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Algebraic-Complex Scheme for Dirichlet–Neumann Data for Parabolic System

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Abstract

In this paper, we use the Laplace–Laplace transformation and complex analysis to give a systematical scheme to determine the proper boundary conditions for initial-boundary value problems in the half space and to construct exponentially sharp pointwise structures of the boundary data. Here, we have used the boundary value problems with the Robin boundary conditions for the convection heat equations and the linearized compressible Navier–Stokes equation with a constant convection velocity to demonstrate this scheme.

1. Introduction

A parabolic system

$$\partial_t V + \boldsymbol{A}\partial_x V - \boldsymbol{B} V_{xx} = 0 \quad \text{for } x, t > 0,\ \ V \in \mathbb{R}^n \tag{1}$$

is an interesting mathematical model in many contexts of mathematics and physics. In general, for such a half space problem, initial boundary conditions will be posed in order to proceed with further mathematical studies. However, in [1] it is realized that the Dirichlet–Neumann relationship for a homogeneous initial value problem (that is $V(x, 0) \equiv 0$) is more fundamental than the Dirichlet boundary value problem in terms of transform variables. With the transform variables, the differential equations are converted into algebraic systems so that one can algebraically manipulate the differential equations. In the end, one can use complex analysis to revert the solution from the transform variables to the physical variables. This opened a new door towards various evolutionary partial differential equations, and various interesting results for PDEs with different wave propagation characteristics in one-dimensional and multi-dimensional have been successfully analyzed. This paper is one of those activities; it aims at constructing exponentially sharp pointwise structures of the boundary data for the Robin's boundary conditions to the convection heat equations and linearized Navier–Stokes equations. It also defines a standard procedure to solve the problems in terms of the physical variables. We call the procedure the algebraic-complex scheme (a-c scheme).

The Robin's boundary condition for the convection heat equation is

$$\begin{cases} u_t + \Lambda u_x = u_{xx}, \ x, t > 0, \\ u_x(0,t) - \kappa u(0,t) = w(t), \\ u(x,0) \equiv 0, \end{cases} \tag{2}$$

and the linearized compressible Navier–Stokes equation is

$$\begin{cases} V_t + \begin{pmatrix} \Lambda & 1 \\ 1 & \Lambda \end{pmatrix} V_x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} V_{xx}, \ x, t > 0, \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} V_x(0,t) + \begin{pmatrix} 0 & 0 \\ 0 & -\kappa \end{pmatrix} V(0,t) = W(t) \equiv \begin{pmatrix} 0 \\ w(t) \end{pmatrix}, \\ V(x,0) \equiv 0, \end{cases} \tag{3}$$

where the parameter Λ in (2) and (3) is a given parameter to specify the background velocity, $W(t)$ and $w(t)$ are the given controlled boundary data. We use the above two systems to demonstrate the simplicity and efficiency of the a-c scheme, since there were no such exponentially sharp estimates obtained for those problems before. Even for the whole space problem, the construction of the fundamental solution for (3) is very non-trivial, and it was obtained in 1990's, see [2]. Furthermore, the a-c scheme also gives classification of the stability of the initial boundary problem in terms of κ and Λ, and one can clearly realize how to impose a boundary condition for a half space problem.

Here, the homogeneous initial data $u(x,0) \equiv 0$ and $V(x,0) \equiv 0$ are generic, since one always can subtract the solutions of initial value problems from a whole space problem to get inhomogeneous boundary value problems as those in (2) and (3).

The a-c scheme is a procedure given as the following ABCD steps:

A. Convert differential equations into an algebraic system. One considers the Laplace–Laplace transform of (1) with the homogeneous initial condition $V(x,0) \equiv 0$:

$$\begin{cases} \mathbb{L}[V](x,s) \equiv \int_0^\infty \mathrm{e}^{-st} V(x,t)\,\mathrm{d}t, \\ \\ \mathbb{J}[V](\xi,s) \equiv \int_0^\infty \mathrm{e}^{-\xi x} \mathbb{L}[V](x,s)\,\mathrm{d}x, \end{cases}$$

and under this homogeneous initial condition (1) becomes

$$\mathbb{J}[V](\xi,s) = \frac{\operatorname{adj}(s + \xi \boldsymbol{A} - \xi^2 \boldsymbol{B})}{p(\xi,s)} \left((\boldsymbol{A} - \xi \boldsymbol{B})\mathbb{L}[V](x,s) - \mathbb{L}[V_x](x,s)\right)\Big|_{x=0},$$

where

$$p(\xi, s) \equiv \det(s + \xi \boldsymbol{A} - \xi^2 \boldsymbol{B}).$$

B. Analyze the roots of the characteristic polynomial $p(\xi, s)$ in the right half complex plane with $s > 0$.

C. Derive the Dirichlet–Neumann relationship and use the relationship together with a given boundary condition such as the Dirichlet boundary condition, Neumann boundary condition and Robin's boundary condition etc. to construct the full boundary data in the transform variables.

The expression for $\mathbb{J}[V](\xi, s)$ in step A is a rational function in ξ so that one can apply the inverse Laplace transformation from ξ to x by the partial fraction of $\mathbb{J}[V](\xi, s)$ in ξ variable. The boundedness of the solution $\lim_{x\to\infty} V(x,t) < \infty$ yields the following Dirichlet–Neumann relationship

$$\operatorname*{Res}_{\substack{\xi=\lambda_j,\\ p(\lambda_j,s)=0\\ \mathrm{Re}(\lambda_j)>0,\ s>0}} \frac{\operatorname{adj}(s+\xi \boldsymbol{A} - \xi^2 \boldsymbol{B})}{p(\xi, s)} \left((\boldsymbol{A}-\xi \boldsymbol{B})\mathbb{L}[V](x, s) - \mathbb{L}[V_x](x, s)\right)\Big|_{x=0} = 0. \tag{D–N}$$

This is a linear system on the Dirichlet and Neumann Data $\mathbb{L}[V](0, s)$ and $\mathbb{L}[V_x](0, s)$ with coefficients in $\mathbb{C}[s, \lambda_1, \ldots, \lambda_l]$ where $\lambda_i(s)$ are roots of $p(\lambda_i(s), s) = 0$ with the property $\mathrm{Re}(\lambda_i(s)) > 0$ for $s > 0$. With a precise distribution of the roots $\lambda_j(s)$ in the right half complex plane, one can determine the proper boundary conditions required so that the determined boundary condition and (D–N) together can solve $\mathbb{L}[V](0, s)$ and $\mathbb{L}[V_x](0, s)$ uniquely and explicitly in terms of the transform variable $s, \lambda_1(s), \ldots, \lambda_l(s)$.

D. Apply complex analysis to convert $\mathbb{L}[V](0, s)$ and $\mathbb{L}[V_x](0, s)$ into $V(0, t)$ and $V_x(0, t)$. The boundary data $\mathbb{L}[V](0, s)$ and $\mathbb{L}[V_x](0, s)$ are given in terms of the roots of $\xi = \lambda_i(s)$ of $p(\xi, s) = 0$. One will need to obtain the analytic properties of the roots in order to use the complex analysis to yield the exponentially sharp pointwise structure of the boundary data in terms of the physical variable t.

After obtaining the full boundary data for (1), one simply applies the first Green's identity to yield the solution $V(x, t)$:

$$\begin{aligned} V(x,t) = &\int_0^t \mathbb{G}(x, t-\tau)(\boldsymbol{A}V(0,\tau) - \boldsymbol{B}V_x(0,\tau))\,\mathrm{d}\tau \\ &- \int_0^t \mathbb{G}_x(x, t-\tau)\boldsymbol{B}V(0,\tau)\,\mathrm{d}\tau, \end{aligned}$$

where $\mathbb{G}(x, t)$ is the fundamental solution of (1) for the whole space problem, and $\mathbb{G}(x, t)$ is an object that has been well-studied in many cases. After obtaining the full boundary data, the boundary value problem with proper imposed boundary conditions can be considered complete.

For the problems (2) and (3), the a-c scheme gives exponentially sharp pointwise structure of the full boundary data so that it gives the bifurcation from time asymptotically stable behavior to time asymptotically unstable behavior in terms of the coefficients in (2) and (3). We will leave the statements of the results in the relevant sections.

In Section 2, we will prepare a simple toolbox for the Laplace transformation and its inverse transformation. In Sections 3 and 4 we will follow the ABCD steps to develop the full boundary data for (2) and (3).

2. A Simple Toolbox for the Laplace Transformation

Let $f(t)$ be a function defined in $t \geq 0$; its Laplace transformation $F(s)$ and the inverse transformation (the Bromwich integral) are given as follows:

$$\begin{cases} F(s) = \mathbb{L}[f](s) \equiv \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \ \ for\ s \in \{z \in \mathbb{C} | \operatorname{Re}(s) \geq 0\}, \\ f(t) = \mathbb{L}^{-1}[F](t) \equiv \dfrac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} \mathrm{e}^{st} F(s)\,\mathrm{d}s \text{ for } t > 0, \end{cases} \tag{4}$$

where γ is a real number so that the contour path of integration is in the region of convergence of $F(s)$. In particular, if $F(s)$ can be analytically extended to $\operatorname{Re}(s) > 0$, we may choose γ to be 0.

To be compatible with the initial data $V(x, 0) = 0$, the controlled boundary data $w(t)$ is given in the following space:

$$w \in \mathscr{V} \equiv \left\{ f \mid \mathbb{L}[f](s) \text{ exists for } \operatorname{Re}(s) > 0, \text{ and } f^{[n]}(0) = 0 \text{ for } n \in \mathbb{N} \right\}. \tag{5}$$

Lemma 1. *For f and $g \in \mathscr{V}$, their Laplace transformations $F = \mathbb{L}[f]$ and $G = \mathbb{L}[g]$ satisfy*

$$\begin{cases} \mathbb{L}\left[f^{[n]}\right] = s^n [F(s)] \\ (-t)^n f(t) = \mathbb{L}^{-1}\left[\frac{\mathrm{d}^n}{\mathrm{d}s^n} F\right](t) \\ [f * g](t) = \mathbb{L}^{-1}[F(s)G(s)](t). \end{cases} \tag{6}$$

3. Convection Heat Equation with Robin Boundary Condition

The Robin boundary value problem for a convection heat equation with convection velocity Λ and with given control data $w(t) \in \mathscr{V}$ is

$$\begin{cases} u_t + \Lambda u_x = u_{xx}, \quad x,\ t \geq 0 \\ u_x(0, t) - \kappa u(0, t) = w(t), \\ u(x, 0) = 0. \end{cases} \tag{7}$$

In this section we will follow the a-c scheme to construct the boundary data $u(0, t)$ and $u_x(0, t)$.

A. The polynomial system.

$$\mathbb{J}[u](\xi, s) = \frac{1}{p(\xi, s)}\left((\xi - \Lambda)\mathbb{L}[u](0, s) + \mathbb{L}[u_x](0, s)\right), \tag{8}$$

$$p(\xi, s) = \xi^2 - \Lambda\xi - s. \tag{9}$$

B. Root of $p(\xi, s) = 0$.

There is only one root $\xi = \lambda(s)$ of $p(\xi, s) = 0$ with the property $\mathrm{Re}(\lambda(s)) > 0$ for $s > 0$,

$$\lambda(s) = \frac{\sqrt{\Lambda^2 + 4s} + \Lambda}{2}. \tag{10}$$

C. Dirichlet–Neumann relation and the full boundary data in the transform variable.

With above root $\lambda(s)$, the Dirichlet–Neumann relation (D–N) becomes

$$(\lambda(s) - \Lambda)\mathbb{L}[u](0, s) + \mathbb{L}[u_x](0, s) = 0. \tag{11}$$

The Robin's boundary condition gives

$$\mathbb{L}[u_x](0, s) - \kappa\mathbb{L}[u](0, s) = \mathbb{L}[w](s). \tag{12}$$

By solving these two linear equations, the full boundary data in terms of the transform variable are

$$\begin{cases} \mathbb{L}[u](0, s) = -\dfrac{1}{\lambda(s) - \Lambda + \kappa}\mathbb{L}[w](s) \\ \mathbb{L}[u_x](0, s) = \mathbb{L}[w](s) + \kappa\mathbb{L}[u](0, s). \end{cases} \tag{13}$$

D. Conversion from s to t.

We continue to invert the function $1/(\lambda(s) - \Lambda + \kappa)$ in (13) into a function in physical variable t.

Lemma 2. *For $\lambda(s)$ given in (10), the function $1/(\lambda(s) - \Lambda + \kappa)$ satisfies the following properties:*

Case. $\Lambda = 0$

$$\frac{1}{\lambda(s) - \Lambda + \kappa} = \frac{1}{\sqrt{s} + \kappa}. \tag{14}$$

Case. $\Lambda \neq 0$ *and* $\kappa(\kappa - \Lambda) \geq 0$, $\kappa > 0$.

The function $1/(\lambda(s) - \Lambda + \kappa)$ is analytic in $\mathrm{Re}(s) > -\Lambda^2/4$.

Case. $\Lambda \neq 0$, $\kappa(\kappa - \Lambda) \geq 0$, $\kappa < 0$.

The function $1/(\lambda(s) - \Lambda + \kappa)$ is meromorphic in $\mathrm{Re}(s) > -\Lambda^2/4$*; and the pole is at $s = \kappa(\kappa - \Lambda)$ and it is a simple pole with*

$$\operatorname*{Res}_{s=\kappa(\kappa-\Lambda)} \frac{1}{\lambda(s) - \Lambda + \kappa} = \Lambda - 2\kappa.$$

Case. $\Lambda \neq 0$, $\kappa(\kappa - \Lambda) < 0$

There exists $\delta_0 > 0$ such that $1/(\lambda(s) - \Lambda + \kappa)$ is analytic in $\mathrm{Re}(s) > -\delta_0$.

For the case $\Lambda \neq 0$, $\kappa(\kappa-\Lambda) \geq 0$, $\kappa < 0$, we denote $\alpha(s)$ the analytic part of the function $1/(\lambda(s)-\Lambda+\kappa)$:

$$\alpha(s) \equiv \frac{1}{\lambda(s)-\Lambda+\kappa} - \frac{1}{s-\kappa(\kappa-\Lambda)} \operatorname*{Res}_{s=\kappa(\kappa-\Lambda)} \frac{1}{\lambda(s)-\Lambda+\kappa}. \tag{15}$$

The function $\alpha(s)$ is analytic in the region $\mathrm{Re}(s) > -\Lambda^2/4$ and satisfies

$$\lim_{s\to\pm i\infty} |\alpha(s)\sqrt{s}| < \infty. \tag{16}$$

With the information on $1/(\lambda(s)-\Lambda+\kappa)$ in Lemmas 2 and (16), we have the following theorem:

Theorem 1. *Given parameters Λ and κ with given control data $w(t) \in \mathscr{V}$ for the problem (2), the solution $u(x,t)$ at $x=0$ satisfies*
Case. $\Lambda = 0$

$$\begin{aligned} u(0,t) &= -\int_0^t \frac{w(\tau)}{\sqrt{\pi(t-\tau)}}\,\mathrm{d}\tau + \kappa\int_0^t \mathrm{e}^{\kappa^2(t-\tau)}\mathrm{Erfc}\left[\kappa\sqrt{t-\tau}\right] w(\tau)\,\mathrm{d}\tau, \\ u_x(0,t) &= w(t) - \kappa\int_0^t \frac{w(\tau)}{\sqrt{\pi(t-\tau)}}\,\mathrm{d}\tau + \\ &\qquad \kappa^2\int_0^t \mathrm{e}^{\kappa^2(t-\tau)}\mathrm{Erfc}\left[\kappa\sqrt{t-\tau}\right] w(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{17}$$

where

$$\mathrm{Erfc}(z) \equiv \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t. \tag{18}$$

Case. $\Lambda \neq 0$ *and* $\kappa < \min(0,\Lambda)$*, then*

$$\kappa(\kappa-\Lambda) > 0; \tag{19}$$

and

$$\begin{aligned} &\left|u(0,t) + (\Lambda-2\kappa)\int_0^t \mathrm{e}^{\kappa(\kappa-\Lambda)(t-\tau)} w(\tau)\,\mathrm{d}\tau\right| \\ &\leq O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}\,|w(\tau)|\,\mathrm{d}\tau, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left|u_x(0,t) - w(t) + \kappa(\Lambda-2\kappa)\int_0^t \mathrm{e}^{\kappa(\kappa-\Lambda)(t-\tau)} w(\tau)\,\mathrm{d}\tau\right| \\ &\leq O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}\,|w(\tau)|\,\mathrm{d}\tau. \end{aligned} \tag{21}$$

Other Cases.

$$|u(0,t)| \leq O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}\,|w(\tau)|\,\mathrm{d}\tau, \tag{22}$$

$$|u_x(0,t) - w(t)| \leq O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}\,|w(\tau)|\,\mathrm{d}\tau, \tag{23}$$

where $C_0 > 0$ is a constant depends on κ and Λ.

Remark 1. The property (19) for the case $\Lambda \neq 0$ and $\kappa < \min(0, \Lambda)$ yield a time exponentially growing component $e^{\kappa(\kappa-\Lambda)t}$ in the boundary data.

Proof. We only give the proofs of (17) and (20). The proofs for the rest are similar, and they are omitted.
Case. $\Lambda = 0$.

In this case, $1/(\lambda(s) - \Lambda + \kappa) = 1/(\sqrt{s} + \kappa)$. From the fact that

$$\mathbb{L}^{-1}\left[1/(\lambda(s) - \Lambda + \kappa)\right](t) = \mathbb{L}^{-1}\left[\frac{1}{\sqrt{s} + \kappa}\right](t) = \frac{1}{\sqrt{\pi t}} + \kappa \mathrm{e}^{\kappa t^2}\mathrm{Erfc}[\kappa\sqrt{t}],$$

formula (6) together with (13), one has (17).
Case. $\Lambda \neq 0$ and $\kappa \leq \min(0, \Lambda)$.

Next we consider case $\Lambda \neq 0$ and $\kappa \leq \min(0, \Lambda)$. From (13) and (6) together, one has

$$u(0,t) = -\mathbb{L}^{-1}\left[\frac{1}{\lambda(s) - \Lambda + \kappa}\right](t) * w(t).$$

This yields that

$$\begin{aligned}
u(0,t) &= -(\Lambda - 2\kappa)\int_0^t \mathbb{L}^{-1}\left[\frac{1}{s - \kappa(\kappa - \Lambda)}\right](t-\tau)w(\tau)\,\mathrm{d}\tau \\
&\quad - \int_0^t \mathbb{L}^{-1}[\alpha(s)](\tau)(t-\tau)w(\tau)\,\mathrm{d}\tau \\
&= -(\Lambda - 2\kappa)\int_0^t \mathrm{e}^{\kappa(\kappa-\Lambda)(t-\tau)}w(\tau)\,\mathrm{d}\tau \\
&\quad - \int_0^t \mathbb{L}^{-1}[\alpha(s)](\tau)(t-\tau)w(\tau)\,\mathrm{d}\tau.
\end{aligned} \tag{24}$$

Due to the fact that $\alpha(s)$ is analytic in $\mathrm{Re}(s) > -\Lambda^2/4$,

$$\begin{aligned}
\mathbb{L}^{-1}[\alpha](t) &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=0} \mathrm{e}^{st}\alpha(s)\,\mathrm{d}s \\
&= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=-\Lambda^2/8} \mathrm{e}^{\mathrm{st}}\alpha(s)\,\mathrm{d}s \\
&= \frac{\mathrm{e}^{-\Lambda^2 t/8}}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{iyt}\alpha(-\Lambda^2/8 + iy)\,\mathrm{d}y.
\end{aligned} \tag{25}$$

Together with the asymptotic nature of $\alpha(s)$, we have that

$$\mathbb{L}^{-1}[\alpha](t) \leq O(1)\frac{\mathrm{e}^{-\Lambda^2 t/8}}{\sqrt{t}}. \tag{26}$$

From (26) and (24), one concludes (20). $\square$

4. Linearized Compressible Navier–Stokes Equation in One-Dimensional with Robin Condition

We consider the following linearized Navier–Stokes equation in one-dimensional with the Robin boundary condition,

$$
V(x,t) \equiv \begin{pmatrix} \rho \\ u \end{pmatrix} (x,t),
$$

$$
\begin{cases}
V_t + \begin{pmatrix} \Lambda & 1 \\ 1 & \Lambda \end{pmatrix} V_x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} V_{xx}, \ \ x, t > 0, \\
\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} V_x(0,t) + \begin{pmatrix} 0 & 0 \\ 0 & -\kappa \end{pmatrix} V(0,t) = \begin{pmatrix} 0 \\ w(t) \end{pmatrix}, \\
V(x,0) \equiv 0,
\end{cases} \tag{27}
$$

where $\Lambda \in (-1,1)$ is the parameter for the background fluid velocity, and $w(t) \in \mathscr{V}$ is a given input for the Robin boundary condition.

A. The polynomial system.

$$
\mathbb{J}[V](\xi,s) = \frac{1}{p(\xi,s)} \begin{pmatrix} s + \Lambda\xi - \xi^2 - \xi \\ -\xi s + \Lambda\xi \end{pmatrix}
$$

$$
\begin{pmatrix} \Lambda\mathbb{L}[\rho](0,s) + \mathbb{L}[u](0,s) \\ \mathbb{L}[\rho](s) + (\Lambda - \xi)\mathbb{L}[u](0,s) - \mathbb{L}[u_x](0,s) \end{pmatrix}, \tag{28}
$$

and

$$
\begin{aligned}
p(\xi,s) &= s^2 + 2\Lambda s\xi + \left(\Lambda^2 - 1 - s\right)\xi^2 - \Lambda\xi^3 \\
&= -\Lambda\left(\xi^3 + \frac{1 + s - \Lambda^2}{\Lambda}\xi^2 - 2s\xi - \frac{s^2}{\Lambda}\right).
\end{aligned} \tag{29}
$$

B. Roots of $p(\xi,s) = 0$ in ξ.

Lemma 3. *For the characteristic polynomial $p(\xi,s)$ in (29), the roots of $p(\xi,s)$ in ξ satisfy the following properties:*

Case. $\Lambda = 0$. *The two roots of the polynomial are*

$$
\lambda(s) = \pm\frac{s}{\sqrt{1+s}}. \tag{30}
$$

Case. $\Lambda \in (-1,1)\backslash\{0\}$.

There are three roots and they are analytic in a region $\mathrm{Re}(s) > -\delta_0$*; for some* $\delta_0 > 0$*, and the three roots satisfy the asymptotic: for* $|s| \ll 1$

$$
\begin{cases}
\lambda(s) = \dfrac{-s}{\Lambda \pm 1} + O(1)s^2, \\
\xi_\Lambda(s) = \dfrac{-1 + \Lambda^2}{\Lambda} + \dfrac{s(1+\Lambda^2)}{\Lambda(-1+\Lambda^2)} + O(1)s^2;
\end{cases} \tag{31}
$$

for $|s| \to \infty$

$$\begin{cases} \lambda(s) = \pm\sqrt{s} + \dfrac{\Lambda}{2} \pm \dfrac{-4+\Lambda^2}{8}\dfrac{1}{\sqrt{s}} + \dfrac{O(1)}{s^{3/2}}, \\ \xi_\Lambda(s) = -\dfrac{s}{\Lambda} - \dfrac{1}{\Lambda} + \dfrac{O(1)}{s^2}. \end{cases} \tag{32}$$

For any given $s > 0$*, the number of roots of* $p(\xi, s) = 0$ *in* $\mathrm{Re}(\xi) > 0$ *is*

$$\begin{cases} one\ for\ x\Lambda > 0, \\ two\ for\ \Lambda < 0. \end{cases}$$

C. Dirichlet–Neumann relation and the boundary data.

Take the Laplace transformation of the Robin boundary condition:

$$\mathbb{L}[u_x](0, s) - \kappa\mathbb{L}[u](0, s) = \mathbb{L}[w](s). \tag{33}$$

The Dirichlet–Neumann relationship is subject to the sign of Λ (number of roots in right half complex plane), and we will give the relations and solve the full boundary data of u according to the sign of Λ.

Case. $\Lambda = 0$**.**

Due to lemma 3, $\xi = \lambda(s)(\equiv s/\sqrt{s+1})$ is the only root of $p(\xi, s) = 0$ in $\mathrm{Re}(\lambda(s)) > 0$ for $s > 0$. Then, (D–N) yields

$$-s\mathbb{L}[\rho](0, s) + (s+1)\lambda(s)\mathbb{L}[u](0, s) + s\mathbb{L}[u_x](0, s) = 0. \tag{34}$$

Take the Laplace transformation of the equation $\rho_t + \Lambda\rho_x + u_x = 0$ in (27) with respect to t only and let $x = 0$ to yield that

$$s\mathbb{L}[\rho](0, s) + \mathbb{L}[u_x](0, s) = 0. \tag{35}$$

By (33), (34) and (35), one has the full boundary data in the transform variable:

$$\begin{cases} \mathbb{L}[\rho](0, s) = \left(-\dfrac{1}{s} + \dfrac{\kappa}{s(\kappa + \lambda(s))}\right)\mathbb{L}[w](s), \\ \mathbb{L}[u](0, s) = -\dfrac{1}{\kappa + \lambda(s)}\mathbb{L}[w](s), \\ \mathbb{L}[u_x](0, s) = \left(1 - \dfrac{\kappa}{\kappa + \lambda(s)}\right)\mathbb{L}[w](s). \end{cases} \tag{36}$$

Here, $1/(\kappa + \lambda(s))$ has the explicit form

$$\frac{1}{\kappa + \lambda(s)} = \frac{\sqrt{s+1}}{s + \kappa\sqrt{s+1}}. \tag{37}$$

Lemma 4. *When* $\kappa > 0$*, the function* $1/(\kappa + \lambda(s))$ *is an analytic function in* $\mathrm{Re}(s) > -1$*. When* $\kappa \le 0$*, the function* $1/(\kappa + \lambda(s))$ *is a meromorphic function in* $\mathrm{Re}(s) > -1$ *and* $s = -\kappa(\sqrt{\kappa^2 + 4} - \kappa)/2$ *is the only pole in* $\mathrm{Re}(s) > -1$*, and the pole is a simple pole with*

$$\operatorname*{Res}_{\xi = -\kappa(\sqrt{\kappa^2+4}-\kappa)/2} \frac{1}{\kappa + \lambda(s)} = -\left(\kappa - \frac{2}{\sqrt{4+\kappa^2}} - \frac{\kappa^2}{\sqrt{4+\kappa^2}}\right).$$

Furthermore, the analytic part $\alpha(s)$ *of* $1/(\kappa+\lambda(s))$ *satisfies*

$$\lim_{s\to\pm i\infty}\left|\alpha(s)\sqrt{s}\right|<\infty,$$

where

$$\begin{aligned}\alpha(s)\equiv&\frac{1}{\kappa+\lambda(s)}-\\&\frac{1}{s+\kappa(\sqrt{\kappa^2+4}-\kappa)/2}\operatorname*{Res}_{\xi=-\kappa(\sqrt{\kappa^2+4}-\kappa)/2}\frac{1}{\kappa+\lambda(s)}.\end{aligned}$$

Case. $\Lambda\in(0,1)$**.**

Due to Lemma 3, there is one root of $p(\xi,s)=0$ in $Re(\xi)>0$ when $s>0$. Then the Dirichlet–Neumann relation (D–N) gives

$$\begin{aligned}&(s+\Lambda\lambda-\lambda^2)(\Lambda\mathbb{L}[\rho](0,s)+\mathbb{L}[u](0,s))\\&-\lambda\mathbb{L}[\rho](0,s)-\lambda(\Lambda-\lambda)\mathbb{L}[u](0,s)+\lambda\mathbb{L}[u_x](0,s)\\&=0.\end{aligned}\tag{38}$$

Together with (33), this yield that

$$\begin{cases}\mathbb{L}[u](0,s)&=-\dfrac{\lambda}{s+\kappa\lambda}\mathbb{L}[w]+\left(\Lambda\dfrac{\lambda^2-s}{s+\kappa\lambda}+(1-\Lambda^2)\dfrac{\lambda}{s+\kappa\lambda}\right)\mathbb{L}[\rho](0,s),\\\mathbb{L}[u_x](0,s)&=\mathbb{L}[w](s)+\kappa\mathbb{L}[u](0,s).\end{cases}\tag{39}$$

Lemma 5. *For each given* $\kappa\in\mathbb{R}$*, there exists* $\delta_0>0$ *such that the functions* $\lambda/(s+\kappa\lambda)$ *and* $(\lambda^2-s)/(s+\kappa\lambda)$ *in region* $\mathrm{Re}(s)>-\delta_0$ *are:*

$$\begin{cases}\text{analytic functions when } \kappa>\Lambda-1;\\\text{meromorphic functions with only one pole in } \mathrm{Re}(s)>-\delta_0 \text{ when } \kappa\le\Lambda-1;\\\text{and the pole is a simple pole at } s=-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))\in\{\mathrm{Re}(s)>0\};\end{cases}$$

and

$$\begin{cases}\operatorname*{Res}_{\xi=-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))}\dfrac{\lambda}{s+\kappa\lambda}=\left(\Lambda-2\kappa-\dfrac{\Lambda}{(\Lambda-\kappa)^2}\right),\\\operatorname*{Res}_{\xi=-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))}\dfrac{\lambda^2-s}{s+\kappa\lambda}=\left(\Lambda-\dfrac{1}{\Lambda-\kappa}\right)\left(\Lambda-2\kappa-\dfrac{\Lambda}{(\Lambda-\kappa)^2}\right).\end{cases}\tag{40}$$

Furthermore, for any $\kappa\in\mathbb{R}$ *the functions* $\lambda/(s+\kappa\lambda)$ *and* $(\lambda^2-s)/(s+\kappa\lambda)$ *satisfy*

$$\lim_{s\to\pm i\infty}\left|\frac{\lambda}{s+\kappa\lambda}\sqrt{s}\right|,\ \lim_{s\to\pm i\infty}\left|\frac{\lambda^2-s}{s+\kappa\lambda}\sqrt{s}\right|<\infty.$$

When $\kappa < \Lambda - 1$, we denote the analytic parts of the meromorphic functions by $\alpha(s)$ and $\beta(s)$:

$$\alpha(s) \equiv \frac{\lambda}{s+\kappa\lambda} - \frac{1}{s+\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))} \operatorname*{Res}_{\xi=-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))} \frac{\lambda}{s+\kappa\lambda}, \tag{41}$$

$$\beta(s) \equiv \frac{\lambda^2-s}{s+\kappa\lambda} - \frac{1}{s+\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))} \operatorname*{Res}_{\xi=-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))} \frac{\lambda^2-s}{s+\kappa\lambda}. \tag{42}$$

The functions $\alpha(s)$ and $\beta(s)$ are analytic in $\mathrm{Re}(s) > -\delta_0$ and satisfy

$$\lim_{s\to\pm i\infty} |\alpha(s)\sqrt{s}|,\ |\beta(s)\sqrt{s}| < \infty.$$

In particular, when $\kappa = \Lambda - 1$,

$$\frac{\lambda}{s+\kappa\lambda} = \frac{2(1-\Lambda)}{s} + \left(\frac{\lambda}{s+\kappa\lambda} - \frac{2(1-\Lambda)}{s}\right), \tag{43}$$

$$\frac{\lambda^2-s}{s+\kappa\lambda} = \frac{-2(1-\Lambda)^2}{s} + \left(\frac{\lambda^2-s}{s+\kappa\lambda} - \frac{-2(1-\Lambda)^2}{s}\right). \tag{44}$$

Case. $\Lambda \in (-1, 0)$**.**

Due to Lemma 3, there are two roots of $p(\xi, s) = 0$ in $\mathrm{Re}(\xi) > 0$ when $s > 0$. One denotes these two roots by $\xi_\Lambda(s)$ and $\lambda_+(s)$, and $\lambda_-(s)$ is the root in $\mathrm{Re}(\lambda_-(s)) < 0$. The asymptotic of $\lambda_-(s)$ are

$$\begin{cases} \lambda_-(s) = -\dfrac{s}{1+\Lambda} + O(1)s^2 & \text{as } |s| \to 0, \\ \lambda_-(s) = -\sqrt{s} + \dfrac{\Lambda}{2} - \dfrac{-4+\Lambda^2}{8\sqrt{s}} + \dfrac{O(1)}{s^{3/2}} & \text{as } |s| \to \infty. \end{cases} \tag{45}$$

The properties that $\mathrm{Re}(\xi_\Lambda) > 0$ and $\mathrm{Re}(\lambda_+) > 0$ lead to (D–N):

$$\begin{cases} (s+\Lambda\lambda_+ - \lambda_+^2)(\Lambda\mathbb{L}[\rho](0,s) + \mathbb{L}[u](0,s)) - \\ \lambda_+\mathbb{L}[\rho](0,s) - \lambda_+(\Lambda-\lambda_+)\mathbb{L}[u](0,s) + \lambda_+\mathbb{L}[u_x](0,s) = 0, \\ (s+\Lambda\xi_\Lambda - \xi_\Lambda^2)(\Lambda\mathbb{L}[\rho](0,s) + \mathbb{L}[u](0,s)) - \\ \xi_\Lambda\mathbb{L}[\rho](0,s) - \xi_\Lambda(\Lambda-\xi_\Lambda)\mathbb{L}[u](0,s) + \xi_\Lambda\mathbb{L}[u_x](0,s) = 0. \end{cases} \tag{46}$$

By combining (D–N) and the Robin boundary condition, and using Vieta's relation, one obtains the full boundary data in the transform variable

$$\begin{cases} \mathbb{L}[\rho](0,s) &= -\dfrac{\lambda_-}{(\lambda_- - \kappa)(s+\Lambda\lambda_-)}\mathbb{L}[w], \\ \mathbb{L}[u](0,s) &= \dfrac{1}{\lambda_- - \kappa}\mathbb{L}[w], \\ \mathbb{L}[u_x](0,s) &= \left(1 + \kappa\dfrac{1}{\lambda_- - \kappa}\right)\mathbb{L}[w]. \end{cases} \tag{47}$$

By similar arguments as in Lemma 5, we obtain that:

Lemma 6. *The functions $\lambda_-/((\lambda_- - \kappa)(s + \Lambda\lambda_-))$ and $1/(\lambda_- - \kappa)$ are meromorphic functions in* $\mathrm{Re}(s) > -\delta_0$. *Furthermore when $\kappa > 0$ it is analytic ; when $\kappa \le 0$ it has a simple pole at $s = -\kappa\left(2\Lambda + \sqrt{4+\kappa^2} - \kappa\right)/2$. They both satisfy*

$$\lim_{s\to\pm i\infty}\left|\frac{\lambda_-}{(\lambda_- - \kappa)(s+\Lambda\lambda_-)}\, s\right|, \ \lim_{s\to\pm i\infty}\left|\frac{1}{\lambda_- - \kappa}\,\sqrt{s}\right| < \infty.$$

When $\kappa \le 0$, let

$$\begin{aligned}\alpha(s) \equiv{}& \frac{\lambda_-}{(\lambda_- - \kappa)(s+\Lambda\lambda_-)} - \left(1 + \frac{\Lambda\kappa}{2} + \frac{\Lambda\sqrt{4+\kappa^2}}{2} - \frac{\kappa}{\sqrt{4+\kappa^2}}\right)\\ &\times \frac{1}{s + \kappa\left(2\Lambda + \sqrt{4+\kappa^2} - \kappa\right)/2} \qquad (48)\end{aligned}$$

$$\begin{aligned}\beta(s) \equiv{}& \frac{1}{\lambda_- - s} - \left(\kappa - \Lambda - \frac{2}{\sqrt{4+\kappa^2}} - \frac{\kappa^2}{\sqrt{4+\kappa^2}}\right)\\ &\times \frac{1}{s + \kappa\left(2\Lambda + \sqrt{4+\kappa^2} - \kappa\right)/2}. \qquad (49)\end{aligned}$$

The functions $\alpha(s)$ and $\beta(s)$ are analytic in $\mathrm{Re}(s) > -\delta_0$ and satisfy

$$\lim_{s\to\pm i\infty} |s\alpha(s)|,\ |\beta(s)\sqrt{s}| < \infty.$$

D. Conversion from s to t.

With the asymptotic and analytic properties of the full boundary data in the transform variable, we can apply complex analysis such as that in Theorem 1 to obtain all the boundary data in physical variables with exponentially sharp estimates.

Theorem 2. *Let $\Lambda \in (-1, 1)$ and κ be given parameters, and $w(t) \in \mathscr{V}$ be the boundary input for the problem (27). The boundary data of $V(0, t)$ satisfy the following properties:*

Case. $\Lambda \in (0, 1)$, $\kappa > \Lambda - 1$, *and $\rho(0, t)$ is given.*

There exists $C_0 > 0$ such that the Dirichlet and Neumann data of $u(0, t)$ satisfy

$$\begin{aligned}|u(0,t)| &\le O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}(|w(\tau)| + |\rho(0,\tau)|)\,\mathrm{d}\tau\\ \times\ |u_x(0,t) - w(t)| &\le O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}(|w(\tau)| + |\rho(0,\tau)|)\,\mathrm{d}\tau. \qquad (50)\end{aligned}$$

Case. $\Lambda \in (0, 1)$, $\kappa \le \Lambda - 1$, $\rho(0, t)$ *is given. Then*

$$\mathrm{Re}\,(-\kappa(\Lambda - \kappa - 1/(\Lambda - \kappa))) > 0,$$

and there exists $C_0 > 0$ *such that*

$$\begin{aligned}
&\Bigg| u(0,t) + \left(\Lambda - 2\kappa - \frac{\Lambda}{(\Lambda-\kappa)^2}\right); \\
&\quad \times \int_0^t \mathrm{e}^{-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))\tau} \left(-w(t-\tau) + \left(1 - \frac{\Lambda}{\Lambda-\kappa}\right)\rho(0,t-\tau)\right) \mathrm{d}\tau \Bigg| \\
&\le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} (|w(\tau)| + |\rho(0,\tau)|)\, \mathrm{d}\tau, \qquad (51)\\
&\Bigg| u_x(0,t) - w(t) + \kappa\left(\Lambda - 2\kappa - \frac{\Lambda}{(\Lambda-\kappa)^2}\right); \\
&\quad \times \int_0^t \mathrm{e}^{-\kappa(\Lambda-\kappa-1/(\Lambda-\kappa))\tau} \left(-w(t-\tau) + \left(1 - \frac{\Lambda}{\Lambda-\kappa}\right)\rho(0,t-\tau)\right) \mathrm{d}\tau \Bigg| \\
&\le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} (|w(\tau)| + |\rho(0,\tau)|)\, \mathrm{d}\tau. \qquad (52)
\end{aligned}$$

Case. $\Lambda \in (-1, 0]$, $\kappa > 0$.

There exists $C_0 > 0$ *such that*

$$\begin{aligned}
|\rho(0,t)| &\le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} |w(\tau)|\, \mathrm{d}\tau, \\
|u(0,t)| &\le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} |w(\tau)|\, \mathrm{d}\tau, \\
|u_x(0,t) - w(t)| &\le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} |w(\tau)|\, \mathrm{d}\tau. \qquad (53)
\end{aligned}$$

Case. $\Lambda \in (-1, 0]$, $\kappa \le 0$. *Then*

$$\mathrm{Re}\left(-\kappa/2\left(-\kappa + 2\Lambda + \sqrt{4+\kappa^2}\right)\right) > 0,$$

and there exists $C_0 > 0$ *such that*

$$\begin{aligned}
&\Bigg| \rho(0,t) - \left(1 + \frac{\Lambda\kappa}{2} + \frac{\Lambda\sqrt{4+\kappa^2}}{2} - \frac{\kappa}{\sqrt{4+\kappa^2}}\right) \\
&\quad \times \int_0^t \mathrm{e}^{-\kappa/2\left(-\kappa+2\Lambda+\sqrt{4+\kappa^2}\right)\tau} w(t-\tau)\, \mathrm{d}\tau \Bigg| \le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} |w(\tau)|\, \mathrm{d}\tau, \qquad (54)\\
&\Bigg| u(0,t) - \left(\kappa - \Lambda - \frac{2}{\sqrt{4+\kappa^2}} - \frac{\kappa^2}{\sqrt{4+\kappa^2}}\right) \\
&\quad \times \int_0^t \mathrm{e}^{-\kappa/2\left(-\kappa+2\Lambda+\sqrt{4+\kappa^2}\right)\tau} w(t-\tau)\, \mathrm{d}\tau \Bigg| \le O(1) \int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}} |w(\tau)|\, \mathrm{d}\tau, \qquad (55)
\end{aligned}$$

$$\left|u_x(0,t) - w(t) - \kappa\left(\kappa - \Lambda - \frac{2}{\sqrt{4+\kappa^2}} - \frac{\kappa^2}{\sqrt{4+\kappa^2}}\right)\right.$$
$$\left.\times \int_0^t \mathrm{e}^{-\kappa/2\left(-\kappa+2\Lambda+\sqrt{4+\kappa^2}\right)\tau} w(t-\tau)\,\mathrm{d}\tau\right| \le O(1)\int_0^t \frac{\mathrm{e}^{-\frac{t-\tau}{C_0}}}{\sqrt{t-\tau}}|w(\tau)|\,\mathrm{d}\tau. \quad (56)$$

Remark 2. When $\Lambda \le 0$, one can not impose conditions on $w(t)$ and $\rho(0,t)$ simultaneously.

Remark 3. Notice that in the cases $\{\Lambda \in (0,1)\} \wedge \{\kappa < \Lambda - 1\}$ and case $\{\Lambda \in (-1,0]\} \wedge \{\kappa < 0\}$, the boundary data contain a time exponentially growing component; one can consider $\kappa = \Lambda - 1$ and $\kappa = 0$ as bifurcation points for $\Lambda \in (0,1)$ and $\Lambda \in (-1,0]$, respectively.

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