, v) = \int_{\mathbb{R}^3} e^{i\eta \cdot x + (-iv \cdot \eta + L)t} \widehat{f}_0(\eta, v) d\eta. \tag{3.1}$$

We can decompose the solution f into the long wave part f_L and the short wave part f_S given, respectively, by

$$\begin{aligned}
 f_L &= \int_{|\eta| < \delta} e^{i\eta \cdot x + (-iv \cdot \eta + L)t} \widehat{f}_0(\eta, v) d\eta, \\
 f_S &= \int_{|\eta| > \delta} e^{i\eta \cdot x + (-iv \cdot \eta + L)t} \widehat{f}_0(\eta, v) d\eta.
 \end{aligned}
 \tag{3.2}$$

The following short wave analysis relies on spectral analysis (Lemma 2.3).

PROPOSITION 3.2 (Short wave f_S). Assume that $\gamma \geq 1$ and $f_0 \in L^2$. Then

$$\|f_S\|_{L^2} \lesssim e^{-a(\tau)t} \|f_0\|_{L^2}.
 \tag{3.3}$$

In order to study the long wave part f_L for $\gamma \geq 1$, we need to further decompose the long wave part as the fluid part and the non-fluid part, i.e., $f_L = f_{L;0} + f_{L;\perp}$, where

$$\begin{aligned}
 f_{L;0} &= \int_{|\eta| < \delta} e^{\lambda(\eta)t} e^{i\eta \cdot x} \langle e_D(-\eta), \widehat{f}_0 \rangle_v e_D(\eta) d\eta, \\
 f_{L;\perp} &= \int_{|\eta| < \delta} e^{i\eta \cdot x} e^{(-iv \cdot \eta + L)t} \Pi_\eta^{D\perp} \widehat{f}_0 d\eta.
 \end{aligned}
 \tag{3.4}$$

Using Lemma 2.3, we obtain the exponential decay of the non-fluid long wave part.

PROPOSITION 3.3 (Non-fluid long wave $f_{L;\perp}$). Assume that $\gamma \geq 1$ and $f_0 \in L^2$. Then for $s > 0$,

$$\|f_{L;\perp}\|_{H_x^s L_v^2} \lesssim e^{-a(\tau)t} \|f_0\|_{L^2}.
 \tag{3.5}$$

For the fluid part, we have the following structure.

PROPOSITION 3.4 (Fluid long wave $f_{L;0}$). For $\gamma \geq 3/2$ and any given $M > 1$, there exists $C > 0$ such that for $|x| \leq Mt$,

$$|f_{L;0}(t, x, v)|_{L_v^2} \leq C \left[(1+t)^{-3/2} e^{-\frac{|x|^2}{C(t+1)}} + e^{-t/C} \right] \|f_0\|_{L_x^1 L_v^2}.
 \tag{3.6}$$

On the other hand, for $1 \leq \gamma < 3/2$ and any given positive integer N , there exists a positive constant C depending on N such that

$$|f_{L;0}(t, x, v)|_{L_v^2} \leq C \left[(1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t}\right)^{-N} + e^{-t/C} \right] \|f_0\|_{L_x^1 L_v^2}.
 \tag{3.7}$$

Proof. Before the proof of this proposition, we need the following two lemmas.

LEMMA 3.5 (Lemma 7.11 of [28]). Suppose that $g(t, \eta, v)$ is analytic in η for $|\eta| < \delta \ll 1$ and satisfies

$$|g(t, \eta, v)|_{L_v^2} \lesssim e^{-A|\eta|^2 t + O(|\eta|^4)}$$

for some constant $A > 0$. Then in the region of $|x| < (\mathfrak{M} + 1)t$, where \mathfrak{M} is any given positive constant, there exists a constant $C > 0$ such that the following inequality holds:

$$\left| \int_{|\eta| < \delta} e^{ix \cdot \eta} g(t, \eta, v) d\eta \right|_{L_v^2} \leq C \left[(1+t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{Ct}} + e^{-t/C} \right].$$

LEMMA 3.6 (Lemma 2.2 of [25]). Let $x, \eta, v \in \mathbb{R}^3$. Suppose $g(t, \eta, v)$ is smooth and has compact support in the variable η , and there exists a constant $b > 0$ such that $g(t, \eta, v)$ satisfies

$$|D_\eta^\beta(g(t, \eta, v))|_{L_v^2} \leq C_\beta(1+t)^{|\beta|/2} e^{-b|\eta|^2 t}$$

for any multi-indexes β with $|\beta| \leq 2N$. Then there exists positive constants C_N such that

$$\left| \int_{\mathbb{R}^3} e^{ix \cdot \eta} g(t, \eta, v) d\eta \right|_{L_v^2} \leq C_N \left[(1+t)^{-3/2} B_N(|x|, t) \right],$$

where N is any fixed integer and

$$B_N(|x|, t) = \left(1 + \frac{|x|^2}{1+t} \right)^{-N}.$$

We now return to the proof of this proposition. Notice that

$$f_{L;0}(t, x, v) = \int_{|\eta| < \delta} e^{i\eta \cdot x} e^{\lambda(\eta)t} \langle e_D(-\eta), \hat{f}_0 \rangle_v e_D(\eta) d\eta.$$

Let

$$g(t, \eta, v) = e^{\lambda(\eta)t} \langle e_D(-\eta), \hat{f}_0 \rangle_v e_D(\eta) \cdot \mathbf{1}_{\{|\eta| < \delta\}},$$

where $\mathbf{1}_D$ is the characteristic function of the domain D . When $\gamma \geq 3/2$, the eigenvalue $\lambda(\eta)$ and eigenvector $e_D(\eta)$ are analytic in η . Owing to the asymptotic expansion of $\lambda(\eta)$ in (2.6), we have

$$|g(t, \eta, v)|_{L_v^2} \leq e^{-a_\gamma |\eta|^2 t + O(|\eta|^4)t} \|f_0\|_{L_x^1 L_v^2}.$$

From Lemma 3.5 it follows

$$|f_{L;0}(t, x, v)|_{L_v^2} \lesssim \left[(1+t)^{-3/2} e^{-\frac{|x|^2}{C(t+1)}} + e^{-t/C} \right] \|f_0\|_{L_x^1 L_v^2}.$$

As for $1 \leq \gamma < 3/2$, the eigenvalue and eigenvector are only smooth in η . In this case, one can see that

$$|D_\eta^\beta g(t, \eta, v)|_{L_v^2} \lesssim \left(1 + t^{|\beta|/2} \right) \left(1 + |\eta|^2 t \right)^{|\beta|/2} e^{-a_\gamma |\eta|^2 t/2} \|f_0\|_{L_x^1 L_v^2},$$

since f_0 has compact support in the x variable. Note that the polynomial growth $\left(1 + |\eta|^2 t \right)^{|\beta|/2}$ can be absorbed by the exponential decay, hence we can conclude that

$$|f_{L;0}(t, x, v)|_{L_v^2} \lesssim \left[(1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-N} \right] \|f_0\|_{L_x^1 L_v^2},$$

in accordance with Lemma 3.6. □

We define the fluid part as $f_F = f_{L,0}$ and the non-fluid part as $f_* = f - f_F = f_{L;\perp} + f_S$. By the fluid–non-fluid decomposition and the wave-remainder decomposition, we have

$$f = f_F + f_* = W^{(3)} + \mathcal{R}^{(3)}.$$

We can define the tail part as $f_R = \mathcal{R}^{(3)} - f_F$ and so f can be written as $f = W^{(3)} + f_F + f_R$.

It follows from Propositions 3.1 and 3.4 that the estimates of wave part $W^{(3)}$ and the fluid part f_F inside the time-like region is completed. Hence, it remains to study the tail

part f_R . From Lemma 2.7 and the fact that \mathbb{G}^t, K are bounded operators on L^2 , $\mathcal{R}^{(3)}$ has the following estimate:

$$\|\mathcal{R}^{(3)}(t)\|_{H_x^2 L_v^2} \leq \int_0^t \|h^{(3)}(s)\|_{H_x^2 L_v^2} ds \lesssim \|f_0\|_{L^2}. \tag{3.8}$$

In view of (3.3), (3.5), (3.8), and Lemma 2.6, there exists $C > 0$ such that

$$\|f_R\|_{L^2} = \|f_* - W^{(3)}\|_{L^2} \lesssim e^{-Ct} \|f_0\|_{L^2}$$

and

$$\|f_R\|_{H_x^2 L_v^2} = \|\mathcal{R}^{(3)} - f_F\|_{H_x^2 L_v^2} \lesssim \|f_0\|_{L^2}.$$

The Sobolev inequality [1, Theorem 5.8] implies

$$\|f_R\|_{L_v^2} \leq \|f_R\|_{L_x^\infty L_v^2} \lesssim \|f_R\|_{H_x^2 L_v^2}^{3/4} \|f_R\|_{L^2}^{1/4} \lesssim e^{-Ct} \|f_0\|_{L^2} \tag{3.9}$$

for some constant $C > 0$. In conclusion, we have that for the time-like region, if $\gamma \geq 3/2$, there exists a constant $C > 0$ such that

$$\left| \mathcal{R}^{(3)} \right|_{L_v^2} \lesssim \left[(1+t)^{-3/2} e^{-C\frac{|x|^2}{t+1}} + e^{-Ct} \right] \|f_0\|_{L_x^\infty L_v^2}; \tag{3.10}$$

and if $1 \leq \gamma < 3/2$, any given $N > 0$, there exists a constant $C > 0$ such that

$$\left| \mathcal{R}^{(3)} \right|_{L_v^2} \lesssim \left[(1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-N} + e^{-Ct} \right] \|f_0\|_{L_x^\infty L_v^2}. \tag{3.11}$$

Combining Proposition 3.1, (3.9), (3.10), and (3.11), we obtain the pointwise estimate for the solution in the time-like region.

THEOREM 3.7 (Time-like region for $\gamma \geq 1$). Let $\gamma \geq 1$ and let f be the solution to equation (1.2). Assume that the initial condition f_0 has compact support in the x variable and is bounded in L_v^2 . Then for any given $M > 1$ and $|x| \leq Mt$,

(i) as $\gamma \geq 3/2$, there exists a positive constant C such that

$$|f|_{L_v^2} \lesssim \left[(1+t)^{-3/2} e^{-C\frac{|x|^2}{t+1}} + (1+t^{-9/4}) e^{-Ct} \right] \|f_0\|_{L_x^\infty L_v^2}; \tag{3.12}$$

(ii) as $1 \leq \gamma < 3/2$, any given $N > 0$, there exists a constant $C > 0$ such that

$$|f|_{L_v^2} \lesssim \left[(1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-N} + (1+t^{-9/4}) e^{-Ct} \right] \|f_0\|_{L_x^\infty L_v^2}. \tag{3.13}$$

3.2. The case $0 < \gamma < 1$. First, we introduce the L^2 estimate and the pointwise estimate of the wave part.

PROPOSITION 3.8. Assume that $0 < \gamma < 1$. Then for $0 \leq j \leq 3$, and $t > 0$, there exists $c_\gamma > 0$ such that

$$\left\| h^{(j)} \right\|_{L^2} \lesssim t^j e^{-c_\gamma \alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{(j+1)\alpha(v)\gamma})} \tag{3.14}$$

and

$$\left| h^{(j)}(t, x, v) \right|_{L_v^2} \lesssim t^{j-\frac{9}{4}} e^{-\frac{1}{4}c_\gamma \alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{(j+1)\alpha(v)\gamma})}. \tag{3.15}$$

Proof. We first consider the L^2 estimate for $h^{(0)}$. It is easy to see that (2.19) is still valid if setting $\mu(t, x, v) = e^{\alpha(v)^\gamma}$, namely,

$$\frac{d}{dt} \|h^{(0)}\|_{L^2}^2 + C \|h^{(0)}\|_{L^2_\sigma}^2 \leq 0 \tag{3.16}$$

and

$$\frac{d}{dt} \|e^{\alpha(v)^\gamma} h^{(0)}\|_{L^2}^2 + C \|e^{\alpha(v)^\gamma} h^{(0)}\|_{L^2_\sigma}^2 \leq 0. \tag{3.17}$$

Hence, $\|h^{(0)}\|_{L^2} \leq \|f_0\|_{L^2}$ for $t \geq 0$ and it suffices to show that for $t \geq 1$,

$$\|h^{(0)}\|_{L^2} \lesssim e^{-c_\gamma \alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{\alpha(v)^\gamma})}.$$

As in the work of Caffisch [4], we consider a time-dependent low velocity part

$$E = \{\langle v \rangle \leq \beta t^{p'}\},$$

and its complementary high velocity part $E^c = \{\langle v \rangle > \beta t^{p'}\}$, where $p' > 0$ and $\beta > 0$ will be determined later. Following the argument as in Section 5 of [33], together with (3.16) and (3.17), we obtain

$$\|h^{(0)}\|_{L^2} \lesssim e^{-c_\gamma \alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{\alpha(v)^\gamma})},$$

for some constant $c_\gamma > 0$, after choosing $p' = \frac{1}{2-\gamma}$ in the Fokker-Planck case and $\beta > 0$ sufficiently large. This completes the L^2 estimate for $h^{(0)}$.

Through the Duhamel Principle, we immediately obtain (3.14) for $1 \leq j \leq 3$. Furthermore, the Sobolev inequality, together with Lemma 2.7 and (3.14), implies the desired pointwise estimate for the wave part $h^{(j)}$, $0 \leq j \leq 3$. \square

Next, we are concerned with the pointwise behavior of the remainder part in the time-like region. By virtue of the lack of the spectral analysis for $0 < \gamma < 1$, we will instead use the method of the weighted L^2 estimate in the Fourier transformed variable and the interpolation argument to deal with the time decay of the solution f to equation (1.2) in this case. The main idea is to construct the desired weighted time-frequency Lyapunov functional to capture the total energy dissipation rate. In the course of the proof we have to take great care to estimate the microscopic and macroscopic parts for $|\eta| \leq 1$ and $|\eta| > 1$, respectively. Consider (1.2); taking the Fourier transform with respect to the x variable leads to

$$\partial_t \widehat{f} + iv \cdot \eta \widehat{f} = L \widehat{f}. \tag{3.18}$$

We first calculate the L^2 estimate.

PROPOSITION 3.9 (L^2 estimate). Let f be the solution to equation (1.2). Then there exists a time-frequency functional $\mathcal{E}(t, \eta)$ such that

$$\mathcal{E}(t, \eta) \approx \left| \widehat{f}(t, \eta, v) \right|_{L^2_v}^2, \tag{3.19}$$

where for any $t > 0$ and $\eta \in \mathbb{R}^3$, we have

$$\partial_t \mathcal{E}(t, \eta) + \sigma \widehat{\rho}(\eta) \left| \widehat{f}(t, \eta, v) \right|_{L^2_{\gamma-1}}^2 \leq 0. \tag{3.20}$$

Here we use the notation $\widehat{\rho}(\eta) := \min\{1, |\eta|^2\}$.

Proof. We multiply equation (3.18) by $\overline{\widehat{f}(t, \eta, v)}$ and integrate over v to obtain

$$\frac{1}{2} \frac{d}{dt} \left| \widehat{f}(t, \eta, v) \right|_{L^2_v}^2 - \operatorname{Re} \langle Lf, \widehat{f} \rangle = 0.$$

From the coercivity in Lemma 2.1, it follows that

$$\frac{1}{2} \frac{d}{dt} \left| \widehat{f}(t, \eta, v) \right|_{L^2_v}^2 + \nu_0 \left| P_1 \widehat{f} \right|_{L^2_v}^2 \leq 0. \tag{3.21}$$

Now, we need the estimate of $P_0 \widehat{f}$. In what follows, we will apply Strain’s argument to estimate the macroscopic dissipation, in the spirit of Kawashima’s work on dissipation of the hyperbolic-parabolic system. Let $a = \langle \mathcal{M}^{1/2}, f \rangle_v$ and $b = (b_1, b_2, b_3)$ with $b_i = \langle v_i \mathcal{M}^{1/2}, f \rangle_v = \langle v_i \mathcal{M}^{1/2}, P_1 f \rangle_v$. Then $P_0 f = a \mathcal{M}^{1/2}$ and from (1.2), a and b satisfy the fluid-type system

$$\begin{cases} \partial_t a + \nabla_x \cdot b = 0, \\ \partial_t b + \alpha \nabla_x a + \nabla_x \cdot \Gamma(P_1 f) = - \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 f dv, \end{cases} \tag{3.22}$$

where

$$\alpha = \frac{1}{3} \int |v|^2 \mathcal{M} dv > 0,$$

and $\Gamma = (\Gamma_{ij})_{3 \times 3}$ is the moment function defined by

$$\Gamma_{ij}(g) = \left\langle (v_i v_j - 1) \mathcal{M}^{1/2}, g \right\rangle_v, \quad 1 \leq i, j \leq 3.$$

Note by the definition of P_0 that $\Gamma(P_1 f) = \int (v \otimes v) \mathcal{M}^{1/2} P_1 f dv$. Taking the Fourier transform with respect to x of (3.22), we have

$$\begin{aligned} |\eta|^2 |\widehat{a}|^2 &= (i\eta \widehat{a}, i\eta \widehat{a}) = \frac{1}{\alpha} \left(i\eta \widehat{a}, -\partial_t \widehat{b} - i\Gamma(P_1 \widehat{f}) \eta - \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 \widehat{f} dv \right) \\ &= \frac{1}{\alpha} \left[- (i\eta \widehat{a}, \widehat{b})_t + |\eta \cdot \widehat{b}|^2 - (i\eta \widehat{a}, i\Gamma(P_1 \widehat{f}) \eta) - \left(i\eta \widehat{a}, \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 \widehat{f} dv \right) \right]. \end{aligned}$$

Invoking on the rapid decay of $\mathcal{M}^{1/2}$ and using the Cauchy-Schwartz inequality, we have

$$\left| \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 \widehat{f} dv \right|^2 \leq \left| \mathcal{M}^{1/2} v \langle v \rangle^{-1} \right|_{L^2_v}^2 \left| \langle v \rangle^{\gamma-1} P_1 \widehat{f} \right|_{L^2_v}^2 \leq 3\alpha \left| P_1 \widehat{f} \right|_{L^2_{\gamma-1}}^2$$

and

$$\left| (i\eta \widehat{a}, i\Gamma(P_1 \widehat{f}) \eta) \right| \leq \epsilon |\eta|^2 |\widehat{a}|^2 + C_\epsilon |\eta|^2 \left| P_1 \widehat{f} \right|_{L^2_{\gamma-1}}^2$$

for any small $\epsilon > 0$. Therefore, we can conclude

$$\partial_t \operatorname{Re} \frac{(i\eta \widehat{a}, \widehat{b})}{1 + |\eta|^2} + \frac{\sigma |\eta|^2}{1 + |\eta|^2} |\widehat{a}|^2 \leq C \left| P_1 \widehat{f} \right|_{L^2_{\gamma-1}}^2 \tag{3.23}$$

for some $\sigma > 0$. Now, we define

$$\mathcal{E}(t, \eta) = \left| \widehat{f}(t, \eta, v) \right|_{L_v^2}^2 + \kappa_3 \operatorname{Re} \frac{(i\eta \widehat{a}, \widehat{b})}{1 + |\eta|^2} \tag{3.24}$$

for a constant $\kappa_3 > 0$ to be determined later. One can fix $\kappa_3 > 0$ small enough such that $\mathcal{E}(t, \eta) \approx \left| \widehat{f}(t, \eta, v) \right|_{L_v^2}^2$. Furthermore, according to Lemma 2.1 and (3.23), we choose $\kappa_3 > 0$ sufficiently small such that

$$\partial_t \mathcal{E}(t, \eta) + \sigma \left| P_1 \widehat{f} \right|_{L_{\gamma-1}^2}^2 + \frac{2\sigma |\eta|^2}{1 + |\eta|^2} |\widehat{a}|^2 \leq 0 \tag{3.25}$$

for some $\sigma > 0$. In conclusion, we now have

$$\partial_t \mathcal{E}(t, \eta) + \sigma \widehat{\rho}(\eta) \left| \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 \leq 0.$$

Here we use the notation $\widehat{\rho}(\eta) := \min\{1, |\eta|^2\}$. □

Since $\gamma - 1 < 0$, it is insufficient to gain the time decay of the total energy of the solution f . Therefore, in order to capture the total energy dissipation rate, we need to make further energy estimates on the microscopic part $P_1 f$ and the macroscopic part $P_0 f$.

PROPOSITION 3.10. Let f be the solution to equation (1.2). Then there exists a weighted time-frequency functional $\widetilde{\mathcal{E}}(t, \eta)$ such that

$$\widetilde{\mathcal{E}}(t, \eta) \approx \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L_v^2}^2, \tag{3.26}$$

where $0 < \alpha\gamma < 1/20$ and for any $t > 0$ and $\eta \in \mathbb{R}^3$ we have

$$\partial_t \widetilde{\mathcal{E}}(t, \eta) + \sigma \widehat{\rho}(\eta) \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 \leq 0. \tag{3.27}$$

Here we use the notation $\widehat{\rho}(\eta) := \min\{1, |\eta|^2\}$.

Proof. First, we shall prove the following Lyapunov inequality with a velocity weight $e^{\alpha\langle v \rangle^\gamma}$, $0 < \alpha\gamma < 1/20$:

$$\frac{d}{dt} \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L_v^2}^2 + \sigma \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 \leq C_\sigma |\eta|^2 \left| \widehat{f} \right|_{L_{\gamma-1}^2}^2 + C_\gamma \left| P_1 \widehat{f} \right|_{L_v^2(B_{2R})}^2, \tag{3.28}$$

where the constants $C_\gamma > 0$ and $R > 0$ are dependent only upon γ . We split the solution f into two parts: $f = P_0 f + P_1 f$, and then apply P_1 to equation (3.18):

$$\partial_t P_1 \widehat{f} + iv \cdot \eta P_1 \widehat{f} - LP_1 \widehat{f} = -P_1 (iv \cdot \eta P_0 \widehat{f}) + P_0 (iv \cdot \eta P_1 \widehat{f}).$$

Multiply the above equation by $e^{\alpha\langle v \rangle^\gamma} \overline{P_1 \widehat{f}}$ and integrate with respect to v to obtain

$$\frac{1}{2} \frac{d}{dt} \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L_v^2}^2 - \operatorname{Re} \left\langle e^{\alpha\langle v \rangle^\gamma} LP_1 \widehat{f}, P_1 \widehat{f} \right\rangle_v = \Gamma,$$

where

$$\Gamma = -\operatorname{Re} \left\langle P_1 (iv \cdot \eta P_0 \widehat{f}), e^{\alpha\langle v \rangle^\gamma} P_1 \widehat{f} \right\rangle + \operatorname{Re} \left\langle P_0 (iv \cdot \eta P_1 \widehat{f}), e^{\alpha\langle v \rangle^\gamma} P_1 \widehat{f} \right\rangle.$$

Owing to the rapid decay of $\mathcal{M}^{1/2}$, we obtain

$$|\Gamma| \leq \epsilon \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L^2_{\gamma-1}}^2 + C_\epsilon |\eta|^2 \left(\left| P_1 \widehat{f}(t, \eta, v) \right|_{L^2_{\gamma-1}}^2 + \left| P_0 \widehat{f} \right|_{L^2_v}^2 \right),$$

which holds for any small $\epsilon > 0$. On the other hand, we rewrite $L = -\Lambda + K$, $K = \varpi \chi_R(|v|)$, where $R > 0$ and $\varpi > 0$ are chosen sufficiently large such that

$$\frac{|v|^2 \langle v \rangle^{2\gamma-4}}{4} - \frac{3}{2} \langle v \rangle^{\gamma-2} - \frac{(\gamma-2)}{2} |v|^2 \langle v \rangle^{\gamma-4} + \varpi \chi_R(|v|) \geq \frac{1}{5} \langle v \rangle^{2\gamma-2}.$$

Hence, we have

$$\begin{aligned} & -\operatorname{Re} \left\langle e^{\alpha\langle v \rangle^\gamma} LP_1 \widehat{f}, P_1 \widehat{f} \right\rangle_v \\ &= \operatorname{Re} \int e^{\alpha\langle v \rangle^\gamma} [(\Lambda - K) P_1 \widehat{f}] P_1 \overline{\widehat{f}} dv \\ &\geq \int e^{\alpha\langle v \rangle^\gamma} \left| \nabla_v P_1 \widehat{f} \right|^2 + \operatorname{Re} \int \alpha \gamma \langle v \rangle^{\gamma-2} e^{\alpha\langle v \rangle^\gamma} (v \cdot \nabla_v P_1 \widehat{f}) P_1 \overline{\widehat{f}} dv \\ &\quad + \frac{1}{5} \int \langle v \rangle^{2\gamma-2} e^{\alpha\langle v \rangle^\gamma} \left| P_1 \widehat{f} \right|^2 dv - C' \left| P_1 \widehat{f} \right|_{L^2_v(B_{2R})}^2, \end{aligned}$$

where $C' = C'(\alpha, \gamma, R)$. Note that $\alpha \gamma < 1/20$. The Cauchy-Schwartz inequality implies

$$\begin{aligned} & \left| \operatorname{Re} \int \alpha \gamma \langle v \rangle^{\gamma-2} e^{\alpha\langle v \rangle^\gamma} (v \cdot \nabla_v P_1 \widehat{f}) P_1 \overline{\widehat{f}} dv \right| \\ &\leq \int \alpha \gamma \langle v \rangle^{\gamma-1} e^{\alpha\langle v \rangle^\gamma} \left| \nabla_v P_1 \widehat{f} \right| \left| P_1 \widehat{f} \right| dv \\ &\leq \int e^{\alpha\langle v \rangle^\gamma} \left| \nabla_v P_1 \widehat{f} \right|^2 dv + \frac{1}{80} \int \langle v \rangle^{2\gamma-2} e^{\alpha\langle v \rangle^\gamma} \left| P_1 \widehat{f} \right|^2 dv, \end{aligned}$$

so we deduce

$$-\operatorname{Re} \left\langle \langle v \rangle^{2\ell} LP_1 \widehat{f}, P_1 \widehat{f} \right\rangle_v \geq \frac{1}{6} \int \langle v \rangle^{2\gamma-2} e^{\alpha\langle v \rangle^\gamma} \left| P_1 \widehat{f} \right|^2 dv - \widetilde{C}(R, \gamma, \alpha) \left| P_1 \widehat{f} \right|_{L^2_v(B_{2R})}^2,$$

where $\widetilde{C}(R, \ell, \alpha) > 0$. Consequently,

$$\frac{d}{dt} \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L^2_v}^2 + \sigma \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L^2_{\gamma-1}}^2 \leq C_\sigma |\eta|^2 \left| \widehat{f} \right|_{L^2_{\gamma-1}}^2 + C_\gamma \left| P_1 \widehat{f} \right|_{L^2_v(B_{2R})}^2$$

for some constant $\sigma > 0$.

In addition, if we multiply (3.18) with $e^{\alpha\langle v \rangle^\gamma} \overline{\widehat{f}(t, \eta, v)}$, integrate in v , and use the same procedure as above, we also obtain

$$\frac{1}{2} \frac{d}{dt} \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L^2_v}^2 + \sigma \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L^2_{\gamma-1}}^2 \leq C_\gamma \left| \widehat{f} \right|_{L^2(B_{2R})}^2. \tag{3.29}$$

To do the weighted estimate, we introduce a new energy as follows:

$$\widetilde{\mathcal{E}}(t, \eta) := \widetilde{\mathcal{E}}^0(t, \eta) + \widetilde{\mathcal{E}}^1(t, \eta)$$

with

$$\begin{aligned} \tilde{\mathcal{E}}^0(t, \eta) &= 1_{|\eta| \leq 1} \left(\mathcal{E}(t, \eta) + \kappa_4 \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_1 \widehat{f}(t, \eta, v) \right|_{L_v^2}^2 \right), \\ \tilde{\mathcal{E}}^1(t, \eta) &= 1_{|\eta| > 1} \left(\mathcal{E}(t, \eta) + \kappa_5 \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L_v^2}^2 \right), \end{aligned}$$

where $\mathcal{E}(t, \eta)$ is defined as in (3.24) and the constants $\kappa_4, \kappa_5 > 0$ will be chosen small enough. Notice further that $|\widehat{a}|^2 = \left| P_0 \widehat{f} \right|_{L_v^2}^2 \gtrsim \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_0 \widehat{f} \right|_{L_v^2}^2$ for all $0 < \alpha\gamma < 1/20$, and so $\tilde{\mathcal{E}}(t, \eta) \approx \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f} \right|_{L_v^2}^2$.

For $\tilde{\mathcal{E}}^1(t, \eta)$, we combine (3.25) and (3.29) for $|\eta| > 1$ to obtain

$$\partial_t \tilde{\mathcal{E}}^1(t, \eta) + \sigma \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 1_{|\eta| > 1} \leq 0 \tag{3.30}$$

for $\kappa_5 > 0$ small enough, since $|\eta|^2 / (1 + |\eta|^2) \geq \frac{1}{2}$.

For $\tilde{\mathcal{E}}^0(t, \eta)$, since $|\eta|^2 / (1 + |\eta|^2) \geq \frac{|\eta|^2}{2}$ for $|\eta| \leq 1$ and $|\widehat{a}|^2 \gtrsim \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} P_0 \widehat{f} \right|_{L_v^2}^2$ for all $\alpha\gamma < 1/20$, combining (3.25) and (3.28) for $|\eta| \leq 1$ gives

$$\partial_t \tilde{\mathcal{E}}^0(t, \eta) + \sigma |\eta|^2 \left| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 1_{|\eta| \leq 1} \leq 0 \tag{3.31}$$

for $\kappa_4 > 0$ small enough. This completes the proof. □

Now, it is enough to prove the estimate in the time-like region. We apply the Hölder inequality to obtain that for $j \geq 1$,

$$\begin{aligned} \mathcal{E}(t, \eta) &\lesssim \left| \widehat{f}(t, \eta, v) \right|_{L_v^2}^2 = \int \left(\left| \widehat{f}(t, \eta, v) \right|^2 e^{\alpha\langle v \rangle^\gamma} \right)^{\frac{1}{j+1}} \left(\left| \widehat{f}(t, \eta, v) \right|^2 e^{-\frac{\alpha}{j}\langle v \rangle^\gamma} \right)^{\frac{j}{j+1}} dv \\ &\leq \left(\int e^{-\frac{\alpha}{j}\langle v \rangle^\gamma} \left| \widehat{f}(t, \eta, v) \right|^2 dv \right)^{j/(j+1)} \left(\int e^{\alpha\langle v \rangle^\gamma} \left| \widehat{f}(t, \eta, v) \right|^2 dv \right)^{1/(j+1)} \\ &\lesssim \left| \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^{2j/(j+1)} \tilde{\mathcal{E}}^{1/(j+1)}(t, \eta). \end{aligned}$$

Thus we conclude that

$$\mathcal{E}^{(j+1)/j}(t, \eta) \lesssim \left| \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 \tilde{\mathcal{E}}^{1/j}(t, \eta) \lesssim \left| \widehat{f}(t, \eta, v) \right|_{L_{\gamma-1}^2}^2 \tilde{\mathcal{E}}^{1/j}(0, \eta).$$

Now we can rewrite (3.25), for any $\eta \in \mathbb{R}^3$, as

$$\partial_t \mathcal{E}(t, \eta) + \sigma \widehat{\rho}(\eta) \mathcal{E}^{(j+1)/j}(t, \eta) \tilde{\mathcal{E}}^{-1/j}(0, \eta) \leq 0.$$

Integrating this over time, we obtain

$$j\mathcal{E}^{-1/j}(0, \eta) - j\mathcal{E}^{-1/j}(t, \eta) \lesssim -t\widehat{\rho}(\eta) \tilde{\mathcal{E}}^{-1/j}(0, \eta).$$

As a consequence, for any $j \geq 1$, uniformly in $\eta \in \mathbb{R}^3$, we get

$$\mathcal{E}(t, \eta) \lesssim \tilde{\mathcal{E}}(0, \eta) \left(\frac{t\widehat{\rho}(\eta)}{j} + 1 \right)^{-j}.$$

Recall that the long wave part f_L and the short wave part f_S of the solution f are given, respectively, by

$$f_L = \int_{|\eta| \leq 1} e^{i\eta \cdot x + (-iv \cdot \eta + L)t} \widehat{f}_0(\eta, v) \, d\eta,$$

$$f_S = \int_{|\eta| > 1} e^{i\eta \cdot x + (-iv \cdot \eta + L)t} \widehat{f}_0(\eta, v) \, d\eta.$$

When $|\eta| \leq 1$ and $k \in \mathbb{N} \cup \{0\}$, since

$$\int_{|\eta| \leq 1} |\eta|^{2k} \left(\frac{t|\eta|^2}{j} + 1 \right)^{-j} \, d\eta \lesssim (1+t)^{-\frac{3}{2}-k} \text{ if } j > \frac{3}{2} + k,$$

we obtain

$$\begin{aligned} \int_{|\eta| \leq 1} |\eta|^{2k} \mathcal{E}(t, \eta) \, d\eta &\lesssim \int_{|\eta| \leq 1} |\eta|^{2k} \left(\frac{t|\eta|^2}{j} + 1 \right)^{-j} \widetilde{\mathcal{E}}(0, \eta) \, d\eta \\ &\lesssim (1+t)^{-\frac{3}{2}-k} \left\| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} f_0 \right\|_{L_x^1 L_v^2}^2, \end{aligned} \tag{3.32}$$

which implies that

$$\left\| \nabla_x^k f_L \right\|_{L^2} \lesssim (1+t)^{-\frac{3}{4}-\frac{k}{2}} \left\| e^{\frac{\alpha}{2}\langle v \rangle^\gamma} f_0 \right\|_{L_x^1 L_v^2}.$$

By the Sobolev inequality ([35]), we get

$$\|f_L\|_{L_x^\infty L_v^2} \lesssim \|\nabla_x^2 f_L\|_{L^2}^{1/2} \|\nabla_x f_L\|_{L^2}^{1/2} \lesssim (1+t)^{-\frac{3}{2}} \|f_0\|_{L_x^1 L_v^2(e^{\alpha\langle v \rangle^\gamma})}.$$

When $|\eta| > 1$, we note that the equations (3.20) and (3.27) for f_S are similar to (3.16) and (3.17) for $h^{(0)}$. Then following a similar procedure of the proof, it implies

$$\|f_S\|_{L^2} \lesssim e^{-c_\gamma \alpha^{\frac{2(1-\gamma)}{2-\gamma}} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{\alpha\langle v \rangle^\gamma})}, \quad t \geq 0,$$

for some constant $c_\gamma > 0$.

To sum up, we have the following proposition.

PROPOSITION 3.11. Let $0 < \gamma < 1$ and let f be the solution of equation (1.2). For any $\alpha > 0$ small with $\alpha\gamma < 1/20$, we have:

(i) (Long wave f_L).

$$\|f_L\|_{L_x^\infty L_v^2} \lesssim (1+t)^{-\frac{3}{2}} \|f_0\|_{L_x^1 L_v^2(e^{\alpha\langle v \rangle^\gamma})}. \tag{3.33}$$

(ii) (Short wave f_S). There exists $c_\gamma > 0$ such that

$$\|f_S\|_{L^2} \lesssim e^{-c_\gamma \alpha^{\frac{2(1-\gamma)}{2-\gamma}} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{\alpha\langle v \rangle^\gamma})}. \tag{3.34}$$

Based on the long wave-short wave decomposition and wave-remainder decomposition, i.e.,

$$f = f_L + f_S = W^{(3)} + \mathcal{R}^{(3)},$$

we now define the tail part as $f_R = \mathcal{R}^{(3)} - f_L = f_S - W^{(3)}$, which leads to that f can be written as $f = W^{(3)} + f_L + f_R$. From Lemma 2.7,

$$\left\| \mathcal{R}^{(3)}(t) \right\|_{H_x^2 L_v^2} \lesssim \int_0^t \left\| h^{(3)}(s) \right\|_{H_x^2 L_v^2} ds \lesssim (1 + t^4) \|f_0\|_{L^2}, \tag{3.35}$$

and so

$$\|f_R\|_{H_x^2 L_v^2} = \left\| \mathcal{R}^{(3)} - f_L \right\|_{H_x^2 L_v^2} \lesssim (1 + t^4) \|f_0\|_{L^2}, \quad t > 0.$$

In view of Propositions 3.8 and 3.11,

$$\|f_R\|_{L^2} = \left\| f_S - W^{(3)} \right\|_{L^2} \lesssim e^{-\frac{c\gamma}{2}\alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{4\alpha\langle v \rangle^\gamma})}, \quad t > 0.$$

The Sobolev inequality implies

$$|f_R|_{L_v^2} \leq \|f_R\|_{L_x^\infty L_v^2} \lesssim \|f_R\|_{H_x^2 L_v^2}^{3/4} \|f_R\|_{L^2}^{1/4} \lesssim e^{-\frac{c\gamma}{16}\alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \|f_0\|_{L^2(e^{4\alpha\langle v \rangle^\gamma})}, \quad t > 0. \tag{3.36}$$

Combining (3.15), (3.33), and (3.36), we obtain the pointwise estimate for the solution in the time-like region.

THEOREM 3.12 (Time-like region for $0 < \gamma < 1$). Let $0 < \gamma < 1$ and let f be the solution to equation (1.2). Assume that the initial condition f_0 has compact support in the x variable and is bounded in $L_v^2(e^{4\alpha\langle v \rangle^\gamma})$. Then for $\alpha > 0$ is small enough, there exists a positive constant c_γ such that

$$|f|_{L_v^2} \lesssim \left[(1 + t)^{-3/2} + (1 + t^{-9/4}) e^{-c_\gamma \alpha \frac{2(1-\gamma)}{2-\gamma} t^{\frac{\gamma}{2-\gamma}}} \right] \|f_0\|_{L_x^\infty L_v^2(e^{4\alpha\langle v \rangle^\gamma})}. \tag{3.37}$$

4. In the space-like region. We have finished the estimate of solution inside the time-like region. To have the global picture of the space-time structure of solution, we still need to investigate the solution in the space-like region. To this end, we shall estimate the wave part $W^{(3)}$ and the remainder part $\mathcal{R}^{(3)}$ separately. Here, the weighted energy estimate plays a decisive role.

4.1. *The case $\gamma \geq 3/2$: Exponential decay.*

PROPOSITION 4.1. Consider the weight functions

$$w(x, t) = e^{\frac{\langle x \rangle - Mt}{2D}}, \quad \mu(x) = e^{\frac{\langle x \rangle}{D}},$$

where D and M are chosen sufficiently large. Then for $0 \leq j \leq 3$, we have

$$\|wh^{(j)}\|_{H_x^2 L_v^2} \lesssim t^{-3+j} \|f_0\|_{L^2(\mu)}, \quad 0 < t \leq 1, \tag{4.1}$$

$$\|wh^{(j)}\|_{H_x^2 L_v^2} \lesssim e^{-Ct} \|f_0\|_{L^2(\mu)}, \quad t > 1, \tag{4.2}$$

and

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \lesssim \|f_0\|_{L^2(\mu)}, \quad t > 0. \tag{4.3}$$

Proof. In view of that $w(x, t)$ is non-increasing in t , it is not hard to verify that

$$\|wg(t)\|_{H_x^2 L_v^2} \lesssim \|g(t)\|_{H_x^2 L_v^2(\mu)}.$$

Then the weighted energy inequalities (4.1) and (4.2) follow from Lemma 2.7 directly.

It remains to show the weighted energy estimates for the remainder part $\mathcal{R}^{(3)}$, $t > 0$. We shall demonstrate that

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_t^2} \lesssim (1+t)\|f_0\|_{L^2(\mu)}, \quad t > 0.$$

To see this, let $u = w\mathcal{R}^{(3)}$ and then $\partial_x^\beta u$, where β is a multi-index, solves the equation

$$\begin{aligned} \partial_t (\partial_x^\beta u) &= -v \cdot \nabla_x (\partial_x^\beta u) - \frac{1}{2D} \left(M - \frac{x \cdot v}{\langle x \rangle} \right) \partial_x^\beta u + L \partial_x^\beta u + K \partial_x^\beta (wh^{(3)}) \\ &+ \frac{1}{2D} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} \binom{\beta}{\beta_1 \beta_2} \partial_x^{\beta_1} \left(\frac{x}{\langle x \rangle} \right) \cdot v \partial_x^{\beta_2} u. \end{aligned}$$

The energy estimate gives

$$\begin{aligned} \frac{1}{2} \partial_t \|\partial_x^\beta u\|_{L^2}^2 &= -\frac{1}{2D} \int \left(M - \frac{x \cdot v}{\langle x \rangle} \right) |\partial_x^\beta u|^2 dx dv + \int (L \partial_x^\beta u) \partial_x^\beta u dx dv \\ &+ \frac{1}{2D} \int \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} \binom{\beta}{\beta_1 \beta_2} \partial_x^{\beta_1} \left(\frac{x}{\langle x \rangle} \right) \cdot v \partial_x^{\beta_2} u \partial_x^\beta u dx dv \\ &+ \int \partial_x^\beta u K \partial_x^\beta (wh^{(3)}) dx dv. \end{aligned}$$

Note that $2\gamma - 2 \geq 1$ if $\gamma \geq 3/2$, and recall that $\Lambda = -L + K$, hence

$$\begin{aligned} \left| \int \frac{x \cdot v}{\langle x \rangle} |\partial_x^\beta u|^2 dx dv \right| &\leq \int \langle v \rangle^{2\gamma-2} |\partial_x^\beta u|^2 dx dv \lesssim \int (\Lambda \partial_x^\beta u) \partial_x^\beta u dx dv \\ &\lesssim - \int (L \partial_x^\beta u) \partial_x^\beta u dx dv + \int |\partial_x^\beta u|^2 dx dv, \end{aligned}$$

and

$$\begin{aligned} &\left| \int \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} \binom{\beta}{\beta_1 \beta_2} \partial_x^{\beta_1} \left(\frac{x}{\langle x \rangle} \right) \cdot v \partial_x^{\beta_2} u \partial_x^\beta u dx dv \right| \\ &\lesssim \int \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} \langle v \rangle^{2\gamma-2} |\partial_x^{\beta_2} u \partial_x^\beta u| dx dv \\ &\lesssim \int \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} ((-L \partial_x^{\beta_2} u) \partial_x^{\beta_2} u + (-L \partial_x^\beta u) \partial_x^\beta u) dx dv \\ &+ \int \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} (|\partial_x^{\beta_2} u|^2 + |\partial_x^\beta u|^2) dx dv. \end{aligned}$$

Also,

$$\left| \int \partial_x^\beta u K \partial_x^\beta (wh^{(3)}) dx dv \right| \lesssim \|\partial_x^\beta u\|_{L^2} \|\partial_x^\beta (wh^{(3)})\|_{L^2}.$$

After choosing D and M large enough, we have

$$\frac{d}{dt} \|u\|_{H_x^2 L_v^2} \lesssim \|wh^{(3)}\|_{H_x^2 L_v^2}.$$

Hence, it follows from (4.1) and (4.2) that

$$\|u\|_{H_x^2 L_v^2}(t) \lesssim \int_0^t \|wh^{(3)}\|_{H_x^2 L_v^2}(s) ds \lesssim \|f_0\|_{L^2(\mu)}. \quad \square$$

Note that $w(x, t) \geq e^{\frac{\langle x \rangle + 2Mt}{8D}}$ if $\langle x \rangle \geq 2Mt$, hence for $\gamma \geq 3/2$, the Sobolev inequality implies

$$\begin{aligned} e^{\frac{\langle x \rangle + 2Mt}{8D}} |f|_{L_v^2} &\leq \sum_{j=0}^3 \left| wh^{(j)} \right|_{L_v^2} + \left| w\mathcal{R}^{(3)} \right|_{L_v^2} \\ &\lesssim \sum_{j=0}^3 \left\| wh^{(j)} \right\|_{H_x^2 L_v^2}^{3/4} \left\| wh^{(j)} \right\|_{L^2}^{1/4} + \left\| w\mathcal{R}^{(3)} \right\|_{H_x^2 L_v^2}^{3/4} \left\| w\mathcal{R}^{(3)} \right\|_{L^2}^{1/4} \\ &\lesssim \left(t^{-9/4} + 1 \right) \|f_0\|_{L^2(\mu)} \\ &\lesssim \left(t^{-9/4} + 1 \right) \|f_0\|_{L^2}. \end{aligned}$$

The last inequality is due to the compact support assumption of the initial data.

THEOREM 4.2 (Space-like region for $\gamma \geq 3/2$). Let $\gamma \geq 3/2$ and let f be the solution to equation (1.2). Assume that the initial condition f_0 has compact support in the x variable and is bounded in L_v^2 . Then there exists a large positive constant M such that if $\langle x \rangle > 2Mt$, we have

$$|f|_{L_v^2} \lesssim (1 + t^{-9/4}) e^{-C(\langle x \rangle + t)} \|f_0\|_{L^2},$$

here $C = C(M)$ is a positive constant.

4.2. The case $0 < \gamma < 3/2$: Subexponential decay. If $0 < \gamma < 3/2$, we consider the weight functions

$$w(t, x, v) = e^{\frac{\alpha \rho(t, x, v)}{2}}, \quad \mu(x, v) = e^{\alpha c(x, v)},$$

where

$$\begin{aligned} \rho(t, x, v) &= 5(\delta(\langle x \rangle - Mt))^{\frac{\gamma}{3-\gamma}} \left(1 - \chi \left(\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right) \right) \\ &+ \left[\left(1 - \chi \left(\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right) \right) \delta(\langle x \rangle - Mt) \langle v \rangle^{2\gamma-3} + 3 \langle v \rangle^\gamma \right] \chi \left(\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right), \end{aligned}$$

and

$$\begin{aligned} c(x, v) &= 5(\delta \langle x \rangle)^{\frac{\gamma}{3-\gamma}} \left(1 - \chi \left(\delta \langle x \rangle \langle v \rangle^{\gamma-3} \right) \right) \\ &+ \left[\left(1 - \chi \left(\delta \langle x \rangle \langle v \rangle^{\gamma-3} \right) \right) \delta \langle x \rangle \langle v \rangle^{2\gamma-3} + 3 \langle v \rangle^\gamma \right] \chi \left(\delta \langle x \rangle \langle v \rangle^{\gamma-3} \right). \end{aligned}$$

Here M is a large positive constant, δ, α are small positive constants; all of them will be chosen later. We introduce the following space-velocity decomposition:

$$\begin{aligned} H_+ &= \{(x, v) : [\delta(\langle x \rangle - Mt)] \geq 2 \langle v \rangle^{3-\gamma}\}, \\ H_0 &= \{(x, v) : \langle v \rangle^{3-\gamma} < [\delta(\langle x \rangle - Mt)] < 2 \langle v \rangle^{3-\gamma}\}, \end{aligned}$$

and

$$H_- = \{(x, v) : [\delta(\langle x \rangle - Mt)] \leq \langle v \rangle^{3-\gamma}\}.$$

PROPOSITION 4.3. Consider the weight functions

$$w(t, x, v) = e^{\frac{\alpha\rho(t,x,v)}{2}} \quad \text{and} \quad \mu(x, v) = e^{\alpha c(x,v)},$$

where $\alpha > 0$ is sufficiently small with $\alpha\gamma < 1/20$. Then:

(i) For $0 \leq j \leq 3$,

$$\|wh^{(j)}\|_{H_x^2 L_v^2} \lesssim t^{-3+j} \|f_0\|_{L^2(\mu)}, \quad 0 < t \leq 1,$$

and

$$\|wh^{(j)}\|_{H_x^2 L_v^2} \lesssim (1+t)^j \|f_0\|_{L^2(\mu)}, \quad t > 1.$$

(ii) For $1 \leq \gamma < 3/2$,

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \lesssim t(1+t) \|f_0\|_{L^2(\mu)}, \quad t > 0,$$

and for $0 < \gamma < 1$,

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \lesssim t(1+t^4) \|f_0\|_{L^2(\mu)}, \quad t > 0.$$

Proof. It is similar to Proposition 4.1 that the weighted energy estimate of the wave parts $h^{(j)}$ is a consequence of Lemma 2.7 by virtue of $\rho(t, x, v)$ being non-increasing in t and $\rho(0, x, v) = c(x, v)$.

We shall focus on the weighted energy estimate for the remainder part $\mathcal{R}^{(3)}$, $t > 0$. We want to show that for $1 \leq \gamma < 3/2$,

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \lesssim t(1+t) \|f_0\|_{L^2(\mu)}, \quad t > 0,$$

and for $0 < \gamma < 1$,

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \lesssim t(1+t^4) \|f_0\|_{L^2(\mu)}, \quad t > 0.$$

Let $u = w\mathcal{R}^{(3)} = e^{\frac{\alpha\rho}{2}} \mathcal{R}^{(3)}$, and then $\partial_x^\beta u$, where β is a multi-index, solves the equation

$$\begin{aligned} \partial_t (\partial_x^\beta u) &= -v \cdot \nabla_x (\partial_x^\beta u) + \frac{\alpha}{2} (\partial_t \rho + v \cdot \nabla_x \rho) \partial_x^\beta u + e^{\frac{\alpha\rho}{2}} L \left(e^{-\frac{\alpha\rho}{2}} \partial_x^\beta u \right) \\ &+ \frac{\alpha}{2} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} \binom{\beta}{\beta_1 \beta_2} (\partial_t \partial_x^{\beta_1} \rho + v \cdot \nabla_x \partial_x^{\beta_1} \rho) \partial_x^{\beta_2} u \\ &+ \sum_{\substack{\beta_1 + \beta_2 + \beta_3 = \beta \\ |\beta_3| < |\beta|}} \binom{\beta}{\beta_1 \beta_2 \beta_3} (\partial_x^{\beta_1} e^{\frac{\alpha\rho}{2}}) L \left((\partial_x^{\beta_2} e^{-\frac{\alpha\rho}{2}}) \partial_x^{\beta_3} u \right) \\ &+ K \partial_x^\beta (wh^{(3)}). \end{aligned}$$

The energy estimate gives

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\partial_x^\beta u\|_{L^2}^2 \\
 &= \int_{\mathbb{R}^3} \left\langle e^{\frac{\alpha\rho}{2}} L \left(e^{-\frac{\alpha\rho}{2}} \partial_x^\beta u \right), \partial_x^\beta u \right\rangle_v dx + \frac{\alpha}{2} \int_{\mathbb{R}^3} \left\langle (\partial_t \rho + v \cdot \nabla_x \rho) \partial_x^\beta u, \partial_x^\beta u \right\rangle_v dx \\
 &+ \frac{\alpha}{2} \sum_{\substack{\beta_1 + \beta_2 = \beta \\ |\beta_1| \geq 1}} \binom{\beta}{\beta_1 \beta_2} \int_{\mathbb{R}^3} \left\langle (\partial_t \partial_x^{\beta_1} \rho + v \cdot \nabla_x \partial_x^{\beta_1} \rho) \partial_x^{\beta_2} u, \partial_x^\beta u \right\rangle_v dx \\
 &+ \sum_{\substack{\beta_1 + \beta_2 + \beta_3 = \beta \\ |\beta_3| < |\beta|}} \binom{\beta}{\beta_1 \beta_2 \beta_3} \int_{\mathbb{R}^3} \left\langle (\partial_x^{\beta_1} e^{\frac{\alpha\rho}{2}}) L \left((\partial_x^{\beta_2} e^{-\frac{\alpha\rho}{2}}) \partial_x^{\beta_3} u \right), \partial_x^\beta u \right\rangle_v dx \\
 &+ \int_{\mathbb{R}^3} \left\langle K \partial_x^\beta (wh^{(3)}), \partial_x^\beta u \right\rangle_v dx \\
 &:= (I_1) + (I_2) + (I_3) + (I_4) + (I_5).
 \end{aligned}$$

We shall estimate (I_i) , $i = 1, \dots, 5$, term by term.

For (I_1) , it is easy to see that

$$\left\langle g, e^{\frac{\alpha\rho}{2}} L \left(e^{-\frac{\alpha\rho}{2}} g \right) \right\rangle_v = \left\langle g, e^{-\frac{\alpha\rho}{2}} L \left(e^{\frac{\alpha\rho}{2}} g \right) \right\rangle_v = \langle g, Lg \rangle_v + \frac{\alpha^2}{4} \langle g^2, |\nabla_v \rho|^2 \rangle_v.$$

In addition, direct calculation gives

$$\begin{aligned}
 \nabla_v \rho &= \left[(\gamma - 3)(1 - 2\chi) \delta(\langle x \rangle - Mt) \langle v \rangle^{2\gamma-3} + 3(\gamma - 3) \langle v \rangle^\gamma - 5(\gamma - 3) (\delta(\langle x \rangle - Mt))^{\frac{\gamma}{3-\gamma}} \right] \\
 &\times \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-4} \right] \frac{v}{\langle v \rangle} \chi' \\
 &+ \left[(2\gamma - 3) \delta(\langle x \rangle - Mt) \langle v \rangle^{2\gamma-4} \right] \frac{v}{\langle v \rangle} (1 - \chi) \chi + 3\gamma \langle v \rangle^{\gamma-1} \frac{v}{\langle v \rangle} \chi.
 \end{aligned}$$

This implies

$$|\nabla_v \rho| \lesssim \langle v \rangle^{\gamma-1} \quad \text{on } H_0 \cup H_-$$

and

$$\nabla_v \rho = 0 \quad \text{on } H_+.$$

Therefore,

$$\begin{aligned}
 (I_1) &= \int_{\mathbb{R}^3} \left\langle e^{\frac{\alpha\rho}{2}} L \left(e^{-\frac{\alpha\rho}{2}} \partial_x^\beta u \right), \partial_x^\beta u \right\rangle_v dx \\
 &\leq - \left(\nu_0 - \frac{\alpha^2 C}{4} \right) \int_{\mathbb{R}^3} |\mathbb{P}_1 \partial_x^\beta u|_{L^2_\sigma}^2 dx + \frac{\alpha^2 C}{4} \int_{H_0 \cup H_-} |\mathbb{P}_0 \partial_x^\beta u|^2 dx dv
 \end{aligned} \tag{4.4}$$

for some constant $\nu_0 > 0$.

For (I_2) and (I_3) , we need the estimates of derivatives of $\rho(t, x, v)$. Direct computation gives

$$\begin{aligned} \partial_t \rho &= -\delta M \langle v \rangle^{2\gamma-3} \left(\frac{5\gamma}{3-\gamma} \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right]^{\frac{2\gamma-3}{3-\gamma}} (1-\chi) + \chi(1-\chi) \right) \\ &\quad + \delta M \left(5 \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right]^{\frac{\gamma}{3-\gamma}} - (1-2\chi) \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right] - 3 \right) \langle v \rangle^{2\gamma-3} \chi' \\ &\leq 0, \end{aligned}$$

(the constants 5 and 3 are chosen artificially such that the quantity in the latter bracket is non-negative on H_0) and

$$\begin{aligned} \nabla_x \rho &= \delta (\nabla_x \langle x \rangle) \langle v \rangle^{2\gamma-3} \left(\frac{5\gamma}{3-\gamma} \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right]^{\frac{2\gamma-3}{3-\gamma}} (1-\chi) + \chi(1-\chi) \right) \\ &\quad - \delta (\nabla_x \langle x \rangle) \left(5 \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right]^{\frac{\gamma}{3-\gamma}} - (1-2\chi) \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma-3} \right] - 3 \right) \langle v \rangle^{2\gamma-3} \chi', \end{aligned}$$

so

$$\partial_t \rho = v \cdot \nabla_x \rho = 0 \quad \text{on } H_-,$$

$$|\partial_t \rho| \lesssim \delta M \langle v \rangle^{2\gamma-3}, \quad |v \cdot \nabla_x \rho| \lesssim \delta \langle v \rangle^{2\gamma-2} \quad \text{on } H_0,$$

$$\partial_t \rho = -\frac{5\delta M \gamma}{3-\gamma} [\delta(\langle x \rangle - Mt)]^{\frac{2\gamma-3}{3-\gamma}}, \quad v \cdot \nabla_x \rho = \frac{5\delta \gamma}{3-\gamma} \frac{v \cdot x}{\langle x \rangle} [\delta(\langle x \rangle - Mt)]^{\frac{2\gamma-3}{3-\gamma}} \quad \text{on } H_+.$$

Furthermore, we can also obtain that for $|\beta_1| \geq 1$,

$$\partial_t \partial_x^{\beta_1} \rho = \nabla_x \partial_x^{\beta_1} \rho = 0 \quad \text{on } H_-,$$

$$|\partial_t \partial_x^{\beta_1} \rho| \lesssim \delta^2 M \langle v \rangle^{\gamma+(|\beta_1|+1)(\gamma-3)}, \quad |\nabla_x \partial_x^{\beta_1} \rho| \lesssim \delta^2 \langle v \rangle^{\gamma+(|\beta_1|+1)(\gamma-3)} \quad \text{on } H_0 \cup H_+.$$

From these, there exist constants $C > 0$ and $C' > 0$ such that

$$\begin{aligned} \alpha \left| \int_{\mathbb{R}^3} \langle v \cdot \nabla_x \rho \partial_x^\beta u, \partial_x^\beta u \rangle_v dx \right| &\leq \alpha \delta C \left(\int_{\mathbb{R}^3} |\langle v \rangle^{\gamma-1} \mathbf{P}_1 \partial_x^\beta u|_{L_v^2}^2 dx + \int_{H_0} |\mathbf{P}_0 \partial_x^\beta u|^2 dx dv \right. \\ &\quad \left. + \int_{H_+} [\delta(\langle x \rangle - Mt)]^{\frac{2\gamma-3}{3-\gamma}} |\mathbf{P}_0 \partial_x^\beta u|^2 dx dv \right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \alpha \int_{\mathbb{R}^3} \langle (\partial_t \rho) \partial_x^\beta u, \partial_x^\beta u \rangle_v dx &\leq -\alpha \delta M C' \int_{H_+} [\delta(\langle x \rangle - Mt)]^{\frac{2\gamma-3}{3-\gamma}} |\mathbf{P}_0 \partial_x^\beta u|^2 dx dv \\ &\quad + \alpha \delta M C \left(\int_{\mathbb{R}^3} |\langle v \rangle^{\gamma-1} \mathbf{P}_1 \partial_x^\beta u|_{L_v^2}^2 dx + \int_{H_0} |\mathbf{P}_0 \partial_x^\beta u|^2 dx dv \right), \end{aligned} \tag{4.6}$$

and for $|\beta_1| \geq 1$,

$$\begin{aligned} & \frac{\alpha}{2} \left| \int_{\mathbb{R}^3} \langle (\partial_t \partial_x^{\beta_1} \rho + v \cdot \nabla_x \partial_x^{\beta_1} \rho) \partial_x^{\beta_2} u, \partial_x^\beta u \rangle_v dx \right| \\ & \leq \frac{\alpha \delta^2 M}{2} C \left[\int_{\mathbb{R}^3} \left(|\langle v \rangle^{\gamma-1} P_1 \partial_x^{\beta_2} u|_{L_v^2}^2 + |\langle v \rangle^{\gamma-1} P_1 \partial_x^\beta u|_{L_v^2}^2 \right) dx \right. \\ & \quad + \int_{H_0} \left(|P_0 \partial_x^{\beta_2} u|^2 + |P_0 \partial_x^\beta u|^2 \right) dx dv \\ & \quad \left. + \int_{H_+} [\delta(\langle x \rangle - Mt)]^{\frac{2\gamma-3}{3-\gamma}} \left(|P_0 \partial_x^{\beta_2} u|^2 + |P_0 \partial_x^\beta u|^2 \right) dx dv \right]. \end{aligned} \tag{4.7}$$

As for (I_4) , $|\beta_1| + |\beta_2| \geq 1$, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} \langle (\partial_x^{\beta_1} e^{\frac{\alpha \rho}{2}}) L \left((\partial_x^{\beta_2} e^{-\frac{\alpha \rho}{2}}) \partial_x^{\beta_3} u \right), \partial_x^\beta u \rangle_v dx \right| \\ & \leq \frac{\alpha \delta C}{2} \left[\int_{\mathbb{R}^3} \left(|P_1 \partial_x^{\beta_3} u|_{L_v^2}^2 + |P_1 \partial_x^\beta u|_{L_v^2}^2 \right) dx + \int_{H_0} |P_0 \partial_x^{\beta_3} u|^2 + |P_0 \partial_x^\beta u|^2 dx dv \right. \\ & \quad \left. + \int_{H_+} [\delta(\langle x \rangle - Mt)]^{\frac{2\gamma-3}{3-\gamma}} \left(|P_0 \partial_x^{\beta_3} u|^2 + |P_0 \partial_x^\beta u|^2 \right) dx dv \right]. \end{aligned} \tag{4.8}$$

Finally,

$$(I_5) \leq \left| \int_{\mathbb{R}^3} \langle K \partial_x^\beta (wh^{(3)}), \partial_x^\beta u \rangle_v dx \right| \lesssim \|\partial_x^\beta u\|_{L^2} \left\| \partial_x^\beta (wh^{(3)}) \right\|_{L^2}. \tag{4.9}$$

Gathering the terms (4.4)–(4.9), we find

$$\begin{aligned} \frac{d}{dt} \|u\|_{H_x^2 L_v^2}^2 & \lesssim \|u\|_{H_x^2 L_v^2} \|wh^{(3)}\|_{H_x^2 L_v^2} + \int_{H_0 \cup H_-} \left(|P_0 \nabla_x^2 u|^2 + |P_0 \nabla_x u|^2 + |P_0 u|^2 \right) dx dv \\ & \lesssim \|u\|_{H_x^2 L_v^2} \|wh^{(3)}\|_{H_x^2 L_v^2} + \|u\|_{H_x^2 L_v^2} \|\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \\ & \lesssim \|u\|_{H_x^2 L_v^2} \left(\|h^{(3)}\|_{H_x^2 L_v^2(\mu)} + \|\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} \right), \end{aligned}$$

after choosing $\delta, \alpha > 0$ small and M large enough with $\alpha\gamma < 1/20$. Hence, it follows from Lemmas 2.6 and 2.7 that for $1 \leq \gamma < 3/2$,

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} = \|u\|_{H_x^2 L_v^2} \lesssim t(1+t) \|f_0\|_{L^2(\mu)},$$

and for $0 < \gamma < 1$,

$$\|w\mathcal{R}^{(3)}\|_{H_x^2 L_v^2} = \|u\|_{H_x^2 L_v^2} \lesssim t(1+t^4) \|f_0\|_{L^2(\mu)}.$$

This completes the proof of the proposition. □

Observe that for $\langle x \rangle > 2Mt$,

$$\rho(t, x, v) \gtrsim (\delta(\langle x \rangle - Mt))^{\frac{\gamma}{3-\gamma}}$$

and

$$\langle x \rangle - Mt > \frac{\langle x \rangle}{3} + \frac{Mt}{3}.$$

The Sobolev inequality immediately gets the following.

THEOREM 4.4 (Space-like region for $0 < \gamma < 3/2$). Let $0 < \gamma < 3/2$ and let f be the solution to equation (1.2). Assume that the initial condition f_0 has compact support in the x variable and is bounded in $L_v^2(e^{4\alpha\langle v \rangle^\gamma})$ for $\alpha > 0$ small enough. Then there exists a positive constant $C = C(M)$ such that for $\langle x \rangle \geq 2Mt$,

$$|f|_{L_v^2} \lesssim (1+t^{-9/4})e^{-C(\langle x \rangle+t)^{\frac{\gamma}{3-\gamma}}} \|f_0\|_{L^2(e^{4\alpha\langle v \rangle^\gamma})}.$$

5. Conclusion. In this paper, we obtain the quantitative pointwise behavior of the solutions of the Fokker-Planck equation with potential $\Phi(v)$, where

$$\Phi = \frac{1}{\gamma} \langle v \rangle^\gamma + \Phi_0, \quad \gamma > 0, \quad \Phi_0 \text{ is a fixed constant.}$$

The structure of the solution sensitively depends on the potential function. For hard potentials, we extend the result [10] with the Gaussian velocity weight $e^{\alpha|\xi|^2}$ to more general exponential velocity weights $e^{\alpha|\xi|^p}$, $0 < p \leq 2$. For Maxwellian molecules and soft potentials, our result is the first attempt aiming at the pointwise structure of the solution.

In the time-like region (t large), the solution is the heat kernel-type $(1+t)^{-3/2}e^{-C\frac{|x|^2}{1+t}}$ for $\gamma \geq 3/2$, almost heat kernel-type $(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N}$ for $1 \leq \gamma < 3/2$, and has polynomial decay $(1+t)^{-3/2}$ for $0 < \gamma < 1$.

In the space-like region ($|x|$ large), the solution has exponential decay $e^{-C|x|}$ for $\gamma > 3/2$ and sub-exponential decay $e^{-C|x|^{\frac{\gamma}{3-\gamma}}}$ for $0 < \gamma < 3/2$, respectively.

Lastly, we would like to remark that our results can be easily generalized to arbitrary dimension d . In fact, we only need to modify the wave-remainder decomposition to guarantee the remainder part owns enough regularity, so that the weighted energy estimate for space-like region can be applied. It turns out the time decay will change from $(1+t)^{-3/2}$ to $(1+t)^{-d/2}$ for the time-like region, while the behavior in the space-like region remains the same.

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