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Global Well-Posedness of Compressible Navier–Stokes Equation with $BV \cap L^1$ Initial Data

Haitao Wang, Shih-Hsien Yu & Xiongtao Zhang

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Abstract

The purpose of this paper is to study the well-posedness problem for weak solutions of Navier–Stokes equations in gas dynamics. We consider rough initial data, in $BV \cap L^1$. The well-posedness theory of Liu and Yu (Commun Pure Appl Math 75(2):223–348, 2022) for the isentropic Navier–Stokes equations is extended to the Navier–Stokes equations with an additional equation for the conservation of energy. A key step is to treat the energy equation as mainly for the dissipation of the temperature. The dissipation is analyzed through the heat kernel with BV variable coefficient constructed in Liu and Yu (2022). This step is natural from the physical point of view, but estimates for the temperature are required to be sufficiently robust for the validity of the conservation of energy in the weak sense; for this, we establish the regularity of the solutions, particularly the estimates of their time derivatives through refined estimates of the heat kernel.

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1. Introduction

Consider the compressible Navier–Stokes (NS for short) equations in Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\frac{\mu u_x}{v}\right)_x, \\ (e + \frac{1}{2}u^2)_t + (pu)_x = \left(\frac{\kappa}{v}\theta_x + \frac{\mu}{v}uu_x\right)_x. \end{cases} \tag{1.1}$$

Here v is the specific volume, u is the velocity, e is the specific internal energy, θ is the temperature, μ and κ are the viscosity and heat conductivity coefficients, respectively, and are assumed to be positive constants. We consider the ideal gases

$$p(v,\theta) = \frac{K\theta}{v}, \qquad e = c_v\theta, \tag{1.2}$$

where K, and heat capacity c_v are both positive constants. This system can also be written in terms of (v, u, θ):

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\frac{\mu u_x}{v}\right)_x, \\ \theta_t + \frac{p}{c_v}u_x - \frac{\mu}{c_v v}(u_x)^2 = \left(\frac{\kappa}{c_v v}\theta_x\right)_x. \end{cases} \tag{1.3}$$

The main purpose of this work is to study the well-posedness and time-asymptotic behavior of system (1.3) with initial data being a rough perturbation around a constant state. Without loss of generality, the constant state is assumed to be $(v, u, \theta) = (1, 0, 1)$. The initial data $(v, u, \theta)(x, 0) = (v_0, u_0, \theta_0)(x)$ is given to satisfy

$$\|v_0 - 1\|_{L^1_x} + \|v_0\|_{BV} + \|u_0\|_{L^1_x} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{L^1_x} + \|\theta_0\|_{BV} < \delta \ll 1, \tag{1.4}$$

where L^1_x denotes the L^1 norm in the space variable x, and $\|\cdot\|_{BV}$ denotes the total variation norm.

Our main results are stated as follows:

Theorem 1.1. (Local existence and regularity, Theorem 4.1) *Suppose that the initial data for* (1.3) *satisfies* (1.4). *Then, there exist positive constants $t_\sharp$ and $C_\sharp$ such that the system* (1.3) *admits a weak solution (v, u, θ) for $t \in (0, t_\sharp)$ satisfying*

$$\begin{cases} \max\left\{ \begin{array}{l} \|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x}, \\ \sqrt{t}\,\|u_t(\cdot,t)\|_{L^1_x},\ t\,\|u_t(\cdot,t)\|_{L^\infty_x} \end{array} \right\} \leqq 2C_\sharp\delta, \\ \max\left\{ \begin{array}{l} \|\theta(\cdot,t)-1\|_{L^1_x},\ \|\theta(\cdot,t)-1\|_{L^\infty_x},\ \|\theta_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|\theta_x(\cdot,t)\|_{L^\infty_x}, \\ \sqrt{t}\,\|\theta_t(\cdot,t)\|_{L^1_x},\ t\,\|\theta_t(\cdot,t)\|_{L^\infty_x} \end{array} \right\} \leqq 2C_\sharp\delta, \\ \max\left\{\|v(\cdot,t)\|_{BV},\ \|v(\cdot,t)-1\|_{L^1_x},\ \|v(\cdot,t)-1\|_{L^\infty_x},\ \sqrt{t}\,\|v_t(\cdot,t)\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ v-1 = v_c^* + v_d^*, \quad v_d^*(x,t) = \sum_{z<x, z\in\mathscr{D}} [v](z)h(x-z), \quad v_c^* \text{ is continuous}, \\ \left| v(\cdot,t)\Big|_{x=z^-}^{x=z^+} \right| \leqq 2\left| v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+} \right|, \quad z\in\mathscr{D}, \end{cases}$$

where $h(x)$ is the Heaviside step function, $\mathscr{D}$ is the discontinuity set of v_0; Moreover, the fluxes of u and θ, (i.e. $\frac{\mu u_x}{v} - p$ and $\frac{\kappa}{c_v v}\theta_x - \int_{-\infty}^x \left(\frac{p}{c_v}u_z - \frac{\mu}{c_v v}(u_z)^2\right) dz$), are both globally Lipschitz continuous with respect to x for any $t>0$; and the specific volume $v(x,t)$ has the following Hölder continuous properties for $0 \leqq s < t$,

$$\begin{cases} \|v(\cdot,t)-v(\cdot,s)\|_{BV} \leqq 2C_\sharp\delta \dfrac{(t-s)|\log(t-s)|}{\sqrt{t}}, \\ \|v(\cdot,t)-v(\cdot,s)\|_{L^\infty} \leqq 2C_\sharp\delta \dfrac{t-s}{\sqrt{t}}, \\ \|v(\cdot,t)-v(\cdot,s)\|_{L^1} \leqq 2C_\sharp\delta(t-s). \end{cases}$$

Theorem 1.2. (Stability and uniqueness, Theorem 5.1)

Suppose there are two solutions (v^a,u^a,θ^a) and (v^b,u^b,θ^b) to the Navier–Stokes equations (1.3) with the regularity properties stated in Proposition 2.1, and for a small δ_* their initial data both satisfy

$$\|v_0\|_{BV} + \|u_0\|_{BV} + \|\theta_0\|_{BV} + \|v_0-1\|_{L^1_x} + \|u_0\|_{L^1_x} + \|\theta_0-1\|_{L^1_x} < \delta_*.$$

Then, there exist $t_*>0$ and $C_\flat>0$ such that, for $0<t<t_*$,

$$\begin{aligned} \|v^a-v^b\|_{L^1_x} + \|u^a-u^b\|_{L^1_x} + \|\theta^a-\theta^b\|_{L^1_x} &\leqq C_\flat\Big(\|\theta_0^a-\theta_0^b\|_{L^\infty_x} + \|\theta_0^a-\theta_0^b\|_{L^1_x} \\ &+ \|u_0^a-u_0^b\|_{L^\infty_x} + \|u_0^a-u_0^b\|_{L^1_x} + \|v_0^a-v_0^b\|_{L^1_x} + \|v_0^a-v_0^b\|_{L^\infty_x} + \|v_0^a-v_0^b\|_{BV}\Big). \end{aligned}$$

Theorem 1.3. (Global existence, Theorem 6.1) *There exist $\delta^*>0$ and $\mathscr{C}>0$ so that for any initial data (v_0,u_0,θ_0) of (1.3) satisfying*

$$\|v_0-1\|_{L^1_x} + \|v_0\|_{BV} + \|u_0\|_{L^1_x} + \|u_0\|_{BV} + \|\theta_0-1\|_{L^1_x} + \|\theta_0\|_{BV} \leqq \varepsilon < \delta^*,$$

the solution constructed in Theorems 4.1 and 5.1 satisfies

$$\begin{aligned} &\left\|\sqrt{t+1}(v(\cdot,t)-1)\right\|_{L^\infty_x} + \left\|\sqrt{t+1}u(\cdot,t)\right\|_{L^\infty_x} + \left\|\sqrt{t+1}(\theta(\cdot,t)-1)\right\|_{L^\infty_x} \\ &+ \left\|\sqrt{t}u_x(\cdot,t)\right\|_{L^\infty_x} + \left\|\sqrt{t}\theta_x(\cdot,t)\right\|_{L^\infty_x} \leqq \mathscr{C}\varepsilon \ \text{ for } t\in(0,+\infty). \end{aligned}$$

The well-posedness problem for the compressible Navier–Stokes equations is established by Nash [10] and Itaya [4] for initial data being Hölder continuous. The system is rewritten as a nonlinear parabolic equation, where the fundamental solution for variable coefficient parabolic equation played a key role in the construction of solution. For Hölder data, the classical frozen coefficient and parametrix method is sufficient for the construction of fundamental solution. Based on the local existence result, Kanel [5] and Kazhikhov–Shelukin [6] derived a priori energy-type estimate and thus obtained the global solution. The energy method is then applied to compressible Navier–Stokes equations in 3-D to obtain the global existence by Matsumura and Nishida [9] for initial data in high order Sobolev's space, where a existence theory is provided based on estimates of constant coefficient linear parabolic system.

On the other hand, the quasi-linear and hyperbolic-parabolic nature of (1.1) allows the initial discontinuities in specific volume v to propagate in later time. Nash and Itaya's theory are not applicable when the coefficients in equations of u and θ cease to be Hölder continuous. Later, the constructions of weak solutions are studied by Hoff [2,3], Lions [7] and Feireisl [1], etc. The piecewise energy estimate is carried out and total variation estimate is obtained by Hoff [2]. The constructions in [7] and [1] are applicable to more general data, for example, in the presence of vacuum. There is no well-posedness theory for the weak solutions obtained by these approaches.

Liu and Yu [8] initiated a new approach of establishing weak solutions in a constructive way and obtained the well-posedness theory as well as properties of the solution for the isentropic Navier–Stokes equations

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\dfrac{\mu u_x}{v}\right)_x . \end{cases} \tag{1.5}$$

This is based on the construction of the fundamental solution to the heat equation with BV coefficient,

$$\begin{cases} (\partial_t - \partial_x \mu \partial_x)\mathsf{H}(x,t;y;\mu) = 0, \\ \mathsf{H}(x,0;y;\mu) = \delta(x-y), \end{cases}$$

where $\mu = \mu(x)$ is a BV function with the property

$$\inf_{x\in\mathbb{R}} \mu(x) > 0, \qquad \|\mu\|_{BV} \ll 1.$$

We will use the system (1.3) and consider the iteration scheme

$$\begin{cases} v_t^{n+1} - u_x^{n+1} = 0, \\ u_t^{n+1} + p(v^n,\theta^n)_x = \left(\dfrac{\mu u_x^{n+1}}{v^n}\right)_x, \\ \theta_t^{n+1} + \dfrac{p(v^n,\theta^n)}{c_v} u_x^n - \dfrac{\mu}{c_v v^n}(u_x^n)^2 = \left(\dfrac{\kappa}{c_v v^n}\theta_x^{n+1}\right)_x, \end{cases} \tag{1.6}$$

and initial step is set to be

$$(v^0, u^0, \theta^0) \equiv (1, 0, 1).$$

With the set-ups in Theorem 1.1 and Theorem 1.2, the second and third equations in (1.6) are inhomogeneous heat equations with BV coefficients. Therefore, we can use the heat kernels approach. The consideration of the dissipation of the temperature θ is natural from the viewpoint of physics. However, in general, the two systems (1.1) and (1.3) are not necessarily equivalent in the weak sense unless the solution fulfills appropriate regularity property. One of our main analytical efforts is to gain the required regularity property. For this, we take full advantage of heat kernel with BV conductivity to study the well-posdenss of (1.3).

Compared to the theory in [8] for the isentropic compressible Navier–Stokes equation (1.5), our theory for the full Navier–Stokes system (1.1) has several physical considerations and analytical difficulties. The second equation in (1.5) is a diffusion equation for u. As the equation is given in a conservative form, when using Green's function as a test function, the weak formulation automatically yields an integral representation of (v, u), which are convenient for transferring the derivative on nonlinear source term, and thus for investigating the time-asymptotic behaviors. However, for full NS (1.1), it is a problem to choose whether $(v, u, e + u^2/2)$ or (v, u, θ) as unknown functions. Considering the diffusion term in the third equation of (1.1), temperature θ would be a good candidate, while a non-conservative form is not convenient for studying time-asymptotic behaviors. If the solution is only constructed in distribution sense, one does not have equivalence between (1.1) and (1.3), and there is a gap between local theory and global existence.

The problem is resolved by carefully investigating the regularity of the weak solution for (1.3). We develop new Hölder in time estimates for heat kernel (see Lemma 2.3). Based on it, we show that θ is Hölder continuos in time, which helps us to prove u_t is in $L^\infty \cap L^1$. Interestingly, this in turn improves θ from Hölder continuity to differentiable in time. With this regularity, the function $(v, u, c_v\theta + u^2/2)$ is a weak solution to conservative form (1.1). This serves as a basis towards the global stability.

It is also worth mentioning that Hölder-type estimates of heat kernel is crucial even in the construction of weak solution for (1.3) due to the pressure term $p(v, \theta)$, unlike isentropic gas, which is not needed.

Another novelty of this paper is the *uniqueness* of the solution. In Theorem 1.2 of [8], the authors proved that the constructed weak solution for isentropic model depends on initial data continuously. In the current paper, from the regularity result, we identify the function space of the constructed weak solution to (1.3). More importantly, we prove *stability* of the solution in this function space, which in turn yields that given any weak solution in distribution sense, it must be identical to the one we constructed as long as it belongs to the aforementioned space.

These results largely rely on the various quantitative estimates of fundamental solution for heat equation with BV variable coefficient, which captures the quasi-linear structure of the equation (1.3), and represents the solution accurately.

The analysis for Theorem 1.3 is done in the same framework as that of [8]. One follows its procedure to replace the Green's function for a linearized 2×2

system by a 3×3 system to build "an effective Green's function", derives an integral representation, and performs a priori estimate to conclude Theorem 1.3. It should be noted that the regularity in time of velocity u and temperature θ plays an important role in the a priori estimate.

The rest of this paper is organized as follows. In Section 2, one introduces preliminary notions and prepares various refined estimates of BV coefficient heat kernel, which serves as basic tools in later analysis. In Section 3, one performs an iteration scheme and proves its convergence to construct the weak solution of (1.3). In Section 4, one studies the regularity and justifies that $(v, u, c_v\theta + u^2/2)$ is a weak solution of (1.1). In Section 5, one establishes stability and uniqueness. In Section 6 one gives the sketch of proof of Theorem 1.3. Some standard but lengthy calculations for necessary estimates are carried out in [12] without obstructing the integrality of this paper and make this paper concise.

2. Preliminaries

In this section, we will provide some preliminary concepts and results that will be used in the later sections. We first give definitions of the weak solutions and a regularity property, then we introduce various estimates of the fundamental solution to heat equation with BV conductivity coefficient, which will serve as basic tools in the construction of local well-posed theory.

Definition 2.1. A tuple (v, u, θ) is a weak solution to the equation (1.3) in the distribution sense if for any test function $\varphi(x,t) \in C_0^1(\mathbb{R}\times[0,+\infty))$,

$$\begin{cases} \int_0^{+\infty}\int_{\mathbb{R}} [\varphi_x u - \varphi_t v]\,\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \varphi(x,0)v(x,0)\mathrm{d}x, \\ \int_0^{+\infty}\int_{\mathbb{R}} \left[\varphi_x\left(\frac{\mu u_x}{v} - p\right) - \varphi_t u\right]\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \varphi(x,0)u(x,0)\mathrm{d}x, \\ \int_0^{+\infty}\int_{\mathbb{R}} \left[\varphi_x\left(\frac{\kappa}{c_v v}\theta_x\right) + \varphi\left(\frac{p}{c_v}u_x - \frac{\mu}{c_v v}(u_x)^2\right) - \varphi_t\theta\right]\mathrm{d}x\mathrm{d}t = \int_{\mathbb{R}} \varphi(x,0)\theta(x,0)\mathrm{d}x. \end{cases} \tag{2.1}$$

The weak solution of the linearized equation (1.6) can be defined similarly.

Proposition 2.1. *For the weak solution (v, u, θ) of (1.3), the function $(v, u, c_v\theta + u^2/2)$ is a weak solution of (1.3) if it satisfies*

$$\begin{cases} v(x,t) - 1 \in C\left([0,t_\sharp); L^1(\mathbb{R})\cap L^\infty(\mathbb{R})\cap BV\right), \\ u(x,t) \in L^\infty\left(0,t_\sharp; W^{1,1}(\mathbb{R})\cap L^\infty(\mathbb{R})\right), \quad \sqrt{t}u_x(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right), \\ \sqrt{t}u_t(x,t) \in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\right), \quad tu_t(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right), \\ \theta(x,t) \in L^\infty\left(0,t_\sharp; W^{1,1}(\mathbb{R})\cap L^\infty(\mathbb{R})\right), \quad \sqrt{t}\theta_x(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right), \\ \sqrt{t}\theta_t(x,t) \in L^\infty\left(0,t_\sharp; L^1(\mathbb{R})\right), \quad t\theta_t(x,t) \in L^\infty\left(0,t_\sharp; L^\infty(\mathbb{R})\right); \end{cases} \tag{2.2}$$

see Corollary 4.1 for the proof.

Next, one provides estimates for the fundamental solution $H(x,t;y,t_0;\rho)$ (heat kernel) of a heat equation with a BV function coefficient ρ in x,

$$(\partial_t + \partial_x \rho(x,t)\partial_x)u = 0.$$

A BV function f can be decomposed as the sum of a continuous part and a discrete part,

$$f(x) = f_c(x) + f_d(x), \quad f_c \text{ is continuous}, \quad f_d(x) = \sum_{\alpha\in\mathscr{D},\alpha<x} f(y)\Big|_{y=\alpha-}^{y=\alpha+} h(x-\alpha),$$

where $\mathscr{D}$ is the discontinuity set of $f(x)$ and $h(\cdot)$ is the Heaviside function. The total variation can be represented as

$$\|f\|_{BV} = \int_{\mathbb{R}} |\mathrm{d}f_c| + \sum_{\alpha\in\mathscr{D}} \left| f_d(x)\big|_{x=\alpha^-}^{x=\alpha^+} \right|. \tag{2.3}$$

Here $\int_{\mathbb{R}}|\mathrm{d}f_c|$ stands for the Lebesgue–Stieltjes integral. According to Lebesgue decomposition, the continuous function f_c contains an absolutely continuous part and a singular part (for example, the Cantor function is continuous and has only singular part). In particular, when f_c is absolutely continuous, one has

$$\int_{\mathbb{R}} |\mathrm{d}f_c| = \int_{\mathbb{R}} |\partial_x f_c|\mathrm{d}x. \tag{2.4}$$

Lebesgue–Stieltjes integral works for both absolutely continuous and singular part of f_c. The estimates of singular part usually rely on analysis of Stieltjes sum, which requires lengthy computations. Nevertheless, we found that in our calculations the estimates of the absolutely part and singular part would be similar, and the calculations for absolutely continuous part can be adapted to the singular part with only minor modifications (see (3.12), (3.18) and Remark 3.1 (3) for example). Therefore, for the readability, throughout this paper our analysis will focus on the absolutely continuous part, since the integral form (2.4) is much more convenient for estimates. Then, we claim similar estimates for the singular case without detailed proof. In the following, we adopt the following notation for the total variation of the continuous part of a BV function

$$\int_{\mathbb{R}\setminus\mathscr{D}} |\partial_x f|\mathrm{d}x \equiv \int_{\mathbb{R}} |\mathrm{d}f_c|. \tag{2.5}$$

The fundamental solution $H(x,t;y,t_0;\rho)$ is defined as a weak solution of the initial value problem:

$$\begin{cases} (\partial_t - \partial_x\rho(x,t)\partial_x)\, H(x,t;y,t_0;\rho) = 0, \quad t > t_0, \\ H(x,t_0;y,t_0;\rho) = \delta(x-y), \end{cases}; \tag{2.6}$$

i.e.

$$\int_{t_0}^{\infty}\int_{\mathbb{R}} (-\phi_t(x,t)H(x,t;y,t_0;\rho)$$

$$+\phi_x(x,t)\rho(x,t)H_x(x,t;y,t_0;\rho))\,\mathrm{d}x\mathrm{d}t = -\phi(y,t_0)$$

for all test functions ϕ; and $\rho(x,t)$ is assumed to satisfy for some positive constants $\bar{\rho}$ and δ_*,

$$\begin{cases} \|\rho(\cdot)-\bar{\rho}\|_{L^1} \leqq \delta_*, \quad \|\rho(\cdot,t)\|_{BV} \leqq \delta_*, \quad \|\rho_t(\cdot,t)\|_\infty \leqq \delta_* \max\left(\frac{1}{\sqrt{t}},1\right), \quad 0<\delta_* \ll 1, \\ \mathscr{D} \equiv \{z \mid \rho(z,t) \text{ is not continuous at } z\} \text{ is invariant in } t. \end{cases} \tag{2.7}$$

The construction $H(x,t;y,t_0;\rho)$ with ρ as a BV-function in x was introduced in [8] and we summarize the results from [8] for the construction in this paper. The new estimates obtained by the construction of [8] are listed in Lemmas 2.2 to 2.5. The detailed calculations for the new estimates will be presented in the “Appendix”.

Lemma 2.1. (Liu and Yu [8]) *Suppose the conditions of ρ in (2.7) hold. Then, there exist positive constants C_* and $t_\sharp \ll 1$ such that the weak solution of (2.6) exists and satisfies the following estimates for $t \in (t_0, t_0+t_\sharp)$*

$$|H(x,t;y,t_0;\rho)| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \tag{2.8}$$

$$|H_x(x,t;y,t_0;\rho)| + \left|H_y(x,t;y,\rho)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}, \tag{2.9}$$

$$\left|\int_{t_0}^t H_x(x,\tau;y,t_0;\rho)d\tau\right|, \left|\int_{t_0}^t H_x(x,t;y,s;\rho)ds\right| \leqq C_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}}. \tag{2.10}$$

Lemma 2.2. *Under the same consideration as in Lemma 2.1, for $t \in (t_0, t_0+t_\sharp)$ the weak solution of (2.6) satisfies*

$$\left|H_{xy}(x,t;y,t_0;\rho)\right| + |H_t(x,t;y,t_0;\rho)| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{\frac{3}{2}}}, \tag{2.11}$$

$$\left|H_{ty}(x,t;y,t_0;\rho)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^2}, \tag{2.12}$$

$$\begin{aligned} &\bigg|\int_{t_0}^t H_{xy}(x,\tau;y,t_0;\rho)d\tau - \frac{\delta(x-y)}{\rho(x,t_0)} \\ &\quad - \int_{t_0}^t \frac{\rho(x,t_0)-\rho(x,\tau)}{\rho(x,t_0)} H_{xy}(x,\tau;y,t_0;\rho)d\tau\bigg| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\bigg|\int_{t_0}^t H_{xy}(x,t;y,s;\rho)ds + \frac{\delta(x-y)}{\rho(y,t)} \\ &\quad - \int_{t_0}^t \frac{\rho(y,t)-\rho(y,s)}{\rho(y,t)} H_{xy}(x,t;y,s;\rho)ds\bigg| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \end{aligned} \tag{2.14}$$

$$\int_{t_0}^{t} H_{xx}(x,\tau;y,t_0;\rho)d\tau = -\frac{\delta(x-y)}{\rho(x,t_0)} - \frac{1}{\rho(x,t_0)}\partial_x\left[\int_{t_0}^{t}(\rho(x,\tau)-\rho(x,t_0))\,H_x(x,\tau;y,t_0;\rho)d\tau\right] + O(1)\left(|\partial_x\rho(x,t_0)|e^{-\frac{(x-y)^2}{C_*(t-t_0)}} + \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\right), \qquad \textit{for } x \notin \mathscr{D}, \tag{2.15}$$

$$\int_{t_0}^{t} H_{xxy}(x,\tau;y,t_0;\rho)d\tau = \frac{1}{\rho(x,t_0)} \times \left[\delta'(x-y) - \int_{t_0}^{t}\partial_x\Big[\big(\rho(x,\tau)-\rho(x,t_0)\big)H_{xy}(x,\tau;y,t_0;\rho)\Big]d\tau\right] - \frac{\partial_x\rho(x,t_0)}{\rho^2(x,t_0)}\left[\delta(x-y) - \int_{t_0}^{t}\big(\rho(x,\tau)-\rho(x,t_0)\big)H_{xy}(x,\tau;y,t_0;\rho)d\tau\right] + O(1)\left(|\partial_x\rho(x,t_0)|\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} + \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\right), \qquad \textit{for } x \notin \mathscr{D}, \tag{2.16}$$

$$\int_{t_0}^{t} H_t(x,t;y,s;\rho)ds = H(x,t-t_0;y;\mu^t) - \delta(x-y) + O(1)\delta_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}}, \quad \textit{where } \mu^t(\cdot) \equiv \rho(\cdot,t). \tag{2.17}$$

Notice that the estimates for the terms involving twice $x-$derivatives do not hold when $x \in \mathscr{D}$, which is due to the presence of Dirac-delta functions in H_{xx} if $x \in \mathscr{D}$. Moreover, the zeroth order estimate actually can be extended to be global in time, while the higher order estimates are obtained only for local time so far.

In addition to the pointwise estimates of heat kernel itself, the following Hölder continuity property of heat kernel is also needed in construction of local solution:

Lemma 2.3. (Hölder continuity in time) *Suppose the conditions in* (2.7) *hold for* ρ. *Then the following estimates hold when* $t_0 < s < t \ll 1$*:*

$$\|H_x(\cdot,t;y,t_0;\rho) - H_x(\cdot,s;y,t_0;\rho)\|_\infty \leqq C_*\frac{(t-s)\,|\log(t-s)|}{(s-t_0)(t-t_0)},$$

$$\|H_x(\cdot,t;y,t_0;\rho) - H_x(\cdot,s;y,t_0;\rho)\|_1 \leqq C_*\frac{(t-s)\,|\log(t-s)|}{\sqrt{s-t_0}(t-t_0)},$$

$$\left\|H_{xy}(\cdot,t;y,t_0;\rho) - H_{xy}(\cdot,s;y,t_0;\rho)\right\|_\infty \leqq C_*\frac{(t-s)\,|\log(t-s)|}{(s-t_0)^{3/2}(t-t_0)},$$

$$\left\|H_{xy}(\cdot,t;y,t_0;\rho) - H_{xy}(\cdot,s;y,t_0;\rho)\right\|_1 \leqq C_*\frac{(t-s)\,|\log(t-s)|}{(s-t_0)(t-t_0)}.$$

An iteration scheme for proving the existence of local solution leads to a consideration of the solutions to heat equations (2.6) with different heat conductivities

ρ^a and ρ^b. Thus, we need the comparison estimates between two heat kernels with different conductivity coefficients. In what follows we denote that

$$\begin{aligned} |||f|||_\infty &\equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{L^\infty_x}, \\ |||f|||_1 &\equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{L^1_x}, \\ |||f|||_{BV} &\equiv \sup_{\sigma\in(0,t_\sharp)} \|f(\cdot,\sigma)\|_{BV}. \end{aligned} \tag{2.18}$$

Lemma 2.4. (Comparison estimates, Corollaries 4.4 and 4.5 in [8]) *Suppose that the conditions in* (2.7) *hold for ρ^a and ρ^b. Then, there exist positive constants $t_\sharp \ll 1$ and C_* such that for $t \in (t_0, t_0 + t_\sharp)$*

$$\begin{aligned} &\left|H(x,t;y,t_0;\rho^b) - H(x,t;y,t_0;\rho^a)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_\infty, \\ &\left|H_x(x,t;y,t_0;\rho^a) - H_x(x,t;y,t_0;\rho^b)\right|, \\ &\left|H_y(x,t;y,t_0;\rho^a) - H_y(x,t;y,t_0;\rho^b)\right| \\ &\quad\leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0} \Big[|\log(t-t_0)| \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_\infty + \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_{BV} \\ &\qquad + \sqrt{t-t_0}\Big(\left|\left|\left|\rho^a-\rho^b\right|\right|\right|_1 + |\log t| \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\right|\right|\right|_\infty\Big)\Big], \\ &\left|\int_{t_0}^t \left[H_x(x,\tau;y,t_0;\rho^a) - H_x(x,\tau;y,t_0;\rho^b)\right] d\tau\right| \\ &\quad\leqq C_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}} \Big[\left|\left|\left|\rho^a-\rho^b\right|\right|\right|_\infty + \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_{BV} \\ &\qquad + \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_1 + \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\right|\right|\right|_\infty\Big]. \end{aligned}$$

Lemma 2.5. *Under the considerations in Lemma* 2.4*, there exist positive constants $t_\sharp \ll 1$ and C_* such that, for $t \in (t_0, t_0 + t_\sharp)$,*

$$\begin{aligned} &\left|H_{xy}(x,t;y,t_0;\rho^a) - H_{xy}(x,t;y,t_0;\rho^b)\right|, \\ &\left|H_t(x,t;y,t_0;\rho^a) - H_t(x,t;y,t_0;\rho^b)\right| \\ &\quad\leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}} \Big[|\log(t-t_0)| \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_\infty + \left|\left|\left|\rho^a-\rho^b\right|\right|\right|_{BV} \\ &\qquad + \sqrt{t-t_0}\Big(\left|\left|\left|\rho^a-\rho^b\right|\right|\right|_1 + |\log t| \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\right|\right|\right|_\infty\Big)\Big], \\ &\left|\int_{t_0}^t \left[H_y(x,t;y,s;\rho^a) - H_y(x,t;y,s;\rho^b)\right] ds\right| \end{aligned}$$

$$\leqq C_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}} \Big[\big\|\big\|\rho^a - \rho^b\big\|\big\|_\infty + \big\|\big\|\rho^a - \rho^b\big\|\big\|_{BV} + \big\|\big\|\rho^a - \rho^b\big\|\big\|_1 + \Big\|\Big\|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a - \rho^b)\Big\|\Big\|_\infty \Big],$$

$$\left| \int_{t_0}^t \Big[H_x(x,t;y,s;\rho^a) - H_x(x,t;y,s;\rho^b) \Big] ds \right|,$$

$$\left| \int_{t_0}^t \Big[H_y(x,\tau;y,t_0;\rho^a) - H_y(x,\tau;y,t_0;\rho^b) \Big] d\tau \right|$$

$$\leqq C_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}} \Big[\big\|\big\|\rho^a - \rho^b\big\|\big\|_\infty + \big\|\big\|\rho^a - \rho^b\big\|\big\|_{BV} + \big\|\big\|\rho^a - \rho^b\big\|\big\|_1 + \Big\|\Big\|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a - \rho^b)\Big\|\Big\|_\infty \Big],$$

$$\int_{t_0}^t \Big[H_{xy}(x,\tau;y,t_0;\rho^a) - H_{xy}(x,\tau;y,t_0;\rho^b) \Big] d\tau$$

$$= \left[\frac{1}{\rho^a(x,t_0)} - \frac{1}{\rho^b(x,t_0)} \right] \delta(x-y)$$

$$- \int_{t_0}^t \left[\frac{\rho^a(x,\tau) - \rho^a(x,t_0)}{\rho^a(x,t_0)} H_{xy}(x,\tau;y,t_0;\rho^a) - \frac{\rho^b(x,\tau) - \rho^b(x,t_0)}{\rho^b(x,t_0)} H_{xy}(x,\tau;y,t_0;\rho^b) \right] d\tau$$

$$+ \ O(1) \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} \Big[|\log(t-t_0)| \big\|\big\|\rho^a - \rho^b\big\|\big\|_\infty + \big\|\big\|\rho^a - \rho^b\big\|\big\|_{BV} + \big\|\big\|\rho^a - \rho^b\big\|\big\|_1 + \Big\|\Big\|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a - \rho^b)\Big\|\Big\|_\infty \Big],$$

$$\int_{t_0}^t \Big[H_{xy}(x,t;y,s;\rho^a) - H_{xy}(x,t;y,s;\rho^b) \Big] ds$$

$$= \left[\frac{1}{\rho^a(y,t)} - \frac{1}{\rho^b(y,t)} \right] \delta(x-y)$$

$$+ \int_{t_0}^t \left[\frac{\rho^a(y,t) - \rho^a(y,s)}{\rho^a(y,t)} H_{xy}(x,t;y,s;\rho^a) - \frac{\rho^b(y,t) - \rho^b(y,s)}{\rho^b(y,t)} H_{xy}(x,t;y,s;\rho^b) \right] ds$$

$$+ O(1) \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} \Big[|\log(t-t_0)| \big\|\big\|\rho^a - \rho^b\big\|\big\|_\infty + \big\|\big\|\rho^a - \rho^b\big\|\big\|_{BV} + \big\|\big\|\rho^a - \rho^b\big\|\big\|_1 + \Big\|\Big\|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a - \rho^b)\Big\|\Big\|_\infty \Big].$$

Lastly, according to the symmetry of the heat equation, the following symmetric properties of the heat kernel hold; their proofs are direct consequences of the proper integrals of the equation (2.6) and are therefore omitted.

Lemma 2.6. *For the heat equation in conservative form* (2.6)*, the solution has the following properties:*

$$\begin{cases} \int_{\mathbb{R}} H(x,t;y,\tau;\rho)dx = \int_{\mathbb{R}} H(x,t;y,\tau;\rho)dy = 1, \\ \int_{\mathbb{R}} H_x(x,t;y,\tau;\rho)dx = \int_{\mathbb{R}} H_x(x,t;y,\tau;\rho)dy = 0, \\ \int_{\mathbb{R}} H_y(x,t;y,\tau;\rho)dx = \int_{\mathbb{R}} H_y(x,t;y,\tau;\rho)dy = 0, \\ \int_{\mathbb{R}} H_t(x,t;y,\tau;\rho)dx = \int_{\mathbb{R}} H_t(x,t;y,\tau;\rho)dy = 0, \\ \int_{\mathbb{R}} H_\tau(x,t;y,\tau;\rho)dx = \int_{\mathbb{R}} H_\tau(x,t;y,\tau;\rho)dy = 0. \end{cases}$$

Remark 2.1. Following the observations in [8], we make the following two remarks:

(1) In order to balance the equation (2.6), $\rho(x,t)H_x(x,t;y,t_0;\rho)$ is actually continuous with respect to x. When consider the backward equation, one also has $\rho(y,t_0)H_y(x,t;y,t_0;\rho)$ is continuous with respect to y.
(2) The weak solution of the heat equation (2.6) can be defined similarly as Definition 2.1. In fact, if the equation (2.6) has a source term in the following conservative form,

$$u_t(x,t) = \left(\rho(x,t)u_x(x,t) + g(x,t)\right)_x,$$

then the mild solution constructed by heat kernel and Duhamel's principle is also a weak solution to the above equation in the distribution sense, provided $g(x,t)$ is a BV function with respect to x. Furthermore, the flux term $(\rho(x,t)u_x(x,t) + g(x,t))$ is continuous with respect to x if one of the following two properties holds:
(a) $g(x,t)$ is Lipschitz continuous with respect to x, i.e.,

$$\|g_x(\cdot,t)\|_\infty < +\infty.$$

(b) $g(x,t)$ is Hölder continuous with respect to t in the sense that

$$|g(x,t) - g(x,s)| \leqq C\frac{(t-s)^\alpha}{s^\alpha}, \quad 0 < s < t, \quad 0 < \alpha < 1.$$

3. Local Existence

In this section, we will use the linear system (1.6) to design an iteration scheme to approximate the solution of (1.3). By making full use of the refined estimates of heat kernel in Section 2, we are able to prove that the sequence of approximate solutions form a Cauchy sequence in an appropriate topology, which yields a weak solution to (1.3). Here we remark that in general, a weak solution of (1.3) is not necessarily a weak solution of (1.1) due to the nonlinearity. In next section, after obtaining enough regularity of the weak solution obtained here, one can show it is indeed a solution to (1.1).

Consider

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \left(\dfrac{\mu u_x}{v}\right)_x, \\ \theta_t + \dfrac{p}{c_v} u_x - \dfrac{\mu}{c_v v}(u_x)^2 = \left(\dfrac{\kappa}{c_v v}\theta_x\right)_x, \\ (v, u, \theta)\Big|_{t=0} = (v_0, u_0, \theta_0), \end{cases} \tag{3.1}$$

where the initial data is a small perturbation around the constant state $(1, 0, 1)$. We set

$$v_0 = 1 + v_0^*, \quad u_0 = u_0^*, \quad \theta_0 = 1 + \theta_0^*,$$

and assume the initial perturbation $(v_0^*, u_0^*, \theta_0^*)$ satisfies the smallness condition

$$\|v_0^*\|_{BV\cap L^1} + \|u_0^*\|_{BV\cap L^1} + \|\theta_0^*\|_{BV\cap L^1} \leqq \delta \ll 1. \tag{3.2}$$

The L^∞ norm of a L^1-integrable function can be bounded by its BV norm. One actually knows from (3.2) that the L^∞ norm of the initial data is bounded by δ as well.

3.1. Iteration scheme

Following Itaya [4], we use (1.6) and (3.1) to construct an iteration as follows:

$$\begin{cases} V_t^{n+1} - U_x^{n+1} = 0, \\ U_t^{n+1} - \left(\dfrac{\mu U_x^{n+1}}{1+V^n}\right)_x = -p(1+V^n, 1+\Theta^n)_x, \\ \Theta_t^{n+1} - \left(\dfrac{\kappa \Theta_x^{n+1}}{c_v(1+V^n)}\right)_x = -\dfrac{p(1+V^n, 1+\Theta^n)}{c_v} U_x^n + \dfrac{\mu}{c_v(1+V^n)}\left(U_x^n\right)^2, \\ (V^{n+1}, U^{n+1}, \Theta^{n+1})\Big|_{t=0} = (v_0^*, u_0^*, \theta_0^*), \\ (V^0, U^0, \Theta^0) = (0, 0, 0). \end{cases} \tag{3.3}$$

The last equality in (3.3) means that we choose the initial step to be the unperturbed constant state. Clearly the right-side of (3.3) vanishes for $n = 0$, so one has

$$\begin{cases} V_t^1 - U_x^1 = 0, \\ U_t^1 - \left(\mu U_x^1\right)_x = 0, \\ \Theta_t^1 - \left(\dfrac{\kappa \Theta_x^1}{c_v}\right)_x = 0, \\ (V^1, U^1, \Theta^1)\Big|_{t=0} = (v_0^*, u_0^*, \theta_0^*). \end{cases} \tag{3.4}$$

This equation is solved by using heat kernel with constant heat conductivity, and one immediately has the following estimates:

Lemma 3.1. *Suppose the initial data* $(v_0^*, u_0^*, \theta_0^*)$ *satisfies the condition* (3.2). *Then, there exists a positive constant* $C_\sharp$ *such that, the following estimates hold for the solution to equations* (3.4)*:*

$$\begin{aligned}
&\max\left\{\left\|U^1(\cdot,t)\right\|_{L_x^1},\ \left\|U^1(\cdot,t)\right\|_{L_x^\infty},\ \left\|U_x^1(\cdot,t)\right\|_{L_x^1},\right.\\
&\qquad\left.\sqrt{t}\left\|U_x^1(\cdot,t)\right\|_{L_x^\infty},\ t\left\|U_t^1(\cdot,t)\right\|_{L_x^\infty}\right\} \leqq C_\sharp\delta, && 0 < t < t_\sharp,\\
&\max\left\{\left\|\Theta^1(\cdot,t)\right\|_{L_x^1},\ \left\|\Theta^1(\cdot,t)\right\|_{L_x^\infty},\ \left\|\Theta_x^1(\cdot,t)\right\|_{L_x^1},\right.\\
&\qquad\left.\sqrt{t}\left\|\Theta_x^1(\cdot,t)\right\|_{L_x^\infty},\ t\left\|\Theta_t^1(\cdot,t)\right\|_{L_x^\infty}\right\} \leqq C_\sharp\delta, && 0 < t < t_\sharp,\\
&\max\left\{\sqrt{t}\left\|V_t^1(\cdot,t)\right\|_{L_x^\infty},\ \|V^1(\cdot,t)\|_{BV},\right.\\
&\qquad\left.\left\|V^1(x,t)\right\|_{L_x^1},\ \left\|V^1(x,t)\right\|_{L_x^\infty}\right\} \leqq C_\sharp\delta, && 0 < t < t_\sharp,\\
&\|V^1(\cdot,t) - V^1(\cdot,s)\|_{BV} \leqq C_\sharp \frac{t-s}{\sqrt{t}}\delta, && 0 \leqq s \leqq t < t_\sharp,\\
&\left|V^1(\cdot,t)\Big|_{x=z^-}^{x=z^+}\right| = \left|v_0^*(\cdot)\Big|_{x=z^-}^{x=z^+}\right|, && z \in \mathscr{D},\quad 0 < t < t_\sharp,
\end{aligned}$$

where $t_\sharp$ *is a sufficiently small positive number constructed in Lemma 2.1,* $\mathscr{D}$ *is the discontinuity set of* v_0^**.*

To obtain a uniform estimates for the sequence of approximate solutions to (3.3) by induction, we propose the ansatz that, for all $n \leqq k$, the equation (3.3) admits a solution (V^n, U^n, Θ^n) with the following estimates:

$$\begin{cases} 0<\delta,\ t_\sharp \ll 1, \quad 1 \leqq n \leqq k, \quad 0<t<t_\sharp, \\ \max\left\{ \left\| U^n(\cdot,t)\right\|_{L^1_x},\ \left\| U^n(\cdot,t)\right\|_{L^\infty_x},\ \left\| U^n_x(\cdot,t)\right\|_{L^1_x},\ \sqrt{t}\left\| U^n_x(\cdot,t)\right\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ \max\left\{ \left\| \Theta^n(\cdot,t)\right\|_{L^1_x},\ \left\| \Theta^n(\cdot,t)\right\|_{L^\infty_x},\ \left\| \Theta^n_x(\cdot,t)\right\|_{L^1_x},\ \sqrt{t}\left\| \Theta^n_x(\cdot,t)\right\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ \max\left\{ \| V^n(\cdot,t)\|_{BV},\ \left\| V^n(\cdot,t)\right\|_{L^1_x},\ \left\| V^n(\cdot,t)\right\|_{L^\infty_x},\ \sqrt{t}\left\| V^n_t(\cdot,t)\right\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ \| V^n(\cdot,t)-V^n(\cdot,s)\|_{BV} \leqq 2C_\sharp\delta\dfrac{(t-s)|\log(t-s)|}{\sqrt{t}}, \quad 0 \leqq s<t, \\ \left| V^n(\cdot,t)\Big|_{x=z^-}^{x=z^+}\right| \leqq 2\left| v^*_0(\cdot)\Big|_{x=z^-}^{x=z^+}\right|, \quad z\in\mathscr{D}. \end{cases} \tag{3.5}$$

The above ansatz is motivated by Lemma 3.1, which implies that it holds for the initial step. In what follows, we will show that $(V^{k+1}, U^{k+1}, \Theta^{k+1})$ also satisfies the ansatz (3.5). For simplicity of presentation, we introduce the following notations for (3.3):

$$\begin{aligned} \mu^k &\equiv \frac{\mu}{1+V^k}, \quad \mathscr{N}^k_1(x,t) \equiv -\partial_x p(1+V^k, 1+\Theta^k), \\ \kappa^k &\equiv \frac{\kappa}{c_v\left(1+V^k\right)}, \\ \mathscr{N}^k_2(x,t) &\equiv -\frac{p(1+V^k,1+\Theta^k)}{c_v}U^k_x + \frac{\mu}{c_v\left(1+V^k\right)}\left(U^k_x\right)^2. \end{aligned} \tag{3.6}$$

With the ansatz (3.5), U^{k+1} and Θ^{k+1} of (3.3) are governed by heat equation with BV conductivity and source. We will study this by applying the estimates of heat kernel in Section 2. According to Lemma 2.1 and Remark 2.1, we can apply Duhamel's principle to construct the weak solution $(V^{k+1}, U^{k+1}, \Theta^{k+1})$ to equation (3.3) as follows:

$$\begin{aligned} U^{k+1}(x,t) &= \int_{\mathbb{R}} H\left(x,t;y,0;\mu^k\right)u^*_0(y)\mathrm{d}y \\ &\quad + \int_0^t\int_{\mathbb{R}\setminus\mathscr{D}} H_y\left(x,t;y,s;\mu^k\right)p(1+V^k,1+\Theta^k)\mathrm{d}y\mathrm{d}s, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \Theta^{k+1}(x,t) &= \int_{\mathbb{R}} H\left(x,t;y,0;\kappa^k\right)\theta^*_0(y)\mathrm{d}y \\ &\quad + \int_0^t\int_{\mathbb{R}} H\left(x,t;y,s;\kappa^k\right)\mathscr{N}^k_2(y,s)\mathrm{d}y\mathrm{d}s. \end{aligned} \tag{3.8}$$

From Lemma 2.1 and Remark 2.1, the integral representations (3.7) and (3.8) yield a weak solution of (3.3).

The following Lemmas 3.2, 3.3, and 3.4 are devoted to justify the ansatz (3.5) for Θ^{k+1}, U^{k+1} and V^{k+1} respectively. We assume that the initial data $(v^*_0, u^*_0, \theta^*_0)$ satisfy the condition (3.2), and that the ansatz (3.5) holds for $n \leqq k$.

Lemma 3.2. *For sufficiently small δ and $t_\sharp$, the ansatz* (3.5) *and the following time difference estimates hold for Θ^{k+1} when $0<s\leqq t<t_\sharp$:*

$$\|\Theta^{k+1}(\cdot,t)-\Theta^{k+1}(\cdot,s)\|_{L^\infty_x}$$

$$\leqq O(1)\left((\delta+\delta^2)\frac{t-s}{\sqrt{s}\sqrt{t}}+\delta\frac{t-s}{\sqrt{t}}+\delta\sqrt{t-s}+\delta^2\frac{\sqrt{t-s}}{\sqrt{s}}+\delta^2\frac{(t-s)^{\frac{1}{4}}}{s^{\frac{1}{4}}}\right),$$

$$\|\Theta^{k+1}(\cdot,t)-\Theta^{k+1}(\cdot,s)\|_{L^1_x}$$

$$\leqq O(1)\left(\delta\frac{t-s}{\sqrt{s}\sqrt{t}}+\delta(t-s)+\delta^2\frac{t-s}{\sqrt{t}}+\sqrt{t-s}\sqrt{s}\delta+\sqrt{t-s}\delta^2\right),$$

$$\|\Theta^{k+1}_x(\cdot,t)-\Theta^{k+1}_x(\cdot,s)\|_{L^1_x}$$

$$\leqq O(1)\left(\frac{\sqrt{t-s}|\log(t-s)|}{\sqrt{t}\sqrt{s}}\delta+\sqrt{t-s}\delta+\frac{\sqrt{t-s}}{\sqrt{t}}\delta^2+\sqrt{t-s}|\log(t-s)|\delta^2\right).$$

Proof. The proof consists of several estimates:

- (**Estimate of** $\|\Theta^{\mathbf{k+1}}_{\mathbf{x}}\|_{\mathbf{L}^\infty_{\mathbf{x}}}$) By (3.8),

$$\Theta^{k+1}_x(x,t)=\int_{\mathbb{R}}H_x\left(x,t;y,0;\kappa^k\right)\theta_0^*(y)\mathrm{d}y + \int_0^t\int_{\mathbb{R}}H_x\left(x,t;y,s;\kappa^k\right)\mathcal{N}_2^k(y,s)\mathrm{d}y\mathrm{d}s. \tag{3.9}$$

By ansatz (3.5), μ^k satisfies (2.7) for $\bar{\rho}=\frac{\kappa}{c_v}$, so, from Lemma 2.1,

$$\int_{\mathbb{R}}\left|H_x\left(x,t;y,0;\mu^k\right)\right|\left|\theta_0^*(y)\right|\mathrm{d}y \leqq C_*\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{C_*t}}}{t}\mathrm{d}y\|\theta_0^*\|_{L^\infty_x}\leqq\frac{C_\sharp\delta}{\sqrt{t}} \tag{3.10}$$

for $C_\sharp$ properly large. For the second term in (3.9), one combines the constitutive relation (1.2) and the ansatz (3.5) to have

$$\begin{aligned}&\left|\int_0^t\int_{\mathbb{R}}H_x\left(x,t;y,s;\kappa^k\right)\mathcal{N}_2^k(y,s)\mathrm{d}y\mathrm{d}s\right|\\ &=\int_0^t\int_{\mathbb{R}}\left|H_x\left(x,t;y,s;\kappa^k\right)\right|\\ &\quad\times\left(\left|\frac{K\left(1+\Theta^k(y,s)\right)}{c_v\left(1+V^k(y,s)\right)}U_y^k(y,s)\right|+\left|\frac{\mu\left(U_y^k(y,s)\right)^2}{c_v\left(1+V^k(y,s)\right)}\right|\right)\mathrm{d}y\mathrm{d}s\\ &\leqq O(1)\delta\int_0^t\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{t-s}\left(\frac{1}{\sqrt{s}}+\frac{\left|U_y^k(y,s)\right|}{\sqrt{s}}\right)\mathrm{d}y\mathrm{d}s\\ &\leqq O(1)\left(\delta+\frac{\delta^2}{\sqrt{t}}\right)\leqq\frac{C_\sharp\delta}{\sqrt{t}}\end{aligned}$$

for sufficiently small δ. Together with (3.9) and (3.10), one obtains

$$\|\Theta^{k+1}_x(x,t)\|_{L^\infty_x}\leqq\frac{2C_\sharp\delta}{\sqrt{t}}.$$

- (**Estimates of** $\|\Theta_{\mathbf{x}}^{\mathbf{k+1}}\|_{\mathbf{L}_{\mathbf{x}}^{\mathbf{1}}}$, $\|\Theta^{\mathbf{k+1}}\|_{\mathbf{L}_{\mathbf{x}}^{\infty}}$ and $\|\Theta^{\mathbf{k+1}}\|_{\mathbf{L}_{\mathbf{x}}^{\mathbf{1}}}$) Use the representation (3.9) of Θ_x^{k+1} again to obtain

$$\int_{\mathbb{R}} |\Theta_x^{k+1}| \mathrm{d}x \leqq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} H_x \left(x, t; y, 0; \kappa^k \right) \theta_0^*(y) \mathrm{d}y \right| \mathrm{d}x + \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} H_x \left(x, t; y, s; \kappa^k \right) \mathcal{N}_2^k(y, s) \mathrm{d}y \mathrm{d}s \right| \mathrm{d}x. \tag{3.11}$$

For the first term of (3.11), since $\int_{\mathbb{R}} H_x(x, t; w, 0; \kappa^k) dw = 0$ by Lemma 2.6, we can introduce the anti-derivative of $H_x(x, t; y, 0; \kappa^k)$ with respect to y,

$$W(x, t; y, 0; \kappa^k) = \begin{cases} \int_{-\infty}^{y} H_x(x, t; w, 0; \kappa^k) \mathrm{d}w & \text{for } y < x, \\ -\int_y^{\infty} H_x(x, t; w, 0; \kappa^k) \mathrm{d}w, & \text{for } y \geqq x. \end{cases}$$

As θ_0^* is a BV function, one can apply integration by parts for Stieltjes integral to have

$$\begin{aligned} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathrm{d}W \left(x, t; y, 0; \kappa^k \right) \theta_0^*(y) \right| \mathrm{d}x &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} W(x, t; y, 0; \kappa^k) d\theta_0^*(y) \right| \mathrm{d}x \\ &\leqq \int_{\mathbb{R}} \int_{\mathbb{R}} \left| W(x, t; y, 0; \kappa^k) \right| \left| \mathrm{d}\theta_0^*(y) \right| \mathrm{d}x \\ &\leqq O(1) \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_* t}}}{\sqrt{t}} \mathrm{d}x \cdot \|\theta_0^*\|_{BV} \leqq C_\sharp \delta \end{aligned} \tag{3.12}$$

for properly large $C_\sharp$. Now for the second term of (3.11), one uses Lemma 2.1, ansatz (3.5) to obtain that

$$\begin{aligned} &\int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} H_x \left(x, t; y, s; \kappa^k \right) \mathcal{N}_2^k(y, s) \mathrm{d}y \mathrm{d}s \right| \mathrm{d}x \\ &= \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \left| H_x \left(x, t; y, s; \kappa^k \right) \right| \\ &\quad \left(\left| \frac{K \left(1 + \Theta^k(y, s) \right)}{c_v \left(1 + V^k(y, s) \right)} U_y^k(y, s) \right| + \left| \frac{\mu \left(U_y^k(y, s) \right)^2}{c_v \left(1 + V^k(y, s) \right)} \right| \right) \mathrm{d}y \mathrm{d}s \mathrm{d}x \\ &\leqq O(1) \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{t-s} \left| U_y^k(y, s) \right| \left(1 + \frac{\delta}{\sqrt{s}} \right) \mathrm{d}y \mathrm{d}s \mathrm{d}x \\ &\leqq O(1) \left(\delta \sqrt{t} + \delta^2 \right) \leqq C_\sharp \delta \end{aligned}$$

for small δ and $t_\sharp$. One combines the above estimates, (3.12) and (3.11), to yield that

$$\int_{\mathbb{R}} |\Theta_x^{k+1}| \mathrm{d}x \leqq 2C_\sharp \delta.$$

The estimates for zeroth order terms are straightforward,

$$\|\Theta^{k+1}\|_{L^\infty_x} \leqq 2C_\sharp\delta, \qquad \|\Theta^{k+1}\|_{L^1_x} \leqq 2C_\sharp\delta.$$

- (**Estimates of** $\|\Theta^{\mathbf{k+1}}(\cdot,\mathbf{t}) - \Theta^{\mathbf{k+1}}(\cdot,\mathbf{s})\|_{\mathbf{L^\infty_x}}$) Form the representation (3.8), one has

$$\begin{aligned}
&\Theta^{k+1}(y,t) - \Theta^{k+1}(y,s)\\
&\quad= \int_{\mathbb{R}} \left(H\left(y,t;z,0;\kappa^k\right) - H\left(y,s;z,0;\kappa^k\right)\right)\theta_0^*(z)\mathrm{d}z\\
&\qquad+ \int_s^t\int_{\mathbb{R}} H\left(y,t;z,\tau;\kappa^k\right)\mathcal{N}_2^k(z,\tau)\mathrm{d}z\mathrm{d}\tau + \int_0^s\int_{\mathbb{R}}\left(H\left(y,t;z,\tau;\kappa^k\right)\right.\\
&\qquad\left.- H\left(y,s;z,\tau;\kappa^k\right)\right)\mathcal{N}_2^k(z,\tau)\mathrm{d}z\mathrm{d}\tau\\
&\quad= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned} \tag{3.13}$$

For the first term $\mathcal{I}_1$, in view of the estimate of H_t in (2.11) of Lemma 2.2 and initial condition (3.2), one directly obtains that

$$\begin{aligned}
|\mathcal{I}_1| &= \left|\int_{\mathbb{R}}\int_s^t H_\sigma\left(y,\sigma;z,0;\kappa^k\right)\theta_0^*(z)\mathrm{d}\sigma\mathrm{d}z\right| \leqq O(1)\int_{\mathbb{R}}\int_s^t \frac{e^{\frac{-(y-z)^2}{C_*\sigma}}}{\sigma^{\frac{3}{2}}}\delta\mathrm{d}\sigma\mathrm{d}z\\
&\leqq \int_s^t \frac{O(1)\delta}{\sqrt{s}\sqrt{\sigma}}\mathrm{d}\sigma \leqq \frac{O(1)\delta(t-s)}{\sqrt{s}\sqrt{t}}.
\end{aligned}$$

For the second term $\mathcal{I}_2$, one combines the form of $\mathcal{N}_2$ in (3.6), Lemma 2.1 and the ansatz (3.5) to obtain that

$$\begin{aligned}
|\mathcal{I}_2| &= \left|\int_s^t\int_{\mathbb{R}} H(y,t;z,\tau;\kappa^k)\left(-\frac{p(1+V^k,1+\Theta^k)}{c_v}U_z^k(z,\tau)\right.\right.\\
&\qquad\left.\left.+\frac{\mu}{c_v\left(1+V^k\right)}\left(U_z^k(z,\tau)\right)^2\right)\mathrm{d}z\mathrm{d}\tau\right|\\
&\leqq O(1)\int_s^t\int_{\mathbb{R}}\frac{e^{\frac{-(y-z)^2}{C_*(t-\tau)}}}{\sqrt{t-\tau}}\left(\frac{\delta}{\sqrt{\tau}}+\frac{\delta^2}{\tau}\right)\mathrm{d}z\mathrm{d}\tau\\
&\leqq O(1)\int_s^t\left(\frac{\delta}{\sqrt{\tau}}+\frac{\delta^2}{\sqrt{s}\sqrt{\tau}}\right)\mathrm{d}\tau\\
&\leqq O(1)\left(\delta\frac{t-s}{\sqrt{t}}+\delta^2\frac{t-s}{\sqrt{s}\sqrt{t}}\right).
\end{aligned}$$

Next, for the third term $\mathcal{I}_3$, in view of the estimates of H_t in (2.11) and the ansatz (3.5), one has

$$|\mathcal{I}_3| = \left|\int_0^s \int_{\mathbb{R}} \left(H\left(y,t;z,\tau;\kappa^k\right) - H\left(y,s;z,\tau;\kappa^k\right)\right)\mathcal{N}_2^k(z,\tau)\mathrm{d}z\mathrm{d}\tau\right|$$
$$\leqq \int_0^s \int_{\mathbb{R}} \left|\int_s^t H_\sigma\left(y,\sigma;z,\tau;\kappa^k\right)\mathrm{d}\sigma\right|\left|\mathcal{N}_2^k(z,\tau)\right|\mathrm{d}z\mathrm{d}\tau$$
$$\leqq O(1)\delta\left(\frac{t-s}{\sqrt{t}} + \sqrt{t-s}\right) + O(1)\delta^2\left(\frac{\sqrt{t-s}}{\sqrt{s}} + \frac{(t-s)^{\frac{1}{4}}}{s^{\frac{1}{4}}}\right). \quad (3.14)$$

The estimates of $\mathcal{I}_1, \mathcal{I}_2$ and $\mathcal{I}_3$ together yield that

$$\left|\Theta^{k+1}(y,t) - \Theta^{k+1}(y,s)\right|$$
$$\leqq O(1)\left((\delta+\delta^2)\frac{t-s}{\sqrt{s}\sqrt{t}} + \delta\frac{t-s}{\sqrt{t}} + \delta\sqrt{t-s}\right.$$
$$\left. + \delta^2\frac{\sqrt{t-s}}{\sqrt{s}} + \delta^2\frac{(t-s)^{\frac{1}{4}}}{s^{\frac{1}{4}}}\right).$$

With a similar argument, one can obtain the Hölder continuity in time of $\Theta^{k+1}(x,t)$ and $\Theta_x^{k+1}(x,t)$ in L^1 sense, and the details are omitted. □

Lemma 3.3. *For sufficiently small δ and $t_\sharp$, the ansatz* (3.5) *holds for U^{k+1} when* $0 < t < t_\sharp$.

Proof. We split the proof into several estimates.

- (**Estimate of $\|\mathbf{U_x^{k+1}}\|_{\mathbf{L_x^\infty}}$**) From (3.7),

$$U_x^{k+1} = \int_{\mathbb{R}} H_x\left(x,t;y,0;\mu^k\right)u_0^*(y)\mathrm{d}y$$
$$+ \int_0^t \int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)p(1+V^k, 1+\Theta^k)\mathrm{d}y\mathrm{d}s. \quad (3.15)$$

By the ansatz (3.5) for V^k, μ^k in (3.6) satisfies the condition (2.7) with $\bar{\rho} = \mu$. Thus, one can apply Lemma 2.1 to obtain

$$\int_{\mathbb{R}} \left|H_x\left(x,t;y,0;\mu^k\right)\right|\left|u_0^*(y)\right|\mathrm{d}y \leqq C_* \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_* t}}}{t}\mathrm{d}y\|u_0^*\|_{L_x^\infty} \leqq \frac{C_\sharp \delta}{\sqrt{t}}, \quad (3.16)$$

where C_* is constructed in Lemma 2.1, and $C_\sharp$ is properly large. One then substitutes the constitutive relation (1.2) into $\mathcal{N}_1^k$ and splits it into three parts:

$$\int_0^t \int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)p(1+V^k, 1+\Theta^k)\mathrm{d}y\mathrm{d}s$$
$$= \int_0^t \int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)\frac{K\left(1+\Theta^k(y,t)\right)}{1+V^k(y,t)}\mathrm{d}y\mathrm{d}s$$

$$
\begin{aligned}
&+K\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)\frac{\Theta^k(y,s)-\Theta^k(y,t)}{1+V^k(y,t)}\mathrm{d}y\mathrm{d}s\\
&+K\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)\left(1+\Theta^k(y,s)\right)\\
&\left(\frac{1}{1+V^k(y,s)}-\frac{1}{1+V^k(y,t)}\right)\mathrm{d}y\mathrm{d}s\\
&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.
\end{aligned}
\tag{3.17}
$$

For the estimate of $\mathcal{I}_1$, one applies integration by parts to get

$$
\begin{aligned}
\mathcal{I}_1&=\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)\frac{K\left(1+\Theta^k(y,t)\right)}{1+V^k(y,t)}\mathrm{d}y\mathrm{d}s\\
&=K\int_0^t\left[\int_{\mathbb{R}\backslash\mathscr{D}} H_x\left(x,t;y,s;\mu^k\right)\partial_y\left(\frac{-\left(1+\Theta^k(y,t)\right)}{1+V^k(y,t)}\right)\mathrm{d}y\right.\\
&\left.+\sum_{z\in\mathscr{D}} H_x\left(x,t;y,s;\mu^k\right)\frac{\left(1+\Theta^k(y,t)\right)}{1+V^k(y,t)}\Big|_{y=z^+}^{y=z^-}\right]\mathrm{d}s\\
&=\mathcal{I}_{11}+\mathcal{I}_{12}.
\end{aligned}
$$

Recall the discussion in Section 2 and our adoption (2.5), the above integration by part is actually in Stieltjes sense. As V^k is away from zero and bounded, it is easy to check that $1/V^k$ is also a BV function. Then, if the continuous part of V^k is absolutely continuous, one applies the ansatz (3.5), the estimates of the heat kernel in Lemma 2.1 to obtain the following estimates of $\mathcal{I}_{11}$ for $0<t<t_\sharp$,

$$
\begin{aligned}
|\mathcal{I}_{11}|&=\left|\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} H_x\right.\\
&\quad\left.\times\left(x,t;y,s;\mu^k\right)\left(\frac{K\Theta_y^k(y,t)\left(1+V^k(y,t)\right)-KV_y^k\left(1+\Theta^k(y,t)\right)}{\left(1+V^k(y,t)\right)^2}\right)\mathrm{d}y\mathrm{d}s\right|\\
&\leqq\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}} C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{t-s}\left(\frac{K\frac{2C_\sharp\delta}{\sqrt{t}}\left(1+2C_\sharp\delta\right)}{\left(1-2C_\sharp\delta\right)^2}\right)\mathrm{d}y\mathrm{d}s\\
&\quad+\int_{\mathbb{R}\backslash\mathscr{D}} C_*e^{-\frac{|x-y|^2}{C_*(t-s)}}\left(\frac{K|V_y^k|\left(1+2C_\sharp\delta\right)}{\left(1-2C_\sharp\delta\right)^2}\right)\mathrm{d}y\\
&=O(1)\frac{\delta}{\sqrt{t}}\int_0^t\frac{1}{\sqrt{t-s}}\mathrm{d}s+O(1)\int_{\mathbb{R}\backslash\mathscr{D}}|V_y^k|\mathrm{d}y\leqq O(1)\delta\leqq\frac{C_\sharp\delta}{4\sqrt{t}}.
\end{aligned}
\tag{3.18}
$$

On the other hand, if V^k contains singular part, then we have the following estimate of Riemann–Stieltjes sum:

$$
\sum_{i=1}^N\left|\frac{1}{1+V^k(x_i)}-\frac{1}{1+V^k(x_{i-1})}\right|=\sum_{i=1}^N\left|\frac{V^k(x_i)-V^k(x_{i-1})}{(1+V^k(x_i))(1+V^k(x_{i-1}))}\right|
$$

$$\leqq \frac{\sum_{i=1}^{N} |V^k(x_i) - V^k(x_{i-1})|}{(1-2C_\sharp\delta)^2}$$
$$\leqq \frac{||V^k||_{BV}}{(1-2C_\sharp\delta)^2}.$$

This is similar as the estimates of $\left|\partial_y\left(\frac{1}{1+V^k(y)}\right)\right|$ in (3.18), where we assume the continuous part of V^k is absolutely continuous. Thus we claim the estimates (3.18) exactly holds for BV function V^k with singular part. This verifies the validation of our adoption (2.5) and discussions in Section 2.

For $\mathcal{I}_{12}$, by the ansatz (3.5), $\Theta^k(\cdot, t)$ is Lipschitz continuous with respect to x. Additionally, it follows from Lemma 2.1 that $H_x\left(x, t; y, s; \mu^k\right)$ is continuous with respect to y. Therefore one combines Lemma 2.1, ansatz (3.5) and initial condition (3.2) to have for $0 < t < t_\sharp$ that

$$|\mathcal{I}_{12}| = \left|\sum_{z\in\mathscr{D}} \int_0^t H_x\left(x, t; y, s; \mu^k\right) \frac{K\left(1+\Theta^k(y,t)\right)}{1+V^k(y,t)}\Big|_{y=z^+}^{y=z^-} \mathrm{d}s\right|$$
$$\leqq C_* e^{-\frac{|x-y|^2}{C_* t}} \frac{K(1+2C_\sharp\delta)}{(1-2C_\sharp\delta)^2} \sum_{z\in\mathscr{D}} \left|V^k(y,t)\Big|_{y=z^+}^{y=z^-}\right| \leqq O(1)\|v_0^*\|_{BV} \leqq \frac{C_\sharp\delta}{4\sqrt{t}}$$

for sufficiently small $t_\sharp$. This finishes the estimates of $\mathcal{I}_1$ in (3.10).

Next, for $\mathcal{I}_2$ in (3.17), by Lemma 2.2 and the time difference estimate of Θ^k in Lemma 3.2, one obtains

$$|\mathcal{I}_2| \leqq K \int_0^t \int_{\mathbb{R}\backslash\mathscr{D}} \left|H_{xy}\left(x, t; y, s; \mu^k\right)\right| \frac{\left|\Theta^k(y,s) - \Theta^k(y,t)\right|}{1+V^k(y,t)} \mathrm{d}y\mathrm{d}s$$
$$\leqq O(1)\left((\delta+\delta^2) + 2\delta\sqrt{t} + 2\delta^2\right)$$
$$\leqq \frac{C_\sharp\delta}{4\sqrt{t}}.$$

For the last part $\mathcal{I}_3$ in (3.17), again by Lemma 2.1 and the ansatz (3.5),

$$|\mathcal{I}_3| \leqq K \int_0^t \int_{\mathbb{R}\backslash\mathscr{D}} \int_s^t \left|H_{xy}\left(x, t; y, s; \mu^k\right)\right|$$
$$\left|1+\Theta^k(y,s)\right| \left(\frac{\left|V_\tau^k(y,\tau)\right|}{\left(1-\left|V^k(y,\tau)\right|\right)^2}\right) \mathrm{d}\tau\mathrm{d}y\mathrm{d}s \leqq O(1)\delta\sqrt{t}$$
$$\leqq \frac{C_\sharp\delta}{4\sqrt{t}}.$$

This, together with (3.16), (3.17), the estimates of $\mathcal{I}_{11}, \mathcal{I}_{12}, \mathcal{I}_2$ and the representation of U_x^{k+1}, gives that

$$\begin{aligned}\|U_x^{k+1}\|_{L_x^\infty} &\leqq \frac{C_\sharp\delta}{\sqrt{t}} + |\mathcal{I}_1| + |\mathcal{I}_2| + |\mathcal{I}_3| \\ &\leqq \frac{C_\sharp\delta}{\sqrt{t}} + \frac{C_\sharp\delta}{2\sqrt{t}} + \frac{C_\sharp\delta}{4\sqrt{t}} + \frac{C_\sharp\delta}{4\sqrt{t}} \leqq \frac{2C_\sharp\delta}{\sqrt{t}}.\end{aligned} \tag{3.19}$$

- (**Estimate of** $\|\mathbf{U_x^{k+1}}\|_{\mathbf{L^1_x}}$) From (3.15),

$$\begin{aligned}\int_{\mathbb{R}}|U_x^{k+1}|\mathrm{d}x \leqq& \int_{\mathbb{R}}\left|\int_{\mathbb{R}} H_x\left(x,t;y,0;\mu^k\right)u_0^*(y)\mathrm{d}y\right|\mathrm{d}x \\ &+\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)p\left(1+V^k,1+\Theta^k\right)\mathrm{d}y\mathrm{d}s\right|\mathrm{d}x.\end{aligned} \tag{3.20}$$

Following the estimates as in (3.12), the first term is bounded by

$$\int_{\mathbb{R}}\left|\int_{\mathbb{R}} H_x\left(x,t;y,0;\mu^k\right)u_0^*(y)\mathrm{d}y\right|\mathrm{d}x \leqq O(1)\|u_0^*\|_{BV} \leqq C_\sharp\delta. \tag{3.21}$$

For the second term of (3.20), one uses Lemma 2.1, ansatz (3.5), the adoption (2.5) and similar integration by parts as in (3.17) to have

$$\begin{aligned}&\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}} H_{xy}\left(x,t;y,s;\mu^k\right)p\left(1+V^k,1+\Theta^k\right)\mathrm{d}y\mathrm{d}s\right|dx \\ &= K\int_{\mathbb{R}}\left|\int_0^t\left[\int_{\mathbb{R}\setminus\mathscr{D}} H_x\left(x,t;y,s;\mu^k\right)\partial_y\left(\frac{-\left(1+\Theta^k(y,s)\right)}{1+V^k(y,s)}\right)\mathrm{d}y\right.\right. \\ &\left.\left.+\sum_{z\in\mathscr{D}} H_x\left(x,t;y,s;\mu^k\right)\frac{\left(1+\Theta^k(y,s)\right)}{1+V^k(y,s)}\Big|_{y=z^+}^{y=z^-}\right]\mathrm{d}s\right|\mathrm{d}x \\ &\leqq \int_0^t\frac{O(1)}{\sqrt{t-s}}\left(\|\Theta_x^k(\cdot,s)\|_{L^1_x}+\int_{\mathbb{R}\setminus\mathscr{D}}|V_y^k(y,s)|\mathrm{d}y+2\sum_{z\in\mathscr{D}}\left|v_0^*(y)\Big|_{y=z^+}^{y=z^-}\right|\right)\mathrm{d}s \\ &\leqq O(1)\sqrt{t}\delta \leqq C_\sharp\delta.\end{aligned} \tag{3.22}$$

Combining (3.20), (3.21) and (3.22), we have, for sufficiently small $t_\sharp$, that

$$\|U_x^{k+1}(\cdot,t)\|_{L^1_x} \leqq 2C_\sharp\delta, \qquad 0<t<t_\sharp. \tag{3.23}$$

The estimates of zeroth order terms are more straightforward, thus we omit the details and obtain

$$\|U^k(\cdot,t)\|_{L^1_x}+\|U^k(\cdot,t)\|_{L^\infty_x} \leqq O(1)\sqrt{t}\delta \leqq 2C_\sharp\delta, \qquad 0<t<t_\sharp \tag{3.24}$$

for $t_\sharp$ sufficiently small. Finally, we combine (3.19), (3.23) and (3.24) to finish the proof. □

Lemma 3.4. *For sufficiently small δ and $t_\sharp$, the ansatz* (3.5) *holds for V^{k+1} when $0 \leqq s < t < t_\sharp$. Moreover, the following Lipschitz continuity in time property holds,*

$$\begin{aligned}&\sum_{z\in\mathscr{D}}\left|V_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+}-V_x^{k+1}(\cdot,s)\Big|_{z^-}^{z^+}\right| \\ &\quad\leqq O(1)\int_s^t\left(1+\frac{1}{\sqrt{\tau}}\right)d\tau\sum_{z\in\mathscr{D}}\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| \leqq O(1)\delta\frac{t-s}{\sqrt{t}}.\end{aligned}$$

Proof. We split the proof into several parts.

- (**Estimates of** $\|\mathbf{V_t^{k+1}}\|_{\mathbf{L_x^\infty}}$, $\|\mathbf{V^{k+1}}\|_{\mathbf{L_x^\infty}}$ and $\|\mathbf{V^{k+1}}\|_{\mathbf{L_x^1}}$,) One combines the initial condition (3.2), the first equation in (3.3), the estimates (3.19) and (3.23) in Lemma 3.3 to obtain for $0 < t < t_\sharp \ll 1$ that,

$$\begin{cases} \|V_t^{k+1}(\cdot,t)\|_{L_x^\infty} = \|U_x^{k+1}(\cdot,t)\|_{L_x^\infty} \leqq \dfrac{2C_\sharp\delta}{\sqrt{t}}, \\ \|V^{k+1}(\cdot,t)\|_{L_x^\infty} \leqq \|v_0^*\|_{L_x^\infty} + \displaystyle\int_0^t \|V_s^{k+1}(\cdot,s)\|_{L_x^\infty} \mathrm{d}s \\ \qquad\qquad\qquad \leqq \delta + 4C_\sharp\sqrt{t}\delta \leqq 2C_\sharp\delta, \\ \|V^{k+1}(\cdot,t)\|_{L_x^1} \leqq \|v_0^*\|_{L_x^1} + \displaystyle\int_0^t \|U_x^{k+1}(\cdot,s)\|_{L_x^1} \mathrm{d}s \leqq \delta + 2C_\sharp t\delta \leqq 2C_\sharp\delta. \end{cases} \tag{3.25}$$

- (**Estimate of** $\int_{\mathbb{R}\setminus\mathscr{D}} |\mathbf{V_x^{k+1}}(\mathbf{x},\mathbf{t})|\mathrm{d}\mathbf{x}$) We will only calculate the case when the continuous part of V^k is absolutely continuous. Then, we claim that similar estimates also hold when V^k contains singular part, and explain in Remark 3.1 (3) how to adapt our computations for that case. Now for V^k without singular part, one uses the equation (3.3) and the representation of U^{k+1} in (3.7) to construct the following estimate for the integration of $|V_x^{k+1}(x,t)|$ on $x \notin \mathscr{D}$:

$$\begin{aligned} &\int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t)\right| \mathrm{d}x \\ &\leqq \int_{\mathbb{R}\setminus\mathscr{D}} \left|(v_0^*)_x\right| \mathrm{d}x + \int_{\mathbb{R}\setminus\mathscr{D}} \left|\int_0^t \int_{\mathbb{R}} H_{xx}\left(x,s;y,0;\mu^k\right) u_0^*(y)\mathrm{d}y\mathrm{d}s\right| \mathrm{d}x \\ &\quad + \int_{\mathbb{R}\setminus\mathscr{D}} \left|\int_0^t \int_0^s \int_{\mathbb{R}\setminus\mathscr{D}} H_{xxy}\left(x,s;y,\tau;\mu^k\right) p(1+V^k, 1+\Theta^k)\mathrm{d}y\mathrm{d}\tau\mathrm{d}s\right| \mathrm{d}x \\ &= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned} \tag{3.26}$$

The term $\mathcal{I}_1$ is controlled by $\|v_0^*\|_{BV}$. For the second term $\mathcal{I}_2$, one combines the initial condition (3.2), the heat kernel estimate (2.15) in Lemma 2.2 to obtain for $0 < t < t_\sharp \ll 1$ that

$$\begin{aligned} \mathcal{I}_2 &\leqq C_* \int_{\mathbb{R}\setminus\mathscr{D}} \int_{\mathbb{R}} \left(\frac{\mu}{(1-\delta)^2} |\partial_x v_0^*(x)| e^{-\frac{|x-y|^2}{C_* t}} + \frac{e^{-\frac{(x-y)^2}{C_* t}}}{\sqrt{t}} \right) \left|u_0^*(y)\right| \mathrm{d}y\mathrm{d}x \\ &\quad + \frac{1}{\mu} \int_{\mathbb{R}\setminus\mathscr{D}} (1+v_0^*(x)) \left| \int_{\mathbb{R}} \left(\delta(x-y) + \partial_x \left[\int_0^t \frac{\mu\left(v_0^*(x) - V^k(x,s)\right)}{(1+v_0^*(x))(1+V^k(x,s))} \right.\right.\right. \\ &\quad \left.\left.\left. \times H_x(x,s;y,0;\mu^k)\mathrm{d}s \right]\right) u_0^*(y)\mathrm{d}y \right| \mathrm{d}x \\ &\leqq C_*\sqrt{C_*\pi}\|u_0^*\|_{L^1} + O(1)\|v_0^*\|_{BV}\|u_0^*\|_{L^1} + \frac{1+\delta}{\mu}\|u_0^*\|_{L^1} + \frac{1}{\mu}\int_{\mathbb{R}\setminus\mathscr{D}} (1+v_0^*(x)) \\ &\quad \times \left| \int_{\mathbb{R}} \int_0^t \frac{\mu\left(\partial_x v_0^*(x) - V_x^k(x,s)\right)(1+v_0^*(x))(1+V^k(x,s))}{(1+v_0^*(x))^2(1+V^k(x,s))^2} H_x(x,s;y,0;\mu^k)\mathrm{d}s u_0^*(y)\mathrm{d}y \right| \mathrm{d}x \\ &\quad + \frac{1}{\mu} \int_{\mathbb{R}\setminus\mathscr{D}} (1+v_0^*(x)) \\ &\quad \times \left| \int_{\mathbb{R}} \int_0^t \frac{\mu\left(v_0^*(x) - V^k(x,s)\right)\left((1+v_0^*(x))V_x^k(x,s)\right)}{(1+v_0^*(x))^2(1+V^k(x,s))^2} H_x(x,s;y,0;\mu^k)\mathrm{d}s u_0^*(y)\mathrm{d}y \right| \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\mu}\int_{\mathbb{R}\setminus\mathscr{D}}(1+v_0^*(x))\\
&\times\left|\int_{\mathbb{R}}\int_0^t\frac{\mu\left(v_0^*(x)-V^k(x,s)\right)\left(\partial_x v_0^*(x)(1+V^k(x,s))\right)}{(1+v_0^*(x))^2(1+V^k(x,s))^2}\quad H_x(x,s;y,0;\mu^k)\mathrm{d}s u_0^*(y)\mathrm{d}y\right|\mathrm{d}x\\
&+\frac{1}{\mu}\int_{\mathbb{R}\setminus\mathscr{D}}(1+v_0^*(x))\\
&\times\left|\int_{\mathbb{R}}\left(\delta(x-y)+\left[\int_0^t\frac{\mu\left(v_0^*(x)-V^k(x,s)\right)}{(1+v_0^*(x))(1+V^k(x,s))}H_{xx}(x,s;y,0;\mu^k)\mathrm{d}s\right]\right)u_0^*(y)\mathrm{d}y\right|\mathrm{d}x\\
&\leqq C_*\sqrt{C_*\pi}\|u_0^*\|_{L^1}+O(1)\|v_0^*\|_{BV}\|u_0^*\|_{L^1}+\frac{1+\delta}{\mu}\|u_0^*\|_{L^1}\\
&+O(1)\int_0^t\frac{1}{\sqrt{s}}\left(\|v_0^*\|_{BV}+\|V^k(\cdot,s)\|_{BV}\right)\|u_0^*\|_{L^\infty}\mathrm{d}s+\frac{1+\delta}{\mu}\|u_0^*\|_{L^1}\\
&+\frac{1}{\mu}\int_{\mathbb{R}\setminus\mathscr{D}}\left|\int_{\mathbb{R}}\int_0^t\mu\left(v_0^*(x)-V^k(x,s)\right)\right.\\
&\times\left.\left(H_s(x,s;y,0;\mu^k)+\frac{V_x^k(x,s)}{(1+V^k(x,s))^2}H_x(x,s;y,0;\mu^k)\right)\mathrm{d}s u_0^*(y)\mathrm{d}y\right|\mathrm{d}x\\
&\leqq\left(C_*\sqrt{C_*\pi}+\frac{2+2\delta}{\mu}\right)\delta+O(1)\delta^2\\
&+\int_{\mathbb{R}\setminus\mathscr{D}}\left|\int_{\mathbb{R}}\int_0^t\int_0^s V_\sigma^k(x,\sigma)\mathrm{d}\sigma H_s(x,s;y,0;\mu^k)\mathrm{d}s u_0^*(y)\mathrm{d}y\right|\mathrm{d}x\\
&\leqq\left(C_*\sqrt{C_*\pi}+\frac{2+2\delta}{\mu}\right)\delta+O(1)\delta^2.
\end{aligned}
$$

For the third term $\mathcal{I}_3$, one switches the integration order of τ and s, applies (2.16) in Lemma 2.2 and ansatz (3.5) to obtain that

$$
\begin{aligned}
\mathcal{I}_3 &\leqq\int_{\mathbb{R}\setminus\mathscr{D}}\left|\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\int_\tau^t H_{xxy}\left(x,s;y,\tau;\mu^k\right)\mathrm{d}s\frac{K\left(\Theta^k(y,\tau)-V^k(y,\tau)\right)}{1+V^k(y,\tau)}\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&\leqq O(1)\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\left(\int_{\mathbb{R}\setminus\mathscr{D}}\left(\left|V_x^k(x,\tau)\right|\frac{e^{-\frac{|x-y|^2}{C_*(t-\tau)}}}{\sqrt{t-\tau}}+\frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau}\right)\mathrm{d}x\right)\\
&\times\left(\left|\Theta^k(y,\tau)\right|+\left|V^k(y,\tau)\right|\right)\mathrm{d}y\mathrm{d}\tau\\
&+\int_{\mathbb{R}\setminus\mathscr{D}}\left|\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\frac{V_x^k(x,\tau)}{\mu}\delta(x-y)\right.\\
&\times\left.\frac{K\left(\Theta^k(y,\tau)-V^k(y,\tau)\right)}{1+V^k(y,\tau)}\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\setminus\mathscr{D}}\left|\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\frac{V_x^k(x,\tau)}{\mu}\int_\tau^t\left(\mu^k(x,s)-\mu^k(x,\tau)\right)H_{xy}(x,s;y,\tau;\mu^k)\mathrm{d}s\right.\\
&\times\left.\frac{K\left(\Theta^k(y,\tau)-V^k(y,\tau)\right)}{1+V^k(y,\tau)}\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\
&+\int_{\mathbb{R}\setminus\mathscr{D}}\left|\int_0^t\int_{\mathbb{R}\setminus\mathscr{D}}\frac{1+V^k(x,\tau)}{\mu}\delta'(x-y)\right.\\
&\times\left.\frac{K\left(\Theta^k(y,\tau)-V^k(y,\tau)\right)}{1+V^k(y,\tau)}\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{1+V^k(x,\tau)}{\mu} \int_\tau^t \left(\mu_x^k(x,s) - \mu_x^k(x,\tau)\right) H_{xy}(x,s;y,\tau;\mu^k) \mathrm{d}s \right. \\
&\times \left. \frac{K\left(\Theta^k(y,\tau) - V^k(y,\tau)\right)}{1+V^k(y,\tau)} \mathrm{d}y\mathrm{d}\tau \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{1+V^k(x,\tau)}{\mu} \int_\tau^t \left(\mu^k(x,s) - \mu^k(x,\tau)\right) H_{xxy}(x,s;y,\tau;\mu^k) \mathrm{d}s \right. \\
&\times \left. \frac{K\left(\Theta^k(y,\tau) - V^k(y,\tau)\right)}{1+V^k(y,\tau)} \mathrm{d}y\mathrm{d}\tau \right| \mathrm{d}x \\
&\leqq O(1)\delta^2 + O(1)t\delta.
\end{aligned}
$$

Now we combine the estimates of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$, representation (3.26), and choose $C_\sharp$ to be properly large to get

$$
\int_{\mathbb{R}\setminus\mathscr{D}} \left| V_x^{k+1}(x,t) \right| \mathrm{d}x \leqq 2C_\sharp\delta, \quad 0 < t < t_\sharp \ll 1.
$$

- (**Estimate of** $\left|\mathbf{V^{k+1}(\cdot, t)}\Big|_{\mathbf{z^-}}^{\mathbf{z^+}}\right|$) Since (V^k, U^k, Θ^k) is a weak solution to (3.3) at k-th step, in view of Remark 2.1, both Θ^k and the flux

$$
\left(\frac{\mu U_x^k}{1+V^{k-1}} - p\left(1+V^{k-1}, 1+\Theta^{k-1}\right) \right)
$$

are continuous with respect to x, which implies that

$$
\begin{aligned}
V_t^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} = U_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} &= \frac{V^k}{\mu}\Big|_{z^-}^{z^+} \left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k, 1+\Theta^k) \right) \\
&+ \frac{K\left(1+V^k\right)^2}{\mu\left(1+\Theta^k\right)}\Big|_{z^-}^{z^+}.
\end{aligned}
$$

Now, one integrates the above equality with respect to time from 0 to t, applies the ansatz (3.5) and estimates in Lemma 3.3 to obtain that

$$
\begin{aligned}
&\left| V^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} \right| \\
&\leqq \left| v_0^*(\cdot)\Big|_{z^-}^{z^+} \right| + \frac{1}{\mu}\int_0^t \left(\frac{\mu\left|U_x^{k+1}\right|}{1-\|V^k\|_{L_x^\infty}} + \left| \frac{K\left(1+\|V^k\|_{L_x^\infty}\right)}{1-\|\Theta^k\|_{L_x^\infty}} \right| + \frac{\left(2+2\|V^k\|_{L_x^\infty}\right)}{1-\|\Theta^k\|_{L_x^\infty}} \right) \mathrm{d}s \\
&\quad \times \sup_{0\leqq\sigma\leqq t} \left| V^k(\cdot,\sigma)\Big|_{z^-}^{z^+} \right| \\
&\leqq \left(1 + O(1)t + O(1)\sqrt{t}\right) \left| v_0^*(\cdot)\Big|_{z^-}^{z^+} \right| \leqq 2\left| v_0^*(\cdot)\Big|_{z^-}^{z^+} \right|, \qquad 0 < t < t_\sharp,
\end{aligned}
\tag{3.27}
$$

for sufficiently small $t_\sharp$.

- (**Estimate of** $\int_{\mathbb{R}\setminus\mathscr{D}} |\mathbf{V_x^{k+1}(x,t)} - \mathbf{V_x^{k+1}(x,s)}|\mathrm{d}\mathbf{x}$) In this step, we show the Hölder continuity in time of the BV norm of the specific volume V^{k+1}. Similar to (3.26), we have the following estimate:

$$\begin{aligned}
&\int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t) - V_x^{k+1}(x,s)\right| \mathrm{d}x \\
&\quad \leqq \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_s^t \int_{\mathbb{R}} H_{xx}(x,\sigma;y,0;\mu^k) u_0^*(y) \mathrm{d}y \mathrm{d}\sigma \right| \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_s^t \int_0^\sigma \int_{\mathbb{R}} H_{xxy}(x,\sigma;y,\tau;\mu^k) p(1+V^k, 1+\Theta^k) \mathrm{d}y \mathrm{d}\tau \mathrm{d}\sigma \right| \mathrm{d}x \\
&\quad = \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{3.28}$$

For the first term $\mathcal{I}_1$ in (3.28), one defines the anti-derivative of $H_{xx}(x,\sigma;z,0;\mu^k)$ with respect to z and applies integration by parts to get

$$\begin{aligned}
&\int_s^t \int_{\mathbb{R}} H_{xx}(x,\sigma;y,0;\mu^k) u_0^*(y) \mathrm{d}y \mathrm{d}\sigma \\
&\quad = -\int_{\mathbb{R}\setminus\mathscr{D}} \left[\int_{-\infty}^y \int_s^t H_{xx}(x,\sigma;z,0;\mu^k) \mathrm{d}\sigma \mathrm{d}z \right] \partial_y u_0^*(y) \mathrm{d}y \\
&\qquad - \sum_{\alpha\in\mathscr{D}} \left[\int_{-\infty}^y \int_s^t H_{xx}(x,\sigma;z,0;\mu^k) \mathrm{d}\sigma \mathrm{d}z\, u_0^*(y) \right]_{y=\alpha^-}^{y=\alpha^+}.
\end{aligned} \tag{3.29}$$

Next, one recalls the estimate of (A.37), and applies the similar argument to find the representation of the time integral of $H_{xx}(x,\sigma;z,0;\mu^k)$ as follows:

$$\begin{aligned}
&\int_s^t H_{xx}(x,\sigma;z,\tau;\mu^k) \mathrm{d}\sigma \\
&\quad = V_x^k(x,\tau) \int_{-\infty}^x \Big(H(w,t;z,\tau;\mu^k) - H(w,s;z,\tau;\mu^k) \Big) dw \\
&\qquad + V_x^k(x,\tau) \int_s^t \left(\frac{1}{1+V^k(x,\sigma)} - \frac{1}{1+V^k(x,\tau)} \right) H_x(x,\sigma;z,\tau;\mu^k) \mathrm{d}\sigma \\
&\qquad + \Big(1+V^k(x,\tau)\Big) \Big(H(x,t;z,\tau;\mu^k) - H(x,s;z,\tau;\mu^k) \Big) + \Big(1+V^k(x,\tau)\Big) \\
&\qquad \times \int_s^t \left(\frac{1}{1+V^k(x,\sigma)} - \frac{1}{1+V^k(x,\tau)} \right) H_{xx}(x,\sigma;z,\tau;\mu^k) \mathrm{d}\sigma \\
&\qquad + \Big(1+V^k(x,\tau)\Big) \int_s^t \partial_x \left(\frac{1}{1+V^k(x,\sigma)} - \frac{1}{1+V^k(x,\tau)} \right) H_x(x,\sigma;x,\tau;\mu^k) \mathrm{d}\sigma.
\end{aligned} \tag{3.30}$$

When $y < x$, one integrates the above representation with respect to z from $-\infty$ to y, and applies the estimates of H in Lemma 2.1 to obtain that

$$\begin{aligned}
&\left| \int_{-\infty}^y \int_s^t H_{xx}(x,\sigma;z,\tau;\mu^k) \mathrm{d}\sigma \mathrm{d}z \right| \\
&\quad \leqq O(1) \int_s^t \Big(\left|\partial_x V^k(x,\sigma)\right| + \left|\partial_x V^k(x,0)\right| \Big) \frac{e^{-\frac{(x-y)^2}{C_*\sigma}}}{\sqrt{\sigma}} \mathrm{d}\sigma + O(1) \int_s^t \frac{e^{-\frac{(x-y)^2}{C_*\sigma}}}{\sigma} \mathrm{d}\sigma.
\end{aligned} \tag{3.31}$$

Meanwhile, when $y \geqq x$, one uses the symmetric property of $H_{xx}(x, \sigma; z, 0; \mu^k)$:

$$\left|\int_{-\infty}^{y} \int_{s}^{t} H_{xx}(x, \sigma; z, \tau; \mu^k) \mathrm{d}\sigma \mathrm{d}z\right| = \left|-\int_{y}^{\infty} \int_{s}^{t} H_{xx}(x, \sigma; z, \tau; \mu^k) \mathrm{d}\sigma \mathrm{d}z\right|$$
$$\leqq O(1) \int_{s}^{t} \left(\left|\partial_x V^k(x, \sigma)\right| + \left|\partial_x V^k(x, 0)\right|\right) \frac{e^{-\frac{(x-y)^2}{C_*\sigma}}}{\sqrt{\sigma}} \mathrm{d}\sigma + O(1) \int_{s}^{t} \frac{e^{-\frac{(x-y)^2}{C_*\sigma}}}{\sigma} \mathrm{d}\sigma. \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.29), and integrating with respect to x, we obtain

$$\mathcal{I}_1 \leqq O(1)\,(1 + \delta)\,\left\|u_0^*\right\|_{BV} \frac{t - s}{\sqrt{t}} \leqq O(1)\delta \frac{t - s}{\sqrt{t}}.$$

Next consider $\mathcal{I}_2$ in (3.28). One changes the order of the integration, and applies integration by parts to obtain that

$$\begin{aligned}
\mathcal{I}_2 \leqq & \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\backslash\mathscr{D}} \left[\int_s^t H_{xx}(x, \sigma; y, \tau; \mu^k) \mathrm{d}\sigma\right]\right. \\
& \left. \times \partial_y p\left(1 + \Theta^k(y, \tau), 1 + V^k(y, \tau)\right) \mathrm{d}y \mathrm{d}\tau\right| \mathrm{d}x \\
& + \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_0^s \sum_{\alpha\in\mathscr{D}} \left[\int_s^t H_{xx}(x, \sigma; y, \tau; \mu^k) \mathrm{d}\sigma p\left(1 + \Theta^k(y, \tau), 1 + V^k(y, \tau)\right)\right]_{y=\alpha^-}^{y=\alpha^+} \mathrm{d}\tau\right| \mathrm{d}x \\
& + \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_s^t \int_{\mathbb{R}\backslash\mathscr{D}} \left[\int_\tau^t H_{xx}(x, \sigma; y, \tau; \mu^k) \mathrm{d}\sigma\right] \partial_y\right. \\
& \left. \times p\left(1 + \Theta^k(y, \tau), 1 + V^k(y, \tau)\right) \mathrm{d}y \mathrm{d}\tau\right| \mathrm{d}x \\
& + \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_s^t \sum_{\alpha\in\mathscr{D}} \left[\int_\tau^t H_{xx}(x, \sigma; y, \tau; \mu^k) \mathrm{d}\sigma\right.\right. \\
& \left.\left. \times p\left(1 + \Theta^k(y, \tau), 1 + V^k(y, \tau)\right)\right]_{y=\alpha^-}^{y=\alpha^+} \mathrm{d}\tau\right| \mathrm{d}x \\
\equiv & T_1 + T_2 + T_3 + T_4.
\end{aligned} \tag{3.33}$$

In the remaining part of the proof, one denotes the pressure term $p\big(1+\Theta^k(y, \tau), 1 + V^k(y, \tau)\big)$ by $p(y, \tau)$ for simplicity. For T_1, in view of (3.30) and Hölder continuity in time of V^k, one has the following estimates:

$$\begin{aligned}
T_1 = & \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\backslash\mathscr{D}} \left[\int_s^t H_{xx}(x, \sigma; y, \tau; \mu^k) \mathrm{d}\sigma\right]\right. \\
& \left. \times \partial_y p(y, \tau) \mathrm{d}y \mathrm{d}\tau\right| \mathrm{d}x \\
\leqq & \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\backslash\mathscr{D}} V_x^k(x, \tau) \left[\int_{-\infty}^x \left(H(z, t; y, \tau; \mu^k) - H(z, s; y, \tau; \mu^k)\right) \mathrm{d}z\right]\right. \\
& \left. \times \partial_y p(y, \tau) \mathrm{d}y \mathrm{d}\tau\right| \mathrm{d}x \\
& + \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\backslash\mathscr{D}} V_x^k(x, \tau) \left[\int_s^t \frac{\left(\mu^k(x, \sigma) - \mu^k(x, \tau)\right)}{\mu} H_x(x, \sigma; y, \tau; \mu^k) \mathrm{d}\sigma\right]\right. \\
& \left. \times \partial_y p(y, \tau) \mathrm{d}y \mathrm{d}\tau\right| \mathrm{d}x \\
& + \int_{\mathbb{R}\backslash\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\backslash\mathscr{D}} \left(1 + V^k(x, \tau)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left[H(x,t;y,\tau;\mu^k) - H(x,s;y,\tau;\mu^k)\right]\partial_y p(y,\tau)\mathrm{d}y\mathrm{d}\tau\Big|\,\mathrm{d}x \\
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\setminus\mathscr{D}} \left(1+V^k(x,\tau)\right)\right. \\
&\times \left[\int_s^t \frac{\left(\mu^k(x,\sigma)-\mu^k(x,\tau)\right)}{\mu} H_{xx}(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma\right]\partial_y p(y,\tau)\mathrm{d}y\mathrm{d}\tau\Big|\,\mathrm{d}x \\
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \left|\int_0^s \int_{\mathbb{R}\setminus\mathscr{D}} \left(1+V^k(x,\tau)\right)\right. \\
&\times \left[\int_s^t \partial_x \frac{\left(\mu^k(x,\sigma)-\mu^k(x,\tau)\right)}{\mu} H_x(x,\sigma;y,\tau;\mu^k)\mathrm{d}\sigma\right]\partial_y p(y,\tau)\mathrm{d}y\mathrm{d}\tau\Big|\,\mathrm{d}x \\
&\leqq O(1)\delta\left[(t-s)\,|\log(t-s)| + \frac{(t-s)s}{\sqrt{t}}\delta\right].
\end{aligned} \tag{3.34}
$$

T_2 is estimated in a similar way:

$$
T_2 \leqq O(1)\delta\left[(t-s)\,|\log(t-s)| + \frac{(t-s)s}{\sqrt{t}}\delta\right]. \tag{3.35}
$$

Next, for T_3 and T_4, applying (3.30) again and following the analysis of T_1, we obtain that

$$
T_3 \leqq O(1)\delta(t-s), \qquad T_4 \leqq O(1)\delta(t-s). \tag{3.36}
$$

Substituting the estimates of T_j, $j = 1\dots4$ in (3.34), (3.35) and (3.36) into (3.33), and using the fact $\delta \ll 1$ and $0 < s < t \ll 1$ to obtain

$$
\mathcal{I}_2 \leqq O(1)\delta(t-s)|\log(t-s)|.
$$

Now we plug the estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$ into (3.28) to obtain

$$
\begin{aligned}
&\int_{\mathbb{R}\setminus\mathscr{D}} \left|V_x^{k+1}(x,t) - V_x^{k+1}(x,s)\right|\mathrm{d}x \leqq O(1)\delta\left(\frac{t-s}{\sqrt{t}} + (t-s)|\log(t-s)|\right) \\
&\leqq 2C_\sharp\delta\frac{(t-s)|\log(t-s)|}{\sqrt{t}},
\end{aligned} \tag{3.37}
$$

where the last inequality is due to the smallness of t and s.

- (**Estimate of** $\sum_{\mathbf{z}\in\mathscr{D}}\left|\mathbf{V_x^{k+1}}(\cdot,\mathbf{t})\Big|_{\mathbf{z}^-}^{\mathbf{z}^+} - \mathbf{V_x^{k+1}}(\cdot,\mathbf{s})\Big|_{\mathbf{z}^-}^{\mathbf{z}^+}\right|$) One uses the representation of V^{k+1} and follows the estimates of (3.27) to obtain that

$$
\begin{aligned}
&\sum_{z\in\mathscr{D}}\left|V_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} - V_x^{k+1}(\cdot,s)\Big|_{z^-}^{z^+}\right| \\
&\qquad \leqq O(1)\int_s^t\left(1+\frac{1}{\sqrt{\tau}}\right)\mathrm{d}\tau\sum_{z\in\mathscr{D}}\left|v_0^*(\cdot)\Big|_{z^-}^{z^+}\right| \\
&\qquad \leqq O(1)\delta\frac{t-s}{\sqrt{t}}.
\end{aligned}
$$

This, together with (3.37), implies that the BV norm of V^n is Hölder continuous with respect to t. The proof of this lemma is therefore complete. □

Remark 3.1. Several remarks are given as follows:

(1) The construction of $C_\sharp$ depends only on C_* (which appears in the estimates of heat kernel in Section 2) and the coefficients of the initial step, i.e., μ, κ and c_v. When $\delta < \delta^*$ for some fixed positive number δ^*, $\frac{1}{1+V^n}$ is uniformly bounded according to the ansatz (3.5). Thus the coefficient C_* is uniformly bounded when we apply Lemmas 2.1, 2.2, 2.4 and 2.5 for the estimates of the solution to (3.3), due to the fact that the heat equations in (3.3) have uniform bounded heat conductivity $\frac{1}{1+V^n}$. In conclusion, $C_\sharp$ and C_* are both uniformly bounded when δ is small. As the choice of the small time $t_\sharp$ depends only on $C_\sharp$, we know that $t_\sharp$ is small but uniform with respect to $\delta < \delta^*$.

(2) From Lemma 3.4 one shows that V^{k+1} is a BV function. In fact, from (3.27) we have the expression of U_x^k:

$$V_t^{k+1}(\cdot,t)\Big|_{z^-}^{z^+} = U_x^{k+1}(\cdot,t)\Big|_{z^-}^{z^+}$$
$$= \frac{1+V^k}{\mu}\left(\frac{\mu U_x^{k+1}}{1+V^k} - p(1+V^k,1+\Theta^k) + p(1+V^k,1+\Theta^k)\right)\Bigg|_{z^-}^{z^+}.$$

As both Θ^k and $\left(\frac{\mu U_x^{k+1}}{1+V^k} - p\left(1+V^k,1+\Theta^k\right)\right)$ are continuous, the above identity immediately implies that V^{k+1} is BV if and only if V^k is BV by virtue of (2.3). Therefore, we only need to show V^1 is BV. As U^1 is a solution of a homogeneous heat equation with constant coefficient, U^1 is smooth. Thus V^1 is BV as soon as v_0^* is BV, and it owns the same discontinuities as the initial data v_0^*. Therefore, we can combine the estimates in Lemma 3.4 to conclude that V^{k+1} is BV.

(3) In the proof of Lemma 3.4, V^{k+1} is decomposed into continuous part and discrete part, and its total variation is bounded by the sum of $\int_{\mathbb{R}\setminus\mathscr{D}}|\partial_x V^{k+1}(\cdot,t)|\mathrm{d}x$ and $\sum_{z\in\mathscr{D}}\left|V^{k+1}(\cdot,t)\Big|_{z^-}^{z^+}\right|$. It appears that one needs the continuous part of V^{k+1} to be absolutely continuous. This requirement is actually unnecessary, as we can estimate the total variation alternatively. Given a partition $\mathscr{P} = \{x_j,\ j\in\mathbb{Z}\}$ of $\mathbb{R}$, by the representation of V^{k+1}, one has

$$\sum_{j\in\mathbb{Z}}|V^{k+1}(x_j,t) - V^{k+1}(x_{j-1},t)|$$
$$\leqq \sum_{j\in\mathbb{Z}}|v_0^*(x_j) - v_0^*(x_{j-1})| + \sum_{j\in\mathbb{Z}}\Big|\int_0^t\int_{\mathbb{R}}$$
$$\times\left(H_x(x_j,\tau;y,0;\mu^k) - H_x(x_{j-1},\tau;y,0;\mu^k)\right)u_0^*(y)\mathrm{d}y\mathrm{d}\tau\Big|$$
$$+\sum_{j\in\mathbb{Z}}\Big|\int_0^t\int_0^\tau\int_{\mathbb{R}}\left(H_{xy}(x_j,\tau;y,s;\mu^k) - H_{xy}(x_{j-1},\tau;y,s;\mu^k)\right)$$
$$\times\ p(1+V^k,1+\Theta^k)(y,s)\mathrm{d}y\mathrm{d}s\mathrm{d}\tau\Big|.$$

Following the similar estimates as for (3.26), one can conclude the right hand side is bounded by $O(1)\delta$ uniformly. Taking supremum over the partition, one finishes the BV estimate. The Hölder continuity in time of BV norm for V^{k+1} can also be obtained in this way.

3.2. Convergence of the scheme

In this part, we will show that the sequence of approximate solutions (V^n, U^n, Θ^n) constructed from the iteration (3.3) is a Cauchy sequence in an appropriate topology, and the topology is strong enough that the limit is a weak solution to system (3.1).

By taking the difference of the solutions at $n+1$-th and n-th steps, one gets the equation for difference functions in two consecutive steps:

$$\begin{cases} \partial_t\left(V^{n+1}-V^n\right)-\partial_x\left(U^{n+1}-U^n\right)=0, \\ \partial_t\left(U^{n+1}-U^n\right)-\partial_x\left(\dfrac{\mu\left(U^{n+1}-U^n\right)_x}{1+V^n}\right)=-\partial_x\left(\dfrac{\mu U_x^n\left(V^n-V^{n-1}\right)}{(1+V^n)(1+V^{n-1})}\right)+\mathcal{N}_1^n-\mathcal{N}_1^{n-1}, \\ \partial_t\left(\Theta^{n+1}-\Theta^n\right)-\partial_x\left(\dfrac{\kappa\left(\Theta^{n+1}-\Theta^n\right)_x}{c_v(1+V^n)}\right)=-\partial_x\left(\dfrac{\kappa\Theta_x^n\left(V^n-V^{n-1}\right)}{c_v(1+V^n)(1+V^{n-1})}\right)+\mathcal{N}_2^n-\mathcal{N}_2^{n-1}, \\ V^{n+1}(x,0)-V^n(x,0)=U^{n+1}(x,0)-U^n(x,0)=\Theta^{n+1}(x,0)-\Theta^n(x,0)=0, \end{cases} \tag{3.38}$$

where, for brevity of presentation, we have used the notations in (3.6).

From the expression (1.2) of pressure p and the estimates of iteration scheme (3.5), one infers that

$$\mathcal{N}_1^n-\mathcal{N}_1^{n-1}=-\partial_x\left(\frac{K(\Theta^n-\Theta^{n-1})}{1+V^n}-\frac{K(V^{n-1}-V^n)(1+\Theta^{n-1})}{(1+V^{n-1})(1+V^n)}\right), \tag{3.39}$$

$$\begin{aligned} \mathcal{N}_2^n-\mathcal{N}_2^{n-1}&=-\frac{U_x^n}{c_v}\left(p^n-\frac{\mu U_x^n}{1+V^n}\right)+\frac{U_x^{n-1}}{c_v}\left(p^{n-1}-\frac{\mu U_x^{n-1}}{1+V^{n-1}}\right) \\ &=O(1)\Big[\left(\left|V^n-V^{n-1}\right|+\left|\Theta^n-\Theta^{n-1}\right|\right)\left|U_x^n\right|+\left|V^n-V^{n-1}\right|\left(U_x^n\right)^2 \\ &\quad+\left(1+\left|U_x^n+U_x^{n-1}\right|\right)\left|U_x^n-U_x^{n-1}\right|\Big]. \end{aligned} \tag{3.40}$$

For the estimate of the difference of Θ^n, we use (3.38) to express the difference in terms of the variable coefficient heat kernel

$$\begin{aligned} \left(\Theta^{n+1}-\Theta^n\right)(x,t)&=\int_0^t\int_{\mathbb{R}}H_y(x,t;y,\tau;\kappa^n)\frac{\kappa\Theta_y^n\left(V^n-V^{n-1}\right)}{(1+V^n)\left(1+V^{n-1}\right)}(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &\quad+\int_0^t\int_{\mathbb{R}}H(x,t;y,\tau;\kappa^n)\left(\mathcal{N}_2^n-\mathcal{N}_2^{n-1}\right)(y,\tau)\mathrm{d}y\mathrm{d}\tau. \end{aligned} \tag{3.41}$$

Lemma 3.5. ($\mathbf{\Theta^{n+1}-\Theta^n}$) *There exists a positive constant $C_\flat$ such that, for sufficiently small δ and $t_\sharp$, and for $0<t<t_\sharp$,*

$$\left\|\frac{\Theta^{n+1}(\cdot,t)-\Theta^n(\cdot,t)}{|\log t|}\right\|_\infty$$

$$
\begin{aligned}
&\leqq C_\flat\left(\sqrt{t_\sharp}+\delta\right)\left(\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_1\right),\\
&\left\|\Theta^{n+1}(\cdot,t)-\Theta^n(\cdot,t)\right\|_1\\
&\leqq C_\flat\left(\sqrt{t_\sharp}+\delta\right)\left(\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_1+\left|\!\left|\!\left|\Theta^n-\Theta^{n-1}\right|\!\right|\!\right|_1+\left|\!\left|\!\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_1\right),\\
&\frac{\sqrt{t}}{|\log t|}\left\|\Theta_x^{n+1}(\cdot,t)-\Theta_x^n(\cdot,t)\right\|_\infty\\
&\leqq C_\flat\left(\sqrt{t_\sharp}+\delta\right)\left(\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_{BV}+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_1\right.\\
&\quad\left.+\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_1+\left|\!\left|\!\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_\infty\right),\\
&\frac{\left\|\Theta_x^{n+1}(\cdot,t)-\Theta_x^n(\cdot,t)\right\|_1}{|\log t|}\\
&\leqq C_\flat\left(\sqrt{t_\sharp}+\delta\right)\left(\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_{BV}+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_1\right.\\
&\quad\left.+\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_1+\left|\!\left|\!\left|\Theta^n-\Theta^{n-1}\right|\!\right|\!\right|_1\right).
\end{aligned}
$$

Moreover, the following Hölder continuity in time estimates hold:

$$
\begin{aligned}
&\left|\left(\Theta^{n+1}(x,t)-\Theta^{n+1}(x,s)\right)-\left(\Theta^n(x,t)-\Theta^n(x,s)\right)\right|\\
&\leqq O(1)(\sqrt{t}+\delta)\frac{\sqrt{t-s}}{\sqrt{s}}\left(|\log s|\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_\infty\right.\\
&\quad+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_{BV}+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_1\\
&\quad+\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\!\right|\!\right|_\infty+|\log s|\left|\!\left|\!\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_1\\
&\quad\left.+\sqrt{s}|\log s|\left|\!\left|\!\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_\infty\right),\\
&\left\|\left(\Theta^{n+1}(\cdot,t)-\Theta^{n+1}(\cdot,s)\right)-\left(\Theta^n(\cdot,t)-\Theta^n(\cdot,s)\right)\right\|_1\\
&\leqq O(1)(\sqrt{t}+\delta)\frac{\sqrt{t-s}}{\sqrt{s}}\left(|\log s|\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_\infty\right.\\
&\quad+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_{BV}+\left|\!\left|\!\left|V^n-V^{n-1}\right|\!\right|\!\right|_1\\
&\quad+\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\!\right|\!\right|_\infty+\sqrt{s}|\log s|\left|\!\left|\!\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_1\\
&\quad\left.+\sqrt{s}|\log s|\left|\!\left|\!\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\!\right|\!\right|_\infty\right).
\end{aligned}
$$

Here the norm $|\!|\!|\cdot|\!|\!|$ is defined in (2.18).

Proof. The proofs are split into several parts.

- (**Estimates of** $\left|\left|\left|\boldsymbol{\Theta^{n+1}-\Theta^{n}}\right|\right|\right|_{\infty}$ **and** $\left|\left|\left|\boldsymbol{\Theta^{n+1}-\Theta^{n}}\right|\right|\right|_{\mathbf{1}}$) By (3.41), Lemma 2.1, estimates of iteration scheme (3.5), and (3.40), one obtains that

$$\begin{aligned}&\left|\left(\Theta^{n+1}-\Theta^{n}\right)(x,t)\right|\\&\quad\leqq O(1)\delta\left|\left|\left|V^{n}-V^{n-1}\right|\right|\right|_{\infty}\\&\qquad+O(1)\delta\sqrt{t}\left(\left|\left|\left|V^{n}-V^{n-1}\right|\right|\right|_{\infty}+|\log t|\left|\left|\left|\frac{\Theta^{n}-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_{\infty}\right)\\&\qquad+O(1)\delta^{2}\left|\left|\left|V^{n}-V^{n-1}\right|\right|\right|_{\infty}+O(1)\left(\sqrt{t}+\delta\right)|\log t|\left|\left|\left|\frac{U_{x}^{n}-U_{x}^{n-1}}{|\log\tau|}\right|\right|\right|_{1}\\&\quad\leqq O(1)\left(\sqrt{t}+\delta\right)\left(\left|\left|\left|V^{n}-V^{n-1}\right|\right|\right|_{\infty}+\left|\left|\left|\frac{\Theta^{n}-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_{\infty}\right.\\&\qquad\left.+|\log t|\left|\left|\left|\frac{U_{x}^{n}-U_{x}^{n-1}}{|\log\tau|}\right|\right|\right|_{1}\right).\end{aligned}$$

In a similar manner, one can get the L^1 estimate of zeroth order,

$$\begin{aligned}&\int_{\mathbb{R}}\left|\left(\Theta^{n+1}-\Theta^{n}\right)(x,t)\right|\mathrm{d}x\\&\quad\leqq O(1)\left(\sqrt{t}+\delta\right)\\&\qquad\times\left(\left|\left|\left|V^{n}-V^{n-1}\right|\right|\right|_{1}+\left|\left|\left|\Theta^{n}-\Theta^{n-1}\right|\right|\right|_{1}+\sqrt{t}|\log t|\left|\left|\left|\frac{U_{x}^{n}-U_{x}^{n-1}}{|\log\tau|}\right|\right|\right|_{1}\right).\end{aligned}$$

- (**Estimate of** $|||\frac{\sqrt{\mathbf{t}}(\boldsymbol{\Theta_x^{n+1}-\Theta_x^{n}})}{|\log\mathbf{t}|}|||_{\infty}$) The equation for difference between consecutive iterations (3.41) is not sufficient for first order estimate due to presence of high order derivatives in the source term. Hence we start with the integral representation from (3.9)

$$\begin{aligned}&\Theta^{n+1}(x,t)-\Theta^{n}(x,t)\\&\quad=\int_{\mathbb{R}}\left(H\left(x,t;y,0;\kappa^{n}\right)-H\left(x,t;y,0;\kappa^{n-1}\right)\right)\theta_{0}^{*}(y)\mathrm{d}y\\&\qquad+\int_{0}^{t}\int_{\mathbb{R}}H\left(x,t;y,s;\kappa^{n-1}\right)\left(\mathcal{N}_{2}^{n}(y,s)-\mathcal{N}_{2}^{n-1}(y,s)\right)\mathrm{d}y\mathrm{d}s\\&\qquad+\int_{0}^{t}\int_{\mathbb{R}}\left(H\left(x,t;y,s;\kappa^{n}\right)-H\left(x,t;y,s;\kappa^{n-1}\right)\right)\mathcal{N}_{2}^{n}(y,s)\mathrm{d}y\mathrm{d}s.\end{aligned}\tag{3.42}$$

The differentiation of (3.42) is put on the heat kernel:

$$\begin{aligned}&\Theta_{x}^{n+1}(x,t)-\Theta_{x}^{n}(x,t)\\&\quad=\int_{\mathbb{R}}\left(H_{x}\left(x,t;y,0;\kappa^{n}\right)-H_{x}\left(x,t;y,0;\kappa^{n-1}\right)\right)\theta_{0}^{*}(y)\mathrm{d}y\\&\qquad+\int_{0}^{t}\int_{\mathbb{R}}H_{x}\left(x,t;y,s;\kappa^{n-1}\right)\left(\mathcal{N}_{2}^{n}(y,s)-\mathcal{N}_{2}^{n-1}(y,s)\right)\mathrm{d}y\mathrm{d}s\end{aligned}$$

$$+\int_0^t \int_{\mathbb{R}} \left(H_x\left(x,t;y,s;\kappa^n\right) - H_x\left(x,t;y,s;\kappa^{n-1}\right)\right)\mathcal{N}_2^n(y,s)\mathrm{d}y\mathrm{d}s. \tag{3.43}$$

For the first term, one recalls the definition of κ^n in (3.6), then applies the initial condition (3.2), Lemma 2.4 to obtain that

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \left(H_x\left(x,t;y,0;\kappa^n\right) - H_x\left(x,t;y,0;\kappa^{n-1}\right)\right)\theta_0^*(y)\mathrm{d}y\right| \\
&\leqq \int_{\mathbb{R}} \left|H_x\left(x,t;y,0;\kappa^n\right) - H_x\left(x,t;y,0;\kappa^{n-1}\right)\right|\mathrm{d}y\|\theta_0^*(y)\|_{L_x^\infty} \\
&\leqq O(1)\delta\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_* t}}}{t}\left[|\log t|\left\|\left|V^n - V^{n-1}\right|\right\|_\infty + \left\|\left|V^n - V^{n-1}\right|\right\|_{BV}\right. \\
&\quad \left. + \left\|\left|V^n - V^{n-1}\right|\right\|_1 + \left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n - U_x^{n-1}\right)\right|\right\|_\infty\right]\mathrm{d}y \\
&\leqq O(1)\frac{\delta}{\sqrt{t}}\left[|\log t|\left\|\left|V^n - V^{n-1}\right|\right\|_\infty + \left\|\left|V^n - V^{n-1}\right|\right\|_{BV}\right. \\
&\quad \left. + \left\|\left|V^n - V^{n-1}\right|\right\|_1 + \left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n - U_x^{n-1}\right)\right|\right\|_\infty\right].
\end{aligned} \tag{3.44}$$

Next, for the second term, one recalls the expression of $\mathcal{N}_2$ in (3.6), (3.40), and combines the ansatz (3.5), Lemma 2.1 to obtain that

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} H_x\left(x,t;y,\tau;\kappa^{n-1}\right)\left(\mathcal{N}_2^n(y,\tau) - \mathcal{N}_2^{n-1}(y,\tau)\right)\mathrm{d}y\mathrm{d}s\right| \\
&\leqq O(1)\delta\left(\left\|\left|V^n - V^{n-1}\right|\right\|_\infty + |\log t|\left\|\left|\frac{\Theta^n - \Theta^{n-1}}{|\log\tau|}\right|\right\|_\infty\right) \\
&\quad + O(1)\frac{\delta^2}{\sqrt{t}}\left\|\left|V^n - V^{n-1}\right|\right\|_\infty + O(1)(\delta + \sqrt{t})\frac{|\log t|}{\sqrt{t}}\left\|\left|\frac{U_x^n - U_x^{n-1}}{|\log\tau|}\right|\right\|_1 \\
&\quad + O(1)(\delta + \sqrt{t})\frac{|\log t|}{\sqrt{t}}\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n - U_x^{n-1}\right)\right|\right\|_\infty.
\end{aligned} \tag{3.45}$$

For the last term in (3.43), applying the comparison estimate in Lemma 2.4, and using the estimates in (3.5) to obtain that

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} \left(H_x\left(x,t;y,\tau;\kappa^n\right) - H_x\left(x,t;y,\tau;\kappa^{n-1}\right)\right)\mathcal{N}_2^n(y,\tau)\mathrm{d}y\mathrm{d}\tau\right| \\
&\leqq O(1)\frac{\delta\left(\sqrt{t}+\delta\right)}{\sqrt{t}}\left[|\log t|\left\|\left|V^n - V^{n-1}\right|\right\|_\infty + \left\|\left|V^n - V^{n-1}\right|\right\|_{BV}\right. \\
&\quad \left. + \left\|\left|V^n - V^{n-1}\right|\right\|_1 + \left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n - U_x^{n-1}\right)\right|\right\|_\infty\right].
\end{aligned} \tag{3.46}$$

Now, considering that δ and $t_\sharp$ are sufficiently small, one combines (3.43), (3.44), (3.45), and (3.46) to conclude that

$$\begin{aligned}
&\frac{\sqrt{t}}{|\log t|}\left|\Theta_x^{n+1}(x,t)-\Theta_x^n(x,t)\right| \\
&\leqq C_\flat\left(\sqrt{t}+\delta\right)\left[\left\|\left|V^n-V^{n-1}\right|\right\|_\infty+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}\right. \\
&\quad+\left\|\left|V^n-V^{n-1}\right|\right\|_1 \\
&\quad+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\right\|_\infty+\left\|\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\right\|_1 \\
&\quad\left.+\left\|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right\|_\infty\right],
\end{aligned}$$

where $C_\flat$ is a positive constant. Note that according to Remark 3.1, $C_\flat$ is uniform bounded due to the uniform boundedness of $C_\sharp$ and C_* when δ is small.

- (**Estimate of** $\left\|\left|\frac{\Theta_{\mathbf{x}}^{\mathbf{n+1}}-\Theta_{\mathbf{x}}^{\mathbf{n}}}{|\log\tau|}\right|\right\|_{\mathbf{1}}$) Integrate (3.43) with respect to x to yield

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\Theta_x^{n+1}(x,t)-\Theta_x^n(x,t)\right|\mathrm{d}x \\
&\leqq\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left(H_x\left(x,t;y,0;\kappa^n\right)-H_x\left(x,t;y,0;\kappa^{n-1}\right)\right)\theta_0^*(y)\mathrm{d}y\right|\mathrm{d}x \\
&\quad+\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}H_x\left(x,t;y,\tau;\kappa^{n-1}\right)\left(\mathcal{N}_2^n(y,\tau)-\mathcal{N}_2^{n-1}(y,\tau)\right)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x \\
&\quad+\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}\left(H_x\left(x,t;y,\tau;\kappa^n\right)-H_x\left(x,t;y,\tau;\kappa^{n-1}\right)\right)\mathcal{N}_2^n(y,\tau)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x.
\end{aligned}\tag{3.47}$$

For the first integral, by Lemma 2.4 and integration by parts, one follows the arguments as in (3.12) to obtain that

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left(H_x\left(x,t;y,0;\kappa^n\right)-H_x\left(x,t;y,0;\kappa^{n-1}\right)\right)\theta_0^*(y)\mathrm{d}y\right|\mathrm{d}x \\
&\leqq O(1)\delta\left(|\log t|\left\|\left|V^n-V^{n-1}\right|\right\|_\infty+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}\right. \\
&\quad\left.+\left\|\left|V^n-V^{n-1}\right|\right\|_1+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\right\|_\infty\right).
\end{aligned}\tag{3.48}$$

For the second term, one follows the estimates in (3.45) and integrates with respect to x to get

$$\begin{aligned}
&\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}H_x\left(x,t;y,\tau;\kappa^{n-1}\right)\left(\mathcal{N}_2^n(y,\tau)-\mathcal{N}_2^{n-1}(y,\tau)\right)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x \\
&\leqq O(1)\delta\left(\left\|\left|V^n-V^{n-1}\right|\right\|_1+\left\|\left|\Theta^n-\Theta^{n-1}\right|\right\|_1\right)+O(1)\delta^2\left\|\left|V^n-V^{n-1}\right|\right\|_\infty \\
&\quad+O(1)\left(\sqrt{t}+\delta\right)|\log t|\left\|\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right|\right\|_1.
\end{aligned}\tag{3.49}$$

Finally, according to comparison estimates in Lemma 2.4 and (3.5), the last term can be estimated as follows,

$$\begin{aligned}&\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}\left(H_x\left(x,t;y,\tau;\kappa^n\right)-H_x\left(x,t;y,\tau;\kappa^{n-1}\right)\right)\mathcal{N}_2^n(y,\tau)\mathrm{d}y\mathrm{d}\tau\right|\mathrm{d}x\\&\quad\leqq O(1)\delta\left(\sqrt{t}+\delta\right)\left(|\log t|\left\|\left|V^n-V^{n-1}\right\|\right|_\infty+\left\|\left|V^n-V^{n-1}\right\|\right|_{BV}\right.\\&\qquad\left.+\left\|\left|V^n-V^{n-1}\right\|\right|_1+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right\|\right|_\infty\right).\end{aligned}\tag{3.50}$$

Combining (3.47), (3.48), (3.49) and (3.50), we conclude

$$\begin{aligned}&\frac{1}{|\log t|}\int_{\mathbb{R}}\left|\Theta_x^{n+1}(x,t)-\Theta_x^n(x,t)\right|\mathrm{d}x\\&\quad\leqq C_\flat\left(\sqrt{t}+\delta\right)\left(\left\|\left|V^n-V^{n-1}\right\|\right|_\infty+\left\|\left|V^n-V^{n-1}\right\|\right|_{BV}+\left\|\left|V^n-V^{n-1}\right\|\right|_1\right.\\&\qquad\left.+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right\|\right|_\infty+\left\|\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right\|\right|_1+\left\|\left|\Theta^n-\Theta^{n-1}\right\|\right|_1\right).\end{aligned}$$

- (**Estimate of** $\|\left[\Theta^{\mathbf{n+1}}(\cdot,\mathbf{t})-\Theta^{\mathbf{n+1}}(\cdot,\mathbf{s})\right]-\left[\Theta^{\mathbf{n}}(\cdot,\mathbf{t})-\Theta^{\mathbf{n}}(\cdot,\mathbf{s})\right]\|_{\mathbf{L}^\infty_{\mathbf{x}}}$) According to the representation (3.13), one has

$$\begin{aligned}&\left(\Theta^{n+1}(x,t)-\Theta^{n+1}(x,s)\right)-\left(\Theta^n(x,t)-\Theta^n(x,s)\right)\\&\quad=\int_{\mathbb{R}}\int_s^t\left(H_\sigma\left(x,\sigma;y,0;\kappa^n\right)-H_\sigma\left(x,\sigma;y,0;\kappa^{n-1}\right)\right)\theta_0^*(y)\mathrm{d}\sigma\mathrm{d}y\\&\qquad+\int_s^t\int_{\mathbb{R}}\left(H\left(x,t;y,\tau;\kappa^n\right)-H\left(x,t;y,\tau;\kappa^{n-1}\right)\right)\mathcal{N}_2^n(y,\tau)\mathrm{d}y\mathrm{d}\tau\\&\qquad+\int_s^t\int_{\mathbb{R}}H\left(x,t;y,\tau;\kappa^{n-1}\right)\left(\mathcal{N}_2^n(y,\tau)-\mathcal{N}_2^{n-1}(y,\tau)\right)\mathrm{d}y\mathrm{d}\tau\\&\qquad+\int_0^s\int_{\mathbb{R}}\int_s^t\left(H_\sigma\left(x,\sigma;y,\tau;\kappa^n\right)-H_\sigma\left(x,\sigma;y,\tau;\kappa^{n-1}\right)\right)\mathcal{N}_2^n(y,\tau)\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\\&\qquad+\int_0^s\int_{\mathbb{R}}\int_s^t H_\sigma\left(x,\sigma;y,\tau;\kappa^{n-1}\right)\left(\mathcal{N}_2^n(y,\tau)-\mathcal{N}_2^{n-1}(y,\tau)\right)\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\\&\quad=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5.\end{aligned}\tag{3.51}$$

The estimates of $\mathcal{I}_i$'s are very similar to the previous arguments, thus we omit the details and directly list the results as below. For any $\alpha\in(0,1]$,

$$\begin{aligned}|\mathcal{I}_1|&\leqq O(1)\delta\frac{t-s}{t}\left(\frac{t}{s}\right)^\alpha|\log s|\left\|\left|V^n-V^{n-1}\right\|\right|_\infty\\&\quad+O(1)\delta\frac{t-s}{t}\left(\frac{t}{s}\right)^\alpha\left(\left\|\left|V^n-V^{n-1}\right\|\right|_{BV}+\left\|\left|V^n-V^{n-1}\right\|\right|_1\right.\\&\quad\left.+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right\|\right|_\infty\right),\\|\mathcal{I}_2|&\leqq O(1)\frac{\delta(t-s)}{t}\left(\sqrt{t}+\delta\left(\frac{t}{s}\right)^\alpha\right)\left\|\left|V^n-V^{n-1}\right\|\right|_\infty,\\|\mathcal{I}_3|&\leqq O(1)\frac{\left(\sqrt{t}+\delta\right)(t-s)|\log t|}{\sqrt{t}}\left(\left\|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right\|\right|_\infty+\left\|\left|\frac{U_x^n-U_x^{n-1}}{|\log\tau|}\right\|\right|_1\right)\end{aligned}$$

$$
\begin{aligned}
&\quad + O(1)\frac{\delta(t-s)}{\sqrt{t}}\left(1+\frac{\delta}{s^{\alpha}t^{\frac{1}{2}-\alpha}}\right)\left\|\left|V^{n}-V^{n-1}\right\|\right|_{\infty},\\
|\mathcal{I}_4| &\leqq O(1)\delta\left(\sqrt{s}+\delta\right)\left[\left(\frac{t-s}{s}\right)^{1-\alpha}+\frac{t-s}{t}\left(\frac{t}{s}\right)^{\alpha}\right]|\log s|\left\|\left|V^{n}-V^{n-1}\right\|\right|_{\infty}\\
&\quad + O(1)\delta\left(\sqrt{s}+\delta\right)\left[\left(\frac{t-s}{s}\right)^{1-\alpha}+\frac{t-s}{t}\left(\frac{t}{s}\right)^{\alpha}\right]\\
&\quad \times\left(\left\|\left|V^{n}-V^{n-1}\right\|\right|_{BV}+\left\|\left|V^{n}-V^{n-1}\right\|\right|_{1}\right)\\
&\quad + O(1)\delta\left(\sqrt{s}+\delta\right)\left[\left(\frac{t-s}{s}\right)^{1-\alpha}+\frac{t-s}{t}\left(\frac{t}{s}\right)^{\alpha}\right]\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n}-U_x^{n-1}\right)\right\|\right|_{\infty},\\
|\mathcal{I}_5| &\leqq O(1)\delta\sqrt{s}\left[\left(\frac{t-s}{s}\right)^{1-\alpha}+\frac{t-s}{t}\left(\frac{t}{s}\right)^{\alpha}\right]\\
&\quad \times\left(\left\|\left|V^{n}-V^{n-1}\right\|\right|_{\infty}+|\log s|\left\|\left|\frac{\Theta^{n}-\Theta^{n-1}}{|\log\tau|}\right\|\right|_{\infty}\right)\\
&\quad + O(1)\delta^{2}\left[\left(\frac{t-s}{s}\right)^{1-\alpha}+\frac{t-s}{t}\left(\frac{t}{s}\right)^{\alpha}\right]\left\|\left|V^{n}-V^{n-1}\right\|\right|_{\infty}\\
&\quad + O(1)\left(\sqrt{s}+\delta\right)\frac{t-s}{t}|\log s|\left\|\left|\frac{U_x^{n}-U_x^{n-1}}{|\log\tau|}\right\|\right|_{1}\\
&\quad + O(1)\left(\sqrt{s}+\delta\right)s|\log s|\left[\left(\frac{t-s}{s}\right)^{1-\alpha}+\frac{t-s}{t}\left(\frac{t}{s}\right)^{\alpha}\right]\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n}-U_x^{n-1}\right)\right\|\right|_{\infty}.
\end{aligned}
\tag{3.52}
$$

Now, we combine (3.51), (3.52) and set $\alpha=\frac{1}{2}$ to obtain

$$
\begin{aligned}
&\left|\left(\Theta^{n+1}(x,t)-\Theta^{n+1}(x,s)\right)-\left(\Theta^{n}(x,t)-\Theta^{n}(x,s)\right)\right|\\
&\leqq O(1)(\sqrt{t}+\delta)\frac{\sqrt{t-s}}{\sqrt{s}}\Bigg(|\log s|\left\|\left|V^{n}-V^{n-1}\right\|\right|_{\infty}\\
&\quad+\left\|\left|V^{n}-V^{n-1}\right\|\right|_{BV}+\left\|\left|V^{n}-V^{n-1}\right\|\right|_{1}\\
&\quad+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n}-U_x^{n-1}\right)\right\|\right|_{\infty}+|\log s|\left\|\left|\frac{U_x^{n}-U_x^{n-1}}{|\log\tau|}\right\|\right|_{1}\\
&\quad+\sqrt{s}|\log s|\left\|\left|\frac{\Theta^{n}-\Theta^{n-1}}{|\log\tau|}\right\|\right|_{\infty}\Bigg).
\end{aligned}
$$

With a similar argument as above, one can also obtain that

$$
\begin{aligned}
&\left\|\left(\Theta^{n+1}(\cdot,t)-\Theta^{n+1}(\cdot,s)\right)-\left(\Theta^{n}(\cdot,t)-\Theta^{n}(\cdot,s)\right)\right\|_{L_x^1}\\
&\leqq O(1)(\sqrt{t}+\delta)\frac{\sqrt{t-s}}{\sqrt{s}}\Bigg(|\log s|\left\|\left|V^{n}-V^{n-1}\right\|\right|_{\infty}\\
&\quad+\left\|\left|V^{n}-V^{n-1}\right\|\right|_{BV}+\left\|\left|V^{n}-V^{n-1}\right\|\right|_{1}
\end{aligned}
$$

$$
+ \left\|\left\| \frac{\sqrt{\tau}}{|\log \tau|} \left(U_x^n - U_x^{n-1}\right) \right\|\right\|_\infty + \sqrt{s}|\log s| \left\|\left\| \frac{U_x^n - U_x^{n-1}}{|\log \tau|} \right\|\right\|_1
+ \sqrt{s}|\log s| \left\|\left\| \frac{\Theta^n - \Theta^{n-1}}{|\log \tau|} \right\|\right\|_\infty \Bigg).
$$

Finally, we have completed the proof of this Lemma by combining all the above estimates. □

Next, we study the iteration difference for U^n. It turns out the difference estimate for Θ^n in Lemma 3.5 plays an important role in studying the difference for U^n through the pressure term.

Lemma 3.6. ($\mathbf{U^{n+1} - U^n}$) *There exists a positive constant $C_\flat$ such that, for sufficiently small δ and $t_\sharp$, when $0 < t < t_\sharp$,*

$$
\begin{aligned}
\left\|\left(U^{n+1} - U^n\right)(\cdot, t)\right\|_\infty &\leqq C_\flat \left(\sqrt{t}|\log t| + \delta\right) \\
&\quad \times \left(\left\|\left\| V^n - V^{n-1} \right\|\right\|_\infty + \left\|\left\| \frac{\Theta^n - \Theta^{n-1}}{|\log \tau|} \right\|\right\|_\infty \right), \\
\left\|\left(U^{n+1} - U^n\right)(\cdot, t)\right\|_1 &\leqq C_\flat \left(\sqrt{t} + \delta\right) \\
&\quad \times \left(\left\|\left\| V^n - V^{n-1} \right\|\right\|_1 + \left\|\left\| \Theta^n - \Theta^{n-1} \right\|\right\|_1 \right), \\
\frac{\sqrt{t}}{|\log t|} \left\|\left(U_x^{n+1} - U_x^n\right)(\cdot, t)\right\|_\infty &\leqq C_\flat \left(\sqrt{t} + \delta\right) \\
&\quad \times \left(\left\|\left\| V^n - V^{n-1} \right\|\right\|_\infty + \left\|\left\| V^n - V^{n-1} \right\|\right\|_{BV} \right. \\
&\quad + \left\|\left\| V^n - V^{n-1} \right\|\right\|_1 \\
&\quad + \left\|\left\| \frac{\left(U_x^n - U_x^{n-1}\right)}{|\log \tau|} \right\|\right\|_1 \\
&\quad + \left\|\left\| \frac{\sqrt{\tau}}{|\log \tau|} \left(U_x^n - U_x^{n-1}\right) \right\|\right\|_\infty \\
&\quad \left. + \left\|\left\| \frac{\left(\Theta^n - \Theta^{n-1}\right)}{|\log \tau|} \right\|\right\|_\infty \right), \\
\frac{\left\|\left(U_x^{n+1} - U_x^n\right)(\cdot, t)\right\|_1}{|\log t|} &\leqq C_\flat \left(\sqrt{t} + \delta\right) \\
&\quad \times \left(\left\|\left\| V^n - V^{n-1} \right\|\right\|_\infty + \left\|\left\| V^n - V^{n-1} \right\|\right\|_{BV} \right. \\
&\quad + \left\|\left\| \frac{\left(U_x^n - U_x^{n-1}\right)}{|\log \tau|} \right\|\right\|_1 \\
&\quad + \left\|\left\| \frac{\sqrt{\tau}}{|\log \tau|} \left(U_x^n - U_x^{n-1}\right) \right\|\right\|_\infty \\
&\quad \left. + \left\|\left\| \frac{\left(\Theta^n - \Theta^{n-1}\right)}{|\log \tau|} \right\|\right\|_\infty \right) \\
&\quad + \frac{C_\flat}{|\log t|} \left(\left\|\left\| V^n - V^{n-1} \right\|\right\|_1 + \left\|\left\| \Theta^n - \Theta^{n-1} \right\|\right\|_1 \right).
\end{aligned}
$$

Proof. We only show the details of the first order error in infinity norm, and the other estimates can be constructed in a similar way. For the estimate of the iteration error of U_x^n, we use (3.7) to obtain the following representation:

$$
\begin{aligned}
&\left(U_x^{n+1}-U_x^n\right)(x,t)\\
&=\int_{\mathbb{R}}\left(H_x\left(x,t;y,0;\mu^n\right)-H_x\left(x,t;y,0;\mu^{n-1}\right)\right)u_0^*(y)\mathrm{d}y\\
&\quad+\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}\left(H_{xy}\left(x,t;y,s;\mu^n\right)-H_{xy}\left(x,t;y,s;\mu^{n-1}\right)\right)\\
&\quad\times\ p(1+V^n,1+\Theta^n)(y,s)\mathrm{d}y\mathrm{d}s\\
&\quad+\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}H_{xy}\left(x,t;y,s;\mu^{n-1}\right)\\
&\quad\left(p(1+V^n,1+\Theta^n)-p(1+V^{n-1},1+\Theta^{n-1})\right)(y,s)\mathrm{d}y\mathrm{d}s\\
&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.
\end{aligned}\tag{3.53}
$$

The estimate of $\mathcal{I}_1$ is similar as (3.44). For $\mathcal{I}_2$, one recalls the Lipschitz continuity of V^n and Hölder continuity of Θ^n with respect to time variable, and follows the estimates in Lemma 3.2 and Lemma 3.3 to obtain

$$
\begin{aligned}
|\mathcal{I}_1|\leqq\ &O(1)\frac{\delta}{\sqrt{t}}\Big(|\log t|\left\|\left|V^n-V^{n-1}\right|\right\|_\infty\\
&+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}+\left\|\left|V^n-V^{n-1}\right|\right\|_1+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\right\|_\infty\Big),
\end{aligned}\tag{3.54}
$$

$$
\begin{aligned}
|\mathcal{I}_2|\leqq\ &O(1)\delta\Big(|\log t|\left\|\left|V^n-V^{n-1}\right|\right\|_\infty\\
&+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}+\left\|\left|V^n-V^{n-1}\right|\right\|_1+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^n-U_x^{n-1}\right)\right|\right\|_\infty\Big).
\end{aligned}\tag{3.55}
$$

For the last term $\mathcal{I}_3$ in (3.53), we use the interpolation method in the estimate for U_x^n in Lemma 3.3 and split $\mathcal{I}_3$ into five parts:

$$
\begin{aligned}
|\mathcal{I}_3|\leqq\ &\bigg|\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}H_{xy}(x,t;y,s;\mu^{n-1})\\
&\times\left[\left(\Theta^n(y,s)-\Theta^n(y,t)\right)-\left(\Theta^{n-1}(y,s)-\Theta^{n-1}(y,t)\right)\right]\frac{1}{v^n(y,s)}\mathrm{d}y\mathrm{d}s\bigg|\\
&+\bigg|\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}H_{xy}(x,t;y,s;\mu^{n-1})\left(\Theta^n(y,t)-\Theta^{n-1}(y,t)\right)\\
&\times\left(\frac{1}{v^n(y,s)}-\frac{1}{v^n(y,t)}\right)\mathrm{d}y\mathrm{d}s\bigg|\\
&+\bigg|\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}H_{xy}(x,t;y,s;\mu^{n-1})\left(1+\Theta^{n-1}(y,s)\right)
\end{aligned}
$$

$$\times \left[\left(\frac{1}{v^n(y,s)} - \frac{1}{v^n(y,t)}\right) - \left(\frac{1}{v^{n-1}(y,s)} - \frac{1}{v^{n-1}(y,t)}\right)\right] \mathrm{d}y\mathrm{d}s\Bigg|$$
$$+ \left|\int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} H_{xy}(x,t;y,s;\mu^{n-1}) \left(\Theta^{n-1}(y,s) - \Theta^{n-1}(y,t)\right)\right.$$
$$\left.\times \left(\frac{1}{v^n(y,t)} - \frac{1}{v^{n-1}(y,t)}\right) \mathrm{d}y\mathrm{d}s\right|$$
$$+ \left|\int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} H_{xy}(x,t;y,s;\mu^{n-1}) \left(\frac{1+\Theta^n(y,t)}{v^n(y,t)} - \frac{1+\Theta^{n-1}(y,t)}{v^{n-1}(y,t)}\right) \mathrm{d}y\mathrm{d}s\right|$$
$$\equiv \sum_{j=1}^{5} \mathcal{I}_{3j}. \tag{3.56}$$

For $\mathcal{I}_{32}$, $\mathcal{I}_{34}$ and $\mathcal{I}_{35}$, one can apply Lemma 2.2 and Lemma 3.2, and follow the similar computations as before to yield

$$\begin{aligned}\mathcal{I}_{32} &\leqq O(1) \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{(t-s)^{3/2}} \frac{\delta(t-s)}{\sqrt{t}} |\log t| \left\|\left|\frac{\Theta^n - \Theta^{n-1}}{|\log \tau|}\right|\right\|_\infty \mathrm{d}s \\ &\leqq O(1)\delta\sqrt{t}|\log t| \left\|\left|\frac{\Theta^n - \Theta^{n-1}}{|\log \tau|}\right|\right\|_\infty, \end{aligned} \tag{3.57}$$

$$\begin{aligned}\mathcal{I}_{34} &\leqq O(1) \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{(t-s)^{3/2}} \delta \frac{\sqrt{t-s}}{\sqrt{s}} \left\|\left|V^n - V^{n-1}\right|\right\|_\infty \mathrm{d}y\mathrm{d}s \\ &\leqq O(1)\delta \left\|\left|V^n - V^{n-1}\right|\right\|_\infty, \end{aligned} \tag{3.58}$$

$$\mathcal{I}_{35} \leqq O(1)\Big(|\log t| \left\|\left|\frac{\Theta^n - \Theta^{n-1}}{|\log \tau|}\right|\right\|_\infty + \left\|\left|V^n - V^{n-1}\right|\right\|_\infty\Big). \tag{3.59}$$

For the other two terms $\mathcal{I}_{31}$ and $\mathcal{I}_{33}$, we need the estimates of time differences of Θ^n and v^n. In fact, according to the Θ's Hölder estimate in Lemma 3.5, one has

$$\begin{aligned}\mathcal{I}_{31} &\leqq O(1) \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{(t-s)^{3/2}} \left(\sqrt{t}+\delta\right) \frac{\sqrt{t-s}}{\sqrt{s}} (|\log s| X_1 + X_2)\, \mathrm{d}y\mathrm{d}s \\ &\leqq O(1) \int_0^t \frac{\sqrt{t}+\delta}{\sqrt{s}\sqrt{t-s}} (|\log s| X_1 + X_2)\, \mathrm{d}s \leqq O(1)\left(\sqrt{t}+\delta\right)(|\log t| X_1 + X_2), \\ X_1 &\equiv \left\|\left|V^n - V^{n-1}\right|\right\|_\infty + \left\|\left|\frac{U_x^n - U_x^{n-1}}{|\log \tau|}\right|\right\|_1, \\ X_2 &\equiv \left\|\left|V^n - V^{n-1}\right|\right\|_{BV} + \left\|\left|V^n - V^{n-1}\right|\right\|_1 + \left\|\left|\frac{\sqrt{\tau}}{|\log \tau|}\left(U_x^n - U_x^{n-1}\right)\right|\right\|_\infty \\ &\quad + \left\|\left|\frac{\Theta^n - \Theta^{n-1}}{|\log \tau|}\right|\right\|_\infty. \end{aligned} \tag{3.60}$$

On the other hand, for $\mathcal{I}_{33}$, one follows the similar idea to take advantage of the Lipschitz continuity of V^n with respect to time variable to obtain the estimates as follows:

$$
\begin{aligned}
\mathcal{I}_{33} \leqq\; & O(1) \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{(t-s)^{3/2}} \int_s^t \frac{|\log\tau|}{\sqrt{\tau}} \mathrm{d}\tau \left\|\left| \frac{\sqrt{\tau}}{|\log\tau|} \left(U_x^n - U_x^{n-1}\right) \right|\right\|_\infty \mathrm{d}y\mathrm{d}s \\
& + O(1) \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*(t-s)}}}{(t-s)^{3/2}} \left\|\left| V^n - V^{n-1} \right|\right\|_\infty \frac{\delta(t-s)}{\sqrt{t}} \mathrm{d}y\mathrm{d}s \\
\leqq\; & O(1)\sqrt{t}|\log t| \left\|\left| \frac{\sqrt{\tau}}{|\log\tau|} \left(U_x^n - U_x^{n-1}\right) \right|\right\|_\infty + O(1)\delta\sqrt{t} \left\|\left| V^n - V^{n-1} \right|\right\|_\infty .
\end{aligned} \tag{3.61}
$$

Combine (3.56)-(3.61) to obtain

$$
\begin{aligned}
|\mathcal{I}_3| \leqq\; & O(1)\left(\sqrt{t}+\delta\right)\Bigg(|\log t| \left\|\left| V^n - V^{n-1} \right|\right\|_\infty + \left\|\left| V^n - V^{n-1} \right|\right\|_{BV} \\
& + \left\|\left| V^n - V^{n-1} \right|\right\|_1 \\
& + |\log t| \left\|\left| \frac{\left(U_x^n - U_x^{n-1}\right)}{|\log\tau|} \right|\right\|_1 + |\log t| \left\|\left| \frac{\sqrt{\tau}}{|\log\tau|} \left(U_x^n - U_x^{n-1}\right) \right|\right\|_\infty \Bigg) \\
& + O(1)|\log t| \left\|\left| \frac{\left(\Theta^n - \Theta^{n-1}\right)}{|\log\tau|} \right|\right\|_\infty + O(1) \left\|\left| V^n - V^{n-1} \right|\right\|_\infty .
\end{aligned} \tag{3.62}
$$

Finally, we combine (3.53), (3.54), (3.55) and (3.62) to conclude that, for sufficiently small δ and $t_\sharp$, the following estimate holds for the difference of velocities between two consecutive steps:

$$
\begin{aligned}
& \frac{\sqrt{t}}{|\log t|} \left| U_x^{n+1}(x,t) - U_x^n(x,t) \right| \\
& \leqq O(1)\left(\sqrt{t}+\delta\right)\Bigg(\left\|\left| V^n - V^{n-1} \right|\right\|_\infty + \left\|\left| V^n - V^{n-1} \right|\right\|_{BV} + \left\|\left| V^n - V^{n-1} \right|\right\|_1 \\
& + \left\|\left| \frac{\left(U_x^n - U_x^{n-1}\right)}{|\log\tau|} \right|\right\|_1 + \left\|\left| \frac{\sqrt{\tau}}{|\log\tau|} \left(U_x^n - U_x^{n-1}\right) \right|\right\|_\infty + \left\|\left| \frac{\left(\Theta^n - \Theta^{n-1}\right)}{|\log\tau|} \right|\right\|_\infty \Bigg).
\end{aligned}
$$

□

As for the iteration difference of the BV function V^n, we also need to estimate the evolution at discontinuities of V^n.

Lemma 3.7. ($\mathbf{V^{n+1} - V^n}$) *There exists a positive constant $C_\flat$ such that, for sufficiently small δ and $t_\sharp$ and for $0 < t < t_\sharp$,*

$$
\begin{aligned}
& \left| V^{n+1}(x,t) - V^n(x,t) \right| \\
& \quad \leqq C_\flat \left(\delta + \sqrt{t_\sharp}\right)\left(\left\|\left| V^n - V^{n-1} \right|\right\|_\infty + \left\|\left| \frac{\Theta^n - \Theta^{n-1}}{|\log\tau|} \right|\right\|_\infty \right), \\
& \| V^{n+1}(\cdot,t) - V^n(\cdot,t) \|_{L^1_x} \\
& \quad \leqq C_\flat \left(\delta + \sqrt{t_\sharp}\right)\left(\left\|\left| V^n - V^{n-1} \right|\right\|_1 + \left\|\left| \Theta^n - \Theta^{n-1} \right|\right\|_1 \right),
\end{aligned}
$$

$$
\begin{aligned}
&\left|\left(V^{n+1}(\cdot,t)-V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right| \\
&\quad \leqq C_\flat\left(\delta+\sqrt{t}\right)\sqrt{t}\sup_{0<\tau<t}\left|\left(V^n(\cdot,\tau)-V^{n-1}(\cdot,\tau)\right)\Big|_{z^-}^{z^+}\right| \\
&\qquad + C_\flat\delta\Big(|||V^n-V^{n-1}|||_\infty+\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\right|\right|_\infty+\left|\left|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_\infty\Big), \\
&\|V^{n+1}(\cdot,t)-V^n(\cdot,t)\|_{BV} \\
&\quad \leqq C_\flat\left(\sqrt{t}+\delta\right)\Big(|||V^n-V^{n-1}|||_\infty+|||V^n-V^{n-1}|||_1+|||V^n-V^{n-1}|||_{BV} \\
&\qquad + \left|\left|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_\infty+|||\Theta^n-\Theta^{n-1}|||_1+\left|\left|\left|\frac{\Theta_x^n-\Theta_x^{n-1}}{|\log\tau|}\right|\right|\right|_1\Big).
\end{aligned}
$$

Proof. We will only show the BV estimate, since the L^1 and L^∞ can be similarly obtained via previous estimates of U. Recall again our adoption (2.5) and discussions in previous sections, we will only treat the case when V^k has no singular part. Then, the BV estimate will be split into two parts. The iteration difference of V^n is expressed as below,

$$
\begin{aligned}
&V^{n+1}(x,t)-V^n(x,t) \\
&\quad = \int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}\int_\tau^t H_{xy}\left(x,s;y,\tau;\mu^n\right) \\
&\qquad \times\left(\frac{K(\Theta^n-\Theta^{n-1})}{1+V^n}-\frac{K(V^{n-1}-V^n)(1+\Theta^{n-1})}{(1+V^{n-1})(1+V^n)}\right)(y,\tau)\mathrm{d}s\mathrm{d}y\mathrm{d}\tau \\
&\qquad + \int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}\int_\tau^t H_{xy}\left(x,s;y,\tau;\mu^n\right)\left(\frac{\mu\left(V^n-V^{n-1}\right)U_y^n}{(1+V^n)(1+V^{n-1})}\right)(y,\tau)\mathrm{d}s\mathrm{d}y\mathrm{d}\tau.
\end{aligned}
\tag{3.63}
$$

• (**Case: x** $\notin\mathscr{D}$) For all $x\notin\mathscr{D}$, the derivative of $V^{n+1}-V^n$ can be defined almost everywhere according to the representation (3.63). We have

$$
\begin{aligned}
&\int_{\mathbb{R}\backslash\mathscr{D}}\left|V_x^{n+1}(x,t)-V_x^n(x,t)\right|\mathrm{d}x \\
&\quad \leqq \int_{\mathbb{R}\backslash\mathscr{D}}\Bigg|\int_0^t\int_{\mathbb{R}}\left[\int_\tau^t H_{xxy}(x,\sigma;y,\tau;\frac{1}{1+V^n})\mathrm{d}\sigma\right] \\
&\qquad \times\left[p^n(y,\tau)-p^{n-1}(y,\tau)\right]\mathrm{d}y\mathrm{d}\tau\Bigg|\mathrm{d}x \\
&\qquad + \int_{\mathbb{R}\backslash\mathscr{D}}\Bigg|\int_0^t\int_{\mathbb{R}}\left[\int_\tau^t H_{xxy}(x,\sigma;y,\tau;\frac{1}{1+V^n})\mathrm{d}\sigma\right] \\
&\qquad \times\left[\frac{\left(V^n-V^{n-1}\right)U_y^n}{(1+V^n)(1+V^{n-1})}\right]\mathrm{d}y\mathrm{d}\tau\Bigg|\mathrm{d}x \\
&\quad \equiv \mathcal{I}_1+\mathcal{I}_2.
\end{aligned}
\tag{3.64}
$$

For the estimate of $\mathcal{I}_1$, let $p^n(y,\tau) = \frac{K\theta^n(y,\tau)}{v^n(y,\tau)}$, combine (3.5), (3.64) and the estimate (2.16) of $\int_\tau^t H_{xxy}\mathrm{d}\sigma$ in Lemma 2.2 to obtain

$$\begin{aligned}
\mathcal{I}_1 \leqq\ & O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}} \left|V_x^n(x,\tau)\right| \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{\sqrt{t-\tau}} \left|p^n(y,\tau)-p^{n-1}(y,\tau)\right| \mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t \left|V_x^n(x,\tau)\right|\left|p^n(x,\tau)-p^{n-1}(x,\tau)\right|\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}}\int_\tau^t \left|V_x^n(x,\tau)\right| \frac{\delta(\sigma-\tau)}{\sqrt{\sigma}} \frac{e^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}} \\
& \times \left|p^n(y,\tau)-p^{n-1}(y,\tau)\right|\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau} \left|p^n(y,\tau)-p^{n-1}(y,\tau)\right|\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t \left|p_x^n(x,\tau)-p_x^{n-1}(x,\tau)\right|\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}}\int_\tau^t \frac{\delta(\sigma-\tau)}{\sqrt{\sigma}} \frac{e^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^2} \\
& \times \left|p^n(y,\tau)-p^{n-1}(y,\tau)\right|\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}}\int_\tau^t \left|V_x^n(x,\sigma)-V_x^n(x,\tau)\right| \frac{e^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}} \\
& \times \left|p^n(y,\tau)-p^{n-1}(y,\tau)\right|\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
\leqq\ & O(1)\sqrt{t}\Big(\left\|\left|V^n-V^{n-1}\right|\right\|_\infty + \left\|\left|V^n-V^{n-1}\right|\right\|_1 + \left\|\left|V^n-V^{n-1}\right|\right\|_{BV} \\
& + \left\|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right\|_\infty + \left\|\left|\Theta^n-\Theta^{n-1}\right|\right\|_1 + \left\|\left|\frac{\Theta_x^n-\Theta_x^{n-1}}{|\log\tau|}\right|\right\|_1\Big).
\end{aligned} \tag{3.65}$$

For the estimate of $\mathcal{I}_2$, we again apply the heat kernel estimate (2.16) in Lemma 2.2 to split $\mathcal{I}_2$ into seven parts:

$$\begin{aligned}
\mathcal{I}_2 \leqq\ & O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}} \left|V_x^n(x,\tau)\right| \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{\sqrt{t-\tau}} \left|V^n(y,\tau)-V^{n-1}(y,\tau)\right| \\
& \times \left|U_y^n(y,\tau)\right|\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t \left|V_x^n(x,\tau)\right|\left|V^n(x,\tau)-V^{n-1}(x,\tau)\right|\left|U_x^n(x,\tau)\right|\mathrm{d}\tau\mathrm{d}x \\
& + O(1)\int_{\mathbb{R}\setminus\mathscr{D}}\int_0^t\int_{\mathbb{R}}\int_\tau^t \left|V_x^n(x,\tau)\right| \frac{\delta(\sigma-\tau)}{\sqrt{\sigma}} \frac{e^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}}
\end{aligned}$$

$$\times \left|V^n(y,\tau)-V^{n-1}(y,\tau)\right|\left|U_y^n(y,\tau)\right|\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x$$

$$+\,O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^t\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau}\left|V^n(y,\tau)-V^{n-1}(y,\tau)\right|\left|U_y^n(y,\tau)\right|\mathrm{d}y\mathrm{d}\tau\mathrm{d}x$$

$$+\,O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^t\partial_x\left[\frac{(V^n-V^{n-1})(x,\tau)U_x^n(x,\tau)}{(1+V^n(x,\tau))(1+V^{n-1}(x,\tau))}\right]\mathrm{d}\tau\right|\mathrm{d}x$$

$$+\,O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^t\int_{\mathbb{R}}\int_\tau^t\frac{\delta(\sigma-\tau)}{\sqrt{\sigma}}\frac{e^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^2}$$

$$\times \left|V^n(y,\tau)-V^{n-1}(y,\tau)\right|\left|U_y^n(y,\tau)\right|\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x$$

$$+\,O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^t\int_{\mathbb{R}}\int_\tau^t\left|V_x^n(x,\sigma)-V_x^n(x,\tau)\right|\frac{e^{-\frac{(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{3/2}}$$

$$\times \left|V^n(y,\tau)-V^{n-1}(y,\tau)\right|\left|U_y^n(y,\tau)\right|\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau\mathrm{d}x$$

$$\equiv\sum_{j=1}^{7}\mathcal{I}_{2j}.$$

Except for $\mathcal{I}_{25}$, the other terms can be estimated by applying the ansatz (3.5), and by the similar computations as for (3.65):

$$\begin{aligned}\mathcal{I}_{21},\mathcal{I}_{22}&\leqq O(1)\sqrt{t}\delta^2\left\|\left|V^n-V^{n-1}\right|\right\|_\infty,\\ \mathcal{I}_{23}&\leqq O(1)t\delta^3\left\|\left|V^n-V^{n-1}\right|\right\|_\infty,\\ \mathcal{I}_{24}&\leqq O(1)\sqrt{t}\delta\left\|\left|V^n-V^{n-1}\right|\right\|_\infty,\\ \mathcal{I}_{26}&\leqq O(1)t\delta^2\left\|\left|V^n-V^{n-1}\right|\right\|_\infty,\\ \mathcal{I}_{27}&\leqq O(1)\delta^2 t|\log t|\left\|\left|V^n-V^{n-1}\right|\right\|_\infty.\end{aligned}\tag{3.66}$$

For $\mathcal{I}_{25}$, we split it into three parts. When ∂_x acts on denominator, it can be estimated similarly as before, and it is bounded by $O(1)\sqrt{t}\delta^2|||V^n-V^{n-1}|||_\infty$. It remains to consider the following two terms

$$\begin{aligned}\mathcal{I}_{25,1}=&\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^t\frac{U_x^n(x,\tau)}{(1+V^n(x,\tau))(1+V^{n-1}(x,\tau))}\partial_x\right.\\ &\left.\times\left(V^n(x,\tau)-V^{n-1}(x,\tau)\right)\mathrm{d}\tau\right|\mathrm{d}x,\\ \mathcal{I}_{25,2}=&\int_{\mathbb{R}\backslash\mathscr{D}}\left|\int_0^t\frac{V^n(x,\tau)-V^{n-1}(x,\tau)}{(1+V^n(x,\tau))(1+V^{n-1}(x,\tau))}U_{xx}^n(x,\tau)\mathrm{d}\tau\right|\mathrm{d}x.\end{aligned}$$

By the ansatz (3.5),

$$\mathcal{I}_{25,1}\leqq O(1)\sqrt{t}\delta\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}.\tag{3.67}$$

While it needs more efforts to estimate $\mathcal{I}_{25,2}$. One first has the representation

$$
\begin{aligned}
U^n_{xx}(x,\tau) = &\int_{\mathbb{R}} H_{xx}(x,\tau;y,0;\frac{1}{v^{n-1}})U_0(y)\mathrm{d}y \\
&- \int_0^\tau \int_{\mathbb{R}\setminus\mathscr{D}} H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})\partial_y p^{n-1}(y,s)\mathrm{d}y\mathrm{d}s \\
&- \int_0^\tau \sum_{z\in\mathscr{D}} \left[H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})p^{n-1}(y,s)\right]_{y=z^-}^{y=z^+} \mathrm{d}s.
\end{aligned}
$$

Substitute it into the expression for $\mathcal{I}_{25,2}$ to yield

$$
\begin{aligned}
\mathcal{I}_{25,2} \leqq &\int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \frac{V^n(x,\tau)-V^{n-1}(x,\tau)}{(1+V^n(x,\tau))(1+V^{n-1}(x,\tau))} \right.\\
&\times \left.\left[\int_{\mathbb{R}} H_{xx}(x,\tau;y,0;\frac{1}{v^{n-1}})U_0(y)\mathrm{d}y \right] \mathrm{d}\tau \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \frac{V^n(x,\tau)-V^{n-1}(x,\tau)}{(1+V^n(x,\tau))(1+V^{n-1}(x,\tau))} \right.\\
&\times \left.\left[\int_0^\tau \int_{\mathbb{R}\setminus\mathscr{D}} H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})\partial_y p^{n-1}(y,s)\mathrm{d}y\mathrm{d}s \right] \mathrm{d}\tau \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \left| \int_0^t \frac{V^n(x,\tau)-V^{n-1}(x,\tau)}{(1+V^n(x,\tau))(1+V^{n-1}(x,\tau))} \right.\\
&\times \left.\left[\int_0^\tau \sum_{z\in\mathscr{D}} \left[H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})p^{n-1}(y,s)\right]_{y=z^-}^{y=z^+} \mathrm{d}s \right] \mathrm{d}\tau \right| \mathrm{d}x \\
\equiv\ & T_1 + T_2 + T_3.
\end{aligned}
$$

To estimate T_1, one rewrites the inner integral involving U_0 via integration by parts

$$
\begin{aligned}
&\int_{\mathbb{R}} H_{xx}(x,\tau;y,0;\frac{1}{v^{n-1}})U_0(y)\mathrm{d}y \\
&\quad = -\int_{\mathbb{R}\setminus\mathscr{D}} \left[\int_{-\infty}^y H_{xx}(x,\tau;z,0;\frac{1}{v^{n-1}})\mathrm{d}z \right] \partial_y U_0(y)\mathrm{d}y \\
&\qquad - \sum_{\alpha\in\mathscr{D}} \left[\int_{-\infty}^y H_{xx}(x,\tau;z,0;\frac{1}{v^{n-1}})\mathrm{d}z\, U_0(y) \right]_{y=\alpha^-}^{y=\alpha^+}.
\end{aligned}
$$

Thus, one applies the heat kernel estimates in Lemma 2.1 to have

$$
\begin{aligned}
T_1 \leqq O(1) \left|\left|\left| V^n - V^{n-1} \right|\right|\right|_\infty \Bigg(&\int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*\tau}}}{\tau} \left|\partial_y U_0(y)\right| \mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&+ \int_{\mathbb{R}\setminus\mathscr{D}} \int_0^t \sum_{\alpha\in\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C_*\tau}}}{\tau} \left| [U_0(y)]_{y=\alpha^-}^{y=\alpha^+} \right| \mathrm{d}\tau\mathrm{d}x \Bigg)
\end{aligned}
$$

$$\leqq O(1)\sqrt{t}\,\|U_0\|_{BV}\left\|\left|V^n - V^{n-1}\right|\right\|_{\infty}. \tag{3.68}$$

Next for T_2, to handle the singularity in time integral, we write

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}\backslash\mathscr{D}} H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})\partial_y p^{n-1}(y,s)\mathrm{d}y\mathrm{d}s \\
&\quad= \int_0^\tau \int_{\mathbb{R}\backslash\mathscr{D}} H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})\left[\partial_y p^{n-1}(y,s) - \partial_y p^{n-1}(y,\tau)\right]\mathrm{d}y\mathrm{d}s \\
&\qquad+ \int_{\mathbb{R}\backslash\mathscr{D}} \left[\int_0^\tau H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})\mathrm{d}s\right]\partial_y p^{n-1}(y,\tau)\mathrm{d}y.
\end{aligned}$$

Observe that for $x \notin \mathscr{D}$,

$$\begin{aligned}
&\int_0^\tau H_{xx}(x,\tau;y,s;\frac{1}{v^{n-1}})\mathrm{d}s = \int_0^\tau v^{n-1}(x,\tau) \\
&\quad\times\left[H_\tau(x,\tau;y,s;\frac{1}{v^{n-1}}) - \partial_x\left(\frac{1}{1+V^{n-1}(x,\tau)}\right)H_x(x,\tau;y,s;\frac{1}{v^{n-1}})\right]\mathrm{d}s \\
&= v^{n-1}(x,\tau)\left(H(x,\tau;y;\frac{1}{v^{n-1}(\cdot,\tau)}) - \delta(x-y)\right) \\
&\quad+ O(1)\left(\delta + \left|V_x^{n-1}(x,\tau)\right|\right)e^{-\frac{(x-y)^2}{C_*\tau}},
\end{aligned}$$

where the last equality is due to the estimate (2.17) in Lemma 2.1. We then substitute the above estimate into T_2, and apply the Hölder continuity in time of p_x, given by V_x and Θ_x in Lemma 3.2 to yield

$$\begin{aligned}
T_2 \leqq\ & O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^t\int_0^\tau\int_{\mathbb{R}\backslash\mathscr{D}}\left|V^n(x,\tau) - V^{n-1}(x,\tau)\right| \\
&\times\left(\frac{e^{-\frac{(x-y)^2}{C_*(\tau-s)}}}{(\tau-s)^{3/2}} + \left|V_x^{n-1}(x,\tau)\right|\frac{e^{-\frac{(x-y)^2}{C_*(\tau-s)}}}{(\tau-s)}\right) \\
&\times\left(\delta\frac{\tau-s}{\sqrt{\tau}}\left(\left|V_y^{n-1}(y,s)\right| + \left|\Theta_y^{n-1}(y,s)\right|\right)\right. \\
&+ \left|\Theta^{n-1}(y,\tau) - \Theta^{n-1}(y,s)\right|\left|V_y^{n-1}(y,s)\right| \\
&\left.+ \left|V_y^{n-1}(y,\tau) - V_y^{n-1}(y,s)\right| + \left|\Theta_y^{n-1}(y,\tau) - \Theta_y^{n-1}(y,s)\right|\right)\mathrm{d}y\mathrm{d}s\mathrm{d}\tau\mathrm{d}x \\
&+ O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^t\int_{\mathbb{R}\backslash\mathscr{D}}\left|V^n(x,\tau) - V^{n-1}(x,\tau)\right| \\
&\times\left(\frac{e^{-\frac{(x-y)^2}{C_*\tau}}}{\sqrt{\tau}} + \left(\delta + \left|V_x^{n-1}(x,\tau)\right|\right)e^{-\frac{(x-y)^2}{C_*\tau}}\right) \\
&\times\left(\left|V_y^{n-1}(y,\tau)\right| + \left|\Theta_y^{n-1}(y,\tau)\right|\right)\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&+ O(1)\int_{\mathbb{R}\backslash\mathscr{D}}\int_0^t\left|V^n(x,\tau) - V^{n-1}(x,\tau)\right|
\end{aligned}$$

$$\times\left(\left|V_x^{n-1}(x,\tau)\right|+\left|\Theta_x^{n-1}(x,\tau)\right|\right)\mathrm{d}\tau\,\mathrm{d}x$$
$$\leqq O(1)\left\|\left|V^n-V^{n-1}\right|\right\|_\infty\delta\sqrt{t}|\log t|. \tag{3.69}$$

By similar analysis for the estimate of T_2, we have

$$T_3\leqq O(1)\delta t\left\|\left|V^n-V^{n-1}\right|\right\|_\infty. \tag{3.70}$$

Combine (3.67), (3.68), (3.69) and (3.70) to yield

$$\mathcal{I}_{25}\leqq O(1)\delta\sqrt{t}\left(|\log t|\left\|\left|V^n-V^{n-1}\right|\right\|_\infty+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}\right).$$

This together with the estimates of $\mathcal{I}_{2j},\ j=1\dots7$ in (3.66) gives

$$\mathcal{I}_{2}\leqq O(1)\delta\sqrt{t}\left(|\log t|\left\|\left|V^n-V^{n-1}\right|\right\|_\infty+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}\right).$$

Therefore we conclude from (3.64) and (3.65) that

$$\begin{aligned}&\int_{\mathbb{R}\setminus\mathscr{D}}\left|V_x^{n+1}(x,t)-V_x^n(x,t)\right|\mathrm{d}x\\&\leqq O(1)\left(\sqrt{t}+\delta\right)\left(\left\|\left|V^n-V^{n-1}\right|\right\|_\infty+\left\|\left|V^n-V^{n-1}\right|\right\|_1+\left\|\left|V^n-V^{n-1}\right|\right\|_{BV}\right.\\&\quad\left.+\left\|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right\|_\infty+\left\|\left|\Theta^n-\Theta^{n-1}\right|\right\|_1+\left\|\left|\frac{\Theta_x^n-\Theta_x^{n-1}}{|\log\tau|}\right|\right\|_1\right).\end{aligned} \tag{3.71}$$

• (**Case: x** $\in\mathscr{D}$) From the representation of the jump in (3.27),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(V^{n+1}(\cdot,t)-V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\\&=\left(\frac{V^n}{\mu}-\frac{V^{n-1}}{\mu}\right)\Big|_{z^-}^{z^+}\left(\frac{\mu U_x^{n+1}}{1+V^n}-p(1+V^n,1+\Theta^n)\right)\\&\quad+\frac{V^{n-1}}{\mu}\Big|_{z^-}^{z^+}\left(\frac{\mu\left(U_x^{n+1}-U_x^n\right)}{1+V^n}+\frac{\mu U_x^n\left(V^{n-1}-V^n\right)}{\left(1+V^{n-1}\right)(1+V^n)}\right.\\&\quad\left.+\frac{K\left(1+\Theta^{n-1}\right)}{\left(1+V^{n-1}\right)}-\frac{K\left(1+\Theta^n\right)}{(1+V^n)}\right)\\&\quad+\left(V^n-V^{n-1}\right)\Big|_{z^-}^{z^+}\frac{K\left(2+V^n(z^+)+V^{n-1}(z^+)\right)}{\mu\left(1+\Theta^n\right)}\\&\quad+\left(2+V^n+V^{n-1}\right)\Big|_{z^-}^{z^+}\frac{K\left(V^n(z^-)-V^{n-1}(z^-)\right)}{\mu\left(1+\Theta^n\right)}\\&\quad+V^{n-1}\Big|_{z^-}^{z^+}K\left(2+V^{n-1}(z^+)+V^{n-1}(z^-)\right)\left(\frac{1}{\mu\left(1+\Theta^n\right)}-\frac{1}{\mu\left(1+\Theta^{n-1}\right)}\right).\end{aligned}$$

Since the initial difference $V^{n+1}(\cdot,0)-V^n(\cdot,0)=0$, and we have the BV estimate in Remark 3.1, the integration of the above equality yields

$$
\begin{aligned}
&\left|\left(V^{n+1}(\cdot,t)-V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right| \\
&\quad\lesssim \left(\delta+\sqrt{t}\right)\sqrt{t}\sup_{0<\tau<t}\left|\left(V^n(\cdot,\tau)-V^{n-1}(\cdot,\tau)\right)\Big|_{z^-}^{z^+}\right| \\
&\qquad+\delta\sqrt{t}|\log t|\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\right|\right|_\infty \\
&\qquad+\delta\sqrt{t}\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_\infty+\delta t|\log t|\left|\left|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_\infty.
\end{aligned}
\tag{3.72}
$$

Taking the summation over z, one obtains

$$
\begin{aligned}
&\sum_{z\in\mathscr{D}}\left|\left(V^{n+1}(\cdot,t)-V^n(\cdot,t)\right)\Big|_{z^-}^{z^+}\right| \\
&\lesssim \delta\left(\sqrt{t}+\delta\right)\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_{BV}+\delta\sqrt{t}|\log t|\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\right|\right|_\infty \\
&\quad+\delta^2\sqrt{t}\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_\infty \\
&\quad+\delta t\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_\infty+\delta t|\log t|\left|\left|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_\infty+t\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_{BV} \\
&\quad+\delta t\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_\infty \\
&\lesssim (\delta+t)\left(\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_\infty+\left|\left|\left|V^n-V^{n-1}\right|\right|\right|_{BV}+\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U_x^{n+1}-U_x^n\right)\right|\right|\right|_\infty\right. \\
&\quad\left.+\left|\left|\left|\frac{\Theta^n-\Theta^{n-1}}{|\log\tau|}\right|\right|\right|_\infty\right).
\end{aligned}
\tag{3.73}
$$

The proof of the lemma is completed by combining the estimates (3.71), (3.72) and (3.73). □

Remark 3.2. For the estimate of total variation for $V^{n+1}-V^n$ with singular part, one can also apply the argument as in item (3) of Remark 3.1.

Now, we are able to prove the main result of this section, i.e., the local-in-time existence of a weak solution to the nonlinear Navier–Stokes equation (3.1).

Theorem 3.1. *Suppose the initial data $(v_0^*, u_0^*, \theta_0^*)$ satisfies the condition (3.2) for small δ. Then there exists a positive constant $t_\sharp$ such that, equation (3.1) admits a weak solution*

$$(v,u,\theta)=(v^*+1,u^*,\theta^*+1),\quad t<t_\sharp,$$

satisfying the estimates

$$\begin{cases} \delta > 0, \quad 0 < t < t_\sharp \ll 1, \\ \max\left\{\|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ \max\left\{\|\theta(\cdot,t)-1\|_{L^1_x},\ \|\theta(\cdot,t)-1\|_{L^\infty_x},\ \|\theta_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|\theta_x(\cdot,t)\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ \max\left\{\|v(\cdot,t)\|_{BV},\ \|v(\cdot,t)-1\|_{L^1_x},\ \|v(\cdot,t)-1\|_{L^\infty_x},\ \sqrt{t}\,\|v_t(\cdot,t)\|_{L^\infty_x}\right\} \leqq 2C_\sharp\delta, \\ v^* = v^*_c + v^*_d, \quad v^*_d(x,t) = \displaystyle\sum_{z<x,\, z\in\mathscr{D}} v^*\Big|_{z^-}^{z^+} h(x-z), \quad v^*_c \text{ is continuous}, \\ \left| v(\cdot,t)\Big|_{x=z^-}^{x=z^+}\right| \leqq 2\left| v^*_0(\cdot)\Big|_{x=z^-}^{x=z^+}\right|, \quad z\in\mathscr{D} \end{cases} \tag{3.74}$$

for some positive constant $C_\sharp$, where $h(x)$ is the Heaviside step function. Moreover, the fluxes of u and θ

$$\text{flux of } u = \frac{\mu u_x}{v} - p, \qquad \text{flux of } \theta = \frac{\kappa}{c_v v}\theta_x - \int_{-\infty}^{x}\left(\frac{p}{c_v}u_z - \frac{\mu}{c_v v}(u_z)^2\right)\mathrm{d}z, \tag{3.75}$$

are both continuous with respect to x.

Proof. The proof consists of the following four steps:

- (*Step 1: Strong convergence*) One introduces the following functional of the iteration difference,

$$\begin{aligned} &\mathscr{F}\left[V^{n+1}-V^n, U^{n+1}-U^n, \Theta^{n+1}-\Theta^n\right] \\ &\equiv \left|\left|\left|V^{n+1}-V^n\right|\right|\right|_\infty + \left|\left|\left|V^{n+1}-V^n\right|\right|\right|_1 + \left|\left|\left|V^{n+1}-V^n\right|\right|\right|_{BV} \\ &\quad + \left|\left|\left|U^{n+1}-U^n\right|\right|\right|_\infty + \left|\left|\left|U^{n+1}-U^n\right|\right|\right|_1 + \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U^{n+1}_x-U^n_x\right)\right|\right|\right|_\infty \\ &\quad + \left|\left|\left|\frac{U^{n+1}_x-U^n_x}{|\log\tau|}\right|\right|\right|_1 \\ &\quad + \left|\left|\left|\frac{\Theta^{n+1}-\Theta^n}{|\log\tau|}\right|\right|\right|_\infty + \left|\left|\left|\Theta^{n+1}-\Theta^n\right|\right|\right|_1 + \left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(\Theta^{n+1}_x-\Theta^n_x\right)\right|\right|\right|_\infty \\ &\quad + \left|\left|\left|\frac{\Theta^{n+1}_x-\Theta^n_x}{|\log\tau|}\right|\right|\right|_1. \end{aligned} \tag{3.76}$$

  As before, here

$$\left|\left|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(U^{n+1}_x-U^n_x\right)\right|\right|\right|_\infty = \sup_{0<\tau<t_\sharp}\left\|\frac{\sqrt{\tau}}{|\log\tau|}\left(U^{n+1}_x-U^n_x\right)(\cdot,\tau)\right\|_{L^\infty_x},$$

  and it is similar for other $|||\cdot|||$ norms. Then, we combine Lemmas 3.5, 3.6 and 3.7 to obtain the following contraction property for sufficiently small δ and $t_\sharp$,

$$\mathscr{F}\left[V^{n+1}-V^n, U^{n+1}-U^n, \Theta^{n+1}-\Theta^n\right]$$

$$\leqq C_\flat \left(\delta + \sqrt{t_\sharp} |\log t_\sharp| + \frac{1}{|\log t_\sharp|} \right) \mathscr{F}\left[V^n - V^{n-1}, U^n - U^{n-1}, \Theta^n - \Theta^{n-1} \right].$$

From the previous analysis in Remark 3.1, $C_\flat$ is uniformly bounded when δ is sufficiently small. When δ and $t_\sharp$ are sufficiently small, $\{(V^n, U^n, \Theta^n)\}$ forms a Cauchy sequence. Therefore, the iteration scheme admits a strong limit (v^*, u^*, θ^*) in the function space

$$\begin{cases} v^*(x,t) \in L^\infty \left(0, t_\sharp; L^1(\mathbb{R})\right), \\ u^*(x,t) \in L^\infty \left(0, t_\sharp; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t} u^*_x(x,t) \in L^\infty \left(0, t_\sharp; L^\infty(\mathbb{R})\right), \\ \theta^*(x,t) \in L^\infty \left(0, t_\sharp; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t} \theta^*_x(x,t) \in L^\infty \left(0, t_\sharp; L^\infty(\mathbb{R})\right). \end{cases} \tag{3.77}$$

Now, letting $(v, u, \theta) = (v^* + 1, u^*, \theta^* + 1)$, the strong convergence immediately implies that (v, u, θ) is a weak solution to the original Navier–Stokes equation (3.1) in the distribution sense.

- (*Step 2: Regularity*) From Lemma 3.2, Lemma 3.3 and Lemma 3.4, $\|V^n\|_{BV}$, $\|U^n_x\|_{L^1}$ and $\|\Theta^n_x\|_{L^1}$ are uniformly bounded for n. Moreover, from the above analysis, V^n, U^n and Θ^n are convergent in L^1. Therefore, we apply Helly's selection Theorem and the estimates in Lemma 3.2, Lemma 3.3 and Lemma 3.4 to conclude that, the limit (v^*, u^*, θ^*) has the following properties:

$$\|v^*(\cdot,t)\|_{BV} \leqq 2C_\sharp \delta, \quad \|u^*_x(\cdot,t)\|_{L^1_x} \leqq 2C_\sharp \delta, \quad \|\theta^*_x(\cdot,t)\|_{L^1_x} \leqq 2C_\sharp \delta. \tag{3.78}$$

- (*Step 3: v is BV*) According to Remark 3.1, V^n is a BV function and it can be decomposed as follows:

$$V^n = V^n_c + V^n_d, \quad V^n_d(x,t) = \sum_{z<x, z\in\mathscr{D}} d^n(z,t) h(x-z),$$

$$d^n(z,t)\Big|_{z\in\mathscr{D}} = V^n(\cdot,t)\Big|_{z^-}^{z^+}, \quad V^n_c \text{ is continuous.}$$

From the proof of Lemma 3.7 (see (3.72)), one actually obtained that $\left| \left(V^{n+1}(\cdot,t) - V^n(\cdot,t)\right)\Big|_{z^-}^{z^+} \right|$ is also a Cauchy sequence, and thus the jump at time t admits a limit $d(z,t)$ for $z \in \mathscr{D}$. Now we construct the step function

$$v^*_d(x,t) \equiv \sum_{z<x} d(z,t) h(x-z), \quad |d(z,t)| \leqq 2 \left| v^*_0(\cdot)\Big|_{x=z^-}^{x=z^+} \right|, \quad z \in \mathscr{D}, \tag{3.79}$$

where the estimate of $|d(z,t)|$ is due to the strong convergence and uniform boundedness of the jump at each iteration step. From the above analysis, V^n_d converges to v^*_d pointwisely. By the uniform convergence, the continuous part V^n_c also has a continuous limit v^*_c. We thus conclude that

$$v^* = \lim_{n\to\infty} V^n = \lim_{n\to\infty} V^n_c + \lim_{n\to\infty} V^n_d = v^*_c + v^*_d.$$

v^* is a BV function possessing the same discontinuities as the initial data.

- (*Step 4: Flux continuity*) Putting the obtained solution v and u into the equation for θ, one has an inhomogeneous linear heat equation of $\bar{\theta}$, with BV coefficient $\frac{1}{v}$ and a BV source term as below,

$$\bar{\theta}_t = \left(\frac{\kappa}{c_v v} \bar{\theta}_x - \int_{-\infty}^{x} \left(\frac{p}{c_v} u_z - \frac{\mu}{c_v v} (u_z)^2 \right) \mathrm{d}z \right)_x , \tag{3.80}$$

which is of the same form as in Remark 2.1. Since the anti-derivative form of the source term is obviously Lipschitz continuous with respect to x, one can apply Remark 2.1 to conclude the existence of the solution $\bar{\theta}$, and the continuity of the flux for the linear equation. The equation is linear for $\bar{\theta}$ and thus has a unique weak solution. On the other hand, the θ constructed from the approximate solution sequence is also a weak solution to this linear equation. Therefore, θ coincides with $\bar{\theta}$ and thus has a continuous flux.
For the flux of u, one substitutes v and θ into the equation of u, and obtains an inhomogeneous linear equation of $\bar{u}$ as below

$$\bar{u}_t = \left(\frac{\mu \bar{u}_x}{v} - p(v, \theta) \right)_x ,$$

which is of the same form as in Remark 2.1. Since θ is already the weak solution to the linear equation (3.80), and we have the regularity and corresponding estimates of v and u, we can follow the proof of Lemma 3.2 to show that,

$$|\theta(y,t) - \theta(y,s)| \leqq O(1) \left(\frac{\sqrt{t-s}}{\sqrt{s}} + \frac{(t-s)^{\frac{1}{4}}}{s^{\frac{1}{4}}} \right).$$

This implies that θ is Hölder continuous with respect to t. Since $v_t = u_x$ is uniformly bounded, v is Lipschitz continuous with respect to t. Therefore, by Remark 2.1 and the same reasoning as for θ, we conclude the unique existence of weak solution $\bar{u}$, which coincides with u, and the continuity of its flux.
This completes the proof of the theorem. □

Remark 3.3. This section is closed with three remarks.

(1) Similar as in Remark 3.1, from the proof of convergence of the iteration scheme, the postive constants $C_\sharp$ and $t_\sharp$ are chosen to satisfy the following properties,

$$C_\sharp \geqq O(1), \quad \sqrt{t_\sharp} O(1) \ll 1.$$

Here the $O(1)$ terms are independent of the choice of δ, since they are uniformly bounded when δ becomes sufficiently small according to Remark 3.1. Therefore, there exists a positive constant δ^*, the smallness properties (3.74) hold for all $\delta < \delta^*$ with the same $C_\sharp$ and $t_\sharp$.

(2) From Theorem 3.1, the solution has the following regularity with respect to x.

$$\begin{cases} v^*(x,t) \in L^\infty\left(0, t_\sharp; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV\right), \\ u^*(x,t) \in L^\infty\left(0, t_\sharp; W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t} u_x^*(x,t) \in L^\infty\left(0, t_\sharp; L^\infty(\mathbb{R})\right), \\ \theta^*(x,t) \in L^\infty\left(0, t_\sharp; W^{1,1}(\mathbb{R}) \cap L^\infty(\mathbb{R})\right), \quad \sqrt{t} \theta_x^*(x,t) \in L^\infty\left(0, t_\sharp; L^\infty(\mathbb{R})\right). \end{cases}$$

However, it is not shown that the solution has regularity with respect to time variable t.

(3) v^* can be represented as a sum of a step function and a continuous function, whose total variations are both controlled by $\|v_0^*\|_{BV}$. Moreover, v^* has the same discontinuities as initial data v_0^*.

4. Regularity

From Theorem 3.1 and Remark 3.3, we have obtained the first order regularity with respect to x and the continuity of the fluxes for the weak solution (v, u, θ) to system (3.1). However, a weak solution to (3.1) is not necessarily a weak solution to the original system in conservative form (1.1) due to nonlinearity, unless some more regularity estimates can be established. This section is devoted to further study the regularity of the weak solution constructed via the iteration scheme.

4.1. Improvement of regularity in time

By Theorem 3.1, $\|v\|_{BV}$ is small in a short time $t \in (0, t_\sharp)$, which allows us to construct a corresponding heat kernel $H(x, t; y, t_0; \frac{1}{v})$ and employ it to represent the weak solution. Our strategy is as follows: we first follow Lemma 3.2 to show the Hölder continuity in time of θ, then use it to get the estimate for time derivative of velocity u. Interestingly, this in turn can be used to improve the Hölder continuity in time of θ to differentiability in time.

We assume that the initial data satisfies (3.2), and (v, u, θ) is the weak solution constructed in Theorem 3.1 in the following Lemmas 4.1, 4.2 and 4.3.

Lemma 4.1. *θ satisfies the following Hölder continuity estimates with respect to t,*

$$\begin{cases} |\theta(x,t) - \theta(x,s)| = \left|\theta^*(x,t) - \theta^*(x,s)\right| \leqq O(1)\delta \dfrac{(t-s)|\log(t-s)|}{s}, & 0 < t < t_\sharp, \\ |\theta(x,t) - \theta(x,s)| = \left|\theta^*(x,t) - \theta^*(x,s)\right| \leqq O(1)\delta \left(\dfrac{\sqrt{t-s}}{\sqrt{s}} + \dfrac{(t-s)}{s}\right), & 0 < t < t_\sharp, \\ \displaystyle\int_{\mathbb{R}} |\theta(x,t) - \theta(x,s)|\, dx = \int_{\mathbb{R}} \left|\theta^*(x,t) - \theta^*(x,s)\right| dx \leqq O(1)\delta \dfrac{(t-s)|\log(t-s)|}{\sqrt{s}}, & 0 < t < t_\sharp \\ \displaystyle\int_{\mathbb{R}} |\theta(x,t) - \theta(x,s)|\, dx = \int_{\mathbb{R}} \left|\theta^*(x,t) - \theta^*(x,s)\right| dx \leqq O(1)\delta \left(\sqrt{t-s} + \dfrac{(t-s)}{\sqrt{s}}\right), & 0 < t < t_\sharp. \end{cases}$$

Proof. We consider the L^∞ and L^1 estimates seperately.

- (**$\mathbf{L^\infty}$ estimate**) As we have the heat kernel $H(x, t; y, t_0; \frac{1}{v})$ when $t < t_\sharp$, we can follow the proof of Lemma 3.2 to obtain

$$\begin{aligned} &\theta^*(x,t) - \theta^*(x,s) \\ &= \int_{\mathbb{R}} \left(H\left(x, t; y, 0; \frac{1}{v}\right) - H\left(x, s; y, 0; \frac{1}{v}\right)\right) \theta_0^*(y) \mathrm{d}y \\ &\quad + \int_s^t \int_{\mathbb{R}} H\left(x, t; y, \tau; \frac{1}{v}\right) \mathcal{N}_2(y, \tau) \mathrm{d}y \mathrm{d}\tau + \int_0^s \int_{\mathbb{R}} \left(H\left(x, t; y, \tau; \frac{1}{v}\right)\right. \end{aligned}$$

$$
\begin{aligned}
&-H\left(x,s;y,\tau;\frac{1}{v}\right)\Big)\mathcal{N}_2(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&\equiv \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3. \qquad (4.1)
\end{aligned}
$$

Here $\mathcal{N}_2$ is defined as $\mathcal{N}_2^k$ in (3.6) with replacing (V^k, U^k, Θ^k) by (v^*, u^*, θ^*). For $\mathcal{I}_1$ and $\mathcal{I}_2$ in (4.1), one can follow the arguments as in Lemma 3.2 to have,

$$
|\mathcal{I}_1| \leqq \frac{O(1)\delta(t-s)}{\sqrt{s}\sqrt{t}}, \qquad |\mathcal{I}_2| \leqq O(1)\left(\delta\frac{t-s}{\sqrt{t}}+\delta^2\frac{t-s}{\sqrt{s}\sqrt{t}}\right).
$$

For the third term $\mathcal{I}_3$ in (4.1), the previous estimates in the proof of Lemma 3.2 (see (3.14)) can be improved to

$$
\begin{aligned}
|\mathcal{I}_3| &\leqq O(1)\left(\int_0^{\frac{s}{2}}+\int_{\frac{s}{2}}^{s}\right)\int_{\mathbb{R}}\int_s^t \frac{e^{\frac{-(y-z)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{\frac{3}{2}}}\left(|u_y(y,\tau)|+\frac{\delta|u_y(y,\tau)|}{\sqrt{\tau}}\right)\mathrm{d}\sigma\mathrm{d}y\mathrm{d}\tau \\
&\leqq O(1)\frac{\delta(t-s)|\log(t-s)|}{s},
\end{aligned}
$$

where the last inequality holds since function $t|\log t|$ is decreasing when $t < e^{-1}$. Combine the estimates of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ and the representation (4.1) to finish the L^∞ estimate. On the other hand, we can also have another estimate of $\mathcal{I}_3$ as follows:

$$
\begin{aligned}
\mathcal{I}_3 &= O(1)\frac{\delta(t-s)}{s}+O(1)\delta\int_{\frac{s}{2}}^{s}\int_s^t \frac{1}{(\sigma-\tau)}\frac{1}{s}\mathrm{d}\sigma\mathrm{d}\tau \\
&\leqq O(1)\frac{\delta(t-s)}{s}+O(1)\frac{\delta\sqrt{t-s}}{\sqrt{s}}. \qquad (4.2)
\end{aligned}
$$

This together with $\mathcal{I}_1$ and $\mathcal{I}_2$, gives the second L^∞ Hölder estimate of θ.

- (**$\mathbf{L^1}$ estimate**) One writes

$$
\begin{aligned}
&\int_{\mathbb{R}}\left|\theta^*(x,t)-\theta^*(x,s)\right|\mathrm{d}x \\
&\leqq \int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left(H\left(x,t;y,0;\frac{1}{v}\right)-H\left(x,s;y,0;\frac{1}{v}\right)\right)\theta_0^*(y)\mathrm{d}y\right|\mathrm{d}x \\
&\quad+\int_{\mathbb{R}}\int_s^t\int_{\mathbb{R}}\left|H\left(x,t;y,\tau;\frac{1}{v}\right)\right||\mathcal{N}_2(y,\tau)|\,\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&\quad+\int_{\mathbb{R}}\int_0^s\int_{\mathbb{R}}\left|H\left(x,t;y,\tau;\frac{1}{v}\right)-H\left(x,s;y,\tau;\frac{1}{v}\right)\right||\mathcal{N}_2(y,\tau)|\,\mathrm{d}y\mathrm{d}\tau\mathrm{d}x \\
&=\bar{\mathcal{I}}_1+\bar{\mathcal{I}}_2+\bar{\mathcal{I}}_3. \qquad (4.3)
\end{aligned}
$$

For $\bar{\mathcal{I}}_3$, using estimate of H_t in (2.11) and L^1 estimate of u_y in (3.74), it follows that

$$
|\bar{\mathcal{I}}_3| \leqq \int_{\mathbb{R}}\int_0^s\int_{\mathbb{R}}\left|\int_s^t H_\sigma\left(x,\sigma;y,\tau;\frac{1}{v}\right)\mathrm{d}\sigma\right|\left|\mathcal{N}_2^k(y,\tau)\right|\mathrm{d}y\mathrm{d}\tau\mathrm{d}x
$$

$$\leqq O(1)\left(\int_0^{\frac{s}{2}}+\int_{\frac{s}{2}}^{s}\right)\int_{\mathbb{R}}\int_{\mathbb{R}}\int_s^t \frac{e^{\frac{-(x-y)^2}{C_*(\sigma-\tau)}}}{(\sigma-\tau)^{\frac{3}{2}}}$$
$$\times\left(|U_y(y,\tau)|+\frac{\delta|U_y(y,\tau)|}{\sqrt{\tau}}\right)\mathrm{d}\sigma\mathrm{d}y\mathrm{d}x\mathrm{d}\tau$$
$$\leqq O(1)\int_0^{\frac{s}{2}}\int_s^t\frac{1}{(\sigma-\frac{s}{2})}\left(\delta+\frac{\delta^2}{\sqrt{\tau}}\right)\mathrm{d}\sigma\mathrm{d}\tau$$
$$+\,O(1)\int_{\frac{s}{2}}^{s}\int_s^t\frac{1}{(\sigma-\tau)}\left(\delta+\frac{\delta^2}{\sqrt{\tau}}\right)\mathrm{d}\sigma\mathrm{d}\tau$$
$$\leqq O(1)\delta\int_0^{\frac{s}{2}}\int_s^t\frac{1}{s}\frac{1}{\sqrt{\tau}}\mathrm{d}\sigma\mathrm{d}\tau+O(1)\delta\int_{\frac{s}{2}}^{s}\int_s^t\frac{1}{(\sigma-\tau)}\frac{1}{\sqrt{s}}\mathrm{d}\sigma\mathrm{d}\tau$$
$$\leqq O(1)\frac{\delta(t-s)}{\sqrt{s}}+O(1)\frac{\delta}{\sqrt{s}}\int_{\frac{s}{2}}^{s}(\log(t-\tau)-\log(s-\tau))\,\mathrm{d}\tau$$
$$\leqq O(1)\frac{\delta(t-s)|\log(t-s)|}{\sqrt{s}}.$$

The estimate for $\bar{\mathcal{I}}_2$ is similar, and we have that

$$\left|\bar{\mathcal{I}}_2\right|\leqq O(1)\delta\frac{t-s}{\sqrt{s}}.$$

By Lemma 2.6, $\int_{\mathbb{R}} H_t(x,t;y,t_0)\mathrm{d}y=0$, hence one can introduce the anti-derivative $W(x,t;y,t_0)$ of $H_t(x,t;y,t_0)$ with respect to y. The term $\bar{\mathcal{I}}_1$ is estimated as follows:

$$\bar{\mathcal{I}}_1=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\left(H(x,t;y,0;\frac{1}{v})-H(x,s;y,0;\frac{1}{v})\right)\theta_0^*(y)\mathrm{d}y\right|\mathrm{d}x$$
$$=\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\int_s^t H_\sigma(x,\sigma;y,0;\frac{1}{v})\theta_0^*(y)\mathrm{d}\sigma\mathrm{d}y\right|\mathrm{d}x=\int_{\mathbb{R}}$$
$$\times\left|-\int_s^t\int_{\mathbb{R}}W(x,\sigma;y,0;\frac{1}{v})d\theta_0^*(y)\mathrm{d}\sigma\right|\mathrm{d}x$$
$$\leqq\int_s^t\int_{\mathbb{R}}|W(x,\sigma;y,0;\frac{1}{v})|\int_{\mathbb{R}}|d\theta_0^*(y)|\mathrm{d}x\mathrm{d}\sigma\leqq O(1)\delta\frac{t-s}{\sqrt{t}}.\quad(4.4)$$

Here we used integration by parts for Stieltjes integral and the estimate for H_t in (2.11). From the above estimates, $\bar{\mathcal{I}}_3$ in (4.3) is dominant, and we conclude

$$\int_{\mathbb{R}}\left|\theta^*(x,t)-\theta^*(x,s)\right|\mathrm{d}x\leqq O(1)\frac{\delta(t-s)|\log(t-s)|}{\sqrt{s}}.$$

Similar to the L^∞ estimates for $\mathcal{I}_3$ in (4.2), one has another estimate for $\bar{\mathcal{I}}_3$ without $\log(t-s)$ term,

$$\left|\bar{\mathcal{I}}_3\right|\leqq O(1)\delta\left(\sqrt{t-s}+\frac{(t-s)}{\sqrt{s}}\right).$$

Then combining this with the estimates of $\bar{\mathcal{I}}_1$ and $\bar{\mathcal{I}}_2$ in (4.3), we obtain the L^1 Hölder estimate, and have completed the proof of this lemma. □

With Hölder continuity of θ in time t, we can apply the representation of u to study its regularity with respect to t.

Lemma 4.2. *$u_t(x,t)$ is well-defined almost everywhere for $x \in \mathbb{R}$ when $t > 0$. Moreover, it has the following property:*

$$\|u_t(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqq O(1)\frac{\delta}{t}, \qquad \|u_t(\cdot,t)\|_{L^1(\mathbb{R})} \leqq O(1)\frac{\delta}{\sqrt{t}}.$$

Proof. We split the proof into two parts.

- (**L**$^\infty$ **estimate**) Use the heat kernel to construct the following representation of u_t for $x \notin \mathscr{D}$:

$$\begin{aligned} u_t(x,t) = &\int_{\mathbb{R}} H_t\left(x,t;y,0;\frac{1}{v}\right)u_0(y)\mathrm{d}y \\ &+ \int_{\mathbb{R}\setminus\mathscr{D}} H_y\left(x,t;y,t;\frac{1}{v}\right)p(v(y,t),\theta(y,t))\mathrm{d}y \\ &+ \int_0^{\frac{t}{2}}\int_{\mathbb{R}\setminus\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)p(v(y,s),\theta(y,s))\mathrm{d}y\mathrm{d}s \\ &+ \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}\setminus\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)(p(v(y,s),\theta(y,s)) \\ &-p(v(y,t),\theta(y,t)))\,\mathrm{d}y\mathrm{d}s \\ &+ \int_{\mathbb{R}\setminus\mathscr{D}}\left(\int_{\frac{t}{2}}^{t} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\mathrm{d}s\right)p(v(y,t),\theta(y,t))\mathrm{d}y \\ = &\,\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4+\mathcal{I}_5. \end{aligned} \tag{4.5}$$

By the estimate of $H_t(x,t;y,t_0)$ in Lemma 2.2 and $H_y(x,t;y,t) = -\delta'(x-y)$, one directly obtains the estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$ in (4.5),

$$\begin{aligned} |\mathcal{I}_1| &\leqq O(1)\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{C^*t}}}{t^{\frac{3}{2}}}u_0(y)\mathrm{d}y \leqq O(1)\frac{\delta}{t}, \\ \mathcal{I}_2 &= \int_{\mathbb{R}\setminus\mathscr{D}} H_y\left(x,t;y,t;\frac{1}{v}\right)p(v(y,t),\theta(y,t))\mathrm{d}y = -\partial_x\left(p(v(x,t),\theta(x,t))\right). \end{aligned}$$

For $\mathcal{I}_3$ in (4.5), one applies the estimate of H_{ty} in (2.12) of Lemma 2.2, the zeroth order estimates of θ and v in Theorem 3.1, and the fact that $\int_{\mathbb{R}} H_{ty}\mathrm{d}y = 0$ to obtain

$$|\mathcal{I}_3| = \left|\int_0^{\frac{t}{2}}\int_{\mathbb{R}\setminus\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right)\left(\frac{K\theta(y,s)}{v(y,s)}-K\right)\mathrm{d}y\mathrm{d}s\right|$$

$$\leqq O(1)\delta \int_0^{\frac{t}{2}} \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C^*(t-s)}}}{(t-s)^2} \mathrm{d}y\mathrm{d}s \leqq O(1)\frac{\delta}{\sqrt{t}}.$$

For $\mathcal{I}_4$ in (4.5), by estimate of H_{ty}, time derivative estimate of v in Theorem 3.1 and the Hölder continuity of θ in Lemma 4.1, one has

$$\begin{aligned}
|\mathcal{I}_4| &\leqq O(1) \int_{\frac{t}{2}}^t \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C^*(t-s)}}}{(t-s)^2} \frac{\delta(t-s)\,|\log(t-s)|}{s} \mathrm{d}y\mathrm{d}s \\
&\leqq O(1)\delta \int_{\frac{t}{2}}^t \frac{1}{\sqrt{t-s}} \frac{|\log(t-s)|}{s} \mathrm{d}s \\
&\leqq O(1)\delta \frac{|\log t|}{\sqrt{t}}.
\end{aligned}$$

Next, for $\mathcal{I}_5$ in (4.5), we need to apply integration by parts in Stieltjes sense. Recall similar estimates of (3.17), (3.18), and discussions in Lemma 3.3, we will only handle the case when the continuous part of $v(x,t)$ is absolutely continuous. Then, as $x \notin \mathscr{D}$, apply integration by parts to yield the following representation:

$$\begin{aligned}
\mathcal{I}_5 = &-\int_{\mathbb{R}\setminus\mathscr{D}} \left(\int_{\frac{t}{2}}^t H_t\left(x,t;y,s;\frac{1}{v}\right) \mathrm{d}s + \delta(x-y) \right) \\
&\times \frac{K\left(v(y,t)\theta_y(y,t) - \theta(y,t)v_y(y,t)\right)}{(v(y,t))^2} \mathrm{d}y \\
&- \sum_{z\in\mathscr{D}} \left(\int_{\frac{t}{2}}^t H_t\left(x,t;\cdot,s;\frac{1}{v}\right) \mathrm{d}s + \delta(x-\cdot) \right) p(v(\cdot,t),\theta(\cdot,t))\Big|_{z^-}^{z^+} \\
&+ \partial_x p(v(x,t),\theta(x,t)) \\
= &\ \mathcal{I}_{51} + \mathcal{I}_{52} + \mathcal{I}_{53}.
\end{aligned}$$

We apply the estimate for time integral of H_t with respect to s in (2.17), the L^1 estimates of θ_x, and the BV estimate of v in Theorem 3.1 to obtain that

$$\begin{aligned}
\mathcal{I}_{51} &\leqq O(1) \int_{\mathbb{R}\setminus\mathscr{D}} \frac{e^{-\frac{(x-y)^2}{C^* t}}}{\sqrt{t}} \left(|v_y(y,t)| + |\theta_y(y,t)|\right) \mathrm{d}y \leqq O(1)\frac{\delta}{\sqrt{t}}, \\
\mathcal{I}_{52} &\leqq O(1)\frac{1}{\sqrt{t}} \sum_{z\in\mathscr{D}} \left| p(v(\cdot,t),\theta(\cdot,t))\Big|_{z^-}^{z^+} \right| \leqq O(1)\frac{\delta}{\sqrt{t}}.
\end{aligned}$$

Note that $\mathcal{I}_2$ and $\mathcal{I}_{53}$ cancel with each other. Therefore, we combine the above estimates to conclude that $\mathcal{I}_1$ is dominant in (4.5), and

$$\|u_t(\cdot,t)\|_{L^\infty} \leqq O(1)\frac{\delta}{t}, \quad 0 < t < t_\sharp.$$

Finally, as $\mathscr{D}$ is of measure zero, we can simply use upper limit from left or right to define the value of u_t at $\mathscr{D}$. As the value on zero measure set will not affect the L^∞ norm and L^1 norm, the above estimates still hold and we finish the proof of the first part of the lemma.

- (**$\mathbf{L^1}$ estimate**) By the cancellation of $\mathcal{I}_2$ and $\mathcal{I}_{53}$ in L^∞ estimate, we have the estimate

$$\begin{aligned}
\int_{\mathbb{R}} |u_t(x,t)|\mathrm{d}x \leqq& \int_{\mathbb{R}} \left| \int_{\mathbb{R}} H_t\left(x,t;y,0;\frac{1}{v}\right) u_0(y)\mathrm{d}y \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \int_0^{\frac{t}{2}} \int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right) p(v(y,s),\theta(y,s))\mathrm{d}y\mathrm{d}s \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \Bigg| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right) \\
&\times (p(v(y,s),\theta(y,s)) - p(v(y,t),\theta(y,t)))\,\mathrm{d}y\mathrm{d}s \Bigg| \,\mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \int_{\mathbb{R}\backslash\mathscr{D}} \left(\int_{\frac{t}{2}}^{t} H_t\left(x,t;y,s;\frac{1}{v}\right)\mathrm{d}s + \delta(x-y)\right) \partial_y p(v(y,t),\theta(y,t))\mathrm{d}y \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \sum_{z\in\mathscr{D}} \left(\int_{\frac{t}{2}}^{t} H_t\left(x,t;\cdot,s;\frac{1}{v}\right)\mathrm{d}s + \delta(x-y)\right) p(v(\cdot,t),\theta(\cdot,t))\Big|_{z^-}^{z^+} \right| \mathrm{d}x \\
=& \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5. \qquad (4.6)
\end{aligned}$$

For the estimates of the last four terms, we only need to add the integration with respect to x on the L^∞ estimate in Step 1. For the estimate of $\mathcal{I}_2$ in (4.6), we have

$$\mathcal{I}_2 = \int_{\mathbb{R}} \left| \int_0^{\frac{t}{2}} \int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right) \left(\frac{K\theta(y,s)}{v(y,s)} - K \right) \mathrm{d}y\mathrm{d}s \right| \mathrm{d}x \leqq O(1)\frac{\delta}{\sqrt{t}}.$$

For the estimate of $\mathcal{I}_3$ in (4.6), we recall the estimates in Theorem 3.1 that $v_t = u_x$ and $\|u_x\|_{L^1_x}$ is of order δ. Therefore, we combine the Hölder estimate of θ in the L^1 sense in Lemma 4.1 to obtain that

$$\begin{aligned}
\mathcal{I}_3 =& \int_{\mathbb{R}} \Bigg| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}\backslash\mathscr{D}} H_{ty}\left(x,t;y,s;\frac{1}{v}\right) \\
&\times (p(v(y,s),\theta(y,s)) - p(v(y,t),\theta(y,t)))\,\mathrm{d}y\mathrm{d}s \Bigg| \,\mathrm{d}x \\
\leqq& O(1)\delta|\log t|.
\end{aligned}$$

For the estimate of $\mathcal{I}_4$ and $\mathcal{I}_5$ in (4.6), the integration of heat kernel with respect to x yields a $\sqrt{t}$ factor. Thus, we have

$$\begin{aligned}
\mathcal{I}_4 &\leqq O(1) \int_{\mathbb{R}\backslash\mathscr{D}} \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C^* t}}}{\sqrt{t}} \mathrm{d}x \left(|v_y(y,t)| + |\theta_y(y,t)| \right) \mathrm{d}y \leqq O(1)\delta, \\
\mathcal{I}_5 &\leqq O(1) \sum_{z\in\mathscr{D}} \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C^* t}}}{\sqrt{t}} \mathrm{d}x \left| p(v(\cdot,t),\theta(\cdot,t))\Big|_{z^-}^{z^+} \right| \leqq O(1)\delta.
\end{aligned}$$

Lastly, for the estimate of $\mathcal{I}_1$ in (4.6), similar to (4.4), one introduces the anti-derivative of $H_t(x,t;y,0;\frac{1}{v})$ with respect to y and applies integration by parts to yield

$$\mathcal{I}_1 \leqq O(1)\frac{\delta}{\sqrt{t}}.$$

The L^1 estimate follows from the estimates of $\mathcal{I}_i$, $i = 1\dots5$ in (4.6) above. This completes the proof of the lemma. □

In Lemma 4.1, the Hölder continuity in time estimate of θ contains a term $\log(t-s)$, thus it is insufficient to conclude the time derivative of θ is well-defined. Examining the proof of Lemma 4.1, one finds that the $\log(t-s)$ term is resulted from the inhomogeneous term involving u_x^2. Now, with the estimates of u_t, we are able to improve the estimates of θ and yield the differentiability of θ with respect to t.

Lemma 4.3. *$\theta_t(x,t)$ is well-defined for $x \in \mathbb{R}$ when $t > 0$. Moreover, it has the following property:*

$$\|\theta_t(\cdot,t)\|_{L^\infty(\mathbb{R})} \leqq O(1)\frac{\delta}{t}, \qquad \|\theta_t(\cdot,t)\|_{L^1(\mathbb{R})} \leqq O(1)\frac{\delta}{\sqrt{t}}.$$

Proof. Using heat kernel, we have the integral representation of θ_t:

$$\begin{aligned}
\theta_t(x,t) &= \int_{\mathbb{R}} H_t\left(x,t;y,0;\frac{1}{v}\right)\theta_0^*(y)\mathrm{d}y \\
&\quad + \int_0^{\frac{t}{2}}\int_{\mathbb{R}} H_t\left(x,t;y,\tau;\frac{1}{v}\right)\mathcal{N}_2(y,\tau)\mathrm{d}y\mathrm{d}\tau \\
&\quad + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}} H_t\left(x,t;y,\tau;\frac{1}{v}\right)\mathcal{N}_2(y,t)\mathrm{d}y\mathrm{d}\tau \\
&\quad + \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}} H_t\left(x,t;y,\tau;\frac{1}{v}\right)\left(\mathcal{N}_2(y,\tau)-\mathcal{N}_2(y,t)\right)\mathrm{d}y\mathrm{d}\tau \\
&= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.
\end{aligned} \tag{4.7}$$

For the first two terms $\mathcal{I}_1$ and $\mathcal{I}_2$, noticing the time integral region is away from t, one does not need to worry about the singularity of $H_t(x,t;y,\tau)$ when τ is close to t. We have the following estimates:

$$\begin{aligned}
|\mathcal{I}_1| &\leqq C_*\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{C_*t}}}{t^{\frac{3}{2}}}\theta_0^*(y)\mathrm{d}y \leqq O(1)\frac{\delta}{t}, \\
|\mathcal{I}_2| &= \left|\int_0^{\frac{t}{2}}\int_{\mathbb{R}} H_t\left(x,t;y,\tau;\frac{1}{v}\right)u_y(y,\tau)\left(-\frac{p(y,\tau)}{c_v}+\frac{\mu}{c_v v(y,\tau)}u_y(y,\tau)\right)\mathrm{d}y\mathrm{d}\tau\right| \\
&\leqq O(1)\int_0^{\frac{t}{2}}\int_{\mathbb{R}}\frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}|u_y(y,\tau)|\left(1+\frac{\delta}{\sqrt{\tau}}\right)\mathrm{d}y\mathrm{d}\tau \leqq O(1)\left(\frac{\delta}{\sqrt{t}}+\frac{\delta^2}{t}\right).
\end{aligned}$$

For $\mathcal{I}_3$, $\mathcal{N}_2(y,t)$ is independent of τ, thus the integral with respect to τ acts only on H_t. We apply the estimate (2.17) of heat kernel in Lemma 2.2 to obtain that

$$
\begin{aligned}
|\mathcal{I}_3| &\leqq \int_{\mathbb{R}} \left| \int_{\frac{t}{2}}^{t} \left(H_t \left(x,t;y,\tau;\frac{1}{v} \right) \right) \mathrm{d}\tau + \delta(x-y) \right| |u_y(y,t)| \\
&\quad \times \left| -\frac{p(y,t)}{c_v} + \frac{\mu}{c_v v(y,t)} u_y(y,t) \right| \mathrm{d}y \\
&\quad + |u_x(x,t)| \left| -\frac{p(x,t)}{c_v} + \frac{\mu}{c_v v(x,t)} u_x(x,t) \right| \\
&\leqq O(1) \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_* \frac{t}{2}}}}{(\frac{t}{2})^{\frac{1}{2}}} |u_y(y,t)| \left(1 + \frac{\delta}{\sqrt{t}} \right) \mathrm{d}y\mathrm{d}\tau + O(1) \left(\frac{\delta}{\sqrt{t}} + \frac{\delta^2}{t} \right) \\
&\leqq O(1) \left(\frac{\delta}{\sqrt{t}} + \frac{\delta^2}{t} \right).
\end{aligned}
$$

Finally, for $\mathcal{I}_4$, we need to exploit the regularity of u_t obtained in Lemma 4.2. To this end, we rewrite $\mathcal{I}_4$ as follows:

$$
\begin{aligned}
\mathcal{I}_4 &= -\int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_t \left(x,t;y,\tau;\frac{1}{v} \right) (u(y,\tau) - u(y,t)) \frac{u_\tau(y,\tau)}{c_v} \mathrm{d}y\mathrm{d}\tau \\
&\quad - \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_{ty} \left(x,t;y,\tau;\frac{1}{v} \right) \\
&\quad \times (u(y,\tau) - u(y,t)) \left(-\frac{p(y,\tau)}{c_v} + \frac{\mu}{c_v v(y,\tau)} u_y(y,\tau) \right) \mathrm{d}y\mathrm{d}\tau \\
&\quad + \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_t \left(x,t;y,\tau;\frac{1}{v} \right) u_y(y,t) \left(\frac{p(y,t)}{c_v} - \frac{p(y,\tau)}{c_v} \right) \mathrm{d}y\mathrm{d}\tau \\
&\quad + \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_t \left(x,t;y,\tau;\frac{1}{v} \right) u_y(y,t) \left(\left(\frac{\mu}{c_v v(y,\tau)} - \frac{\mu}{c_v v(y,t)} \right) u_y(y,\tau) \right) \mathrm{d}y\mathrm{d}\tau \\
&\quad - \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_{ty} \left(x,t;y,\tau;\frac{1}{v} \right) u_y(y,t) \left(\frac{\mu}{c_v v(y,t)} (u(y,\tau) - u(y,t)) \right) \mathrm{d}y\mathrm{d}\tau \\
&\quad - \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_t \left(x,t;y,\tau;\frac{1}{v} \right) (u_t(y,t) + p_y(y,t)) \left(\frac{1}{c_v} (u(y,\tau) - u(y,t)) \right) \mathrm{d}y\mathrm{d}\tau \\
&= \sum_{j=1}^{6} \mathcal{I}_{4j}.
\end{aligned} \tag{4.8}
$$

Next, we estimate $\mathcal{I}_{4j}$ term by term. For $\mathcal{I}_{41}$, apply the estimate of u_t in Lemma 4.2 to obtain

$$
|\mathcal{I}_{41}| = \left| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_t \left(x,t;y,\tau;\frac{1}{v} \right) (u(y,\tau) - u(y,t)) \frac{u_\tau(y,\tau)}{c_v} \mathrm{d}y\mathrm{d}\tau \right| \leqq O(1) \frac{\delta^2}{t}.
$$

For $\mathcal{I}_{42}$, by the estimate of u_t again,

$$
|\mathcal{I}_{42}| = \left| \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_{ty} \left(x,t;y,\tau;\frac{1}{v} \right) (u(y,\tau) - u(y,t)) \right.
$$

$$\times \left(-\frac{p(y,\tau)}{c_v} + \frac{\mu}{c_v v(y,\tau)} u_y(y,\tau)\right) \mathrm{d}y\mathrm{d}\tau\Bigg|$$
$$\leqq O(1) \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^2} \left|\int_\tau^t u_s(y,s)\mathrm{d}s\right| \left|1 + u_y(y,\tau)\right| \mathrm{d}y\mathrm{d}\tau$$
$$\leqq O(1)\left(\frac{\delta}{\sqrt{t}} + \frac{\delta^2}{t}\right).$$

For $\mathcal{I}_{43}$, we take advantage of the L^∞ Hölder continuity of θ without logarithm terms in Lemma 4.1, and note $v_t = u_x$ to yield the estimates:

$$|\mathcal{I}_{43}| = \left|\int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} H_t\left(x,t;y,\tau;\frac{1}{v}\right) u_y(y,t) \left(\frac{p(y,t)}{c_v} - \frac{p(y,\tau)}{c_v}\right) \mathrm{d}y\mathrm{d}\tau\right|$$
$$\leqq O(1) \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \left|u_y(y,t)\right| (|\theta(y,t) - \theta(y,\tau)|$$
$$+ |v(y,t) - v(y,\tau)|) \,\mathrm{d}y\mathrm{d}\tau$$
$$\leqq O(1) \int_{\frac{t}{2}}^{t} \frac{1}{t-\tau} \frac{\delta}{t} \left(\frac{\delta(t-\tau)}{\tau} + \frac{\delta(\sqrt{t-\tau})}{\sqrt{\tau}} + \frac{\delta(t-\tau)}{\sqrt{t}}\right) \mathrm{d}\tau \leqq O(1)\frac{\delta^2}{t}$$

for small time. The estimates for $\mathcal{I}_{44}, \mathcal{I}_{45}$ and $\mathcal{I}_{46}$ are very similar as the first three terms. Thus, we omit the details and provide the following estimates:

$$|\mathcal{I}_{44}| \leqq O(1)\frac{\delta^3}{\sqrt{t}}, \quad |\mathcal{I}_{45}| \leqq O(1)\frac{\delta^2}{t}, \quad |\mathcal{I}_{46}| \leqq O(1)\frac{\delta^2}{t}.$$

The estimates of $\mathcal{I}_{4j}$, $j = 1\ldots 6$ together yield that

$$\mathcal{I}_4 \leqq O(1)\frac{\delta}{t}.$$

Combine all the estimates of $\mathcal{I}_j$, $j = 1\ldots 4$ to obtain the L^∞ estimate of θ_t as

$$\|\theta_t(\cdot,t)\|_{L^\infty} \leqq O(1)\frac{\delta}{t}.$$

Lastly, we can follow the computations in Lemma 4.2, and apply the L^1 estimates of u_t, u_y and θ_y in Lemma 4.2 and Theorem 3.1 to yield the L^1 estimate of θ_t:

$$\|\theta_t(\cdot,t)\|_{L^1} \leqq O(1)\frac{\delta}{\sqrt{t}}.$$

The details are omitted. □

4.2. Regularity of the weak solution

Now, we are in the position to state our second theorem concerning the regularity of the local-in-time weak solution.

Theorem 4.1. *Suppose the initial data* $(v_0^*, u_0^*, \theta_0^*)$ *satisfy the condition* (3.2) *for small* δ. *Let* (v, u, θ) *be the corresponding local-in-time weak solution constructed in Theorem* 3.1. *The following assertions hold:*

(1) *In addition to the estimates in* (3.74)*, there exists a positive constant* $C_\sharp$ *such that when* $t \in (0, t_\sharp)$*, the solution satisfies*

$$\max\left\{\sqrt{t}\,\|u_t(\cdot,t)\|_{L_x^1},\ t\,\|u_t(\cdot,t)\|_{L_x^\infty},\ \sqrt{t}\,\|\theta_t(\cdot,t)\|_{L_x^1},\ t\,\|\theta_t(\cdot,t)\|_{L_x^\infty}\right\} \leqq 2C_\sharp\delta. \tag{4.9}$$

(2) *The fluxes of* u *and* θ *(defined in* (3.75)*) are both globally Lipschitz continuous with respect to* x *for* $t > 0$.

(3) *The specific volume* $v(x,t)$ *satisfies the following Hölder continuity in time properties for* $0 \leqq s < t$,

$$\begin{cases} \|v(\cdot,t)-v(\cdot,s)\|_{BV} \leqq O(1)\delta\dfrac{(t-s)|\log(t-s)|}{\sqrt{t}} \\ \|v(\cdot,t)-v(\cdot,s)\|_{L^\infty} \leqq O(1)\delta\dfrac{t-s}{\sqrt{t}}, \\ \|v(\cdot,t)-v(\cdot,s)\|_{L^1} \leqq O(1)\delta(t-s). \end{cases} \tag{4.10}$$

Proof. The first assertion follows from Theorem 3.1, Lemmas 4.2 and 4.3. The second assertion is a consequence of the first assertion by the equations

$$u_t = \left(-p + \frac{\mu u_x}{v}\right)_x, \quad \theta_t = \left(\frac{\kappa}{c_v v}\theta_x - \int_{-\infty}^x \left(\frac{p}{c_v}u_z - \frac{\mu}{c_v v}(u_z)^2\right)\mathrm{d}z\right)_x.$$

According to Lemmas 4.2 and 4.3, u_t and θ_t have finite L^∞ norm when $t > 0$. We thus conclude that the fluxes of both u and θ are Lipschitz continuous with respect to x for $t > 0$. Finally, from ansatz (3.5), Lemma 3.4 and (3.25), we know that (4.10) holds for V^n. Then, we apply the strong convergence in Theorem 3.1 to obtain the desired results. □

Since the solution to the equation (1.3) fulfills the regularity estimates, we are able to use (v, u, θ) to construct the weak solution of the original Navier–Stokes equations (1.1) in conservative form.

Corollary 4.1. *Suppose the initial data* $(v_0^*, u_0^*, \theta_0^*)$ *satisfy the condition* (3.2) *for small* δ*, and* (v, u, θ) *is the corresponding weak solution to* (1.3) *constructed in Theorems* 3.1 *and* 4.1 *for* $t < t_\sharp$*, where* $t_\sharp$ *is a sufficiently small positive constant. Let*

$$E(x,t)=\frac{u^2}{2}+c_v\theta=\frac{u^2}{2}+e.$$

Then the tuple (v,u,E) *is a weak solution to the nonlinear equation* (1.1) *with initial data*

$$(v_0,u_0,E_0)=\left(1+v_0^*,\ u_0^*,\ c_v(1+\theta_0^*)+\frac{(u_0^*)^2}{2}\right).$$

Proof. Suppose (v,u,θ) is the weak solution constructed in Theorems 3.1 and 4.1, which satisfies weak formulation in Definition 2.1. Since u_t is defined in strong sense, we can set φu as a test function in $(2.1)_2$. Together with $c_v\times$ $(2.1)_3$ and some elementary manipulations, one shows that E satisfies the weak formulation of (1.1). □

Remark 4.1. Note that the regularity obtained in Theorem 4.1 is the same as (2.2) in Proposition 2.1. In particular, it should be emphasized that v is continuous in $L^1\cap L^\infty\cap BV$ around $t=0$, which holds due to (4.10).

5. Local Stability and Uniqueness

In this section, we continue to study the stability of the weak solution constructed in Theorem 3.1 and Theorem 4.1, which implies the continuous dependence of the solution on initial data and the uniqueness of weak solution. We first prove that any weak solution belonging to the function space (2.2) are small in short time, provided the initial data is small.

Lemma 5.1. *Suppose the initial data satisfy*

$$\|v_0\|_{BV}+\|u_0\|_{BV}+\|\theta_0\|_{BV}+\|v_0-1\|_{L^1}+\|u_0\|_{L^1}+\|\theta_0-1\|_{L^1}<\delta_*\ll 1,$$

and (v,u,θ) *is any weak solution of* (1.3) *belonging to* (2.2) *subjected to the above initial data. Let* $C_\sharp$ *and* δ *be the parameters given in Theorems* 3.1 *and* 4.1. *Then, if* δ_* *is sufficiently small, there exists a small positive constant* t_* *such that*

$$\begin{aligned}
&\max\left\{\|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x}\right\}\\
&\quad\leqq 2C_\sharp\delta,\quad 0<t<t_*,\\
&\max\left\{\|\theta(\cdot,t)\|_{L^1_x},\ \|\theta(\cdot,t)\|_{L^\infty_x},\ \|\theta_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|\theta_x(\cdot,t)\|_{L^\infty_x}\right\}\\
&\quad\leqq 2C_\sharp\delta,\quad 0<t<t_*.
\end{aligned}$$

Proof. From Theorems 3.1 and 4.1, there exists at least one weak solution in the space (2.2) provided the initial data is small. If the given solution is exactly the one as we constructed, then the smallness of the solution immediately follows from (3.74) and (4.9).

In general, suppose a weak solution satisfies condition (2.2) with sufficiently small initial data. By Remark 4.1, there exist a small t_* such that

$$\begin{aligned}&\|v(\cdot,t)-1\|_{L^1_x}\leqq C\delta_*,\quad \|v(\cdot,t)-1\|_{L^\infty_x}\leqq C\delta_*,\\&\|v(\cdot,t)-1\|_{BV}\leqq C\delta_*,\quad 0\leqq t<t_*\ll 1.\end{aligned}$$

As δ_* and t_* are sufficiently small, one can follow the arguments in Section 2 to construct the heat kernel $H(x,t;y,\tau;\frac{\mu}{v})$ and $H(x,t;y,\tau;\frac{\kappa}{c_v v})$. Then, multiplying $H(x,t;y,\tau;\frac{\mu}{v})$ to the second equation in (1.3), using integration by parts, and in light of weak formulation, one has an integral representation of u,

$$u(x,t)=\int_{\mathbb{R}}H(x,t;y,0;\frac{\mu}{v})u(y,0)\mathrm{d}y+\int_0^t\int_{\mathbb{R}}H_y(x,t;y,\tau;\frac{\mu}{v})p(y,\tau)\mathrm{d}y\mathrm{d}\tau. \tag{5.1}$$

Similarly, one also obtains the representation of θ:

$$\begin{aligned}\theta(x,t)=&\int_{\mathbb{R}}H(x,t;y,0;\frac{\kappa}{c_v v})\theta(y,0)\mathrm{d}y\\&+\int_0^t\int_{\mathbb{R}}H(x,t;y,\tau;\frac{\kappa}{c_v v})\left(-\frac{pu_y}{c_v}+\frac{\mu}{c_v v}(u_y)^2\right)\mathrm{d}y\mathrm{d}\tau.\end{aligned}$$

In what follows, we will first show the smallness of $u(x,t)$, which then follows that $\theta(x,t)$ is small as well in the short time.

- (**Smallness of u(x, t)**) The estimates of $u(x,t)$ are very similar to Lemma 3.3. First, for the zeroth order, one directly applies the representation (5.1), the estimates of H in Lemma 2.1 and the regularity of the solution in (2.2) to obtain that, for $t<t_*\ll 1$,

$$\begin{aligned}|u(x,t)|\leqq&\int_{\mathbb{R}}\left|H(x,t;y,0;\frac{\mu}{v})\right||u(y,0)|\mathrm{d}y\\&+\int_0^t\int_{\mathbb{R}}\left|H_y(x,t;y,\tau;\frac{\mu}{v})\right||p(y,\tau)-K|\mathrm{d}y\mathrm{d}\tau\\\leqq&\,O(1)\delta_*+O(1)\sqrt{t},\\\|u(x,t)\|_{L^1}\leqq&\int_{\mathbb{R}}\int_{\mathbb{R}}\left|H(x,t;y,0;\frac{\mu}{v})\right||u(y,0)|\mathrm{d}y\mathrm{d}x\\&+\int_{\mathbb{R}}\int_0^t\int_{\mathbb{R}}\left|H_y(x,t;y,\tau;\frac{\mu}{v})\right||p(y,\tau)-K|\mathrm{d}y\mathrm{d}\tau\mathrm{d}x\\\leqq&\,O(1)\delta_*+O(1)\sqrt{t}.\end{aligned}$$

Then, for the first order estimates of $u(x,t)$, we deal with the L^∞ estimate first. One differentiates the representation (5.1) with respect to x to obtain

$$u_x(x,t) = \int_{\mathbb{R}} H_x(x,t;y,0;\frac{\mu}{v})u(y,0)\mathrm{d}y + \int_0^t \int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})p(y,\tau)\mathrm{d}y\mathrm{d}\tau. \tag{5.2}$$

For the homogeneous term, the initial data will provide the small factor δ_*, and one follows the proof in Lemma 3.3 to have

$$\left|\int_{\mathbb{R}} H_x\left(x,t;y,0;\frac{\mu}{v}\right)u(y,0)\mathrm{d}y\right| \leqq O(1)\frac{\delta_*}{\sqrt{t}}. \tag{5.3}$$

For the inhomogeneous term, we follow the estimates for (3.17) to obtain

$$\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})p(y,\tau)\mathrm{d}y\mathrm{d}\tau\right| \\
&\leqq \left|\int_0^t \int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})p(y,t)\mathrm{d}y\mathrm{d}\tau\right| \\
&\quad + \left|\int_0^t \int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})\frac{K(\theta(y,\tau)-\theta(y,t))}{v(y,t)}\mathrm{d}y\mathrm{d}\tau\right| \\
&\quad + \left|\int_0^t \int_{\mathbb{R}} H_{xy}(x,t;y,\tau;\frac{\mu}{v})K\theta(y,\tau)\left(\frac{1}{v(y,\tau)}-\frac{1}{v(y,t)}\right)\mathrm{d}y\mathrm{d}\tau\right| \\
&\leqq O(1) + O(1)\int_0^{\frac{t}{2}}\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\left(\int_\tau^t |\theta_\sigma(y,\sigma)|\mathrm{d}\sigma\right)\mathrm{d}y\mathrm{d}\tau \\
&\quad + O(1)\int_{\frac{t}{2}}^t\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\left(\int_\tau^t |\theta_\sigma(y,\sigma)|\mathrm{d}\sigma\right)\mathrm{d}y\mathrm{d}\tau \\
&\quad + O(1)\int_0^t\int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}}\left(\int_\tau^t |v_\sigma(y,\sigma)|\mathrm{d}\sigma\right)\mathrm{d}y\mathrm{d}\tau \\
&\leqq O(1)(1+\sqrt{t}).
\end{aligned} \tag{5.4}$$

Now, since $t < t_* \ll 1$, we combine (5.2), (5.3) and (5.4) to obtain that

$$|u_x(x,t)| \leqq O(1)\frac{\delta_*}{\sqrt{t}} + O(1) \leqq O(1)\frac{\delta_* + \sqrt{t_*}}{\sqrt{t}}, \quad 0 < t < t_*.$$

The L^1 estimates are similar as the estimates for (3.20), (3.21), (3.22) and (3.23), and we have

$$\|u_x(\cdot,t)\|_{L^1} \leqq O(1)(\delta_* + \sqrt{t_*}), \quad 0 < t < t_*.$$

Now, combine the above zeroth and first order estimates to conclude that, for sufficiently small δ_* and t_*, the following estimates hold:

$$\max\left\{\|u(\cdot,t)\|_{L^1_x},\ \|u(\cdot,t)\|_{L^\infty_x},\ \|u_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|u_x(\cdot,t)\|_{L^\infty_x}\right\}$$
$$\leqq 2C_\sharp\delta,\quad 0<t<t_*.$$

there $C_\sharp$ and δ are given in Theorem 3.1 and Theorem 4.1.

- (**Smallness of** $\theta(\mathbf{x},\mathbf{t})$) As we have shown the smallness of $u(x,t)$, one can follow the proof of Lemma 3.2 to obtain

$$\max\left\{\|\theta(\cdot,t)\|_{L^1_x},\ \|\theta(\cdot,t)\|_{L^\infty_x},\ \|\theta_x(\cdot,t)\|_{L^1_x},\ \sqrt{t}\,\|\theta_x(\cdot,t)\|_{L^\infty_x}\right\}$$
$$\leqq 2C_\sharp\delta,\quad 0<t<t_*.$$

□

Lemma 5.1 states that, if the initial data is sufficiently small, then for any weak solution in the function space (2.2), we can find a small time t_* such that, the solution will be as small as the solution constructed in Theorem 3.1 in the short time $t\in[0,t_*]$. In next lemma, we will show that the difference between two different small solutions are stable locally in time.

Lemma 5.2. *Suppose there are two weak solutions* (v^a,u^a,θ^a) *and* (v^b,u^b,θ^b) *to the NS equation* (3.1) *satisfying the smallness properties* (3.74) *in Theorem* 3.1 *for common* $C_\sharp$, δ *and* $t_\sharp$, *with initial data fulfilling* (3.2) *respectively. Then, there exists a positive constant* $C_\flat$ *such that*

$$\begin{aligned}
&\mathscr{F}\left[v^a-v^b,u^a-u^b,\theta^a-\theta^b\right]\\
&\leqq C_\flat\Big(\|\theta_0^a-\theta_0^b\|_{L^\infty_x}+\|\theta_0^a-\theta_0^b\|_{L^1_x}\\
&\quad+\|u_0^a-u_0^b\|_{L^\infty_x}+\|u_0^a-u_0^b\|_{L^1_x}\\
&\quad+\|v_0^a-v_0^b\|_{L^1_x}+\|v_0^a-v_0^b\|_{L^\infty_x}+\|v_0^a-v_0^b\|_{BV}\Big),
\end{aligned}$$

where $\mathscr{F}$ *is the functional defined in* (3.76). *In particular, it follows that*

$$\begin{aligned}
&\left|\left|\left|v^a-v^b\right|\right|\right|_{L^1_x}+\left|\left|\left|u^a-u^b\right|\right|\right|_{L^1_x}+\left|\left|\left|\theta^a-\theta^b\right|\right|\right|_{L^1_x}\\
&\leqq C_\flat\Big(\|\theta_0^a-\theta_0^b\|_{L^\infty_x}+\|\theta_0^a-\theta_0^b\|_{L^1_x}+\|u_0^a-u_0^b\|_{L^\infty_x}+\|u_0^a-u_0^b\|_{L^1_x}\\
&\quad+\|v_0^a-v_0^b\|_{L^1_x}+\|v_0^a-v_0^b\|_{L^\infty_x}+\|v_0^a-v_0^b\|_{BV}\Big).
\end{aligned}$$

Therefore, the solution satisfying the properties in Theorem 3.1 *is unique up to a measure zero set.*

Proof. From Theorem 3.1, the existence of weak solutions (v^a, u^a, θ^a) and (v^b, u^b, θ^b) are guaranteed. Let

$$\mu^a = \frac{\mu}{v^a}, \quad \kappa^a = \frac{\kappa}{c_v v^a}, \quad \mu^b = \frac{\mu}{v^b}, \quad \kappa^b = \frac{\kappa}{c_v v^b}.$$

As the two solutions satisfy the requirements in Theorem 3.1, they both have continuous fluxes. We can apply Duhamel's principle to get the integral representations for both.

Then we can follow the proof of convergence of the approximate solutions sequence to construct the difference estimates. For instance, for the first order difference estimates of θ, we have the representation

$$\begin{aligned}
&\theta_x^a(x,t) - \theta_x^b(x,t)\\
&\quad = \int_{\mathbb{R}} \left(H_x\left(x,t;y,0;\kappa^a\right) - H_x\left(x,t;y,0;\kappa^b\right)\right)\theta_0^a(y)\mathrm{d}y\\
&\qquad + \int_{\mathbb{R}} H_x\left(x,t;y,0;\kappa^b\right)\left(\theta_0^a(y) - \theta_0^b(y)\right)\mathrm{d}y\\
&\qquad + \int_0^t\int_{\mathbb{R}} H_x\left(x,t;y,s;\kappa^b\right)\left(\mathcal{N}_2^a(y,s) - \mathcal{N}_2^b(y,s)\right)\mathrm{d}y\mathrm{d}s\\
&\qquad + \int_0^t\int_{\mathbb{R}} \left(H_x\left(x,t;y,s;\kappa^a\right) - H_x\left(x,t;y,s;\kappa^b\right)\right)\mathcal{N}_2^a(y,s)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Compared to the calculations in Lemma 3.5, there are three differences. Firstly, the discontinuous sets $\mathscr{D}_a$ and $\mathscr{D}_b$ for the two solutions might be different. In such a case, we just need to introduce the new discontinuity set $\mathscr{D} = \mathscr{D}_a \cup \mathscr{D}_b$, then all the calculations before still hold for the new discontinuity set $\mathscr{D}$. Secondly, the initial data θ_0^a does not belong to L^1 itself. By Lemma 2.6, $\int_{\mathbb{R}} H_x \mathrm{d}y = 0$, thus,

$$\begin{aligned}
&\int_{\mathbb{R}} \left(H_x\left(x,t;y,0;\kappa^a\right) - H_x\left(x,t;y,0;\kappa^b\right)\right)\theta_0^a(y)\mathrm{d}y\\
&\quad = \int_{\mathbb{R}} \left(H_x\left(x,t;y,0;\kappa^a\right) - H_x\left(x,t;y,0;\kappa^b\right)\right)(\theta_0^a(y) - 1)\mathrm{d}y,
\end{aligned}$$

which is of exactly the same form as in Lemma 3.5. The third difference is from the additional term induced by the initial difference. A direct calculation gives the estimate

$$\left|\int_{\mathbb{R}} H_x\left(x,t;y,0;\kappa^b\right)\left(\theta_0^a(y) - \theta_0^b(y)\right)\mathrm{d}y\right| \leqq O(1)\frac{1}{\sqrt{t}}\|\theta_0^a - \theta_0^b\|_{L_x^1}.$$

Therefore, we combine the above analysis with the estimates in Lemma 3.5 to yield that for sufficiently small δ and $t_\sharp$ that

$$\begin{aligned}&\frac{\sqrt{t}}{|\log t|}\left\|\theta_x^a(\cdot,t)-\theta_x^b(\cdot,t)\right\|_\infty \leqq \frac{O(1)}{|\log(t)|}\|\theta_0^a-\theta_0^b\|_{L_x^\infty}+C_\flat\left(\sqrt{t_\sharp}+\delta\right)\\&\times\Big(\left\|\left|v^a-v^b\right|\right\|_\infty+\left\|\left|v^a-v^b\right|\right\|_{BV}+\left\|\left|v^a-v^b\right|\right\|_1\\&+\left\|\left|\frac{\sqrt{\tau}}{|\log\tau|}\left(u_x^a-u_x^b\right)\right|\right\|_\infty+\left\|\left|\frac{u_x^a-u_x^b}{|\log\tau|}\right|\right\|_1+\left\|\left|\frac{\theta^a-\theta^b}{|\log\tau|}\right|\right\|_\infty\Big).\end{aligned}$$

We then follow the similar procedures as we have done in Lemmas 3.5, 3.6 and 3.7 and include the difference resulted from the initial data, to obtain

$$\begin{aligned}&\mathscr{F}\left[v^a-v^b,u^a-u^b,\theta^a-\theta^b\right]\\&\leqq 11C_\flat\left(\delta+\sqrt{t_\sharp}|\log t_\sharp|\right)\mathscr{F}\left[v^a-v^b,u^a-u^b,\theta^a-\theta^b\right]\\&\quad+\frac{O(1)}{|\log(t)|}\|\theta_0^a-\theta_0^b\|_{L_x^\infty}+O(1)\|\theta_0^a-\theta_0^b\|_{L_x^1}\\&\quad+\frac{O(1)}{|\log(t)|}\|u_0^a-u_0^b\|_{L_x^\infty}+\frac{O(1)\sqrt{t}}{|\log(t)|}\|\theta_0^a-\theta_0^b\|_{L_x^\infty}\\&\quad+O(1)\|u_0^a-u_0^b\|_{L_x^\infty}+O(1)\|u_0^a-u_0^b\|_{L_x^1}\\&\quad+\|v_0^a-v_0^b\|_{L_x^1}+\|v_0^a-v_0^b\|_{L_x^\infty}+\|v_0^a-v_0^b\|_{BV}.\end{aligned}$$

In particular, as δ and $t_\sharp$ are both sufficiently small, the first term on the right hand side can be absorbed by the left hand side, then the desired difference estimate follows.

The L^1 difference is a direct corollary from this. Therefore, two solutions coincide in L^1 sense if they have common initial data, i.e., the solution with initial condition (3.2) and the properties in Theorem 3.1 is unique up to a measure zero set. □

With Lemmas 5.1 and 5.2 in hand, we are ready to show the local-in-time stability and prove the uniqueness of the weak solution in the function space (2.2).

Theorem 5.1. *Suppose there are two weak solutions* (v^a,u^a,θ^a) *and* (v^b,u^b,θ^b) *to the Navier–Stokes equation* (1.3) *both belonging to* (2.2)*, and their initial data both satisfy the following condition for small* δ_**,*

$$\|v_0\|_{BV}+\|u_0\|_{BV}+\|\theta_0\|_{BV}+\|v_0-1\|_{L^1}+\|u_0\|_{L^1}+\|\theta_0-1\|_{L^1}<\delta_*\ll 1,$$

Then, there exist a sufficiently small positive constant t_* *and a properly large positive constant* $C_\flat$ *such that, the following stability holds:*

$$\begin{aligned}&\mathscr{F}\left[v^a-v^b,u^a-u^b,\theta^a-\theta^b\right]\\&\leqq C_\flat\Big(\|\theta_0^a-\theta_0^b\|_{L_x^\infty}+\|\theta_0^a-\theta_0^b\|_{L_x^1}\end{aligned}$$

$$+ \|u_0^a - u_0^b\|_{L_x^\infty} + \|u_0^a - u_0^b\|_{L_x^1} + \|v_0^a - v_0^b\|_{L_x^1}$$
$$+ \|v_0^a - v_0^b\|_{L_x^\infty} + \|v_0^a - v_0^b\|_{BV}\Big),$$

where $\mathscr{F}$ is the functional defined in (3.76). Moreover, this immediately implies the uniqueness of the weak solution. Namely, for sufficiently small initial data, there exists a sufficiently small positive constant t_* such that, the equation (1.3) admits a unique weak solution in the sense (2.2) for $t \in [0, t_*)$.

Proof. First of all, Theorems 3.1 and 4.1 establish the existence of weak solution if $\delta_* < \delta$ and $t < t_\sharp$, where δ and $t_\sharp$ are given in Theorems 3.1 and 4.1. Next, Lemma 5.1 guarantees the smallness of the weak solution for $t \in [0, t_*)$, where t_* is constructed in Lemma 5.1. Therefore, for $t \in [0, t_*)$, we can apply Lemma 5.2 to obtain the difference estimate as

$$\begin{aligned}
&\mathscr{F}\Big[v^a - v^b, u^a - u^b, \theta^a - \theta^b\Big] \\
&\leqq C_\flat\Big(\|\theta_0^a - \theta_0^b\|_{L_x^\infty} + \|\theta_0^a - \theta_0^b\|_{L_x^1} \\
&\quad + \|u_0^a - u_0^b\|_{L_x^\infty} + \|u_0^a - u_0^b\|_{L_x^1} + \|v_0^a - v_0^b\|_{L_x^1} \\
&\quad + \|v_0^a - v_0^b\|_{L_x^\infty} + \|v_0^a - v_0^b\|_{BV}\Big), \quad 0 < t < t_*.
\end{aligned}$$

In particular, if the two solutions have the same initial data, the above difference estimate implies that

$$\mathscr{F}\Big[v^a - v^b, u^a - u^b, \theta^a - \theta^b\Big] = 0,$$

which follows that the two solutions coincide with each other almost everywhere. Then, as the two solutions both belong to (2.2), they have the following continuity properties:

- $v(x, t)$ has both left and right limits at $x \in \mathbb{R}$ when $0 < t < t_*$,
- $u(x, t)$ and $\theta(x, t)$ are continuous for $x \in \mathbb{R}$ when $0 < t < t_*$.

Then, we apply the above continuity of the two solutions to conclude that,

$$(v^a(x, t), u^a(x, t), \theta^a(x, t)) = (v^b(x, t), u^b(x, t), \theta^b(x, t)), \quad x \in \mathbb{R}, \quad 0 < t < t_*.$$

□

6. Global Existence

In previous sections, we have constructed the local-wellposedness of weak solution to (1.3), and obtained its regularity and stability, which are sufficient to guarantee it is also a weak solution to the original system (1.1). In this section, we continue to investigate the global stability and the time asymptotic behavior of the solution.

We first introduce the pointwise structures of Green's function for system (1.1) linearized around a constant state, then construct the "effective Green's function" by interpolation of heat kernel in short time and Green's function in large time. Using this "effective Green's function" and in view of the weak formulation, we derive an "effective" integral representation of the solution, which is convenient to study the global behavior of the solution. Refined analysis then provides us a global a priori estimate, which finally concludes the global existence and the time asymptotic behavior of the solution.

As the construction of Green's function (Lemmas 6.1, 6.2 and 6.3) and the proof of a priori estimate (Lemma 6.6) are technical and lengthy, the details of them are spelled out in another separated paper [12], interested readers are referred to there.

6.1. Green's function

In order to preserve the conservative form of equation (1.1), we define the state variables

$$E = e + \frac{1}{2}u^2, \quad U = (v, u, E), \quad p(v, e(E, u)) \equiv \frac{E - \frac{1}{2}u^2}{v}, \tag{6.1}$$

and thus,

$$e_u = -u, \quad e_E = 1.$$

Then, the system (1.1) is rewritten as following conservation form with variables defined in (6.1),

$$\begin{cases} v_t - u_x = 0 \\ u_t + p_v v_x + p_e e_u u_x + p_e e_E E_x = \left(\frac{\mu u_x}{v}\right)_x \\ E_t + u p_v v_x + (p + u p_e e_u) u_x + u p_e e_E E_x = \left(\frac{\kappa\theta_e e_u + \mu u}{v} u_x + \frac{\kappa\theta_e e_E}{v} E_x\right)_x . \end{cases} \tag{6.2}$$

We can also write the system into a vector form as

$$U_t + F(U)_x = (B(U)U_x)_x \Longleftrightarrow U_t + F'(U)U_x = (B(U)U_x)_x.$$

where U, F, $F'(U)$ and B are defined as

$$U = \begin{pmatrix} v \\ u \\ E \end{pmatrix}, \quad F(U) = \begin{pmatrix} -u \\ p \\ pu \end{pmatrix}, \quad F'(U) = \begin{pmatrix} 0 & -1 & 0 \\ p_v & -p_e u & p_e \\ p_v u & p - p_e u^2 & p_e u \end{pmatrix},$$

$$B(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu}{v} & 0 \\ 0 & \left(\frac{\mu}{v} - \frac{\kappa\theta_e}{v}\right) u & \frac{\kappa\theta_e}{v} \end{pmatrix}.$$

Now we consider the linearization of equations (6.2) around a constant state $\bar{U}$. Let $U = \bar{U} + V$. We have

$$V_t + F'(\bar{U})V_x - B(\bar{U})V_{xx} = [N_1(V; \bar{U}) + N_2(V; \bar{U})]_x, \tag{6.3}$$

where N_1 and N_2 are nonlinear terms from the hyperbolic and parabolic parts, respectively,

$$N_1(V;\bar{U}) = -[F(\bar{U}+V) - F(\bar{U}) - F'(\bar{U})V], \quad N_2(V;\bar{U}) = B(U) - B(\bar{U}). \tag{6.4}$$

The Green's function $\mathbb{G}(x,t;\bar{U})$ for the linearized equation (6.3) is the solution to the system

$$\begin{cases} \partial_t \mathbb{G}(x,t;\bar{U}) = \left(-F'(\bar{U})\partial_x + B(\bar{U})\partial_{xx}\right)\mathbb{G}(x,t;\bar{U}), \\ \mathbb{G}(x,0;\bar{U}) = \delta(x)I, \end{cases} \tag{6.5}$$

where I is the 3×3 identity matrix and $\delta(x)$ is the Dirac-delta function. It is shown in [12], the Green's function $\mathbb{G}(x,t)$ can be decomposed into singular and regular parts, $\mathbb{G}^*(x,t)$ and $\mathbb{G}^\dagger(x,t)$ respectively. Moreover, the singular part $\mathbb{G}^*(x,t)$ can be further decomposed to three parts $\mathbb{G}^{*,j}$, $j=1,2,3$, i.e.,

$$\mathbb{G}(x,t) = \mathbb{G}^*(x,t) + \mathbb{G}^\dagger(x,t) = \sum_{j=1}^{3}\mathbb{G}^{*,j}(x,t) + \mathbb{G}^\dagger(x,t). \tag{6.6}$$

The next two lemmas characterize the singular part $\mathbb{G}^{*,j}$, where $A_{j,1}$, $j=1,2,3$, α_j^* and β_j^*, $j=2,3$ are constants, $M_j^{*,k}$, $j,k=1,2,3$ are constant matrices, whose explicit expressions are listed in "Appendix B".

Lemma 6.1. ([12]) *The singular part of the Green's function consists of three parts. There exist positive constants σ_0^* and σ_0 such that the following pointwise estimates hold. For the first part $\mathbb{G}^{*,1}(x,t)$,*

$$\begin{cases} \mathbb{G}^{*,1}(x,t) = e^{\frac{vp_v}{\mu}t}\delta(x)M_1^{*,0} + O(1)e^{-\sigma_0^* t-\sigma_0|x|}, \\ \partial_x\mathbb{G}^{*,1}(x,t) = e^{\frac{vp_v}{\mu}t}\left(\frac{d}{dx}\delta(x)M_1^{*,0} - \delta(x)M_1^{*,1}\right) + O(1)e^{-\sigma_0^* t-\sigma_0|x|}, \\ \partial_x^2\mathbb{G}^{*,1}(x,t) = e^{\frac{vp_v}{\mu}t}\left(\frac{d^2}{dx^2}\delta(x)M_1^{*,0} - \frac{d}{dx}\delta(x)M_1^{*,1} + \delta(x)\left(-M_1^{*,2} - A_{1,1}tM_1^{*,0}\right)\right) + O(1)e^{-\sigma_0^* t-\sigma_0|x|}, \\ \partial_x^3\mathbb{G}^{*,1}(x,t) = e^{\frac{vp_v}{\mu}t}\left(\frac{d^3}{dx^3}\delta(x)M_1^{*,0} - \frac{d^2}{dx^2}\delta(x)M_1^{*,1} + \frac{d}{dx}\delta(x)\left(-M_1^{*,2} - A_{1,1}tM_1^{*,0}\right)\right) \\ \qquad\qquad + e^{\frac{vp_v}{\mu}t}\delta(x)\left(M_1^{*,3} + tA_{1,1}M_1^{*,1}\right) + O(1)e^{-\sigma_0^* t-\sigma_0|x|}. \end{cases} \tag{6.7}$$

For the other two parts $\mathbb{G}^{*,2}(x,t)$ and $\mathbb{G}^{*,3}(x,t)$, when $t \geqq 1$,

$$\partial_x^k\mathbb{G}^{*,j}(x,t) = O(1)e^{-\sigma_0^* t-\sigma_0|x|}, \quad k=0,1,2, \quad j=2,3, \quad t\geqq 1. \tag{6.8}$$

When $0<t<1$, the following estimates for $\mathbb{G}^{*,2}(x,t)$ and $\mathbb{G}^{*,3}(x,t)$ hold:

$$
\begin{cases}
\mathbb{G}^{*,j}(x,t) &= O(1)e^{-\sigma_0^* t-\sigma_0|x|} + \dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} M_j^{*,0}, \\
\partial_x \mathbb{G}^{*,j}(x,t) &= O(1)e^{-\sigma_0^* t-\sigma_0|x|} + \partial_x \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,0} \\
&\quad - \dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} M_j^{*,1}, \\
\partial_x^2 \mathbb{G}^{*,j}(x,t) &= O(1)e^{-\sigma_0^* t-\sigma_0|x|} + \partial_x^2 \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,0} \\
&\quad - \partial_x \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,1} \\
&\quad - \dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \left(M_j^{*,2} + tA_{j,1} M_j^{*,0} \right), \\
\partial_x^3 \mathbb{G}^{*,j}(x,t) &= O(1)e^{-\sigma_0^* t-\sigma_0|x|} + \partial_x^3 \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,0} \\
&\quad - \partial_x^2 \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,1} \\
&\quad - \partial_x \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] \left(M_j^{*,2} + tA_{j,1} M_j^{*,0} \right) \\
&\quad + \dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \left(M_j^{*,3} + tA_{j,1} M_j^{*,1} \right).
\end{cases}
\tag{6.9}
$$

Lemma 6.2. ([12]) *Let $j = 2, 3$, the time derivatives and the mixed derivatives of the singular parts have the following estimates:*

$$
\begin{aligned}
&\partial_t \mathbb{G}^{*,1}(x,t) = \frac{vp_v}{\mu} e^{\frac{vp_v}{\mu} t} \delta(x) M_1^{*,0} + O(1)e^{-\sigma_0^* t-\sigma_0|x|}, \\
&\partial_{xt} \mathbb{G}^{*,1}(x,t) = \frac{vp_v}{\mu} e^{\frac{vp_v}{\mu} t} \frac{d}{dx} \delta(x) M_1^{*,0} - \frac{vp_v}{\mu} e^{\frac{vp_v}{\mu} t} \delta(x) M_1^{*,1} + O(1)e^{-\sigma_0^* t-\sigma_0|x|}, \\
&\partial_t \mathbb{G}^{*,j}(x,t) = O(1)e^{-\sigma_0^* t-\sigma_0|x|}
\end{aligned}
$$

$$+\begin{cases} \alpha_j^* \partial_x^2 \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,0} - \alpha_j^* \partial_x \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,1} \\ \quad + \dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \left(-\alpha_j^* M_j^{*,2} + \beta_j^* M_j^{*,0} \right), & 0 < t < 1, \\ 0, & t \geqq 1, \end{cases}$$

$$\partial_{xt} \mathbb{G}^{*,j}(x,t) = O(1) e^{-\sigma_0^* t - \sigma_0 |x|}$$

$$+\begin{cases} \alpha_j^* \partial_x^3 \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,0} - \alpha_j^* \partial_x^2 \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] M_j^{*,1} \\ \quad + \partial_x \left[\dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \right] \left(-\alpha_j^* M_j^{*,2} - t\alpha_j^* A_{j,1} M_j^{*,0} + \beta_j^* M_j^{*,0} \right) \\ \quad + \dfrac{e^{\beta_j^* t}}{\sqrt{4\pi\alpha_j^* t}} e^{-\frac{x^2}{4\alpha_j^* t}} \left(\alpha_j^* M_j^{*,3} + t\alpha_j^* A_{j,1} M_j^{*,1} - \beta_j^* M_j^{*,1} \right), & 0 < t < 1, \\ 0, & t \geqq 1. \end{cases}$$

The following lemma describes the regular part $\mathbb{G}^\dagger$:

Lemma 6.3. ([12]) *There exist positive constants σ_0 and σ_0^* such that*

$$\sum_{k=0}^{3} |\partial_x^k \mathbb{G}^\dagger(x,t)| \leqq O(1) t e^{-\sigma_0 |x|}, \qquad 0 < t < 1,$$

$$\begin{aligned} &\left| \partial_x^k \mathbb{G}^\dagger(x,t) - \sum_{j=1}^{3} \partial_x^k \left(\frac{e^{-\frac{(x+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}} \right) M_j^0 - \sum_{j=1}^{3} \partial_x^{k+1} \left(\frac{e^{-\frac{(x+\beta_j t)^2}{4\alpha_j t}}}{2\sqrt{\pi\alpha_j t}} \right) M_j^1 \right| \\ &\quad \leqq \sum_{j=1}^{3} \frac{O(1) e^{-\frac{(x+\beta_j t)^2}{4Ct}}}{t^{\frac{k+2}{2}}} M_j^0 \\ &\qquad + \sum_{j=1}^{3} \frac{O(1) e^{-\frac{(x+\beta_j t)^2}{4Ct}}}{t^{\frac{k+3}{2}}} + O(1) e^{-\sigma_0^* t - \sigma_0 |x|}, \qquad t \geqq 1, \quad 0 \leqq k \leqq 3, \end{aligned}$$

where α_j and β_j are given below,

$$\alpha_1 = \frac{-\kappa\theta_e p_v}{v(pp_e - p_v)}, \quad \alpha_2 = \alpha_3 = \frac{(\kappa p\theta_e p_e + \mu pp_e - \mu p_v)}{2v(pp_e - p_v)},$$
$$\beta_1 = 0, \quad \beta_2 = -\sqrt{pp_e - p_v}, \quad \beta_3 = \sqrt{pp_e - p_v}.$$

The constant matrices M_j^0, M_j^1, $j = 1, 2, 3$ are given in "Appendix B".

6.2. Integral representation of solution

Without loss of generality, we assume that the initial data is a small perturbation around the constant state $(\bar{v}, \bar{u}, \bar{\theta}) = (1, 0, 1)$. Using total energy E as a variable, we have

$$\bar{U} = (\bar{v}, \bar{u}, \bar{E}) = (1, 0, c_v),$$

where c_v is the heat capacity defined in (1.2), and $\bar{U}$ is the equilibrium state in (6.3). The Green's matrix $\mathbb{G}$ can be represented as follows:

$$\mathbb{G} = \mathbb{G}^* + \mathbb{G}^\dagger, \quad \mathbb{G}^* = \begin{pmatrix} \mathbb{G}^*_{11} & \mathbb{G}^*_{12} & \mathbb{G}^*_{13} \\ \mathbb{G}^*_{21} & \mathbb{G}^*_{22} & \mathbb{G}^*_{23} \\ \mathbb{G}^*_{31} & \mathbb{G}^*_{32} & \mathbb{G}^*_{33} \end{pmatrix}, \quad \mathbb{G}^\dagger = \begin{pmatrix} \mathbb{G}^\dagger_{11} & \mathbb{G}^\dagger_{12} & \mathbb{G}^\dagger_{13} \\ \mathbb{G}^\dagger_{21} & \mathbb{G}^\dagger_{22} & \mathbb{G}^\dagger_{23} \\ \mathbb{G}^\dagger_{31} & \mathbb{G}^\dagger_{32} & \mathbb{G}^\dagger_{33} \end{pmatrix}. \tag{6.10}$$

Since $\mathbb{G}(x, t; \bar{U})$ satisfies the forward equation (6.5), it follows that $\mathbb{G}(x - y, t - \tau; \bar{U})$ also satisfies the backward equation,

$$\begin{aligned} &\partial_\tau \mathbb{G}(x - y, t - \tau; \bar{U}) + \partial_y \mathbb{G}(x - y, t - \tau; \bar{U}) F'(\bar{U}) \\ &\quad + \partial_y^2 \mathbb{G}(x - y, t - \tau; \bar{U}) B(\bar{U}) = 0. \end{aligned} \tag{6.11}$$

In the following, we will simply denote the Green's function by $\mathbb{G}(x - y, t - \tau)$. As we already chose $\bar{U} = (\bar{v}, \bar{u}, \bar{E}) = (1, 0, c_v)$, the matrices $F'(\bar{U})$ and $B(\bar{U})$ have the following form:

$$F'(\bar{U}) = \begin{pmatrix} 0 & -1 & 0 \\ -K & 0 & \frac{K}{c_v} \\ 0 & K & 0 \end{pmatrix}, \quad B(\bar{U}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \frac{\kappa}{c_v} \end{pmatrix}.$$

Substitute the above $F'(\bar{U})$ and $B(\bar{U})$ into the backward equation (6.11) to obtain the equations for each component:

$$\begin{pmatrix} \partial_\tau \mathbb{G}_{11} - K\partial_y \mathbb{G}_{12} & \partial_\tau \mathbb{G}_{12} - \partial_y \mathbb{G}_{11} + K\partial_y \mathbb{G}_{13} + \mu\partial_y^2 \mathbb{G}_{12} & \partial_\tau \mathbb{G}_{13} + \frac{K}{c_v}\partial_y \mathbb{G}_{12} + \frac{\kappa}{c_v}\partial_y^2 \mathbb{G}_{13} \\ \partial_\tau \mathbb{G}_{21} - K\partial_y \mathbb{G}_{22} & \partial_\tau \mathbb{G}_{22} - \partial_y \mathbb{G}_{21} + K\partial_y \mathbb{G}_{23} + \mu\partial_y^2 \mathbb{G}_{22} & \partial_\tau \mathbb{G}_{23} + \frac{K}{c_v}\partial_y \mathbb{G}_{22} + \frac{\kappa}{c_v}\partial_y^2 \mathbb{G}_{23} \\ \partial_\tau \mathbb{G}_{31} - K\partial_y \mathbb{G}_{32} & \partial_\tau \mathbb{G}_{32} - \partial_y \mathbb{G}_{31} + K\partial_y \mathbb{G}_{33} + \mu\partial_y^2 \mathbb{G}_{32} & \partial_\tau \mathbb{G}_{33} + \frac{K}{c_v}\partial_y \mathbb{G}_{32} + \frac{\kappa}{c_v}\partial_y^2 \mathbb{G}_{33} \end{pmatrix} = 0. \tag{6.12}$$

In order to construct a new effective integral representation of v, u and θ, we introduce an effective Green's function G similar as in [8]. Define a smooth non-increasing cutoff function as

$$\mathcal{X}(t) \in C^\infty(\mathbb{R}_+), \quad \mathcal{X}'(t) \leqq 0, \quad \|\mathcal{X}'\|_{L^\infty(\mathbb{R}_+)} \leqq 2, \quad \mathcal{X}(t) = \begin{cases} 1, & \text{for } t \in (0, 1], \\ 0, & \text{for } t > 2. \end{cases} \tag{6.13}$$

Then, we choose a small positive constant ν_0 (which will be determined later) such that, the heat kernel $H(x, t; y, \tau; \frac{1}{v})$ and the local weak solution $(v(x, \tau), u(x, \tau), E(x, \tau))$ for (1.1) both exist when $\tau \in (t - 2\nu_0, t]$. We interpolate the heat kernel for short time and Green's function for large time via cutoff function (6.13), thus introduce the effective Green's functions as

$$\begin{cases} G_{22}(x,t;y,\tau) = \mathcal{X}\left(\frac{t-\tau}{\nu_0}\right) H\left(x,t;y,\tau;\frac{\mu}{v}\right) + \left(1-\mathcal{X}\left(\frac{t-\tau}{\nu_0}\right)\right)\mathbb{G}_{22}(x-y;t-\tau), \\ G_{33}(x,t;y,\tau) = \mathcal{X}\left(\frac{t-\tau}{\nu_0}\right) H\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right) + \left(1-\mathcal{X}\left(\frac{t-\tau}{\nu_0}\right)\right)\mathbb{G}_{33}(x-y;t-\tau). \end{cases} \tag{6.14}$$

Now we can represent the solution (v, u, E) in terms of the effective Green's function, which captures both the local-in-time regularity and global-in-time space-time structure of the solution.

Lemma 6.4. *Suppose the weak solution $(v(x,\tau), u(x,\tau), E(x,\tau))$ for (1.1) exists for $\tau \in [0,t]$, and the heat kernel $H(x,t;y,\tau;\frac{\mu}{v})$ exists for $\tau \in (t-2\nu_0, t)$ for a sufficiently small positive constant ν_0 such that $2\nu_0 < t$. Then we have the representation of $u(x,t)$,*

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \mathbb{G}_{21}(x-y,t)(v(y,0)-1)dy + \int_{\mathbb{R}} G_{22}(x,t;y,0)u(y,0)dy \\ &\quad + \int_{\mathbb{R}} \mathbb{G}_{23}(x-y,t)\,(E(y,0)-c_v)\,dy + \sum_{i=1}^{3} \mathcal{R}_i^u, \end{aligned} \tag{6.15}$$

where the inhomogeneous remainders $\mathcal{R}_i$ are listed as

$$\begin{aligned} \mathcal{R}_1^u &= \int_0^{t-2\nu_0} \int_{\mathbb{R}} \partial_y \mathbb{G}_{22}(x-y,t-\tau) \\ &\quad \times \left(\frac{K(v-1)^2}{v} + \frac{K(\theta-1)(1-v)}{v} - \frac{Ku^2}{2c_v} + \frac{\mu u_y(v-1)}{v}\right) dyd\tau \\ &\quad + \int_0^{t-2\nu_0} \int_{\mathbb{R}} \partial_y \mathbb{G}_{23}(x-y,t-\tau) \\ &\quad \times \left(\left(\frac{K(\theta-1)+K(1-v)}{v}\right) u + \frac{\kappa\theta_y(v-1)}{v} + \left(\frac{\kappa}{c_v} - \frac{\mu}{v}\right) uu_y\right) dyd\tau, \\ \mathcal{R}_2^u &= \int_{t-2\nu_0}^{t-\nu_0} \int_{\mathbb{R}} \partial_y \mathbb{G}_{22}(x-y,t-\tau) \\ &\quad \times \left(\frac{K(v-1)^2}{v} + \frac{K(\theta-1)(1-v)}{v} + \frac{\mu u_y(v-1)}{v}(1-\mathcal{X}) - \frac{Ku^2}{2c_v}\right) dyd\tau \\ &\quad + \int_{t-2\nu_0}^{t-\nu_0} \int_{\mathbb{R}} \frac{1}{\nu_0}\mathcal{X}'\left(\frac{t-\tau}{\nu_0}\right)\left(\mathbb{G}_{22}(x-y;t-\tau) - H\left(x,t;y,\tau;\frac{\mu}{v}\right)\right) u(y,\tau)dyd\tau \\ &\quad + \int_{t-2\nu_0}^{t-\nu_0} \int_{\mathbb{R}} \mathcal{X}\left(\frac{t-\tau}{\nu_0}\right) \\ &\quad \times \left(K\partial_y\mathbb{G}_{23}(x-y,t-\tau) - \partial_y\mathbb{G}_{21}(x-y,t-\tau)\right) u(y,\tau)dyd\tau \\ &\quad + \int_{t-2\nu_0}^{t-\nu_0} \int_{\mathbb{R}} \mathcal{X}\left(\frac{t-\tau}{\nu_0}\right)\left(H_y\left(x,t;y,\tau;\frac{\mu}{v}\right) - \partial_y\mathbb{G}_{22}(x-y;t-\tau)\right)\frac{K(\theta-v)}{v} dyd\tau \\ &\quad + \int_{t-2\nu_0}^{t-\nu_0} \int_{\mathbb{R}} \partial_y\mathbb{G}_{23}(x-y;t-\tau) \\ &\quad \times \left(\frac{\kappa\theta_y(v-1)}{v} + \frac{Ku(\theta-v)}{v} + \left(\frac{\kappa}{c_v} - \frac{\mu}{v}\right) uu_y\right) dyd\tau, \\ \mathcal{R}_3^u &= \int_{t-\nu_0}^{t} \int_{\mathbb{R}} K\partial_y\mathbb{G}_{22}(x-y,t-\tau)(v(y,\tau)-1)dyd\tau \end{aligned}$$

$$
\begin{aligned}
&- \int_{t-\nu_0}^{t} \int_{\mathbb{R}} \partial_y \mathbb{G}_{21}(x-y, t-\tau) u(y,\tau) dy d\tau \\
&+ \int_{t-\nu_0}^{t} \int_{\mathbb{R}} H_y \left(x, t; y, \tau; \frac{\mu}{v}\right) \left(\frac{K\theta(y,\tau) - Kv(y,\tau)}{v(y,\tau)} \right) dy d\tau \\
&- \int_{t-\nu_0}^{t} \int_{\mathbb{R}} \partial_y \mathbb{G}_{22}(x-y, t-\tau) \left(K(\theta-1) + \frac{Ku^2}{2c_v} \right) dy d\tau \\
&+ \int_{t-\nu_0}^{t} \int_{\mathbb{R}} \partial_y \mathbb{G}_{23}(x-y, t-\tau) \left((pu) + \frac{\kappa(v-1)\theta_y}{v} + \left(\frac{\kappa}{c_v} - \frac{\mu}{v} \right) u u_y \right) dy d\tau.
\end{aligned}
$$

Proof. We multiply the vector $\big(\mathbb{G}_{21}(x-y, t-\tau), G_{22}(x, t; y, \tau), \mathbb{G}_{23}(x-y, t-\tau)\big)$ to the system (1.1), apply integration by parts, and split the time integral into three parts $[0, t-2\nu_0]$, $[t-2\nu_0, t-\nu_0]$ and $[t-\nu_0, t]$ to have the desired results. Interested readers are referred to [12] for the computational details. □

Lemma 6.5. *Suppose the weak solution $(v(x,\tau), u(x,\tau), E(x,\tau))$ for (1.1) exists for $\tau \in [0, t]$, and the heat kernel $H(x, t; y, \tau; \frac{\kappa}{c_v v})$ exists for $\tau \in (t-2\nu_0, t)$ for a sufficiently small positive constant ν_0 such that $2\nu_0 < t$. Then we have the representation of $v(x,t)$ and $E(x,t)$ as*

$$
\begin{aligned}
&v(x,t) - 1 \\
&\quad = \int_{\mathbb{R}} \mathbb{G}_{11}(x-y, t)(v(y,0)-1) dy + \int_{\mathbb{R}} \mathbb{G}_{12}(x, t; y, 0) u(y,0) dy \\
&\quad\quad + \int_{\mathbb{R}} \mathbb{G}_{13}(x-y, t) \left(E(y,0) - c_v\right) dy \\
&\quad\quad + \int_0^t \int_{\mathbb{R}} \partial_y \mathbb{G}_{12}(x, t; y, \tau) \left(\frac{K(v-1)^2}{v} + \frac{(K\theta - K)(1-v)}{v} \right. \\
&\quad\quad \left. + \frac{\mu u_y (v-1)}{v} - \frac{Ku^2}{2c_v} \right) dy d\tau \\
&\quad\quad + \int_0^t \int_{\mathbb{R}} \partial_y \mathbb{G}_{13}(x-y, t-\tau) \\
&\quad\quad \times \left((\frac{K\theta}{v} - K) u + \left(\kappa - \frac{\kappa}{v}\right) \theta_y + \left(\frac{\kappa}{c_v} - \frac{\mu}{v} \right) u u_y \right) dy d\tau,
\end{aligned} \tag{6.16}
$$

$$
\begin{aligned}
&E(x,t) - c_v \\
&\quad = \int_{\mathbb{R}} \mathbb{G}_{31}(x-y, t)(v(y,0)-1) dy + \int_{\mathbb{R}} \mathbb{G}_{32}(x, t; y, 0) u(y,0) dy \\
&\quad\quad + \int_{\mathbb{R}} G_{33}(x-y, t) \left(E(y,0) - c_v\right) dy + \sum_{i=1}^{3} \mathcal{R}_i^\theta,
\end{aligned} \tag{6.17}
$$

where the remainders $\mathcal{R}_i^\theta$ are listed as

$$
\begin{aligned}
\mathcal{R}_1^\theta = &\int_0^{t-2\nu_0} \int_{\mathbb{R}} \partial_y \mathbb{G}_{32}(x, t; y, \tau) \\
&\times \left(\frac{K(v-1)^2}{v} + \frac{(K\theta - K)(1-v)}{v} + \frac{\mu u_y (v-1)}{v} - \frac{Ku^2}{2c_v} \right) dy d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t-2\nu_0}\int_{\mathbb{R}} \partial_y \mathbb{G}_{33}(x-y,t-\tau)\\
&\times\left((\frac{K\theta}{v}-K)u+\left(\kappa-\frac{\kappa}{v}\right)\theta_y+\left(\frac{\kappa}{c_v}-\frac{\mu}{v}\right)uu_y\right)dyd\tau,\\
\mathcal{R}_2^\theta=&\int_{t-2\nu_0}^{t-\nu_0}\int_{\mathbb{R}}\partial_y\mathbb{G}_{32}(x,t;y,\tau)\left(\frac{K(\theta-1)}{v}+\frac{\mu(v-1)u_y}{v}+\frac{K(v-1)^2}{v}\right)dyd\tau\\
&-\int_{t-2\nu_0}^{t-\nu_0}\int_{\mathbb{R}}\left(1-\mathcal{X}\left(\frac{t-\tau}{\nu_0}\right)\right)\left(\frac{K}{c_v}\partial_y\mathbb{G}_{32}\right)\\
&\times (E(y,\tau)-c_v)\,dyd\tau\\
&+\int_{t-2\nu_0}^{t-\nu_0}\int_{\mathbb{R}}\frac{1}{\nu_0}\mathcal{X}'\left(\frac{t-\tau}{\nu_0}\right)\left(-H\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right)+\mathbb{G}_{33}(x-y;t-\tau)\right)\\
&\times (E(y,\tau)-c_v)\,dyd\tau\\
&+\int_{t-2\nu_0}^{t-\nu_0}\int_{\mathbb{R}}\mathcal{X}\left(\frac{t-\tau}{\nu_0}\right)H_y\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right)\\
&\times\left((pu)+\left(\frac{\kappa}{c_v v}-\frac{\mu}{v}\right)uu_y\right)dyd\tau\\
&+\int_{t-2\nu_0}^{t-\nu_0}\int_{\mathbb{R}}\left(1-\mathcal{X}\left(\frac{t-\tau}{\nu_0}\right)\right)\partial_y\mathbb{G}_{33}(x-y;t-\tau)\\
&\times\left((pu)+\frac{\kappa(v-1)}{v}\theta_y+\left(\frac{\kappa}{c_v}-\frac{\mu}{v}\right)uu_y\right)dyd\tau\\
&+\int_{t-2\nu_0}^{t-\nu_0}\int_{\mathbb{R}}\left(-K\partial_y\mathbb{G}_{33}\right)u(y,\tau)dyd\tau,\\
\mathcal{R}_3^\theta=&\int_{t-\nu_0}^{t}\int_{\mathbb{R}}\partial_y\mathbb{G}_{32}(x-y,t-\tau)\left(K(v(y,\tau)-1)+\mu u_y\frac{v-1}{v}+\frac{K\theta-Kv}{v}\right)dyd\tau\\
&+\int_{t-\nu_0}^{t}\int_{\mathbb{R}}\left(-K\partial_y\mathbb{G}_{33}\right)u(y,\tau)dyd\tau\\
&+\int_{t-\nu_0}^{t}\int_{\mathbb{R}}\partial_y H\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right)\left((pu)+\left(\frac{\kappa}{c_v v}-\frac{\mu}{v}\right)uu_y\right)dyd\tau.
\end{aligned}
$$

Proof. The calculations are similar to those in Lemma 6.4. Multiplying the vector

$$\left(\mathbb{G}_{11}(x-y,t-\tau)\ \mathbb{G}_{12}(x-y,t-\tau)\ \mathbb{G}_{13}(x-y,t-\tau)\right)$$

on the system (1.1), integrating with respect to y and τ, using integration by parts and the backward equation, one can get the representation for $v(x,t)$. For the representation of $E(x,t)$, we just need to multiply

$$\left(\mathbb{G}_{31}(x-y,t-\tau)\ \mathbb{G}_{32}(x-y,t-\tau)\ G_{33}(x,t;y,\tau)\right)$$

on the system (1.1) and follow the similar procedure. As the computations are routine though tedious, they are omitted; see [12] for details. □

Remark 6.1. Although the detailed computations in Lemmas 6.4 and 6.5 are lengthy, the goal is clear, i.e., we want to write the solution into the sum of linear terms and nonlinear terms. Since we are dealing with perturbation problem, one expects the linear terms are dominant. Notice that when $\tau\in(t-2\nu_0,t)$, there are also some linear terms. It turns out they can be controlled either by comparison estimates of Green's function or by the appropriate small parameter ν_0.

Note that in the representation of zeroth order solutions, we need the information of first order derivative of u and θ. If we formally differentiate the expressions (6.15) and (6.17) with respect to x, the derivatives will induce additional singularity in time, so that the time integral is not integrable anymore. Therefore, in order to gain the derivative estimates, we provide another representation for the derivatives, which is a consequence of the previous representations; see [12] for detailed calculations.

Corollary 6.1. *For the first order derivative of $\mathcal{R}_3^u$ and $\mathcal{R}_3^\theta$, we have the following identities:*

$$
\begin{aligned}
\partial_x \mathcal{R}_3^u = &\int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{\mu}\left(\partial_x \mathbb{G}_{22}(x-z,\nu_0) - H_x\left(x,t;z,t-\nu_0;\frac{\mu}{v}\right)\right) \\
&dz(Kv(y,t-\nu_0) - K\theta(y,t-\nu_0))dy \\
&- \int_{t-\nu_0}^{t}\int_{-\infty}^{x}\int_{-\infty}^{y} -\frac{1}{\mu}\left(\partial_x \mathbb{G}_{22}(x-z,t-\tau) - H_x\left(x,t;z,\tau;\frac{\mu}{v}\right)\right) \\
&dz(Kv_\tau(y,\tau) - K\theta_\tau(y,\tau))dyd\tau \\
&- \int_{x}^{+\infty}\int_{y}^{+\infty} \frac{1}{\mu}\left(\partial_x \mathbb{G}_{22}(x-z,\nu_0) - H_x\left(x,t;z,t-\nu_0;\frac{\mu}{v}\right)\right) \\
&dz(Kv(y,t-\nu_0) - K\theta(y,t-\nu_0))dy \\
&+ \int_{t-\nu_0}^{t}\int_{x}^{+\infty}\int_{y}^{+\infty} -\frac{1}{\mu}\left(\partial_x \mathbb{G}_{22}(x-z,t-\tau) - H_x\left(x,t;z,\tau;\frac{\mu}{v}\right)\right) \\
&dz(Kv_\tau(y,\tau) - K\theta_\tau(y,\tau))dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} -\frac{1}{\mu}\left(-\partial_x \mathbb{G}_{21}(x-y,t-\tau) + K\partial_x \mathbb{G}_{23}(x-y,t-\tau)\right) \\
&\times (Kv(y,\tau) - K\theta(y,\tau))dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} \partial_x \mathbb{G}_{21}(x-y,t-\tau)u_y(y,\tau)dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} \partial_x \mathbb{G}_{22}(x-y,t-\tau)\frac{Kuu_y}{c_v}dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} \partial_{xy} \mathbb{G}_{23}(x-y,t-\tau)\left((pu) + \frac{\kappa(v-1)\theta_y}{v} + \left(\frac{\kappa}{c_v} - \frac{\mu}{v}\right)uu_y\right)dyd\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_x \mathcal{R}_3^\theta = &\int_{t-\nu_0}^{t}\int_{\mathbb{R}} \partial_{xy} \mathbb{G}_{32}(x-y,t-\tau)\left(K(v(y,\tau)-1) + \mu u_y \frac{v-1}{v} + \frac{K\theta - Kv}{v}\right)dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} K\left(\partial_x \mathbb{G}_{33}(x-y,t-\tau) - H_x\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right)\right)u_y(y,\tau)dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} H_x\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right)\left(1 - \frac{\kappa}{c_v \mu}\right)uu_\tau dyd\tau \\
&+ \int_{t-\nu_0}^{t}\int_{\mathbb{R}} H_x\left(x,t;y,\tau;\frac{\kappa}{c_v v}\right)\left(1 - \frac{\kappa}{c_v \mu}\right)u_y\left(K - p + \frac{\mu}{v}u_y\right)dyd\tau \\
&+ \int_{-\infty}^{x}\int_{-\infty}^{y} H_x\left(x,t;z,t-\nu_0;\frac{\kappa}{c_v v}\right) \\
&dz\left(\frac{\kappa K}{c_v \mu}\right)\frac{(\theta(y,t-\nu_0) - v(y,t-\nu_0))}{v(y,t-\nu_0)}u(y,t-\nu_0)dy
\end{aligned}
$$

$$
\begin{aligned}
&+ v\left(\frac{\kappa K}{c_v \mu}\right) \frac{(\theta(x,t-\nu_0)-v(x,t-\nu_0))}{v(x,t-\nu_0)} u(x,t-\nu_0) \\
&+ \int_{t-\nu_0}^{t} \int_{-\infty}^{x} \int_{-\infty}^{y} H_x\left(x,t;z,\tau;\frac{\kappa}{c_v v}\right) dz \left(\frac{\kappa K}{c_v \mu}\right) \\
&\times \left(\frac{(\theta_\tau v - \theta v_\tau)}{v^2} u(y,\tau) + \frac{(\theta - v)}{v} u_\tau(y,\tau)\right) dy d\tau \\
&+ \int_{t-\nu_0}^{t} \left(\frac{\kappa K}{c_v \mu}\right) \left(\frac{(\theta_\tau(x,\tau)v(x,\tau) - \theta(x,\tau)v_\tau(x,\tau))}{v^2(x,\tau)} u(x,\tau)\right. \\
&\left. + \frac{(\theta(x,\tau)-v(x,\tau))}{v(x,\tau)} u_\tau(x,\tau)\right) d\tau \\
&- \int_{x}^{+\infty} \int_{y}^{+\infty} H_x\left(x,t;z,t-\nu_0;\frac{\kappa}{c_v v}\right) dz \left(\frac{\kappa K}{c_v \mu}\right) \\
&\times \left(\frac{\theta(y,t-\nu_0)-v(y,t-\nu_0)}{v(y,t-\nu_0)}\right) u(y,t-\nu_0) dy \\
&- \int_{t-\nu_0}^{t} \int_{x}^{+\infty} \int_{y}^{+\infty} H_x\left(x,t;z,\tau;\frac{\kappa}{c_v v}\right) dz \left(\frac{\kappa K}{c_v \mu}\right) \\
&\times \left(\frac{(\theta_\tau v - \theta v_\tau)}{v^2} u(y,\tau) + \frac{(\theta - v)}{v} u_\tau(y,\tau)\right) dy d\tau.
\end{aligned}
$$

Remark 6.2. It should be emphasized that in the integral representations of $\partial_x \mathcal{R}_3^u$ and $\partial_x \mathcal{R}_3^\theta$, one needs the time differentiability estimates of v_τ, θ_τ and u_τ respectively, which are already established in Theorem 4.1.

6.3. Time asymptotic behavior

In this part, we will study the global existence of the solution to the nonlinear Navier–Stokes equation (1.1). According to Theorems 4.1 and 5.1, if the initial data is controlled by a sufficiently small constant δ as in (3.2), there exists a unique weak solution (v, u, θ) to (1.3), or equivalently, (v, u, E) to (1.1), for $t < t_\sharp$. Moreover, the solutions are kept small in the sense of (3.74) and (4.9).

We define a stopping time as

$$
\begin{aligned}
&T = \sup_{t \geqq 0} \left\{ t \,\middle|\, \mathcal{G}(\tau) < \delta, \quad \text{for } 0 < \tau < t \right\}, \\
&\mathcal{G}(\tau) \equiv \|\sqrt{\tau+1}(v(\cdot,\tau)-1)\|_\infty + \|\sqrt{\tau+1}u(\cdot,\tau)\|_\infty + \|\sqrt{\tau+1}(\theta(\cdot,\tau)-1)\|_\infty \\
&\quad + \|v(\cdot,\tau)-1\|_{L^1} + \|u(\cdot,\tau)\|_{L^1} + \|\theta(\cdot,\tau)-1\|_{L^1} \\
&\quad + \|v(\cdot,\tau)-1\|_{BV} + \|u(\cdot,\tau)\|_{BV} + \|\theta(\cdot,\tau)-1\|_{BV} \\
&\quad + \|\sqrt{\tau}u_x(\cdot,\tau)\|_{L^\infty} + \|\sqrt{\tau}\theta_x(\cdot,\tau)\|_{L^\infty}.
\end{aligned} \tag{6.18}
$$

By Theorem 4.1, there exists a positive constant δ_* (smaller than δ) such that, if the initial data satisfy

$$
\begin{aligned}
&\|v_0 - 1\|_{BV} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{BV} + \|v_0 - 1\|_{L^1} \\
&\quad + \|u_0\|_{L^1} + \|\theta_0 - 1\|_{L^1} < \delta^*,
\end{aligned} \tag{6.19}
$$

then the stopping time $T > t_\sharp$. Here $t_\sharp$ is the existence time associated with δ in Theorem 4.1.

Based on the integral representations of the solution in (6.16), (6.15) and (6.17), and of their derivatives in Corollary 6.1, we can prove the following a priori estimate, which then yields a sharper estimates of the solution. This is the key lemma for the proof of global existence. We refer the interested reader to [12] for the details of proof.

Lemma 6.6. (A priori estimate, [12]) *Let* (v, u, E), $t_\sharp$ *and* δ *be the local solution and corresponding parameters constructed in Theorem 4.1. We further suppose that the following properties hold for the solution:*

$$\begin{cases} \|v_0-1\|_{BV}+\|u_0\|_{BV}+\|\theta_0-1\|_{BV}+\|v_0-1\|_{L^1}+\|u_0\|_{L^1}+\|\theta_0-1\|_{L^1}<\delta^*, \\ \mathcal{G}(\tau)<\delta, \quad \textit{for}\ \forall\tau<t, \\ t_\sharp \geqq 4\nu_0. \end{cases} \tag{6.20}$$

where $\mathcal{G}(\tau)$ *is defined in* (6.18), δ^* *is as in* (6.19) *and* ν_0 *is given in* (6.14). *Then,* $u(x,t)$ *has the following estimates for* $t \geqq t_\sharp$*:*

$$\begin{cases} \|u(\cdot,t)\|_{L^1} \leqq C(\nu_0)\delta^* + O(1)\left(\delta^2+\sqrt{\nu_0}\delta^2+\sqrt{\nu_0}\delta+\nu_0\delta+\delta^2\right), \\ \|\sqrt{1+t}u(\cdot,t)\|_{L^\infty} \leqq C(\nu_0)\delta^* + O(1)\left(\sqrt{\nu_0}\delta+\delta^2\right), \\ \|u_x(\cdot,t)\|_{L^1} \leqq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta, \\ \|\sqrt{t}u_x(\cdot,t)\|_{L^\infty} \leqq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta. \end{cases}$$

$\theta(x,t)$ *has the following estimates for* $t \geqq t_\sharp$*:*

$$\begin{cases} \|\theta(\cdot,t)-1\|_{L^1} \leqq O(1)\left(C(\nu_0)\delta^*+\sqrt{\nu_0}\delta+\delta^2\right) + O(1)\left(C(\nu_0)\delta^*+\sqrt{\nu_0}\delta+\delta^2\right)^2, \\ \|\sqrt{1+t}(\theta(\cdot,t)-1)\|_{L^\infty} \leqq O(1)\left(C(\nu_0)\delta^*+\sqrt{\nu_0}\delta+\delta^2\right) + O(1)\left(C(\nu_0)\delta^*+\sqrt{\nu_0}\delta+\delta^2\right)^2, \\ \|\theta_x(\cdot,t)\|_{L^1} \leqq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta, \\ \|\sqrt{t}\theta_x(\cdot,t)\|_{L^\infty} \leqq C(\nu_0)\delta^* + O(1)\dfrac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta. \end{cases}$$

$v(x,t)$ *has the following estimates for* $t \geqq t_\sharp$*:*

$$\begin{cases} \|v(\cdot,t)-1\|_{L^1} \leqq C(\nu_0)\delta^* + O(1)\delta^2, \\ \|\sqrt{1+t}(v(\cdot,t)-1)\|_{L^\infty} \leqq C(\nu_0)\delta^* + O(1)\delta^2, \\ \|v(\cdot,t)\|_{BV} \leqq C(\nu_0)\delta^* + O(1)\dfrac{\delta^2}{\sqrt{\nu_0}}. \end{cases}$$

We are now ready to prove the global existence of the solution constructed in Theorem 4.1 for sufficiently small initial data.

Theorem 6.1. (Global existence) *Suppose initial data* (v_0, u_0, θ_0) *of Navier–Stokes equation* (1.3) *satisfy*

$$\|v_0 - 1\|_{L^1_x} + \|v_0\|_{BV} + \|u_0\|_{L^1_x} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{L^1_x} + \|\theta_0\|_{BV} \leqq \delta^*, \tag{6.21}$$

for δ^* *sufficiently small. Then the solution constructed in Theorems* 4.1 *and* 5.1 *exists globally in time, and there exists positive constant* $\mathscr{C}$ *such that, the solution satisfies*

$$\begin{aligned}&\left\|\sqrt{t+1}(v(\cdot,t)-1)\right\|_{L^\infty_x} + \left\|\sqrt{t+1}u(\cdot,t)\right\|_{L^\infty_x} + \left\|\sqrt{t+1}(\theta(\cdot,t)-1)\right\|_{L^\infty_x}\\ &\quad + \left\|\sqrt{t}u_x(\cdot,t)\right\|_{L^\infty_x} + \left\|\sqrt{t}\theta_x(\cdot,t)\right\|_{L^\infty_x} \leqq \mathscr{C}\delta^* \ \textit{for } t \in (0, +\infty).\end{aligned} \tag{6.22}$$

Proof. We let $C_\sharp$, $t_\sharp$ and δ be the parameters in Theorem 4.1, then Theorems 4.1 and 5.1 guarantee the existence of weak solution (v, u, θ) in $[0, t_\sharp)$. Now we let the initial data satisfy the smallness condition

$$\|v_0 - 1\|_{BV} + \|u_0\|_{BV} + \|\theta_0 - 1\|_{BV} + \|v_0 - 1\|_{L^1} + \|u_0\|_{L^1} + \|\theta_0 - 1\|_{L^1} < \delta^*.$$

For sufficiently small δ^*, we can define a stopping time T as in (6.18), which has the properties

$$T > t_\sharp, \qquad \mathcal{G}(T) \geqq \delta, \qquad \mathcal{G}(\tau) < \delta \ \text{ for all } \tau < T,$$

where $\mathcal{G}(\tau)$ is defined in (6.18). In order to prove the global existence of the solution, it suffices to show $T = \infty$ for sufficiently small δ^*. We will prove this by contradiction. If we suppose not, then

$$T < +\infty, \quad \text{for arbitrary postive } \delta^*. \tag{6.23}$$

On the other hand, by Lemma 6.6, the solution at T has the estimates

$$\begin{aligned}\mathcal{G}(T) &\leqq C(\nu_0)\delta^* + O(1)\frac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta\\ &\quad + \left(C(\nu_0)\delta^* + O(1)\frac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 + O(1)\sqrt{\nu_0}\delta\right)^2,\end{aligned}$$

where ν_0 is a small positive constant such that $\nu_0 \leqq \frac{t_\sharp}{4}$.

It is observed that when we choose δ^* and δ to be smaller, all the other parameters such as $C_\sharp$ and $t_\sharp$ are uniform. Moreover, all the $O(1)$ coefficients in the above formula are independent of ν_0, and thus the $O(1)$ coefficients will be uniform when δ, δ^* and ν_0 are changed to be smaller. Therefore, we can first choose ν_0 to be sufficiently small such that

$$O(1)\sqrt{\nu_0}\delta \leqq \frac{\delta}{6}.$$

Then, we fix ν_0, and let δ to be sufficiently small so that

$$O(1)\frac{|\log(\nu_0)|}{\sqrt{\nu_0}}\delta^2 \leqq \frac{\delta}{6}.$$

Finally, for fixed ν_0 and δ, we let δ^* to be sufficiently small such that

$$C(\nu_0)\delta^* \leqq \frac{\delta}{6}. \tag{6.24}$$

Now, for these well chosen ν_0, δ and δ^*, we combine all the above estimates to obtain that

$$\mathcal{G}(T) \leqq \frac{\delta}{2} + \frac{\delta^2}{4} \leqq \frac{3\delta}{4} < \delta,$$

which obviously contradicts to the assumption (6.23). Thus, for sufficiently small positive constant δ^* such that (6.21) holds, we can find a small positive constant δ such that

$$\mathcal{G}(t) < \delta, \quad \text{for all } t > 0. \tag{6.25}$$

From previous discussion and the definition of $\mathcal{G}(t)$ in (6.18), (6.25) immediately implies the global existence and uniqueness of the weak solution. Moreover, the large time behavior (6.22) directly follows from (6.25). □

Remark 6.3. It is even possible to establish the space-time pointwise estimate of the global solution for BV data, provided the initial data satisfy certain space decay assumption. This will be pursued in the future work. See [11] for the pointwise results for isentropic gas.

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Appendix A. Heat Kernel Estimates

The goal of this section is to provide the proofs for the estimates in Section 2. We consider the following heat equation with the coefficient $\rho(x,t)$ being a BV function of x,

$$\begin{cases} (\partial_t - \partial_x\rho(x,t)\partial_x)\, H(x,t;y,t_0;\rho) = 0, \quad t > t_0, \\ H(x,t_0;y,t_0;\rho) = \delta(x-y). \end{cases}$$

Here the BV coefficient $\rho(x,t)$ satisfies the following properties,

$$\begin{cases} \|\rho(\cdot)-\bar\rho\|_{L^1} \leqq \delta_*, \quad \|\rho(\cdot,t)\|_{BV} \leqq \delta_*, \quad \|\rho_t(\cdot,t)\|_\infty \leqq \delta_* \max\left(\frac{1}{\sqrt{t}},1\right), \quad 0<\delta_* \ll 1, \\ \mathscr{D} \equiv \{z \mid \rho(z,t) \text{ is not continuous at } z\} \text{ is invariant in } t. \end{cases} \tag{A.1}$$

To construct $H(x,t;y,t_0;\rho)$, the strategy is as follows: we first treat the case that ρ is a step function in space variable and independent of time; then we use step function to approximate a general BV function (still time-independent); lastly, we use time-independent solution and time-frozen technique to construct the heat kernel for time-dependent BV coefficient.

A.1. Step function conductivity coefficient

Consider

$$\begin{cases} (\partial_t - \partial_x \mu(x)\partial_x)\, H(x,t;y,t_0;\mu) = 0, \quad t>t_0, \\ H(x,t_0;y,t_0;\mu) = \delta(x-y), \end{cases} \tag{A.2}$$

where $\mu(x)$ is a step function.

Proposition A.1. ([8], Basic estimates) *When step function μ satisfies that*

$$\left|\mu(x)-\mu_0\right| \ll 1, \ \textit{and} \ \left\|\mu\right\|_{BV} \ll 1,$$

the heat kernel for equation (A.2) satisfies the following estimates: for all $x, y \in \mathbb{R}$

$$\begin{cases} H(x,t;y;\mu) = (1+O(1)\|\mu\|_{BV})\, \dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{4t}}}{\sqrt{4\pi t}}, \\ |\partial_x H(x,t;y;\mu)|, \left|\partial_y H(x,t;y;\mu)\right| = O(1)\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{Dt}}}{t}, \\ |\partial_t H(x,t;y;\mu)|, \left|\partial_{xy} H(x,t;y;\mu)\right| = O(1)\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{Dt}}}{t^{3/2}}, \\ |\partial_{tx} H(x,t;y;\mu)|, \left|\partial_{ty} H(x,t;y;\mu)\right| = O(1)\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{Dt}}}{t^2}. \end{cases} \tag{A.3}$$

Here for $x, y \notin \mathscr{D}$ (the discontinuity set of $\mu(x)$), the partial derivatives are standard ones. While for $y=\alpha \in \mathscr{D}$, $H_y(x,t;\alpha+;\mu)$ and $H_y(x,t;\alpha-;\mu)$ both exist and satisfy the estimates.
Moreover, for all $\ell \in \mathbb{N}$, when $t>0$,

$$\begin{cases} \partial_t^\ell H(x,t;y;\mu): & \textit{continuous in } x \in \mathbb{R} \textit{ for all } y \in \mathbb{R}, \ \textit{in } y \in \mathbb{R} \textit{ for all } x \in \mathbb{R}, \\ \mu(x)\partial_t^\ell H_x(x,t;y;\mu): & \textit{continuous in } x \in \mathbb{R} \textit{ for all } y \in \mathbb{R}, \ \textit{in } y \in \mathbb{R} \textit{ for all } x \in \mathbb{R}, \\ \mu(y)\partial_t^\ell H_y(x,t;y;\mu): & \textit{continuous in } x \in \mathbb{R} \textit{ for all } y \in \mathbb{R}, \ \textit{in } y \in \mathbb{R} \textit{ for all } x \in \mathbb{R}. \end{cases} \tag{A.4}$$

Proposition A.2. ([8], Comparison estimates) *Suppose that the steps function μ^a and μ^b satisfy $\|\mu^a\|_{BV} + \|\mu^b\|_{BV} \ll 1$, $\inf_{z\in\mathbb{R}} \mu^a(z), \inf_{z\in\mathbb{R}} \mu^b(z) > \underline{\mu} > 0$. Then for $t \in (0, e^{-1})$, $x, y \in \mathbb{R}$,*

$$|H(x,t;y;\mu^a) - H(x,t;y;\mu^b)| \leqq O(1)\|\mu^a - \mu^b\|_\infty \frac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}}{\sqrt{t}}, \tag{A.5}$$

$$\begin{aligned} &|H_x(x,t;y;\mu^a) - H_x(x,t;y;\mu^b)| + |H_y(x,t;y;\mu^a) - H_y(x,t;y;\mu^b)| \\ &\leqq O(1)\left(|\log t|\|\mu^a - \mu^b\|_\infty + \|\mu^a - \mu^b\|_{BV} + \sqrt{t}\|\mu^a - \mu^b\|_1\right) \\ &\times \frac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}}{t}, \end{aligned} \tag{A.6}$$

$$\begin{aligned} &|H_{xy}(x,t;y;\mu^a) - H_{xy}(x,t;y;\mu^b)| \\ &\leqq O(1)\left(|\log t|\|\mu^a - \mu^b\|_\infty + \|\mu^a - \mu^b\|_{BV} + \sqrt{t}\|\mu^a - \mu^b\|_1\right) \\ &\times \frac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}}{t^{3/2}}. \end{aligned} \tag{A.7}$$

The first one comes from writing $H(x,t;y;\mu^b)$ into an integral equation in terms of $H(x,t;y;\mu^a)$ and direct computations. The derivative comparison are much more subtle. Straightforward differentiating the integral equation will induce non-integrable time singularity. One has to do delicate estimate on the Laplace wave train level, then invert it to physical variable. See [8] for details.

A.2. Time-independent conductivity coefficient

Now consider conductivity coefficient $\mu(x)$ is a general BV function. The strategy is to construct a sequence of step functions $\{\mu^k(x)\}$ to approximate $\mu(x)$ in the following sense

$$\begin{cases} \|\mu^k\|_{BV} \leqq 2\|\mu\|_{BV}, \\ \|\mu^k - \mu\|_\infty < \dfrac{1}{2^k} \to 0, \text{ as } k \to \infty. \end{cases}$$

For each step function $\mu^k(x)$, one can construct the heat kernel $H(x,t;y;\mu^k)$. Then it is shown that

Proposition A.3. ([8], Theorem 3.6) *Suppose $\|\mu\|_{BV} \ll 1$ and $\inf_{z\in\mathbb{R}} \mu(z) > \underline{\mu} > 0$. Let μ^k be the step functions constructed as above. Then*

$$H(x,t;y;\mu) \equiv \lim_{k\to\infty} H(x,t;y;\mu^k) \text{ exists.}$$

$H(x,t;y;\mu)$ is a weak solution of

$$\begin{cases} (\partial_t - \partial_x \mu(x)\partial_x)\, H(x,t;y;\mu) = 0, \quad t>0, \\ H(x,0;y;\mu) = \delta(x-y), \end{cases} \tag{A.8}$$

and satisfies

$$\begin{cases} H(x,t;y;\mu) = (1+O(1)\|\mu\|_{BV})\,\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{4t}}}{\sqrt{4\pi t}}, \\ |\partial_x H(x,t;y;\mu)|, \left|\partial_y H(x,t;y;\mu)\right| = O(1)\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{5t}}}{t}, \\ |\partial_t H(x,t;y;\mu)|, \left|\partial_{xy} H(x,t;y;\mu)\right| = O(1)\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{5t}}}{t^{3/2}}, \\ |\partial_{tx} H(x,t;y;\mu)|, \left|\partial_{ty} H(x,t;y;\mu)\right| = O(1)\dfrac{e^{-\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu(z)}}\right)^2}{5t}}}{t^2}. \end{cases} \tag{A.9}$$

Up to a sub-sequence,

$$\lim_{k\to\infty} \partial^\beta H(x,t;y;\mu^k) = \partial^\beta H(x,t;y;\mu) \text{ exits},$$

where $\partial^\beta \in \{\partial_x, \partial_y, \partial_{xy}, \partial_t, \partial_{tx}, \partial_{ty}\}$. Moreover, for all $\ell \in \mathbb{N}$, when $t>0$,

$$\begin{cases} \partial_t^\ell H(x,t;y;\mu): & \text{continuous in } x\in\mathbb{R} \text{ for all } y\in\mathbb{R},\ \text{in } y\in\mathbb{R} \text{ for all } x\in\mathbb{R}, \\ \mu(x)\partial_t^\ell H_x(x,t;y;\mu): & \text{continuous in } x\in\mathbb{R} \text{ for all } y\in\mathbb{R},\ \text{in } y\in\mathbb{R} \text{ for all } x\in\mathbb{R}, \\ \mu(y)\partial_t^\ell H_y(x,t;y;\mu): & \text{continuous in } x\in\mathbb{R} \text{ for all } y\in\mathbb{R},\ \text{in } y\in\mathbb{R} \text{ for all } x\in\mathbb{R}. \end{cases} \tag{A.10}$$

Proposition A.4. ([8], Comparison estimates) *Suppose that two BV functions μ^a and μ^b satisfy $\|\mu^a\|_{BV} + \|\mu^b\|_{BV} \ll 1$, $\inf\limits_{z\in\mathbb{R}} \mu^a(z), \inf\limits_{z\in\mathbb{R}} \mu^b(z) > \underline{\mu} > 0$. Then for $x, y \in \mathbb{R}$,*

$$|H(x,y;y;\mu^a) - H(x,t;y;\mu^b)| \leqq O(1)\|\mu^a - \mu^b\|_\infty \frac{e^{\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}}{\sqrt{t}}, \tag{A.11}$$

$$\begin{aligned} &|H_x(x,y;y;\mu^a) - H_x(x,t;y;\mu^b)| + |H_y(x,y;y;\mu^a) - H_y(x,t;y;\mu^b)| \\ &\quad \leqq O(1)\left(\left(1+|\log t|\right)\|\mu^a-\mu^b\|_\infty + \|\mu^a-\mu^b\|_{BV} + \sqrt{t}\|\mu^a-\mu^b\|_1\right) \\ &\quad \times \frac{e^{\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}}{t}, \end{aligned} \tag{A.12}$$

$$
\begin{aligned}
&|H_{xy}(x, y; y; \mu^a) - H_{xy}(x, t; y; \mu^b)| \\
&\quad \leqq O(1)\Big(\big(1 + |\log t|\big)\|\mu^a - \mu^b\|_\infty + \|\mu^a - \mu^b\|_{BV} + \sqrt{t}\|\mu^a - \mu^b\|_1\Big) \\
&\qquad \times \frac{e^{\frac{\left(\int_y^x \frac{dz}{\sqrt{\mu^a(z)}\vee\sqrt{\mu^b(z)}}\right)^2}{5t}}}{t^{3/2}}.
\end{aligned} \tag{A.13}
$$

Propositions A.3 and A.4 are followed from comparison estimates of steps function in Proposition A.2 and a limiting procedure.

Proposition A.5. (Estimates involving time integral) *Suppose* $\|\mu\|_{BV} \ll 1$ *and* $\inf_{z\in\mathbb{R}} \mu(z) > \underline{\mu} > 0$. *Let* $H(x, t; y; \mu)$ *be the associated heat kernel. Then there exists positive constant* D *such that*

$$
\begin{cases}
\left|\int_0^t H_x(x, \tau; y; \mu)d\tau\right| \leqq O(1)e^{-\frac{(x-y)^2}{Dt}} & \text{for } x \in \mathbb{R}, y \in \mathbb{R}, \\
\left|\int_0^t H_{xy}(x, \tau; y; \mu)d\tau - \frac{\delta(x-y)}{\mu(x)}\right| \leqq O(1)\frac{e^{-\frac{(x-y)^2}{Dt}}}{\sqrt{t}} & \text{for } x \in \mathbb{R}, y \in \mathbb{R}, \\
\left|\int_0^t H_{xx}(x, \tau; y; \mu)d\tau + \partial_x\left(\frac{H(x-y)}{\mu(x)}\right)\right| \leqq O(1)(\frac{1}{\sqrt{t}} + |\partial_x\mu(x)|)e^{-\frac{(x-y)^2}{Dt}} & \text{for } x \notin \mathscr{D}, y \in \mathbb{R}, \\
\left|\int_0^t H_{xxy}(x, \tau; y; \mu)d\tau - \partial_x\left(\frac{\delta(x-y)}{\mu(x)}\right)\right| \leqq O(1)(\frac{1}{\sqrt{t}} + |\partial_x\mu(x)|)\frac{e^{-\frac{(x-y)^2}{Dt}}}{\sqrt{t}} & \text{for } x \notin \mathscr{D}, y \in \mathbb{R}.
\end{cases}
$$

Proof. By Proposition A.3, $H(x, t; y; \mu)$ is a solution to the heat equation (A.8) satisfying (A.9). Integrating the equation (A.8) with respect to time, and switching the differentiation, one has

$$
H(x, t; y; \mu) - \delta(x-y) = \partial_x\big(\mu(x)\partial_x \int_0^t H(x, \tau; y; \mu)\mathrm{d}\tau\big).
$$

Integrate against x to yield

$$
\begin{aligned}
\int_0^t \mu(x)\partial_x H(x, \tau; y; \mu)\mathrm{d}\tau &= \int_{-\infty}^x H(z, t; y; \mu)\mathrm{d}z - H(x-y) \\
&= \begin{cases} -\int_x^{+\infty} H(z, t; y; \mu)\mathrm{d}z, & \text{for } x > y, \\ \int_{-\infty}^x H(z, t; y; \mu)\mathrm{d}z, & \text{for } x < y. \end{cases}
\end{aligned}
$$

This then implies the following identities,

$$
\begin{aligned}
\int_0^t H_x(x, \tau; y; \mu)\mathrm{d}\tau &= \frac{1}{\mu(x)}\left(\int_{-\infty}^x H(z, t; y; \mu)\mathrm{d}z - H(x-y)\right) \\
&= \begin{cases} -\frac{1}{\mu(x)}\int_x^{+\infty} H(z, t; y)\mathrm{d}z, & \text{for } x > y, \\ \frac{1}{\mu(x)}\int_{-\infty}^x H(z, t; y)\mathrm{d}z, & \text{for } x < y, \end{cases} \\
\int_0^t H_{xy}(x, \tau; y; \mu)\mathrm{d}\tau &= \frac{1}{\mu(x)}\left(\int_{-\infty}^x H_y(z, t; y; \mu)\mathrm{d}z + \delta(x-y)\right), \\
\int_0^t H_{xx}(x, \tau; y; \mu)\mathrm{d}\tau &= \partial_x\left(\frac{1}{\mu(x)}\int_{-\infty}^x H(z, t; y; \mu)\mathrm{d}z\right) - \partial_x\left(\frac{H(x-y)}{\mu(x)}\right),
\end{aligned}
$$

$$\int_0^t H_{xxy}(x,\tau;y;\mu)\mathrm{d}\tau = \partial_x\left(\frac{1}{\mu(x)}\int_{-\infty}^x H_y(z,t;y;\mu)\mathrm{d}z\right) + \partial_x\left(\frac{\delta(x-y)}{\mu(x)}\right). \tag{A.14}$$

By Proposition A.3 and straightforward computations, one completes the proof. □

A.3. Time-dependent conductivity coefficient

Let $\rho(x,t)$ be a function satisfying (A.1). We are now in the position to consider the Green's function $H(x,t;y,t_0;\rho)$ to the following equation,

$$\begin{cases} \partial_t H = \partial_x\left(\rho(x,t)\partial_x H\right), \ t > t_0, \\ H(x,t_0;y,t_0;\rho) = \delta(x-y). \end{cases} \tag{A.15}$$

To establish the estimate for $H(x,t;y,t_0;\rho)$, we shall represent it by an integral equation using heat kernel with time-independent coefficient. We denote $H(x,t;y,t_0;\rho)$ by $\bar{H}(x,t;y,t_0)$ for the brevity of notation. In the sequential, we gather all the estimates of $H(x,t;y,t_0;\rho)$ which are needed in this paper.

Theorem A.1. *Let $\rho(x,t)$ be a function satisfying* (A.1). *Then for δ_* sufficiently small and $t_0 < t \ll 1$, the following estimates for heat kernel $H(x,y;y,t_0;\rho)$ hold*

$$|H(x,t;y,t_0;\rho)| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \tag{A.16}$$

$$|H_x(x,t;y,t_0;\rho)|, \left|H_y(x,t;y,t_0;\rho)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}, \tag{A.17}$$

$$|H_t(x,t;y,t_0;\rho)|, \left|H_{xy}(x,t;y,t_0;\rho)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}}, \tag{A.18}$$

$$\left|H_{ty}(x,t;y,t_0;\rho)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^2}. \tag{A.19}$$

Proof. • (**Estimate of** (A.16) **and** (A.17): H, H_x, H_y)

For fixed $T > 0$, set $\mu(x) \equiv \rho(x,T)$ and consider

$$\int_{t_0}^t H(x,t;z,\sigma;\mu)\left(\partial_\sigma \bar{H}(z,\sigma;y,t_0) - \partial_z\left(\rho(z,\sigma)\partial_z \bar{H}(z,\sigma;y,t_0)\right)\right)\mathrm{d}z\mathrm{d}\sigma = 0.$$

By the fact that $H(x,t;z,\sigma;\mu)$ and $\mu(z)\partial_z H(x,t;z,\sigma;\mu)$ are continuous in z, one performs integration by parts to get the representation of $\bar{H}(x,t;y,t_0)$,

$$\bar{H}(x,t;y,t_0) = H(x,t;y,t_0;\mu) + \int_{t_0}^t\int_{\mathbb{R}} H_z(x,t;z,\sigma)\big(\rho(z,T)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0)\mathrm{d}z\mathrm{d}\sigma. \tag{A.20}$$

Differentiate with respect to x to yield the integral equation of $\bar{H}_x$,

$$\bar{H}_x(x,t;y,t_0) = H_x(x,t;y,t_0;\mu) + \int_{t_0}^{t}\int_{\mathbb{R}} H_{zx}(x,t;z,\sigma)\big(\rho(z,T)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0)\mathrm{d}z\mathrm{d}\sigma. \tag{A.21}$$

Suppose $\delta_* \ll 1$ in (A.1), from Proposition A.3, there exists positive C_* such that

$$|H_x(x,t;y,\tau;\mu)| \leqq C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau}, \quad \big|H_{xy}(x,t;y,\tau;\mu)\big| \leqq C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{(t-\tau)^{3/2}} \quad \text{for } t>\tau,\ x,y\in\mathbb{R}.$$

One thus makes the following weaker ansatz,

$$\big|\bar{H}_x(x,t;y,\tau)\big| \leqq 2C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-\tau)}}}{t-\tau}.$$

Setting $T=t$, one then has

$$|\rho(z,t)-\rho(z,\sigma)| = \left|\int_{\sigma}^{t}\partial_\tau\rho(z,\tau)\mathrm{d}\tau\right| \leqq \sup_{\tau\in[\sigma,t]}\|\sqrt{\tau}\partial_\tau\rho(\cdot,\tau)\|_\infty\int_{\sigma}^{t}\frac{\mathrm{d}\tau}{\sqrt{\tau}} \leqq 2\delta_*\frac{t-\sigma}{\sqrt{t}}.$$

Substitute it and the ansatz into the integral in (A.21) to find

$$\begin{aligned}
&\left|\int_{t_0}^{t}\int_{\mathbb{R}} H_{zx}(x,t;z,\sigma)\big(\rho(z,T)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0)\mathrm{d}z\mathrm{d}\sigma\right| \\
&\leqq O(1)\delta_*C_*^2\int_{t_0}^{t}\int_{\mathbb{R}}\frac{e^{-\frac{(x-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^{3/2}}\frac{t-\sigma}{\sqrt{t}}\frac{e^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{\sigma-t_0}\mathrm{d}z\mathrm{d}\sigma \\
&\leqq O(1)\delta_*C_*^2\int_{t_0}^{t}\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t}\sqrt{t-t_0}\sqrt{\sigma-t_0}}\mathrm{d}\sigma \\
&\leqq O(1)\delta_*C_*^2\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t}}.
\end{aligned}$$

Then one gets

$$\big|\bar{H}_x(x,t;y,t_0)\big| \leqq \left(C_* + O(1)\delta_*C_*^2\frac{t-t_0}{\sqrt{t}}\right)\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}.$$

When $t - t_0 < 1$, and δ_* is sufficiently small, the ansatz is justified. This also follows that

$$\left|\bar{H}(x, t; y, t_0)\right| \leqq 2C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t - t_0}}.$$

If setting $T = t_0$ in (A.20) instead of $T = t$, taking derivative with respect to y, and following the similar argument, one can get the estimate for $\bar{H}_y(x, t; y, t_0)$ as well,

$$\left|\bar{H}_y(x, t; y, t_0)\right| \leqq 2C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t - t_0}.$$

- (**Estimate of the first term in** (A.18): H_t)
  Next we estimate $\bar{H}_t(x, t; y, t_0)$ by difference estimate. By (A.20), we consider

$$\begin{aligned}
&\bar{H}(x, t+h; y, t_0) - \bar{H}(x, t; y, t_0) \\
&\quad = H(x, t+h; y, t_0) - H(x, t; y, t_0) \\
&\qquad + \int_{t_0}^{t} \int_{\mathbb{R}} \left[H_z(x, t+h; z, \sigma) - H_z(x, t; z, \sigma)\right] \\
&\qquad \times \big(\rho(z, T) - \rho(z, \sigma)\big)\bar{H}_z(z, \sigma; y, t_0)\mathrm{d}z\mathrm{d}\sigma \\
&\qquad + \int_{t}^{t+h} \int_{\mathbb{R}} H_z(x, t+h; z, \sigma)\big(\rho(z, T) - \rho(z, \sigma)\big)\bar{H}_z(z, \sigma; y, t_0)\mathrm{d}z\mathrm{d}\sigma.
\end{aligned}$$

  Taking $T = t$ and using Proposition A.3, one can estimate each term on the right-hand-side and get

$$\left|\bar{H}(x, t+h; y, t_0) - \bar{H}(x, t; y, t_0)\right| \leqq O(1)|h| \frac{e^{-\frac{(x-y)^2}{C(t-t_0)}}}{(t - t_0)^{3/2}}.$$

  Therefore one concludes that there exists positive C_* such that

$$\left|\bar{H}_t(x, t; y, t_0; \rho)\right| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t - t_0)^{3/2}} \quad \text{for } 0 < t - t_0 < 1,\ x, y \in \mathbb{R}.$$

- (**Estimate of the second term in** (A.18): H_{xy})
  The representation (A.20) is insufficient to get the higher order estimate $\bar{H}_{xy}$ due to high singularity in time integral. We take advantage of the estimates for time-independent coefficient problem and interpolate the heat kernels of time-independent coefficients frozen at t and t_0 to approximate the heat kernel for time-dependent coefficient, and prove it is indeed a good approximation when $t - t_0 \ll 1$.
  Introduce a smooth non-increasing cutoff function $\chi(s)$ with the property

$$\chi(s) = \begin{cases} 1, & \text{for } 0 \leqq s < \frac{1}{3}, \\ 0, & \text{for } s > \frac{2}{3}. \end{cases}$$

For fixed t_0 and t, set

$$\begin{cases} \mu^t(x) \equiv \rho(x,t), \ \ \mu^{t_0}(x) \equiv \rho(x,t_0), \\ \tilde{H}(x,t;y,\sigma) \equiv \chi\big(\frac{\sigma-t_0}{t-t_0}\big) H(x,t;y,\sigma;\mu^{t_0}) + \Big(1-\chi\big(\frac{\sigma-t_0}{t-t_0}\big)\Big) H(x,t;y,\sigma;\mu^t) \text{ for } \sigma\in[t_0,t]. \end{cases} \tag{A.22}$$

Consider

$$\begin{aligned} \int_{t_0}^t \int_{\mathbb{R}} \tilde{H}(x,t;z,\sigma)\,(\partial_\sigma H(z,\sigma;y,t_0;\rho) \\ -\partial_z\big(\rho(z,\sigma)\partial_z H(z,\sigma;y,t_0;\rho)\big)\big)\,\mathrm{d}z\mathrm{d}\sigma = 0. \end{aligned}$$

Using integration by parts, and in view of the facts that $H(x,t;z,\sigma;\mu^t)$, $\mu^t(z)H(x,t;z,\sigma;\mu^t)$, $H(x,t;z,\sigma;\mu^{t_0})$ and $\mu^{t_0}(z)H_z(x,t;z,\sigma;\mu^{t_0})$ are all continuous in z, one obtains the representation of $H(x,t;y,t_0;\rho)$ for $t>t_0$,

$$\begin{aligned} H(x,t;y,t_0;\rho) = \tilde{H}(x,t;y,t_0) &+ \int_{t_0}^t \int_{\mathbb{R}} \frac{\chi'\big(\frac{\sigma-t_0}{t-t_0}\big)}{t-t_0} \\ &\times \big(H(x,t;z,\sigma;\mu^{t_0}) - H(x,t;z,\sigma;\mu^t)\big) H(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^t \int_{\mathbb{R}} \chi\big(\frac{\sigma-t_0}{t-t_0}\big) H_z(x,t;z,\sigma;\mu^{t_0}) \\ &\times \big(\rho(z,\sigma)-\mu^{t_0}(z)\big) H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^t \int_{\mathbb{R}} \Big(1-\chi\big(\frac{\sigma-t_0}{t-t_0}\big)\Big) \\ &\times H_z(x,t;z,\sigma;\mu^t)\big(\rho(z,\sigma)-\mu^t(z)\big) H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma. \end{aligned} \tag{A.23}$$

Differentiate (A.23) with respect to x and y to yield

$$\begin{aligned} H_{xy}(x,t;y,t_0;\rho) = \tilde{H}_{xy}(x,t;y,t_0) &+ \int_{t_0}^t \int_{\mathbb{R}} \frac{\chi'\big(\frac{\sigma-t_0}{t-t_0}\big)}{t-t_0} \big(H_x(x,t;z,\sigma;\mu^{t_0}) \\ &- H_x(x,t;z,\sigma;\mu^t)\big) H_y(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^t \int_{\mathbb{R}} \chi\big(\frac{\sigma-t_0}{t-t_0}\big) H_{zx}(x,t;z,\sigma;\mu^{t_0}) \\ &\times \big(\rho(z,\sigma)-\mu^{t_0}(z)\big) H_{zy}(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma \\ &+ \int_{t_0}^t \int_{\mathbb{R}} \Big(1-\chi\big(\frac{\sigma-t_0}{t-t_0}\big)\Big) H_{zx}(x,t;z,\sigma;\mu^t) \\ &\times \big(\rho(z,\sigma)-\mu^t(z)\big) H_{zy}(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma. \end{aligned} \tag{A.24}$$

This gives rise to an integral equation for $H_{xy}(\cdot,\cdot;y,t_0;\rho)$. In the σ integral, there are two possible singularities, that is, when $\sigma=t_0$ and $\sigma=t$. The advantage of this representation is that in each integral on the right-hand side of (A.24), only one singularity shows up thanks to cutoff function, and it can be controlled by either $\rho(\cdot,\sigma)-\mu^{t_0}$ or $\rho(\cdot,\sigma)-\mu^t$. By Propositions A.3 and A.4, following the similar arguments as in the estimate of $H_x(x,t;y,t_0;\rho)$, i.e., making weaker ansatz and proving a stronger one, we can conclude the estimate

$$\big|H_{xy}(x,t;y,t_0;\rho)\big| \leqq 2C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}}.$$

- (**Estimate of** (A.19): H_{ty}) In a similar way, one differentiates (A.23) in y to find the integral representation of $H_y(x,t;\, y,t_0;\, \rho)$. Taking the difference between $H_y(x,t+h;\, y,t_0;\, \rho)$ and $H_y(x,t;\, y,t_0;\, \rho)$, and by lengthy computations, we obtain that

$$\begin{aligned}&\left|H_y(x,t+h;\, y,t_0;\, \rho) - H_y(x,t;\, y,t_0;\, \rho)\right|\\ &\leqq O(1)|h|\frac{e^{\frac{(x-y)^2}{C(t-t_0)}}}{(t-t_0)^2} \qquad \text{when } |h| < (t-t_0)/10.\end{aligned}$$

Therefore we arrive at the conclusion that there exists positive C_* such that

$$\left|H_{ty}(x,t;\, y,t_0;\, \rho)\right| \leqq C_* \frac{e^{\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^2}.$$

□

Theorem A.2. (Hölder continuity in time) *When* $t_0 < s < t \ll 1$, *one has Hölder continuity in time estimates*

$$\|H_x(\cdot,t;\, y,t_0;\, \rho) - H_x(\cdot,s;\, y,t_0;\, \rho)\|_\infty \leqq C_* \frac{(t-s)\,|\log(t-s)|}{(s-t_0)(t-t_0)}, \tag{A.25}$$

$$\|H_x(\cdot,t;\, y,t_0;\, \rho) - H_x(\cdot,s;\, y,t_0;\, \rho)\|_1 \leqq C_* \frac{(t-s)\,|\log(t-s)|}{(t-t_0)\sqrt{s-t_0}}, \tag{A.26}$$

$$\left\|H_{xy}(\cdot,t;\, y,t_0;\, \rho) - H_{xy}(\cdot,s;\, y,t_0;\, \rho)\right\|_\infty \leqq C_* \frac{(t-s)\,|\log(t-s)|}{(s-t_0)^{3/2}(t-t_0)}, \tag{A.27}$$

$$\left\|H_{xy}(\cdot,t;\, y,t_0;\, \rho) - H_{xy}(\cdot,s;\, y,t_0;\, \rho)\right\|_1 \leqq C_* \frac{(t-s)\,|\log(t-s)|}{(s-t_0)(t-t_0)}. \tag{A.28}$$

Proof. • (**Hölder continuity in time of** $H_x(x,t;\, y,t_0;\, \rho)$)
Assume $t > s > t_0$ and $t - s < 1$, by (A.21),

$$\begin{aligned}&H_x(x,t;\, y,t_0;\, \rho) = H_x(x,t;\, y,t_0;\, \mu^T)\\ &\quad + \int_{t_0}^t \int_{\mathbb{R}} H_{xz}(x,t;\, z,\sigma;\, \mu^T)\big(\rho(z,T) - \rho(z,\sigma)\big) H_z(z,\sigma;\, y,t_0;\, \rho)\mathrm{d}z\mathrm{d}\sigma,\end{aligned}$$

where $\mu^T = \rho(\cdot,T)$. Set $T = s$ and denote

$$\bar{H}(x,t;\, y,t_0) \equiv H(x,t;\, y,t_0;\, \rho), \quad H(x,t;\, y,t_0) \equiv H(x,t;\, y,t_0;\, \mu^s)$$

for simplicity of notations.
Replacing t by s in the above representation and taking the difference, one gets

$$\begin{aligned}&\bar{H}_x(x,t;\, y,t_0) - \bar{H}_x(x,s;\, y,t_0)\\ &\quad = H_x(x,t;\, y,t_0) - H_x(x,s;\, y,t_0)\\ &\quad\quad + \int_{t_0}^s \int_{\mathbb{R}} \big(H_{zx}(x,t;\, z,\sigma) - H_{zx}(x,s;\, z,\sigma)\big)\\ &\quad\quad \times \big(\rho(z,s) - \rho(z,\sigma)\big)\bar{H}_z(z,\sigma;\, y,t_0)\mathrm{d}z\mathrm{d}\sigma\end{aligned}$$

$$+\int_s^t\int_{\mathbb{R}} H_{zx}(x,t;z,\sigma)\big(\rho(z,s)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0)\mathrm{d}z\mathrm{d}\sigma$$
$$\equiv \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3. \tag{A.29}$$

Furthermore, one rewrites $\mathcal{I}_3$ as

$$\begin{aligned}\mathcal{I}_3 &= \int_s^t\int_{\mathbb{R}} H_{xz}(x,t;z,\sigma)\big(\rho(z,s)-\rho(z,\sigma)\big)\bar{H}_z(z,\sigma;y,t_0)\mathrm{d}z\mathrm{d}\sigma\\ &= -\int_s^t\int_{\mathbb{R}} H_{zx}(x,t;z,\sigma)\big(\rho(z,\sigma)-\rho(z,t)\big)\bar{H}_z(z,\sigma;y,t_0)\mathrm{d}z\mathrm{d}\sigma\\ &\quad -\int_s^t\int_{\mathbb{R}} H_{zx}(x,t;z,\sigma)\big(\rho(z,t)-\rho(z,s)\big)\bar{H}_z(z,t;y,t_0)\mathrm{d}z\mathrm{d}\sigma\\ &\quad -\int_s^t\int_{\mathbb{R}} H_{zx}(x,t;z,\sigma)\big(\rho(z,t)-\rho(z,s)\big)\\ &\quad \times\big(\bar{H}_z(z,\sigma;y,t_0)-\bar{H}_z(z,t;y,t_0)\big)\mathrm{d}z\mathrm{d}\sigma\\ &\equiv \mathcal{I}_{31}+\mathcal{I}_{32}+\mathcal{I}_{33}.\end{aligned} \tag{A.30}$$

We first consider L^∞ estimate. By Proposition A.3,

$$\begin{aligned}|\mathcal{I}_1| &= \left|\int_s^t H_{x\sigma}(x,\sigma;y,t_0)\mathrm{d}\sigma\right|\\ &\leqq O(1)\int_s^t \frac{e^{-\frac{(x-y)^2}{C_*(\sigma-t_0)}}}{(\sigma-t_0)^2}\mathrm{d}\sigma \leqq O(1)\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\frac{t-s}{s-t_0}.\end{aligned}$$

For $\mathcal{I}_2$, one has

$$\begin{aligned}|\mathcal{I}_2| &= \left|\int_{t_0}^s\int_{\mathbb{R}}\int_s^t\int_\sigma^s H_{\tau xz}(x,\tau;z,\sigma)\rho_\chi(z,\chi)\bar{H}_z(z,\sigma;y,t_0)d\chi\mathrm{d}\tau\mathrm{d}z\mathrm{d}\sigma\right|\\ &\leqq O(1)C_*^2\int_{t_0}^s\int_{\mathbb{R}}\int_s^t \frac{e^{-\frac{(x-z)^2}{C_*(\tau-\sigma)}}}{(\tau-\sigma)^{5/2}}\delta_*\frac{s-\sigma}{\sqrt{s}}\frac{e^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{\sigma-t_0}\mathrm{d}\tau\mathrm{d}z\mathrm{d}\sigma\\ &\leqq O(1)\delta_*C_*^2\int_{t_0}^s\int_s^t \frac{e^{-\frac{(x-y)^2}{C_*(\tau-t_0)}}}{\sqrt{s}}\frac{1}{\sqrt{\tau-t_0}}\frac{s-\sigma}{(\tau-\sigma)^2\sqrt{\sigma-t_0}}\mathrm{d}\tau\mathrm{d}\sigma.\end{aligned}$$

Carrying out the σ integral,

$$\begin{aligned}&\int_{t_0}^s \frac{s-\sigma}{(\tau-\sigma)^2\sqrt{\sigma-t_0}}\mathrm{d}\sigma\\ &\quad\leqq O(1)\frac{s-t_0}{(\tau-t_0)^2}\int_{t_0}^{(t_0+s)/2}\frac{\mathrm{d}\sigma}{\sqrt{\sigma-t_0}}\\ &\qquad + O(1)\frac{1}{\sqrt{s-t_0}}\int_{(t_0+s)/2}^s \frac{s-\sigma}{(\tau-s+s-\sigma)^2}\mathrm{d}\sigma\\ &\quad\leqq O(1)\left(\frac{(s-t_0)^{3/2}}{(\tau-t_0)^2}+\frac{\log\left(1+\frac{s-t_0}{2(\tau-s)}\right)}{\sqrt{s-t_0}}-\frac{\sqrt{s-t_0}}{\tau-s+\tau-t_0}\right).\end{aligned}$$

Now one needs to calculate

$$\int_s^t \left(\frac{(s-t_0)^{3/2}}{(\tau-t_0)^{5/2}} + \frac{\log\left(1+\frac{s-t_0}{2(\tau-s)}\right)}{\sqrt{s-t_0}\sqrt{\tau-t_0}} - \frac{\sqrt{s-t_0}}{(\tau-s+\tau-t_0)\sqrt{\tau-t_0}} \right) \mathrm{d}\tau.$$

Straightforward computations show

$$\int_s^t \frac{(s-t_0)^{3/2}}{(\tau-t_0)^{5/2}}\mathrm{d}\tau + \int_s^t \frac{\sqrt{s-t_0}}{(\tau-s+\tau-t_0)\sqrt{\tau-t_0}}\mathrm{d}\tau \leqq O(1)\frac{t-s}{t-t_0}.$$

To calculate

$$\int_s^t \frac{\log\left(1+\frac{s-t_0}{2(\tau-s)}\right)}{\sqrt{s-t_0}\sqrt{\tau-t_0}}\mathrm{d}\tau,$$

one consider two cases: (i). $t-s<\frac{s-t_0}{10}$ and (ii). $t-s>\frac{s-t_0}{10}$. For case (i),

$$\begin{aligned}\int_s^t \frac{\log\left(1+\frac{s-t_0}{2(\tau-s)}\right)}{\sqrt{s-t_0}\sqrt{\tau-t_0}}\mathrm{d}\tau &\leqq O(1)\frac{1}{s-t_0}\int_s^t \log\left(\frac{s-t_0}{\tau-s}\right)\mathrm{d}\tau \\ &\leqq O(1)\frac{t-s}{s-t_0}\log\left(\frac{s-t_0}{t-s}\right) \leqq O(1)\frac{(t-s)\,|\log(t-s)|}{t-t_0}.\end{aligned}$$

For case (ii), one splits the integral to two parts,

$$\begin{aligned}&\int_s^t \frac{\log\left(1+\frac{s-t_0}{2(\tau-s)}\right)}{\sqrt{s-t_0}\sqrt{\tau-t_0}}\mathrm{d}\tau \\ &= \frac{1}{\sqrt{s-t_0}}\int_0^{t-s}\frac{\log\left(1+\frac{s-t_0}{2\sigma}\right)}{\sqrt{\sigma+s-t_0}}\mathrm{d}\sigma \\ &\lesssim \int_0^{(s-t_0)/10}\frac{\log\left(1+\frac{s-t_0}{2\sigma}\right)}{s-t_0}\mathrm{d}\sigma + \int_{(s-t_0)/10}^{t-s}\frac{1}{\sqrt{s-t_0}}\frac{1}{\sqrt{\sigma}}\frac{s-t_0}{\sigma}\mathrm{d}\sigma \\ &\lesssim 1+\left(1-\frac{s-t_0}{10(t-s)}\right).\end{aligned}$$

Combining the above estimates then follows that

$$|\mathcal{I}_2| \leqq O(1)\delta_* C_*^2 \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{s}}\frac{(t-s)\,|\log(t-s)|}{t-t_0}.$$

Now we estimate the first two terms in $\mathcal{I}_3$.

$$\begin{aligned}|\mathcal{I}_{31}| &\leqq O(1)\delta_* C_*^2 \int_s^t\int_{\mathbb{R}} \frac{e^{-\frac{(x-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^{3/2}}\frac{t-\sigma}{\sqrt{t}}\frac{e^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{\sigma-t_0}\mathrm{d}z\mathrm{d}\sigma \\ &\leqq O(1)\delta_* C_*^2 \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\frac{t-s}{\sqrt{t}},\end{aligned}$$

and

$$
\begin{aligned}
|\mathcal{I}_{32}| &= \left| \int_{\mathbb{R}\backslash\mathscr{D}} \left(\int_s^t H_{xz}(x,t;z,\sigma)\mathrm{d}\sigma \right) \big(\rho(z,t)-\rho(z,s)\big) \bar{H}_z(z,t;y,t_0)\mathrm{d}z \right| \\
&\leqq \left| \int_{\mathbb{R}\backslash\mathscr{D}} \frac{\delta(x-z)}{\mu(x)} \big(\rho(z,t)-\rho(z,z)\big) \bar{H}_z(z,t;y,t_0)\mathrm{d}z \right| \\
&\quad + O(1)\delta_* C_*^2 \int_{\mathbb{R}\backslash\mathscr{D}} \frac{e^{-\frac{(x-z)^2}{C_*(t-s)}}}{\sqrt{t-s}} \frac{t-s}{\sqrt{t}} \frac{e^{-\frac{(z-y)^2}{C_*(t-t_0)}}}{t-t_0}\mathrm{d}z \\
&\leqq O(1)\delta_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0} \frac{t-s}{\sqrt{t}} + O(1)\delta_* C_*^2 \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0+t-s)}}}{t-t_0} \frac{t-s}{\sqrt{t}} \\
&\leqq O(1)\delta_* \frac{e^{-\frac{(x-y)^2}{2C_*(t-t_0)}}}{t-t_0} \frac{t-s}{\sqrt{t}}.
\end{aligned}
$$

Combine the estimates of $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_{31}$ and $\mathcal{I}_{32}$ to yield an integral equation

$$
\begin{aligned}
&\bar{H}_x(x,t;y,t_0) - \bar{H}_x(x,s;y,t_0) \\
&\quad = \Phi(x,t;y,t_0) - \int_s^t \int_{\mathbb{R}} H_{xz}(x,t;z,\sigma)\big(\rho(z,t)-\rho(z,s)\big) \\
&\quad \times \big(\bar{H}_z(z,\sigma;y,t_0) - \bar{H}_z(z,t;y,t_0)\big)\mathrm{d}z\mathrm{d}\sigma.
\end{aligned} \tag{A.31}
$$

Here

$$
|\Phi(x,t;y,t_0)| \leqq O(1) e^{-\frac{(x-y)^2}{2C_*(t-t_0)}} \frac{(t-s)\,|\log(t-s)|}{(t-t_0)(s-t_0)}.
$$

Thus there exists positive C_1 such that

$$
\|\Phi(\cdot,t;y,t_0)\|_\infty \leqq C_1 \frac{(t-s)\,|\log(t-s)|}{(t-t_0)(s-t_0)}.
$$

We make the following **ansatz**:

$$
\left\| \bar{H}_x(\cdot,t;y,t_0) - \bar{H}_x(\cdot,s;y,t_0) \right\|_\infty \leqq 2C_1 \frac{(t-s)\,|\log(t-s)|}{(t-t_0)(s-t_0)}.
$$

Substituting the ansatz into $\mathcal{I}_{33}$, one gets

$$
\begin{aligned}
|\mathcal{I}_{33}| &\leqq O(1) \int_s^t \int_{\mathbb{R}} \frac{e^{-\frac{(x-z)^2}{C_*(t-\sigma)}}}{(t-\sigma)^{3/2}} \delta_* \frac{t-s}{\sqrt{t}} \frac{(t-\sigma)\,|\log(t-\sigma)|}{(t-t_0)(\sigma-t_0)}\mathrm{d}z\mathrm{d}\sigma \\
&\leqq O(1)\delta_* \int_s^t \frac{t-s}{\sqrt{t}(t-t_0)} \frac{|\log(t-\sigma)|}{\sigma-t_0}\mathrm{d}\sigma \\
&\leqq O(1)\delta_* \frac{t-s}{t-t_0} \frac{1}{\sqrt{t}} \left(\int_s^{(t+s)/2} \frac{|\log(t-s)|}{\sigma-t_0}\mathrm{d}\sigma + \int_{(t+s)/2}^t \frac{|\log(t-\sigma)|}{t-t_0}\mathrm{d}\sigma \right) \\
&\leqq O(1)\delta_* \frac{t-s}{t-t_0} \frac{|\log(t-s)|}{\sqrt{t}} \left[\log\left(1+\frac{t-s}{s-t_0}\right) + \frac{t-s}{t-t_0} \right]
\end{aligned}
$$

$$\leqq O(1)\delta_* \frac{(t-s)\,|\log(t-s)|}{(t-t_0)(s-t_0)} \frac{s-t_0}{\sqrt{t}} \left(\frac{t-s}{s-t_0} + \frac{t-s}{t-t_0} \right)$$
$$\leqq O(1)\delta_* \frac{(t-s)\,|\log(t-s)|}{(t-t_0)(s-t_0)}. \tag{A.32}$$

Therefore the ansatz is justified provided δ_* is sufficiently small.
The estimates for L^1_x norm is even simpler. Actually, integrating the magnitude of (A.29) with respect to x and using (A.30), we obtain the representation of

$$\int_{\mathbb{R}} \left| \bar{H}_x(x,t;y,t_0) - \bar{H}_x(x,s;y,t_0) \right| \mathrm{d}x.$$

By calculating known terms, making suitable ansatz and justifying it provided δ_* sufficiently, we conclude

$$\left\| \bar{H}_x(\cdot,t;y,t_0) - \bar{H}_x(\cdot,s;y,t_0) \right\|_1 \leqq O(1) \frac{(t-s)\,|\log(t-s)|}{(t-t_0)\sqrt{s-t_0}}.$$

- (**Hölder continuity in time of** $H_{xy}(x,t;y,t_0;\rho)$)
  In this case, due to the high singularity, (A.21) is not appropriate to use, one has to resort to expression (A.24). By applying similar arguments as for Hölder estimates of H_x, we can conclude the proof. □

Theorem A.3. (Estimates involving time integral for time-dependent coefficient) *Let $\rho(x,t)$ be a function satisfying* (A.1). *Then for δ_* sufficiently small and $t_0 < t \ll 1$. The following estimates for heat kernel $H(x,y;y,t_0;\rho)$ hold*

$$\left| \int_{t_0}^{t} H_x(x,\tau;y,t_0;\rho)d\tau \right| \leqq C_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}}, \tag{A.33}$$

$$\left| \int_{t_0}^{t} H_x(x,t;y,s;\rho)\mathrm{d}s \right| \leqq C_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}}, \tag{A.34}$$

$$\Bigg| \int_{t_0}^{t} H_{xy}(x,\tau;y,t_0;\rho)d\tau - \frac{\delta(x-y)}{\rho(x,t_0)}$$
$$- \int_{t_0}^{t} \frac{\rho(x,t_0)-\rho(x,\tau)}{\rho(x,t_0)} H_{xy}(x,\tau;y,t_0;\rho)d\tau \Bigg| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \tag{A.35}$$

$$\Bigg| \int_{t_0}^{t} H_{xy}(x,t;y,s;\rho)\mathrm{d}s + \frac{\delta(x-y)}{\rho(y,t)}$$
$$- \int_{t_0}^{t} \frac{\rho(y,t)-\rho(y,s)}{\rho(y,t)} H_{xy}(x,t;y,s;\rho)\mathrm{d}s \Bigg| \leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}, \tag{A.36}$$

$$\int_{t_0}^{t} H_{xx}(x,\tau;y,t_0;\rho)d\tau = -\frac{\delta(x-y)}{\rho(x,t_0)} - \frac{1}{\rho(x,t_0)}\partial_x$$
$$\times \left[\int_{t_0}^{t} (\rho(x,\tau)-\rho(x,t_0))\, H_x(x,\tau;y,t_0)d\tau \right]$$

$$+ \; O(1)\left(|\partial_x\rho(x,t_0)|e^{-\frac{(x-y)^2}{C_*(t-t_0)}} + \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\right), \; for\; x \notin \mathscr{D}, \tag{A.37}$$

$$\begin{aligned}\int_{t_0}^{t} \bar{H}_{xxy}(x,\tau;y,t_0)d\tau &= \frac{1}{\rho(x,t_0)}\\ &\times\left[\delta'(x-y) - \int_{t_0}^{t}\partial_x\Big[\big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_{xy}(x,\tau;y,t_0)\Big]d\tau\right]\\ &- \frac{\partial_x\rho(x,t_0)}{\rho^2(x,t_0)}\left[\delta(x-y) - \int_{t_0}^{t}\big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_{xy}(x,\tau;y,t_0)d\tau\right]\\ &+ O(1)\left(|\partial_x\rho(x,t_0)|\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} + \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\right) \; for\; x\notin\mathscr{D},\end{aligned} \tag{A.38}$$

$$\int_{t_0}^{t} H_t(x,t;y,s;\rho)\mathrm{d}s = H(x,t-t_0;y;\mu^t) - \delta(x-y) + O(1)\delta_* e^{-\frac{(x-y)^2}{C_*(t-t_0)}}. \tag{A.39}$$

Proof. • (**Estimates of** (A.33)–(A.38))
We begin with

$$\partial_\tau \bar{H}(x,\tau;y,t_0) = \partial_x\left[\rho(x,\tau)\partial_x\bar{H}(x,\tau;y,t_0)\right].$$

Integrate with respect to τ from t_0 to t to yield

$$\bar{H}(x,t;y,t_0) - \delta(x-y) = \partial_x\left[\int_{t_0}^{t}\rho(x,\tau)\partial_x\bar{H}(x,\tau;y,t_0)\mathrm{d}\tau\right].$$

Integrate against x to get

$$\int_{-\infty}^{x}\bar{H}(z,t;y,t_0)\mathrm{d}z - H(x-y) = \int_{t_0}^{t}\rho(x,\tau)\bar{H}_x(x,\tau;y,t_0)\mathrm{d}\tau.$$

Using of this, one can write

$$\begin{aligned}&\int_{t_0}^{t}\bar{H}_x(x,\tau;y,t_0)\mathrm{d}\tau\\ &\quad= \int_{t_0}^{t}\left(\frac{\rho(x,\tau)}{\rho(x,t_0)} - \frac{\rho(x,\tau)-\rho(x,t_0)}{\rho(x,t_0)}\right)\bar{H}_x(x,\tau;y,t_0)\mathrm{d}\tau\\ &\quad= \frac{1}{\rho(x,t_0)}\left[\int_{-\infty}^{x}\bar{H}(z,t;y,t_0)\mathrm{d}z - H(x-y)\right.\\ &\qquad\left. - \int_{t_0}^{t}\big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_x(x,\tau;y,t_0)\mathrm{d}\tau\right].\end{aligned} \tag{A.40}$$

This immediately follows that

$$\begin{aligned}
&\int_{t_0}^{t} \bar{H}_{xx}(x,\tau;y,t_0)\mathrm{d}\tau\\
&\quad = \partial_x\left(\frac{1}{\rho(x,t_0)}\right)\left[\int_{-\infty}^{x} \bar{H}(z,t;y,t_0)\mathrm{d}z - H(x-y)\right.\\
&\qquad \left. - \int_{t_0}^{t} \big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_x(x,\tau;y,t_0)\mathrm{d}\tau\right]\\
&\qquad + \frac{1}{\rho(x,t_0)}\left[\bar{H}(x,t;y,t_0) - \delta(x-y) - \int_{t_0}^{t} \partial_x\right.\\
&\qquad \times, \left.\Big[\big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_x(x,\tau;y,t_0)\Big]\mathrm{d}\tau\right],
\end{aligned}$$

which in turn gives

$$\begin{aligned}
&\int_{t_0}^{t} \bar{H}_{xxy}(x,\tau;y,t_0)\mathrm{d}\tau\\
&\quad = \partial_x\left(\frac{1}{\rho(x,t_0)}\right)\left[\int_{-\infty}^{x} \bar{H}_y(z,t;y,t_0)\mathrm{d}z + \delta(x-y)\right.\\
&\quad \left. - \int_{t_0}^{t} \big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_{xy}(x,\tau;y,t_0)\mathrm{d}\tau\right]\\
&\qquad + \frac{1}{\rho(x,t_0)}\Big[\bar{H}_y(x,t;y,t_0) + \delta'(x-y)\\
&\qquad - \int_{t_0}^{t} \partial_x\Big[\big(\rho(x,\tau)-\rho(x,t_0)\big)\bar{H}_{xy}(x,\tau;y,t_0)\Big]\mathrm{d}\tau\Big].
\end{aligned}$$

From above expressions, and the following identities (which hold for $x \notin \mathscr{D}$),

$$\begin{aligned}
\bar{H}_{xx}(x,\tau;y,t_0) &= \frac{1}{\rho(x,\tau)}\bar{H}_\tau(x,\tau;y,t_0) - \frac{\partial_x\rho(x,\tau)}{\rho(x,\tau)}\bar{H}_x(x,\tau;y,t_0),\\
\bar{H}_{xxy}(x,\tau;y,t_0) &= \frac{1}{\rho(x,\tau)}\bar{H}_{\tau y}(x,\tau;y,t_0) - \frac{\partial_x\rho(x,\tau)}{\rho(x,\tau)}\bar{H}_{xy}(x,\tau;y,t_0),
\end{aligned}$$

We conclude the proof.
The estimate for (A.39) is of more technical difficulties. We first write down the integral equation for $\bar{H}(x,t;y,s)$ in terms of $H(x,t;y,s;\mu^t)$, then represent the time derivative in t, perform the time integral in s, and use comparison estimates in Theorem (A.4) to conclude the proof. □

Theorem A.4. (Comparison Estimates for time-dependent coefficient) *Let $\rho^a(x,t)$ and $\rho^b(x,t)$ be two functions satisfying* (A.1). *Suppose $t_0 < t \ll 1$. Then the following comparison estimates hold:*

$$\begin{aligned}
&\left|H(x,t;y,t_0;\rho^b)-H(x,t;y,t_0;\rho^a)\right|\\
&\quad\leqq C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_\infty, \qquad\text{(A.41)}\\
&\left|H_x(x,t;y,t_0;\rho^a)-H_x(x,t;y,t_0;\rho^b)\right|,\\
&\quad\left|H_y(x,t;y,t_0;\rho^a)-H_y(x,t;y,t_0;\rho^b)\right|\\
&\quad\leqq C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\Big[|\log(t-t_0)|\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_{BV}\\
&\qquad+\sqrt{t-t_0}\Big(\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_1+|\log t|\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\right|\!\right|\!\right|_\infty\Big)\Big], \qquad\text{(A.42)}\\
&\left|H_{xy}(x,t;y,t_0;\rho^a)-H_{xy}(x,t;y,t_0;\rho^b)\right|,\\
&\quad\left|H_t(x,t;y,t_0;\rho^a)-H_t(x,t;y,t_0;\rho^b)\right|\\
&\quad\leqq C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{(t-t_0)^{3/2}}\Big[|\log(t-t_0)|\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_\infty+\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_{BV}\\
&\qquad+\sqrt{t-t_0}\Big(\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_1+|\log t|\left|\!\left|\!\left|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\right|\!\right|\!\right|_\infty\Big)\Big]. \qquad\text{(A.43)}
\end{aligned}$$

Proof. • (**Comparison estimates of** H) Consider

$$\begin{aligned}
0=&\int_{t_0}^t\int_{\mathbb{R}}H(x,t;z,\sigma;\rho^a)\\
&\times\Big[\partial_\sigma H(z,\sigma;y,t_0;\rho^b)-\partial_z\big(\rho^b(z,\sigma)\partial_z H(z,\sigma;y,t_0;\rho^b)\big)\Big]\mathrm{d}z\mathrm{d}\sigma.
\end{aligned}$$

Use integration by parts to find

$$\begin{aligned}
&H(x,t;y,t_0;\rho^b)-H(x,t;y,t_0;\rho^a)=-\int_{t_0}^t\int_{\mathbb{R}}H_z(x,t;z,\sigma;\rho^a)\\
&\times\left[\rho^b(z,\sigma)-\rho^a(z,\sigma)\right]H_z(z,\sigma;y,t_0;\rho^b)\mathrm{d}z\mathrm{d}\sigma.
\end{aligned}$$

It follows from this that

$$\begin{aligned}
&\left|H(x,t;y,t_0;\rho^b)-H(x,t;y,t_0;\rho^a)\right|\\
&\quad\leqq O(1)\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_\infty\int_{t_0}^t\int_{\mathbb{R}}\frac{e^{-\frac{(x-z)^2}{C_*(t-\sigma)}}}{t-\sigma}\frac{e^{-\frac{(z-y)^2}{C_*(\sigma-t_0)}}}{\sigma-t_0}\mathrm{d}z\mathrm{d}\sigma\\
&\quad\leqq O(1)\left|\!\left|\!\left|\rho^a-\rho^b\right|\!\right|\!\right|_\infty\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}.
\end{aligned}$$

- (**Comparison estimates of** H_x) By (A.21) and setting $T = t$,

$$H_x(x,t;y,t_0;\rho) = H_x(x,t;y,t_0;\mu^t) + \int_{t_0}^{t}\int_{\mathbb{R}} H_{xz}(x,t;z,\sigma;\mu^t)\,[\rho(z,t)-\rho(z,\sigma)]\,H_z(z,\sigma;y,t_0;\rho)\mathrm{d}z\mathrm{d}\sigma,$$

where $\mu^t(z) \equiv \rho(z,t)$. Substituting $\rho = \rho^a, \rho^b$ and taking difference, one has

$$\begin{aligned}
&H_x(x,t;y,t_0;\rho^a) - H_x(x,t;y,t_0;\rho^b)\\
&= H_x(x,t;y,t_0;\mu_a^t) - H_x(x,t;y,t_0;\mu_b^t)\\
&+ \int_{t_0}^{t}\int_{\mathbb{R}} \left[H_{xz}(x,t;z,\sigma;\mu_a^t) - H_{xz}(x,t;z,\sigma;\mu_b^t)\right]\\
&\times \left[\rho^a(z,t)-\rho^a(z,\sigma)\right] H_z(z,\sigma;y,t_0;\rho^a)\mathrm{d}z\mathrm{d}\sigma\\
&+ \int_{t_0}^{t}\int_{\mathbb{R}} H_{xz}(x,t;z,\sigma;\mu_b^t)\\
&\times \left[\left(\rho^a(z,t)-\rho^a(z,\sigma)\right) - \left(\rho^b(z,t)-\rho^b(z,\sigma)\right)\right] H_z(z,\sigma;y,t_0;\rho^a)\mathrm{d}z\mathrm{d}\sigma\\
&+ \int_{t_0}^{t}\int_{\mathbb{R}} H_{xz}(x,t;z,\sigma;\mu_b^t)\\
&\times \left[\rho^b(z,t)-\rho^b(z,\sigma)\right]\left[H_z(z,\sigma;y,t_0;\rho^a) - H_z(z,\sigma;y,t_0;\rho^b)\right]\mathrm{d}z\mathrm{d}\sigma\\
&\equiv T_1 + T_2 + T_3 + T_4.
\end{aligned}$$

Calculating out the terms T_1, T_2 and T_3, one has

$$\begin{aligned}
&|T_1| + |T_2| + |T_3|\\
&\leqq C_* \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\left[|\log(t-t_0)|\left\|\rho^a-\rho^b\right\|_\infty + \left\|\rho^a-\rho^b\right\|_{BV}\right.\\
&\left.+\sqrt{t-t_0}\left(\left\|\rho^a-\rho^b\right\|_1 + |\log t|\left\|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\left[\rho^a-\rho^b\right]\right\|_\infty\right)\right].
\end{aligned}$$

Note that to estimate T_4, one needs to bound

$$H_z(z,\sigma;y,t_0;\rho^a) - H_z(z,\sigma;y,t_0;\rho^b),$$

which is exactly the estimates we are seeking for. In other words, we have an integral equation for function

$$H_x(x,t;y,t_0;\rho^a) - H_x(x,t;y,t_0;\rho^b).$$

Making ansatz and substituting into T_4, one can justify the ansatz provided $O(1)\delta_*\sqrt{t-t_0} < C_*/2$.

- (**Comparison estimates of** H_y)
Next, we shall estimate

$$H_y(x,t;y,t_0;\rho^a) - H_y(x,t;y,t_0;\rho^b).$$

From

$$0 = \int_{t_0}^{t}\int_{\mathbb{R}}\Big[\partial_\sigma H(x,t;z,\sigma;\rho) + \partial_z\big[\rho(z,\sigma)\partial_z H(x,t;z,\sigma;\rho)\big]\Big] \times H(z,\sigma;y,t_0;\mu^{t_0})\mathrm{d}z\mathrm{d}\sigma,$$

one can find another representation of $H(x,t;y,t_0;\rho)$ in terms of the time-independent heat kernel,

$$H(x,t;y,t_0;\rho) = H(x,t;y,t_0;\mu^{t_0}) - \int_{t_0}^{t}\int_{\mathbb{R}} H_z(x,t;z,\sigma;\rho) \times \big[\rho(z,\sigma)-\rho(z,t_0)\big]H_z(z,\sigma;y,t_0;\mu^{t_0})\mathrm{d}z\mathrm{d}\sigma.$$

Differentiate it with respect to y to obtain

$$H_y(x,t;y,t_0;\rho) = H_y(x,t;y,t_0;\mu^{t_0}) - \int_{t_0}^{t}\int_{\mathbb{R}} H_z(x,t;z,\sigma;\rho) \times \big[\rho(z,\sigma)-\rho(z,t_0)\big]H_{zy}(z,\sigma;y,t_0;\mu^{t_0})\mathrm{d}z\mathrm{d}\sigma.$$

Then by substituting $\rho = \rho^a, \rho^b$, taking difference and following the similar arguments as above, one can conclude

$$\Big|H_y(x,t;y,t_0;\rho^a) - H_y(x,t;y,t_0;\rho^b)\Big| \leqq C_*\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{t-t_0}\Big[|\log(t-t_0)|\,\Big\|\Big|\rho^a-\rho^b\Big|\Big\|_\infty + \Big\|\Big|\rho^a-\rho^b\Big|\Big\|_{BV} + \sqrt{t-t_0}\Big(\Big\|\Big|\rho^a-\rho^b\Big|\Big\|_1 + |\log t|\,\Big\|\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau\big[\rho^a-\rho^b\big]\Big|\Big\|_\infty\Big)\Big].$$

- (**Comparison estimates of** H_{xy})
  As for

$$H_{xy}(x,t;y,t_0;\rho^a) - H_{xy}(x,t;y,t_0;\rho^b),$$

  thanks to (A.24), one can do much more tedious but similar in spirit estimates as above to complete the estimate.
- (**Comparison estimates of** H_t)
  For comparison of H_t, we shall use

$$H_t(x,t;y,t_0;\rho^a) - H_t(x,t;y,t_0;\rho^b) = \lim_{h\to 0}\frac{1}{h}\Big[\big(H(x,t+h;y,t_0;\rho^a) - H(x,t;y,t_0;\rho^a)\big) - \big(H(x,t+h;y,t_0;\rho^b) - H(x,t;y,t_0;\rho^b)\big)\Big]$$

  and prove the uniform estimate of difference quotient in h. We may assume $0 < h < (t-t_0)/10$. As the estimates are of the same spirit, but lengthy, we omit the details. □

Taking advantage of the equation itself or representing the time-dependent coefficient heat kernel in terms of time-independent heat kernel as in Theorem A.3, we have the following comparison estimates involving time integral.

Theorem A.5. (Comparison estimates involving time integral) *Let $\rho^a(x,t)$ and $\rho^b(x,t)$ be two functions satisfying (A.1). Suppose $t_0 < t \ll 1$. Then the following comparison estimates hold:*

$$
\begin{aligned}
&\left|\int_{t_0}^{t}\Big[H_x(x,\tau;y,t_0;\rho^a)-H_x(x,\tau;y,t_0;\rho^b)\Big]d\tau\right|,\\
&\left|\int_{t_0}^{t}\Big[H_y(x,t;y,s;\rho^a)-H_y(x,t;y,s;\rho^b)\Big]\mathrm{d}s\right|\\
&\quad\leqq C_*e^{-\frac{(x-y)^2}{C_*(t-t_0)}}\Big[\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_\infty+\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_{BV}\\
&\qquad+\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_1+\Big|\Big|\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\Big|\Big|_\infty\Big],\\
&\left|\int_{t_0}^{t}\Big[H_x(x,t;y,s;\rho^a)-H_x(x,t;y,s;\rho^b)\Big]\mathrm{d}s\right|,\\
&\left|\int_{t_0}^{t}\Big[H_y(x,\tau;y,t_0;\rho^a)-H_y(x,\tau;y,t_0;\rho^b)\Big]d\tau\right|\\
&\quad\leqq C_*e^{-\frac{(x-y)^2}{C_*(t-t_0)}}\Big[\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_\infty+\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_{BV}\\
&\qquad+\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_1+\Big|\Big|\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\Big|\Big|_\infty\Big],\\
&\int_{t_0}^{t}\Big[H_{xy}(x,\tau;y,t_0;\rho^a)-H_{xy}(x,\tau;y,t_0;\rho^b)\Big]d\tau\\
&\quad=\Big[\frac{1}{\rho^a(x,t_0)}-\frac{1}{\rho^b(x,t_0)}\Big]\delta(x-y)\\
&\qquad-\int_{t_0}^{t}\Big[\frac{\rho^a(x,\tau)-\rho^a(x,t_0)}{\rho^a(x,t_0)}H_{xy}(x,\tau;y,t_0;\rho^a)\\
&\qquad-\frac{\rho^b(x,\tau)-\rho^b(x,t_0)}{\rho^b(x,t_0)}H_{xy}(x,\tau;y,t_0;\rho^b)\Big]d\tau\\
&\qquad+O(1)\frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}}\Big[|\log(t-t_0)|\,\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_\infty+\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_{BV}\\
&\qquad+\Big|\Big|\Big|\rho^a-\rho^b\Big|\Big|\Big|_1+\Big|\Big|\Big|\frac{\sqrt{\tau}}{|\log\tau|}\partial_\tau(\rho^a-\rho^b)\Big|\Big|\Big|_\infty\Big],\\
&\int_{t_0}^{t}\Big[H_{xy}(x,t;y,s;\rho^a)-H_{xy}(x,t;y,s;\rho^b)\Big]\mathrm{d}s\\
&\quad=\Big[\frac{1}{\rho^a(y,t)}-\frac{1}{\rho^b(y,t)}\Big]\delta(x-y)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{t_0}^{t} \left[\frac{\rho^a(y,t) - \rho^a(y,s)}{\rho^a(y,t)} H_{xy}(x,t;y,s;\rho^a) \right. \\
&\left. - \frac{\rho^b(y,t) - \rho^b(y,s)}{\rho^b(y,t)} H_{xy}(x,t;y,s;\rho^b) \right] d\tau \\
&+ O(1) \frac{e^{-\frac{(x-y)^2}{C_*(t-t_0)}}}{\sqrt{t-t_0}} \left[|\log(t-t_0)| \left\|\left| \rho^a - \rho^b \right\|\right|_\infty + \left\|\left| \rho^a - \rho^b \right\|\right|_{BV} \right. \\
&\left. + \left\|\left| \rho^a - \rho^b \right\|\right|_1 + \left\|\left| \frac{\sqrt{\tau}}{|\log \tau|} \partial_\tau (\rho^a - \rho^b) \right\|\right|_\infty \right].
\end{aligned}
$$

Appendix B. Explicit Expressions of Constants in Green's Function

B.1. Constants in Lemmas 6.1 and 6.2

The following give the explicit expressions of the constants appeared in Lemmas 6.1 and 6.2.

$$
\begin{aligned}
&\alpha_2^* = \frac{\mu}{v}, \quad \alpha_3^* = \frac{\kappa\theta_e}{v}, \quad \beta_2^* = \frac{v\left(\mu p p_e + \kappa\theta_e p_v - \mu p_v\right)}{\mu\left(\mu - \kappa\theta_e\right)}, \quad \beta_3^* = \frac{p v p_e}{\kappa\theta_e - \mu}, \\
&A_{1,1} = -\frac{v^3\left(\kappa\theta_e p_v^2 + \mu p p_e p_v\right)}{\kappa\mu^3\theta_e}, \\
&A_{2,1} = \frac{v^3\left(\mu^3 p^2 p_e^2 - \mu p p_e p_v\left(\kappa^2\theta_e^2 - 3\kappa\mu\theta_e + 2\mu^2\right) + p_v^2\left(\mu - \kappa\theta_e\right)^3\right)}{\mu^3\left(\mu - \kappa\theta_e\right)^3}, \\
&A_{3,1} = \frac{p v^3 p_e\left(\kappa p\theta_e p_e + p_v\left(\kappa\theta_e - \mu\right)\right)}{\kappa\theta_e\left(\kappa\theta_e - \mu\right)^3}.
\end{aligned}
$$

$$
M_1^{*,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_1^{*,1} = \begin{pmatrix} 0 & \frac{v}{\mu} & 0 \\ -\frac{vp_v}{\mu} & 0 & 0 \\ -\frac{uvp_v}{\mu} & 0 & 0 \end{pmatrix}, \quad M_1^{*,2} = \begin{pmatrix} -\frac{v^2 p_v}{\mu^2} & -\frac{uv^2 p_e}{\kappa\mu\theta_e} & \frac{v^2 p_e}{\kappa\mu\theta_e} \\ 0 & \frac{v^2 p_v}{\mu^2} & 0 \\ -\frac{pv^2 p_v}{\kappa\mu\theta_e} & \frac{uv^2 p_v}{\mu^2} & 0 \end{pmatrix},
$$

$$
M_1^{*,3} = \begin{pmatrix} 0 & -\frac{v^3(p\mu p_e + 2\kappa p_v\theta_e)}{\kappa\mu^3\theta_e} & 0 \\ \frac{v^3 p_v(p\mu p_e + 2\kappa p_v\theta_e)}{\kappa\mu^3\theta_e} & \frac{uv^3 p_e p_v}{\kappa\mu^2\theta_e} & -\frac{v^3 p_e p_v}{\kappa\mu^2\theta_e} \\ \frac{uv^3 p_v(p\mu p_e + 2\kappa p_v\theta_e)}{\kappa\mu^3\theta_e} & -\frac{v^3\left(pp_v - u^2 p_e p_v\right)}{\kappa\mu^2\theta_e} & -\frac{uv^3 p_e p_v}{\kappa\mu^2\theta_e} \end{pmatrix}.
$$

$$
M_2^{*,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & u & 0 \end{pmatrix}, \quad M_2^{*,1} = \begin{pmatrix} 0 & -\frac{v}{\mu} & 0 \\ \frac{vp_v}{\mu} & -\frac{uvp_e}{\mu - \kappa\theta_e} & \frac{vp_e}{\mu - \kappa\theta_e} \\ \frac{uvp_v}{\mu} & \frac{v\left(p - u^2 p_e\right)}{\mu - \kappa\theta_e} & \frac{uvp_e}{\mu - \kappa\theta_e} \end{pmatrix},
$$

$$
M_2^{*,2} = \begin{pmatrix} \frac{v^2 p_v}{\mu^2} & -\frac{uv^2 p_e}{\mu^2 - \kappa\mu\theta_e} & \frac{v^2 p_e}{\mu^2 - \kappa\mu\theta_e} \\ 0 & \frac{v^2\left(p\mu^2 p_e - p_v(\mu - \kappa\theta_e)^2\right)}{\mu^2(\mu - \kappa\theta_e)^2} & 0 \\ -\frac{pv^2 p_v}{\mu^2 - \kappa\mu\theta_e} & \frac{uv^2\left(2p\mu^2 p_e - p_v(\mu - \kappa\theta_e)^2\right)}{\mu^2(\mu - \kappa\theta_e)^2} & -\frac{pv^2 p_e}{(\mu - \kappa\theta_e)^2} \end{pmatrix},
$$

$$M_2^{*,3} = \begin{pmatrix} 0 & \frac{v^3\left(2p_v + \frac{p\mu p_e(\kappa\theta_e - 2\mu)}{(\mu-\kappa\theta_e)^2}\right)}{\mu^3} & 0 \\ -\frac{v^3 p_v\left(2p_v + \frac{p\mu p_e(\kappa\theta_e - 2\mu)}{(\mu-\kappa\theta_e)^2}\right)}{\mu^3} & -\frac{uv^3 p_e(2p\mu^2 p_e - p_v(2\mu^2 - 3\kappa\theta_e\mu + \kappa^2\theta_e^2))}{\mu^2(\mu-\kappa\theta_e)^3} & \frac{v^3 p_e(2p\mu^2 p_e - p_v(2\mu^2 - 3\kappa\theta_e\mu + \kappa^2\theta_e^2))}{\mu^2(\mu-\kappa\theta_e)^3} \\ -\frac{uv^3 p_v(2p_v(\mu-\kappa\theta_e)^2 + p\mu p_e(\kappa\theta_e - 2\mu))}{\mu^3(\mu-\kappa\theta_e)^2} & \frac{v^3(p-u^2p_e)(2p\mu^2 p_e - p_v(2\mu^2 - 3\kappa\theta_e\mu + \kappa^2\theta_e^2))}{\mu^2(\mu-\kappa\theta_e)^3} & \frac{uv^3 p_e(2p\mu^2 p_e - p_v(2\mu^2 - 3\kappa\theta_e\mu + \kappa^2\theta_e^2))}{\mu^2(\mu-\kappa\theta_e)^3} \end{pmatrix}.$$

$$M_3^{*,0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -u & 1 \end{pmatrix}, \quad M_3^{*,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{uvp_e}{\mu-\kappa\theta_e} & -\frac{vp_e}{\mu-\kappa\theta_e} \\ 0 & -\frac{v(p-u^2p_e)}{\mu-\kappa\theta_e} & -\frac{uvp_e}{\mu-\kappa\theta_e} \end{pmatrix},$$

$$M_3^{*,2} = \begin{pmatrix} 0 & \frac{uv^2p_e}{\kappa\theta_e(\mu-\kappa\theta_e)} & -\frac{v^2p_e}{\kappa\theta_e(\mu-\kappa\theta_e)} \\ 0 & -\frac{pv^2p_e}{(\mu-\kappa\theta_e)^2} & 0 \\ \frac{pv^2p_v}{\kappa\theta_e(\mu-\kappa\theta_e)} & -\frac{2puv^2p_e}{(\mu-\kappa\theta_e)^2} & \frac{pv^2p_e}{(\mu-\kappa\theta_e)^2} \end{pmatrix},$$

$$M_3^{*,3} = \begin{pmatrix} 0 & \frac{pv^3p_e}{\kappa\theta_e(\mu-\kappa\theta_e)^2} & 0 \\ -\frac{pv^3p_ep_v}{\kappa\theta_e(\mu-\kappa\theta_e)^2} & \frac{uv^3p_e(p_v(\mu-\kappa\theta_e) - 2p\kappa p_e\theta_e)}{\kappa\theta_e(\kappa\theta_e-\mu)^3} & \frac{v^3p_e(2p\kappa p_e\theta_e + p_v(\kappa\theta_e-\mu))}{\kappa\theta_e(\kappa\theta_e-\mu)^3} \\ -\frac{puv^3p_ep_v}{\kappa\theta_e(\mu-\kappa\theta_e)^2} & \frac{v^3(p-u^2p_e)(2p\kappa p_e\theta_e + p_v(\kappa\theta_e-\mu))}{\kappa\theta_e(\kappa\theta_e-\mu)^3} & \frac{uv^3p_e(2p\kappa p_e\theta_e + p_v(\kappa\theta_e-\mu))}{\kappa\theta_e(\kappa\theta_e-\mu)^3} \end{pmatrix}.$$

B.2. Constants in Lemma 6.3

The constant matrices M_j^k in Lemma 6.3 are given as follows.

$$M_1^0 = \begin{pmatrix} \frac{pp_e}{pp_e-p_v} & -\frac{up_e}{pp_e-p_v} & \frac{p_e}{pp_e-p_v} \\ 0 & 0 & 0 \\ -\frac{pp_v}{pp_e-p_v} & \frac{up_v}{pp_e-p_v} & -\frac{p_v}{pp_e-p_v} \end{pmatrix},$$

$$M_2^0 = \begin{pmatrix} -\frac{p_v}{2pp_e-2p_v} & \frac{up_e-\sqrt{pp_e-p_v}}{2pp_e-2p_v} & -\frac{p_e}{2pp_e-2p_v} \\ \frac{p_v}{2\sqrt{pp_e-p_v}} & \frac{1}{2} - \frac{up_e}{2\sqrt{pp_e-p_v}} & \frac{p_e}{2\sqrt{pp_e-p_v}} \\ \frac{(p+u\sqrt{pp_e-p_v})p_v}{2pp_e-2p_v} & \frac{-p_e\sqrt{pp_e-p_v}u^2 - p_vu + p\sqrt{pp_e-p_v}}{2pp_e-2p_v} & \frac{p_e(p+u\sqrt{pp_e-p_v})}{2pp_e-2p_v} \end{pmatrix},$$

$$M_3^0 = \begin{pmatrix} -\frac{p_v}{2pp_e-2p_v} & \frac{up_e+\sqrt{pp_e-p_v}}{2pp_e-2p_v} & -\frac{p_e}{2pp_e-2p_v} \\ -\frac{p_v}{2\sqrt{pp_e-p_v}} & \frac{1}{2}\left(\frac{up_e}{\sqrt{pp_e-p_v}} + 1\right) & -\frac{p_e}{2\sqrt{pp_e-p_v}} \\ \frac{(p-u\sqrt{pp_e-p_v})p_v}{2pp_e-2p_v} & -\frac{-p_e\sqrt{pp_e-p_v}u^2 + p_vu + p\sqrt{pp_e-p_v}}{2pp_e-2p_v} & \frac{p_e(p-u\sqrt{pp_e-p_v})}{2pp_e-2p_v} \end{pmatrix}.$$

$$M_1^1 = \begin{pmatrix} 0 & \frac{p\kappa p_e\theta_e}{v(p_v-pp_e)^2} & 0 \\ -\frac{p\kappa p_ep_v\theta_e}{v(p_v-pp_e)^2} & \frac{u\kappa p_ep_v\theta_e}{v(p_v-pp_e)^2} & -\frac{\kappa p_ep_v\theta_e}{v(p_v-pp_e)^2} \\ -\frac{pu\kappa p_ep_v\theta_e}{v(p_v-pp_e)^2} & -\frac{\kappa(p-u^2p_e)p_v\theta_e}{v(p_v-pp_e)^2} & -\frac{u\kappa p_ep_v\theta_e}{v(p_v-pp_e)^2} \end{pmatrix},$$

$$M_2^1 = (\xi_1, \xi_2, \xi_3),$$

$$\xi_1 = \begin{pmatrix} \frac{p_v(\mu p_v - pp_e(\mu+3\kappa\theta_e))}{4v(pp_e-p_v)^{5/2}} \\ \frac{p\kappa p_ep_v\theta_e}{2v(p_v-pp_e)^2} \\ \frac{pp_v(p_e(p\mu+\kappa(p+2u\sqrt{pp_e-p_v})\theta_e) - p_v(\mu-2\kappa\theta_e))}{4v(pp_e-p_v)^{5/2}} \end{pmatrix},$$

$$\xi_2 = \begin{pmatrix} \frac{p_e(-2p\kappa\sqrt{pp_e-p_v}\theta_e - up_v(\mu-2\kappa\theta_e) + pup_e(\mu+\kappa\theta_e))}{4v(pp_e-p_v)^{5/2}} \\ -\frac{(p_e(p\sqrt{pp_e-p_v}\mu + (2u\kappa p_v - p\kappa\sqrt{pp_e-p_v})\theta_e) - \mu\sqrt{pp_e-p_v}p_v)}{4v(p_v-pp_e)^2} \\ -\frac{(2p^2u(\mu-\kappa\theta_e)p_e^2 + up_v(\kappa(5p+2u\sqrt{pp_e-p_v})\theta_e - 3p\mu)p_e + p_v(u\mu p_v - 2p\kappa\sqrt{pp_e-p_v}\theta_e))}{4v(pp_e-p_v)^{5/2}} \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} -\frac{p_e(pp_e(\mu+\kappa\theta_e)-p_v(\mu-2\kappa\theta_e))}{4v(pp_e-p_v)^{5/2}} \\ \frac{\kappa p_e p_v \theta_e}{2v(p_v-pp_e)^2} \\ \frac{p_e\left(p_e(\mu-\kappa\theta_e)p^2+p_v\left(2\kappa\left(2p+u\sqrt{pp_e-p_v}\right)\theta_e-p\mu\right)\right)}{4v(pp_e-p_v)^{5/2}} \end{pmatrix}.$$

$$M_3^1 = (\zeta_1, \zeta_2, \zeta_3),$$

$$\zeta_1 = \begin{pmatrix} \frac{p_v(pp_e(\mu+3\kappa\theta_e)-\mu p_v)}{4v(pp_e-p_v)^{5/2}} \\ \frac{p\kappa p_e p_v \theta_e}{2v(p_v-pp_e)^2} \\ \frac{p_v\left(pp_v(\mu-2\kappa\theta_e)-pp_e\left(p\mu+\kappa\left(p-2u\sqrt{pp_e-p_v}\right)\theta_e\right)\right)}{4v(pp_e-p_v)^{5/2}} \end{pmatrix},$$

$$\zeta_2 = \begin{pmatrix} -\frac{p_e\left(2p\kappa\sqrt{pp_e-p_v}\theta_e-up_v(\mu-2\kappa\theta_e)+pup_e(\mu+\kappa\theta_e)\right)}{4v(pp_e-p_v)^{5/2}} \\ \frac{\left(p_e\left(p\mu\sqrt{pp_e-p_v}-\kappa\left(\sqrt{pp_e-p_v}p+2up_v\right)\theta_e\right)-\mu\sqrt{pp_e-p_v}p_v\right)}{4v(p_v-pp_e)^2} \\ \frac{\left(2p^2u(\mu-\kappa\theta_e)p_e^2+up_v\left(\kappa\left(5p-2u\sqrt{pp_e-p_v}\right)\theta_e-3p\mu\right)p_e+p_v\left(u\mu p_v+2p\kappa\sqrt{pp_e-p_v}\theta_e\right)\right)}{4v(pp_e-p_v)^{5/2}} \end{pmatrix},$$

$$\zeta_3 = \begin{pmatrix} \frac{p_e(pp_e(\mu+\kappa\theta_e)-p_v(\mu-2\kappa\theta_e))}{4v(pp_e-p_v)^{5/2}} \\ \frac{\kappa p_e p_v \theta_e}{2v(p_v-pp_e)^2} \\ \frac{p_e\left(p_e(\kappa\theta_e-\mu)p^2+p_v\left(p\mu+\left(2u\kappa\sqrt{pp_e-p_v}-4p\kappa\right)\theta_e\right)\right)}{4v(pp_e-p_v)^{5/2}} \end{pmatrix}.$$

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Haitao Wang
School of Mathematical Sciences,
Institute of Natural Sciences, MOE-LSC, CMA-Shanghai,
Shanghai Jiao Tong University,
Shanghai
China.
e-mail: haitallica@sjtu.edu.cn

and

Shih-Hsien Yu
Institute of Mathematics,
Academia Sinica,
Taipei
Taiwan.
e-mail: matysh@gate.sinica.edu.tw

and

Xiongtao Zhang
School of Mathematics and Statistics,
Center for Mathematical Sciences,
Hubei Key Laboratory of Engineering Modeling and Scientific Computing,
Huazhong University of Science and Technology,
Luoyu Road 1037,
Wuhan
430074 China.
e-mail: xtzhang@hust.edu.cn

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