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A class of monotonicity-preserving variable-step discretizations for Volterra integral equations

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Abstract

We study in this paper the monotonicity properties of the numerical solutions to Volterra integral equations with nonincreasing completely positive kernels on nonuniform meshes. There is a duality between the complete positivity and the properties of the complementary kernel being nonnegative and nonincreasing. Based on this, we propose the "complementary monotonicity" to describe the nonincreasing completely positive kernels, and the "right complementary monotone" (R-CMM) kernels as the analogue for nonuniform meshes. We then establish the monotonicity properties of the numerical solutions inherited from the continuous equation if the discretization has the R-CMM property. Such a property seems weaker than log-convexity and there is no restriction on the step size ratio of the discretization for the R-CMM property to hold.

Keywords Resolvent · Convolution · Complete positivity · Nonuniform mesh · Fractional differential equations

Mathematics Subject Classification 65L20 · 65R20

1 Introduction

The time-delay memory is ubiquitous in physical models, which may be resulted from dimension reduction as in the generalized Langevin model for particles in heat bath [13, 19, 40, 41] or may be resulted from viscoelasticity in soft matter [5, 30], or dielectric

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susceptibility for polarization [2, 33], to name a few examples. Due to the causality and time translational invariance [29, Chap. 1], the memory terms are often modeled by a one-sided convolution $\int_0^t a(t-s)f(s)\,ds$ where a is the memory kernel. Causality refers to the fact that the output cannot precede the input, which implies, by Tichmarsh's theorem, that the Fourier transform of a is analytic in the upper half plane so that the real and imaginary parts satisfy the Kramers-Kronig relation (see [29]). Besides the causality, the memory kernel a should also reflect the fading memory principle [30, 36]. A popular model to build in these physical principles would be the nonincreasing, completely positive kernels, which are a class of kernels with nonnegative resolvent kernels (see [4, 27] and see also Sect. 2.1 for the definitions). These kernels have been proved to reflect many important asymptotic properties that the system is expected to have [4]. A special but important class of the nonincreasing completely positive kernels is the completely monotone (CM) functions, which have been well studied in literature [32, 38]. The CM functions have been widely used in physical modeling. For example, the interconversion relationship in the linear viscoelasticity is modeled by a convolution quadrature with completely monotone kernels [25]. There are many interesting models with memory in literature for various applications [1, 6, 31, 35, 39].

A basic model for the memory is the Volterra integral equations (see [11, 25, 28, 37]). In this work, we focus on the Volterra integral equations taking values in $\mathbb{R}$. Let $f : [0,\infty)\times\mathbb{R}\to\mathbb{R}$ be a given smooth function. The integral equation we consider in this work is

$$u(t) = h(t) + \int_0^t a(t-s)f(s,u(s))\,ds, \tag{1.1}$$

where $u : [0,T) \to \mathbb{R}$ is the solution curve. The function $h(t)$ is a given signal function. The function $a(\cdot)$ is the convolution kernel and is assumed to be nonzero. We will allow a to be weakly singular in the sense that $a(0+)$ could be ∞ but it is integrable on $(0, 1)$:

$$0 < \int_0^1 a(t)dt < \infty. \tag{1.2}$$

A special example of the integral Eq. (1.1) is the time fractional ordinary differential equations (FODEs) with Caputo derivative [7] of order $\alpha \in (0, 1)$

$$D_c^\alpha u = f(t,u), \quad u(0) = u_0. \tag{1.3}$$

Here, the Caputo derivative is defined by

$$D_c^\alpha u = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{u'(s)}{(t-s)^\alpha}ds. \tag{1.4}$$

See also [15, 17] for some generalized definitions. The time fractional ODE (1.3) is equivalent to the integral equation (see [7, 17] etc)

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s))ds. \tag{1.5}$$

Hence, the FODEs are Volterra equations with kernel

$$a(t) = \frac{1}{\Gamma(\alpha)} t_+^{\alpha-1}, \quad t_+ = t1_{t\ge 0}. \tag{1.6}$$

If $a = 1_{t\ge 0}$, it reduces to the usual ODE, which is in fact the $\alpha \to 1$ limit of the above time fractional equation.

The Volterra Eq. (1.1) with completely positive kernels (not necessarily nonincreasing) has two important monotonicity properties. The first is the order preserving property for two solution curves (i.e., two solution curves will not cross) for reasonable given input signals h (see Theorem 2.1 below). A second monotonicity property is that the solution to the autonomous equations is monotone for reasonable given signals (see Theorem 2.2 below). Also, as we will remark in Sect. 2.1, the nonincreasing property of the kernel a cannot be implied by the complete positivity, but is required by the fading memory principle.

Due to the memory kernels, especially some weakly singular kernels, the models often exhibit multi-scale behaviors [6, 35, 39], which bring numerical challenges. The adaptive time-stepping is often adopted to address this issue [12, 14, 21, 26, 34]. Suppose that the computational time interval is $[0, T]$. Let $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ be the grid points. We define

$$\tau_n := t_n - t_{n-1}, \quad n \ge 1. \tag{1.7}$$

Let u_n be the numerical solution at t_n. By implicit discretization of the Volterra integral Eq. (1.1), one may obtain

$$u_n = h(t_n) + \sum_{j=1}^{n} \bar{a}_{n-j}^n f(t_j, u_j)\tau_j = h(t_n) + \sum_{j=1}^{n} a_{n-j}^n f(t_j, u_j). \tag{1.8}$$

Here, $\{\bar{a}_{n-j}^n\}$ is an approximation of $a(t_n - s)$ on $[t_{j-1}, t_j]$ while a_{n-j}^n is like the integral of $a(t_n - s)$ on this interval. It is clear that

Lemma 1.1 *Suppose $f(t, \cdot)$ is Lipschitz with Lipschitz constant M uniform in t. If $M \sup_{j\le N} a_0^j < 1$, then the numerical solution to (1.8) is uniquely solvable.*

From the viewpoint of structure-preserving methods, it is desired that the discrete numerical methods preserve the monotonicity properties of the solutions. Our main goal is to investigate this for the scheme above.

In [20], the so-called CM-preserving schemes have been proposed for equations with CM kernels so that these two monotonicity properties can be preserved. These schemes have been shown to enjoy good stability properties. Recently, Chen and Stynes used the CM-preserving property to obtain sharp error estimate for the multi-term time-fractional diffusion equations [3]. However, the CM-preserving schemes are based on the convolution and thus they are restricted on uniform meshes. Moreover, if the time continuous kernel itself is not CM, there is no reason to consider such a class of discretizations.

We aim to identify a suitable class of variable discretizations so that the monotonicity properties for numerical solutions will be preserved. In Sect. 2.2, we find that the complete positivity is enough to preserve the desired properties for uniform meshes. However, for nonuniform meshes, the complete positivity is not enough for the desired properties. We thus turn our attention to more restricted kernels that are also nonincreasing, which is natural due to the fading memory principle and turns out to be suitable for variable step discretizations. Hence our focus in this paper is the nonincreasing completely positive kernels. By a characterization of completely positive kernels due to Clement and Nohel [4], the nonincreasing completely positive kernels have a nice dual symmetry, which we call complementary monotonicity. We also propose "right complementary monotone" (R-CMM) kernels for nonuniform meshes inherited from the continuous kernel, with which we prove that the monotonicity properties for nonuniform meshes hold. The well-known L1 discretization and a piecewise constant integral scheme for the time fractional equations are shown to be R-CMM. The R-CMM property is stronger than the complete positivity but is weaker than the usual condition that a is log-convex. In a subsequent work, we will show that the completely positive discretizations, and thus the R-CMM discretizations, enjoy very good stability results. Moreover, it is found that there is no restriction on the ratios of the stepsizes for the R-CMM property to hold, so it is flexible in applications.

The rest of the paper is organized as follows. In Sect. 2, we first present some concepts and results for the time continuous equations and schemes with uniform meshes. In Sect. 3, we focus on the nonincreasing completely positive kernels. Motivated by the duality between the complete positivity and the properties of the complementary kernel, we propose the concept of "complementary monotonicity" (CMM) and study the discrete analogue on uniform meshes. In Sect. 4, we show properties for the R-CMM kernels on nonuniform meshes. In Sect. 5, we prove the main results in this paper, namely the schemes whose kernels are R-CMM will preserve the monotonicity properties. The application to the FODEs is discussed in Sect. 6 while some illustrating numerical experiments are performed in Sect. 7. Lastly, we conclude the work in Sect. 8.

2 Concepts and preliminaries

In this section, we first introduce some basic concepts and basic results for our later discussion. In particular, we first summarize several properties of the time continuous integral equations in Sect. 2.1. Then, we discuss the monotonicity preserving properties of the numerical solutions on uniform meshes in Sect. 2.2.

2.1 The resolvent kernels and complete positivity

We first recall the standard one-sided convolution for two functions u and v defined on $[0, \infty)$

$$(u * v)(t) = \int_{[0,t]} u(s)v(t-s)\,ds, \tag{2.1}$$

which can be generalized to distributions whose supports are on $[0, \infty)$ (see [15, sections 2.1,2.2]). The convolution is commutative, associative. The identity is the Dirac delta δ. With this convolution, the Volterra integral Eq. (1.1) can be written as

$$u = h + a * f(\cdot, u(\cdot)).$$

The resolvent kernels often play important roles (see [4, 27] for examples).

Definition 2.1 Let $\lambda > 0$. The resolvent kernels r_λ and s_λ for a are defined respectively by

$$r_\lambda + \lambda r_\lambda * a = \lambda a, \quad s_\lambda + \lambda s_\lambda * a = 1_{t \geq 0}. \tag{2.2}$$

Clearly, the resolvent kernel r_λ satisfies

$$(\delta + \lambda a) * (\delta - r_\lambda) = \delta. \tag{2.3}$$

It is clearly that (see [4])

$$s_\lambda = 1 * (\delta - r_\lambda) = 1 - \int_0^t r_\lambda(\tau)\,d\tau. \tag{2.4}$$

Intuitively, $\delta - r_\lambda = \lambda^{-1} r_\lambda * a^{(-1)}$. Considering the complementary kernel a^c that satisfies $a^c * a = a * a^c = 1_{t \geq 0}$, one then has formally that $s_\lambda = \lambda^{-1} r_\lambda * a^c$.

In [4], the so-called "completely positive" kernels were considered by Clement and Nohel.

Definition 2.2 Let $T > 0$. A kernel $a \in L^1(0, T)$ is said to be completely positive if both the resolvent kernels r_λ and s_λ defined in Definition 2.1 are nonnegative for every $\lambda > 0$.

A sufficient condition is the following (see [27]).

Lemma 2.1 *If the kernel $a \in L^1(0, T)$ is nonnegative, nonincreasing and $t \mapsto \log a(t)$ is convex, then a is completely positive.*

In fact, the statement for the log-convexity of a in [27] is that $t \mapsto a(t)/a(t+T)$ is nonincreasing for all $T > 0$. If a is CM, $\log a$ is convex (see [27, Lemma 2]).

The following description of the complete positivity has been proved in [4, Theorem 2.2]. (The second claim has been mentioned in Remark (i) below the main result there.)

Lemma 2.2 *Let $T > 0$. A kernel $a \in L^1(0, T)$ with $a \not\equiv 0$ is completely positive on $[0, T]$ if and only if there exists $\alpha \geq 0$ and $c \in L^1(0, T)$ nonnegative and nonincreasing satisfying*

$$\alpha a + c * a = a * (\alpha\delta + c) = 1_{t \geq 0}. \tag{2.5}$$

Moreover, provided that a is completely positive, $\alpha > 0$ if and only if $a \in L^\infty(0, T)$ and in this case a is in fact absolutely continuous on $[0, T]$.

This result tells us that there is a complementary kernel $a^c = \alpha\delta + c$ for a. Clearly, a^c is a nonnegative and nonincreasing measure on $[0, T]$. We note that a completely positive kernel a is nonnegative (see [4, Proposition 2.1]). However, it has been remarked in [4] that a can increase at some subintervals, and also a does not have to be convex. Hence, the complete positivity does not necessarily satisfy the requirement of fading memory.

Now, we present some monotonicity properties of the time continuous equations with completely positive kernels. We will always assume that $h \in C([0, T], \mathbb{R}) \cap C^1((0, T], \mathbb{R})$ where $T > 0$ is the largest time considered. If $a \in L^1(0, T)$ and f is smooth as assumed, then u is absolutely continuous on $[0, T_b)$ where $T_b \leq T$ is the largest time of existence (see [11, 37]).

The first result is about the order preserving property of two solution curves.

Theorem 2.1 *Suppose the kernel a is completely positive. If the input signals $h_i \in C([0, T], \mathbb{R}) \cap C^1((0, T], \mathbb{R})$ $(i = 1, 2)$ and $\gamma(t) := h_1(t) - h_2(t)$ satisfies that*

$$\beta_\lambda(t) := (\delta - r_\lambda) * \gamma(t) = \gamma(t) - \int_0^t r_\lambda(t - s)\gamma(s)\, ds \geq 0, \quad \forall \lambda > 0,$$

then the two solutions to (1.1) satisfy that $u_1(t) \geq u_2(t)$ for all t on the common interval of existence. If moreover $\gamma(0) > 0$, then $u_1(t) > u_2(t)$ for all t. Consequently, if h_1 and h_2 are two constants with $h_1 > h_2$, then $u_1(t) > u_2(t)$.

The key in the proof is to convolve the integral equation with $\delta - r_\lambda$ to obtain the following relation for $v = u_1 - u_2$:

$$v(t) = \beta_\lambda(t) + \int_0^t r_\lambda(t - s)[1 + \lambda^{-1} g(s)]v(s)\, ds, \tag{2.6}$$

where $g(s) = \int_0^1 \partial_u f(s, zu_1(s) + (1 - z)u_2(s))\, dz$. This will give the nonnegativity of v and see the details in Appendix A. For fractional ODEs, $h_i(t)$'s are constants so $\beta_\lambda \geq 0$ is obvious. Hence, the solution curves of the fractional ODEs never cross each other. This recovers the result in [10, Theorem 4.1].

Next, we consider another monotonicity, the monotonicity of one solution with respect to time.

Theorem 2.2 *Consider the Volterra* Eq. (1.1) *and a solution u corresponding to an input signal $h \in C([0, T], \mathbb{R}) \cap C^1((0, T], \mathbb{R})$. Suppose that a is completely positive.*

(i) If $(\delta - r_\lambda) * h' \geq 0$, $f(0, h(0)) \geq 0$ *and* $\partial_t f(t,u)|_{u=u(t)} \geq 0$ *on the solution curve, then the solution is nondecreasing.*
(ii) If $(\delta - r_\lambda) * h' \leq 0$, $f(0, h(0)) \leq 0$ *and* $\partial_t f(t,u)|_{u=u(t)} \leq 0$ *on the solution curve, then the solution is nonincreasing.*

If any of the inequalities is strict, then u is strictly monotone. As a consequence, if h is a constant and the equation is autonomous (f only depends on u), then any solution curve is monotone.

For the proof, one may refer to Appendix A. The basic idea is again to convolve the equation with $\delta - r_\lambda$ and take the derivative on time to obtain an equation for $u'(t)$:

$$u'(t) = \beta(t) + r_\lambda * ([1 + \lambda^{-1}\partial_u f]u'),$$

Here, β is a term related to h and the initial value, $\partial_t f(t,u)$. If h is a constant and the equation is autonomous, $\beta(t) = \lambda^{-1} r_\lambda(t) f(h(0))$. This clearly is a generalization of the result for autonomous time fractional ODEs in [9, Theorem 3.3]. For the special case $a(t) \equiv 1$, $h(t) \equiv u_0$ and $f(t,u) = f(u)$, it reduces to the autonomous ODE

$$\dot{u} = f(u).$$

Then, $r_\lambda = \lambda e^{-\lambda t}$ and the equaiton for u' above becomes

$$u'(t) = e^{-\lambda t} f(u_0) + \int_0^t e^{-\lambda(t-s)}(\lambda + f'(u(s)))u'(s)\,ds.$$

It is clear that the right hand is in fact equal to $f(u(t))$. Such a form is interesting as one can see easily that $u'(t)$ has the same sign as $f(u_0)$, which implies that u is monotone.

2.2 Monotonicity-preseving property for completely positive kernels on uniform meshes

Consider the discretization (1.8) when it is uniform, i.e. $\tau_n \equiv \tau$ is a constant and $a_{n-j}^n \equiv a_{n-j}$ depends only on the value $n - j$. We recall the usual convolution, which is commutative,

$$(a * b)_n = \sum_{j=0}^{n} a_{n-j} b_j. \tag{2.7}$$

It is clear that $\delta_d = (1, 0, 0, \cdots)$ is the convolution identity and the convolution inverse of a exists if and only if $a_0 \neq 0$.

Corresponding to Definition 2.2, the complete positivity for uniform meshes was introduced in [8] as follows.

Definition 2.3 A sequence $a = (a_0, a_1, \cdots)$ with $a_0 \neq 0$ is said to be completely positive if the resolvent sequence given by

$$r_\lambda + \lambda r_\lambda * a = \lambda a$$

is nonnegative for all $\lambda > 0$ and it holds that $\sum_{i=0}^{n}(r_\lambda)_i \le 1$ for all n.

The following has been proved in [8].

Lemma 2.3 *The sequence a with $a_0 \neq 0$ is completely positive if and only if the convolutional inverse $b = a^{(-1)}$ satisfies*

$$b_0 > 0; \quad b_j \le 0, j \ge 1; \quad \sum_{j=0}^{n} b_j \ge 0, n \ge 1. \tag{2.8}$$

This result actually is an analogue to Lemma 2.2. This is because $a^c = b * (1, 1, \cdots)$. Hence, the nonnegativity and nonincreasing properties of a^c are reflected by (2.8).

We have the following result.

Theorem 2.3 *Suppose that $f(t, \cdot)$ is Lipschitz with constant $M > 0$. Consider the scheme (1.8) on a uniform mesh such that $a^n_{n-j} \equiv a_{n-j}$. If the discrete kernel $a = (a_0, a_1, \cdots)$ is completely positive and $Ma_0 < 1$, then the following property holds:*

(a) If the two input signals $h^{(i)}(t_n)$, $i = 1, 2$ satisfy $\beta_n = [(h^{(1)} - h^{(2)}) * (\delta_d - r_\lambda)]_n \ge 0$ for any $\lambda > 0$, then $u^{(1)}_n \ge u^{(2)}_n$.
(b) Suppose moreover that $f(t, u) = f(u)$ does not depend on t and $h(t) = u_0$ is a constant, then the numerical solution is monotone.

Proof The proof of part (a) is the same as the one for the nonuniform case later in Theorem 5.1, so we skip it here. We focus on the part (b). We assume that $f(u_0) > 0$. The case for $f(u_0) = 0$ is obvious while the case for $f(u_0) < 0$ is similar.

For $n = 1$, one has that $u_1 - a_0 f(u_1) = u_0$. Let $\mu(u) = u - a_0 f(u)$, which is clearly increasing. Clearly, $\mu(u_0) < u_0 = \mu(u_1)$ and thus $u_1 > u_0$. Define

$$v_n := u_{n+1} - u_n, \quad n \ge 0.$$

Hence, $v_0 > 0$ and we aim to show $v_n \ge 0$ for $n \ge 1$.

The scheme (1.8) is then written as $u - u_0 = a * f(u)$, where $u - u_0 := (u_1 - u_0, u_2 - u_0, \cdots)$ and $f(u) = (f(u_1), f(u_2), \cdots)$. Taking convolution with $\delta_d - r_\lambda$, one has

$$u_n - u_0 = r_\lambda * (u - u_0 + f(u)/\lambda)_n,$$

which gives

$$v_n = (r_\lambda)_n [u_1 - u_0 + f(u_1)/\lambda] + \sum_{j=1}^{n} (r_\lambda)_{n-j} \left[u_{j+1} - u_j + \frac{f(u_{j+1})}{\lambda} - \frac{f(u_j)}{\lambda} \right]$$
$$= (r_\lambda)_n [v_0 + f(u_1)/\lambda] + \sum_{j=1}^{n} (r_\lambda)_{n-j} (1 + \frac{g_j}{\lambda}) v_j,$$

where $g_j = \int_0^1 f'(zu_{j+1} + (1-z)u_j)\,dz$. It follows that

$$[1 - (r_\lambda)_0(1 + g_n/\lambda)]v_n = (r_\lambda)_n[v_0 + f(u_1)/\lambda] + \sum_{j=1}^{n-1} (r_\lambda)_{n-j}(1 + \frac{g_j}{\lambda})v_j$$

Choosing λ large enough, $v_0 + f(u_1)/\lambda > 0$ and $1 + g_j/\lambda > 0$ for $j \le n-1$. Since $(r_\lambda)_0 = \lambda a_0/(1+\lambda a_0)$ and $|g_n| \le M$, then for $Ma_0 < 1$, the coefficient of v_n is positive for λ large enough. Then, one can see that $v_n > 0$. □

However, for the nonuniform meshes, the complete positivity introduced in [8] is not enough for the monotonicity as can be seen in the proof of Theorem 5.2. Fortunately, if we focus on the slightly restricted kernels that are also nonincreasing, which is natural in physical models due to the fading memory, then the monotonicity properties can be established for the nonuniform case as well. This is the main motivation of this paper and will be explained in detail in the remaining part.

3 Complementary monotone kernels

As we have discussed above, we will require the original kernels to be nonincreasing and completely positive. This turns out to be a beautiful property which worth separate investigation. In Sect. 3.1, we study this property and then the analogue on uniform meshes in Sect. 3.2.

3.1 Time continuous complementary monotone kernels

In this subsection, we study the properties of the kernels if we require the kernel to be both nonincreasing and completely positive. Lemma 2.2 gives a nice characterization of the duality between the complete positivity and the nonnegativity and nonincreasing property. If the kernel itself is nonincreasing (and it is already known to be nonnegative), then a^c is completely positive. The kernel and its complementary kernel then obey a dual symmetry, which we call "complementary monotonicity" in the sense that both the kernel and the complementary kernel are nonnegative and monotone, and also completely positive.

Motivated by Lemma 2.2, we will consider kernels of the following form:

$$\mathcal{A} := \{a = \alpha\delta + \tilde{a} : \alpha \ge 0, \quad \tilde{a} \text{ is integrable on } [0, T]\}. \tag{3.1}$$

For such kernels, $a \in \mathcal{A}$ is nonincreasing if and only if $\tilde{a}$ is nonincreasing. It is nonnegative if and only if $\tilde{a}$ is nonnegative. We then define the following.

Definition 3.1 A kernel $a \in \mathcal{A}$ with $a \not\equiv 0$ is said to be complementary monotone (CMM) if a is nonnegative and nonincreasing and there exists a kernel $a^c \in \mathcal{A}$ that is nonnegative and nonincreasing such that $a * a^c = 1_{t\geq 0}$.

Using Lemma 2.2, one can show the following characterization of the CMM kernels, where a completely positive kernel is allowed to be in $\mathcal{A}$.

Proposition 3.1 *Fix $T > 0$. The following are equivalent.*

(a) A kernel $a \in \mathcal{A}$ is CMM on $[0, T]$.
(b) The kernel a is nonincreasing and is completely positive on $[0, T]$.
(c) The complementary kernel $a^c \in \mathcal{A}$ exists, and is nonincreasing, completely positive on $[0, T]$.

Note that in [4], the kernel does not have an atom at $t = 0$. Here, we allow an atom at $t = 0$ and no longer require it to be an L^1 function. However, the proof for the properties of the complementary kernels and resolvents in [4, Theorem 2.2] is actually valid as well. We sketch the proof in Appendix B for the convenience of the readers.

The result above indicates that the nonincreasing property of the kernel is kind of crucial for the complete positivity of the complementary kernel. This indicates that the complete positivity of the complementary kernel may have given a description to the fading memory principle.

Remark 3.1 Since the complementary kernel is $a^{(-1)} * 1_{t\geq 0}$. Then, the above result actually indicates that $a^{(-1)}$ exists for CMM kernels in $\mathcal{A}$, given by

$$a^{(-1)} = \alpha^c \delta' + (\tilde{a}^c)',$$

where $a^c = \alpha^c \delta + \tilde{a}^c$ and the derivative is in distributional sense. If $\tilde{a}^c(0+)$ exists and is nonzero, then there is an additional δ in the convolutional inverse.

3.2 Complementary monotone kernels on uniform meshes

We focus on the discrete analogue of complementary monotonicity for the uniform meshes, i.e., $\tau_n \equiv \tau$ and thus $t_j = j\tau$. Though the complete positivity is already enough for the numerical solutions to preserve the desired properties, the complementary monotonicity itself is interesting enough for separate discussion even for uniform meshes.

Similar to Definition 3.1, we introduce the following.

Definition 3.2 A sequence $a = (a_0, a_1, \cdots)$ is complementary monotone (CMM) if it is nonnegative, nonincreasing, and its complementary kernel a^c satisfying $a * a^c = (1, 1, \cdots)$ is also nonnegative and nonincreasing.

Remark 3.2 We note a characterization of the convolution inverse of a completely monotone (CM) sequence in [16] (or [20] for an improved version). A sequence

$v = (v_0, v_1, \ldots)$ is said to be CM if the sequence $\delta^j v := (I - E)^j v$ has nonnegative elements for any integer $j \geq 0$ (i.e., $(\delta^j v)_k \geq 0$ for any integer $k \geq 0$), where E is the shift operator with $(Ev)_j = v_{j+1}$. In [16], it has been shown that the inverse of a CM sequence a can be written as $a^{(-1)} = (a_0^{-1}, -c_1, -c_2, \cdots)$ with $(c_1, c_2, \cdots)$ being CM. Consequently, consider the complementary a^c in the sense that $a * a^c = (1, 1, 1, \cdots)$. Then, $a^c = a^{(-1)} * (1, 1, \cdots)$ and a^c is also CM. This is in fact our original motivation to consider using complementary kernels to investigate the monotonicity preserving properties, before we noticed [4, Theorem 2.2].

Below, we will show that the CMM property is simply the complete positivity plus the nonincreasing property.

Theorem 3.1 *The following are equivalent:*

(a) The sequence $a = (a_0, a_1, \cdots)$ is CMM.
(b) $a_0 \neq 0$ and $a = (a_0, a_1, \cdots)$ is nonincreasing and completely positive.
(c) The sequence $a = (a_0, a_1, \cdots)$ is nonincreasing with $a_0 \neq 0$, and its convolutional inverse $b = a^{(-1)} = (b_0, b_1, \cdots)$ satisfies

$$b_0 > 0, \quad b_j \leq 0, \quad \forall j \geq 1. \tag{3.2}$$

(d) The complementary kernel of a is nonincreasing and completely positive.

Proof (c) $\Rightarrow$ (a): By the relations $a_0 b_0 = 1$ and

$$a_n b_0 = -\sum_{j=1}^{n} a_{n-j} b_j, \quad n \geq 1,$$

it is straightforward to see that a must be nonnegative if $b_0 > 0$ and $b_j \leq 0$ for $j \geq 1$ by induction. See Lemma 4.1 below for the more general version on nonuniform meshes. By $a^c = b * (1, 1, \cdots)$, a^c is nonincreasing. Besides, since a is nonincreasing, the first element in $(a^c)^{(-1)} = a * (1, -1, 0, \cdots)$ is positive and other elements are nonpositive. By the result just proved, a^c is nonnegative, or $\sum_{j=0}^{n} b_j \geq 0$.

(a) $\Rightarrow$ (b): That $a_0 > 0$ and a is nonincreasing are clear. Since a^c is nonnegative and nonincreasing, (2.8) holds. Lemma 2.3 then gives the result.

(b) $\Rightarrow$ (c): This follows directly by Lemma 2.3.

The complementary kernel of a exists if and only if $a_0 \neq 0$. Since a is CMM if and only if the complementary kernel a^c is CMM, then the equivalence between (d) and (a) is then clear as we have established the equivalence between (a) and (c). ◻

Theorem 3.1 indicates that the CMM property is just nonincreasing plus the complete positivity. As we can see from the equivalence between (a) and (c), the nonincreasing property somehow implies the nonnegativity of a^c (or $\sum_{j=0}^{n} b_j \geq 0$). The following tells us that the CMM property is weaker than the log-convexity.

Lemma 3.1 *If $a = (a_0, a_1, \cdots)$ is nonnegative, nonincreasing and is log-convex in the sense $a_{j-1} a_{j+1} \geq a_j^2$, then it is CMM.*

We refer the readers to [22, Lemma 2.3] for the result on the signs of the inverse if the discrete kernel is log-convex. We remark that for a to be CMM, $b_2 \le 0$ is equivalent to $a_0 a_2 \ge a_1^2$. Nevertheless, the log-convexity for all j is clearly strong.

4 Complementary monotone kernels on nonuniform meshes

In the following, we generalize the CMM properties mentioned in Sect. 3 to the the nonuniform meshes. This will be the main tool we use in this paper to prove the monotonicity preserving properties on nonuniform meshes.

4.1 Pseudo-convolution

Let us have brief review of the pseudo-convolution discussed in [8]. We arrange the kernel $\{a_{n-j}^n\}$ into a lower triangular array A of the following form

$$A = \begin{bmatrix} a_0^1 & & & \\ a_1^2 & a_0^2 & & \\ \cdots & \vdots & \vdots & \\ a_{n-1}^n & \cdots & a_1^n & a_0^n \\ \cdots & \vdots & \vdots & & \vdots \end{bmatrix}. \tag{4.1}$$

Once we have arrays of this form, the pseudo-convolution between two such kernels A and B is defined by the "matrix product". In particular, $C := A \bar{*} B$ is given by

$$c_k^n = \sum_{j=0}^{k} a_{k-j}^n b_j^{n+j-k}, \quad \text{or} \quad c_{n-k}^n = \sum_{j=k}^{n} a_{n-j}^n b_{j-k}^j. \tag{4.2}$$

If $a_j^n = a_j$ and $b_j^n = b_j$ are both independent of n, then it reduces to the usual convolution. By the definition, the convolution for $n \le N$ does not depend on the data with $n > N$. Hence, though the discussion here is for infinite arrays, the result can apply to array kernels with finite data.

Consider the following special kernels. Here, I is the generalization of the identity matrix and we expect it to be the identity for pseudo-convolution. The kernel L is the correspondance of $1_{t \ge 0}$ or the kernel $(1, 1, \cdots)$ for uniform meshes, which would be used to define complementary kernels. The kernel $L^{(-1)}$ is expected to be the inverse of L and thus the correspondance of $(1, -1, \cdots)$ for uniform meshes.

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \vdots & \\ & & & 1 \\ & & & \vdots \end{bmatrix}, \quad L = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & \vdots & \vdots & & \\ 1 & 1 & \cdots & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \quad L^{(-1)} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \vdots & 1 \\ & & & \cdots & \vdots \end{bmatrix}. \tag{4.3}$$

It can be verified that I is indeed the identify for the pseudo-convolution, and the following properties hold.

(a) The pseudo-convolution is associative.
(b) For a given A, if a kernel B satisfies $A \bar{*} B = I$, then $B \bar{*} A = I$.

The kernel B is actually the ROC kernel defined in [23]. Clearly, B is both the left inverse and the right inverse of A for pseudo-convolution so we may simply call it the inverse, and denote

$$A^{(-1)} := B, \quad \text{such that } B \bar{*} A = A \bar{*} B = I.$$

The following lemma from [8, Lemma 4.3] is reminiscent of the M-matrices and is often useful.

Lemma 4.1 *Let B be the inverse of A. If B has positive diagonal elements and non-positive off-diagonal elements, then A has nonnegative elements and the entries on the diagonal are positive.*

The kernel L corresponds to the sequence $(1, 1, \cdots)$ in the usual convolution, and $L^{(-1)}$ in (4.3) is clearly the inverse of L. With this, one may define the complementary kernels.

Definition 4.1 For a given A, the kernel C_R with $A \bar{*} C_R = L$ is called the right complementary kernel. The kernel C_L with $C_L \bar{*} A = L$ is called the left complementary kernel.

If A is invertible, then direct verification tells us that $C_R = A^{(-1)} \bar{*} L$ and $C_L = L \bar{*} A^{(-1)}$. Using this fact, one has $C_R^{(-1)} = L^{(-1)} \bar{*} A$ and $C_L^{(-1)} = A \bar{*} L^{(-1)}$. Consequently, one has the following observation.

Lemma 4.2 *Moreover, a_j^n is nonincreasing in n if and only if the inverse of C_R has nonpositive off-diagonals; a_j^n is nonincreasing in j if and only if the inverse of C_L has nonpositive off-diagonals.*

One can also define the pseudo-convolution between a kernel and a vector. Consider

$$V = \{x = (x_1, x_2, \cdots)^T : \quad x_i \in \mathbb{R}\}.$$

Define the pseudo-convolution $\bar{*}$: $K \times V \to V$, $y = A \bar{*} x$ by

$$y_n = \sum_{j=1}^{n} a_{n-j}^n x_j. \tag{4.4}$$

then, it holds that $A \bar{*} (B \bar{*} x) = (A \bar{*} B) \bar{*} x$.

Remark 4.1 Here, the index of the vector x starts with $i = 1$ instead of $i = 0$. This convention is adapted to the fact that there are only $n - 1$ elements "a_{n-j}^n" for fixed n, and consistent for the implicit numerical scheme used in Sect. 6 later. This should be compared to the convolution kernel $a = (a_0, a_1, \cdots)$ above which starts with index $i = 0$. The difference between "vectors" and "kernels" is not essential and they can be unified if we understand the vector x as the first column of an array kernel $b^n_{n-j} \equiv b_{n-j}$ so that $x_j = b^j_{j-1} = b_{j-1}$. Then when $a^n_{n-j} = a_{n-j}$ is a uniform convolutional kernel, $A \bar{*} x = a * b$ reduces to the usual convolution between kernels.

Next, we consider the resolvent kernels for nonuniform meshes using the pseudo-convolution:

$$R_\lambda + \lambda R_\lambda \bar{*} A = \lambda A \Longleftrightarrow A - R_\lambda \bar{*} A = \frac{1}{\lambda} R_\lambda. \tag{4.5}$$

Lemma 4.3 *Suppose the diagonal elements of A are positive and its right complementary kernel is C_R. Then, the resolvent R_λ defined by (4.5) always exists for $\lambda > 0$. Moreover, the following holds:*

(a) $R_\lambda \bar{*} A = A \bar{*} R_\lambda$, $R_\lambda \bar{*} A^{(-1)} = A^{(-1)} \bar{*} R_\lambda$;
(b) $I - R_\lambda = (I + \lambda A)^{(-1)} = \lambda^{-1} R_\lambda \bar{*} A^{(-1)}$;

The following proved in [8] describes the asymptotic behavior of the resolvents.

Lemma 4.4 *Suppose that A is invertible. The resolvent R_λ satisfies the following as $\lambda \to \infty$:*

$$R_\lambda = I - \lambda^{-1} A^{(-1)} + O(\lambda^{-2}).$$

The $O(\lambda^{-2})$ is elementwise under the limit $\lambda \to +\infty$.

4.2 Basic definitions and results

For the array kernels, the monotonicity of the kernels is not very straightforward now. We need to look at the columns and rows.

Definition 4.2 Consider an array kernel $A = (a^n_{n-j})$. We call A column monotone if it has nonnegative entries and $a^{n-1}_{j-1} \ge a^n_j$. We call it to be row monotone, if it has nonnegative entries and $a^n_{j-1} \ge a^n_j$. We call it doubly monotone if it is both column monotone and row monotone.

The column monotonicity actually means that for different time n, the approximation of the kernel a on a fixed interval $I_j = (t_{j-1}, t_j)$ is nonincreasing. The row monotonicity means that for a fixed time n, the approximation of the kernel a is monotone over different intervals I_j.

As a generalization of the uniform mesh case, we propose the following.

Definition 4.3 (a) A column monotone kernel A is called right complementary monotone (R-CMM) if its right complementary kernel C_R is doubly monotone.
(b) A row monotone kernel is called left complementary monotone (L-CMM) if its left complementary kernel is doubly monotone.

(c) A doubly monotone kernel A is complementary monotone (CMM) if it is both R-CMM and L-CMM.

We have seen that the signs of the entries of the inverse together with the non-increasing property can also be used to characterize the CMM property for uniform meshes, but this is not the case for nonuniform meshes. If $A^{(-1)}$ has positive diagonal and nonpositive off-diagonals (so that the kernel is completely positive), then C_R is row monotone but is not necessarily column monotone, even with the column monotonicity of A. An important relation (4.10) we need later is hard to establish. Hence, we use the complementary kernel to define the R-CMM property here. Note that we are not requiring the kernel A itself to be doubly monotone because the row monotonicity is not needed.

The pseudo-convolution for $n \le N$ is not affected by the data with $n > N$. Hence, one may consider the local versions of the CMM concepts.

Definition 4.4 If A is column monotone for $n \le N$ and C_R is doubly monotone for all $n \le N$, then we call A to be "local R-CMM with range N". The local L-CMM and local CMM are similarly defined.

The results below are mainly stated for R-CMM kernels, while the ones for L-CMM can be proved similarly. Moreover, we only study the global CMM properties and the local versions can be easily obtained by the local feature of the pseudo-convolution.

We now give some characterizations of the R-CMM kernels.

Theorem 4.1 *The following are equivalent.*

(a) The array kernel A is R-CMM;

(b) The right complementary kernel C_R is doubly monotone and $C_R^{(-1)}$ has positive diagonals and nonpositive off-diagonals;

(c) A is column monotone, and both $A^{(-1)}$ and $(L^{(-1)} \bar{*} A \bar{*} L)^{(-1)}$ have positive diagonals and nonpositive off-diagonals.

Proof The equivalence between (a) and (b) follows from Definition 4.3 and the fact $C_R^{(-1)} = L^{(-1)} \bar{*} A$. From (b) to (c), one only has to use the fact $C_R^{(-1)} = L^{(-1)} \bar{*} A$ and apply Lemma 4.1. From (c) to (b), one apply directly the observation $A^{(-1)} = C_R \bar{*} L^{(-1)}$ and $(L^{(-1)} \bar{*} A \bar{*} L)^{(-1)} = L^{(-1)} \bar{*} C_R$. We omit the details. □

Clearly, the column monotonicity of A is equivalent to the nonpositivity of off-diagonals in $C_R^{(-1)}$. This, together with the positive diagonals, implies that C_R must have nonnegative elements. We remark, however, the column monotonicity of A is clearly stronger than the nonnegativity of the elements of C_R.

Below, we present some necessary conditions hidden in Theorem 4.1 above.

Corollary 4.1 *Suppose A is R-CMM. Then, the following facts hold.*

(1) The kernel $L^{(-1)} \bar{*} A \bar{*} L$ has nonnegative entries, and it implies that for each $n \ge 1$,

$$\sum_{j=k}^{n+1} a_{n+1-j}^{n+1} \ge \sum_{j=k}^{n} a_{n-j}^{n}, \quad 1 \le k \le n-1. \tag{4.6}$$

Moreover, it is row monotone.

(2) Let $A^{(-1)} = B = (b^n_{n-j})$. Then It holds that

$$b^n_0 > 0, \quad b^n_{n-j} \le 0, \quad \sum_{j=1}^{n} b^n_{n-j} \ge 0, \forall n \ge 1 \quad j < n. \tag{4.7}$$

The claims in (4.7) are just reinterpretation of the signs for the elements of complementary kernels. The condition (4.6) is actually very natural, since

$$\sum_{j=k}^{n} a^n_{n-j} \approx \int_0^{\sum_{j=k}^{n} \tau_j} a(s)\, ds.$$

Below, we give a sufficient condition for A to be R-CMM. We recall a basic result in [22, Lemma 2.3]:

Lemma 4.5 *If a kernel $\tilde{A} = (\tilde{a}^n_{n-j})$ has positive entries such that*

$$\tilde{a}^{n-1}_{j-1}\tilde{a}^n_{j+1} \ge \tilde{a}^n_j \tilde{a}^{n-1}_j, \tag{4.8}$$

then the inverse has positive diagonal elements and nonpositive off-diagonal elements. If moreover, $\tilde{A}$ is column monotone, then the right complementary kernel $\tilde{C}_R$ has nonnegative elements so that $\tilde{C}_R$ is row monotone.

Note that the statement here is slight different from that in [22, Lemma 2.3]. In [22, Lemma 2.3], the conditions of column monotonicity and (4.8) are proposed together. However, if we go over the proof, one can find that the column monotonicity is used for the signs of RCC kernels, namely the last inequality in (4.7). Moreover, in [22, Lemma 2.3], they assumed strict monotonicity along columns, which is not needed by us.

As mentioned, we need C_R to be column monotone. The condition (4.8) imposed on A seems not enough for A to be R-CMM. We need to put conditions on $L^{(-1)} \bar{*} A \bar{*} L$ as well. Hence, a sufficient condition would be the following.

Proposition 4.1 *If the following conditions are satisfied:*

(1) A is column monotone;
(2) the kernels A and $L^{(-1)} \bar{*} A \bar{*} L$ both have positive elements and both satisfy (4.8);

then A is R-CMM.

Note that we are not requiring the column monotonicity for $L^{(-1)} \bar{*} A \bar{*} L$, which may not hold for schemes considered. We believe that the conditions in Proposition 4.1 are kind of strong in general. Nevertheless, we will use Proposition 4.1 for the example in Sect. 6.

The R-CMM could be preserved under a certain scaling transform. In fact, we have the following.

Lemma 4.6 *Suppose A is R-CMM and τ is diagonal as*

$$\tau = \begin{bmatrix} \tau_1 & & & & \\ & \tau_2 & & & \\ & & \vdots & & \\ & & & \tau_j & \\ & & & & \vdots \end{bmatrix}, \tag{4.9}$$

with the property that $0 < \tau_1 \le \tau_2 \le \tau_3 \cdots$. *Then,* $A \bar{*} \tau$ *is R-CMM.*

Proof For any diagonal kernel with $\tau_i > 0$, $A \bar{*} \tau$ is again column monotone. Moreover, $(A \bar{*} \tau)^{(-1)} = \tau^{(-1)} \bar{*} A^{(-1)}$. Hence, the signs of the elements in inverse of $A \bar{*} \tau$ are as desired. Moreover,

$$(L^{(-1)} \bar{*} A \bar{*} \tau \bar{*} L)^{(-1)} = L^{(-1)} \bar{*} \tau^{(-1)} \bar{*} A^{(-1)} \bar{*} L = L^{(-1)} \bar{*} \tau^{(-1)} \bar{*} C_R.$$

Note that C_R is doubly monotone. Clearly, if $\tau_1 \le \tau_2 \le \tau_3 \cdots$, then $\tau^{(-1)} \bar{*} C_R$ is column monotone. Then, $(L^{(-1)} \bar{*} A \bar{*} \tau \bar{*} L)^{(-1)}$ has the desired signs. The result then follows from Theorem 4.1. □

Note that, under this scaling, (4.8) and the column monotonicity of A are invariant (the doubly monotonicity is hard to preserve under the scaling though). Hence, the property R-CMM is kind of robust under such a scaling. (The condition (4.8) for $L^{(-1)} \bar{*} A \bar{*} L$ may be broken, though.)

4.3 R-CMM versus complete positivity

Similar to Definition 2.3, the complete positivity was introduced in [8].

Definition 4.5 An array kernel A is completely positive if $0 < (R_\lambda)_0^n < 1$, $(R_\lambda)_{n-j}^n \ge 0$ and $\sum_{j=1}^n (R_\lambda)_{n-j}^n \le 1$ for all $\lambda > 0$.

It has been shown in [8] that the following holds.

Lemma 4.7 *Let A be an invertible array kernel and $B = A^{(-1)} = (b_{n-j}^n)$. Then, A is completely positive if and only if the conditions (4.7) hold.*

Moreover, similar to Definition 4.4, due to the local property of the pseudo-convolution, we can also introduce the local complete positivity.

Definition 4.6 A kernel is called locally completely positive with range N if for all $n \le N$, $0 < (R_\lambda)_0^n < 1$, $(R_\lambda)_{n-j}^n \ge 0$ and $\sum_{j=1}^n (R_\lambda)_{n-j}^n \le 1$ for all $\lambda > 0$.

The characterization in Lemma 4.7 also holds for the local version, that is conditions (4.7) shold hold for all $n \le N$.

By Corollary 4.1, A being R-CMM is clearly stronger than being completely positive. In fact, the column monotonicity of A implies that C_R is nonnegative and thus

$\sum_{j=1}^{n}(R_\lambda)_{n-j}^{n} \le 1$. However, C_R being nonnegative cannot imply that A is column monotone. Hence, R-CMM is stronger.

Similar to the uniform mesh version in Theorem 3.1, one has the following results.

Theorem 4.2 *The following are equivalent.*

(a) The array kernel A is R-CMM;
(b) The kernel A is column monotone with positive diagonals, and both the resolvents of A and $L^{(-1)}\bar{*}A\bar{*}L$ defined in (4.5) have nonnegative entries for all $\lambda > 0$;
(c) The diagonals of A are positive and $I + \lambda A$ is R-CMM for all $\lambda > 0$.

Proof (a) $\Rightarrow$ (b): Let $B = A^{(-1)}$. Then B has positive diagonals and nonpositive off-diagonals by the R-CMM property. Note that $R_\lambda^{(-1)} = I + \lambda^{-1}A^{(-1)}$ has nonpositive off-diagonals and positive diagonals. Then, the entries of R_λ are nonnegative.

Let $M = \lambda L^{(-1)}\bar{*}A\bar{*}L$ whose inverse has nonpositive off-diagonals by the R-CMM property of A. Writing $(I + M)^{(-1)} = I - N$, then N is the resolvent of $L^{(-1)}\bar{*}A\bar{*}L$. One can similarly find $N^{(-1)} = I + M^{(-1)}$. Hence, N has nonnegative entries by Lemma 4.1.

(b) $\Rightarrow$ (c): clearly, $I + \lambda A$ is column monotone. Note that $(I + \lambda A)^{(-1)} = I - R_\lambda$ has nonpositive off-diagonals. Consider $L^{(-1)}\bar{*}(I + \lambda A)\bar{*}L = I + \lambda L^{(-1)}\bar{*}A\bar{*}L$. The inverse of $L^{(-1)}\bar{*}(I + \lambda A)\bar{*}L$ is thus I minus the resolvent of $L^{(-1)}\bar{*}A\bar{*}L$ with parameter $\lambda > 0$, and thus has nonpositive off-diagonal entries. Theorem 4.1 implies that $I + \lambda A$ is R-CMM.

(c) $\Rightarrow$ (a): the condition implies that $\lambda^{-1}I + A$ is R-CMM for all $\lambda > 0$. Taking $\lambda \to \infty$, one can show that the complementary kernel is continuous in λ^{-1} elementwise. The monotonicity and nonnegativity is preserved in the limit. □

Corollary 4.2 *If A is R-CMM, then it holds that*

$$\sum_{j=k}^{n}(R_\lambda)_{n-j}^{n} \ge \sum_{j=k}^{n-1}(R_\lambda)_{n-1-j}^{n-1}, \forall 1 \le k \le n-1. \tag{4.10}$$

Proof Since $I + \lambda A$ is also R-CMM, then $(I - R_\lambda)\bar{*}L$, as the complementary kernel of $I + \lambda A$, is doubly monotone. Hence, $R_\lambda\bar{*}L$ are nondecreasing along the columns and thus (4.10) holds. □

Remark 4.2 Note that the right complementary kernel of R_λ is $\lambda^{-1}C_R + L$ which is doubly monotone. If we can show that R_λ is column monotone, then R_λ is R-CMM, which is left for future study.

The relation (4.10) is an important property we need in Sect. 5 to show that the numerical solutions to autonomous equation on nonuniform meshes are monotone. This relation is inherently present on uniform meshes derived from complete positivity as r_λ is nonnegative. For nonuniform meshes, it is difficult to obtain using only complete positivity. The R-CMM property is slightly stronger but seems more suitable as it has built in the fading memory principle.

5 Monotonicity properties for the numerical solutions

In this section, we consider the numerical scheme (1.8). In other words

$$u = h + \bar{A}\bar{*}\tau\bar{*}f = h + A\bar{*}f, \tag{5.1}$$

where τ is given in (4.9). We will consider the schemes where A is R-CMM and investigate the two monotonicity properties mentioned in Sect. 2.1 in the discrete level.

The following fact is an analogue of Theorem 2.1. It is about the completely positve kernels and thus also R-CMM kernels.

Theorem 5.1 *Suppose $f(t,\cdot)$ is uniformly Lipschitz continuous on $[0,T]$ with Lipschitz constant $M > 0$. Assume A is locally completely positive with range N (see Definition 4.6). Assume $M\sup_{j\le N} a_0^j = M\sup_{j\le N}\tau_j\bar{a}_0^j < 1$. If $\gamma(t_n) := h^{(1)}(t_n) - h^{(2)}(t_n)$ satisfies $r = \gamma - R_\lambda\bar{*}\gamma \ge 0$ where R_λ is defined in (4.5), then the corresponding numerical solutions satisfy $u_n^{(1)} \ge u_n^{(2)}$ for all $n \le N$. In particular, if $h^{(i)}$'s are two constants and $h^{(1)} \ge h^{(2)}$, then the claim holds.*

Proof Clearly, we can consider completely positive kernels without loss of generality. Basically, if $w_n = u_n^{(1)} - u_n^{(2)}$, then

$$w_n = \gamma(t_n) + \sum_{j=1}^{n} a_{n-j}^n g_i w_i, \quad g_i = \int_0^1 \partial_u f(t_i, z u_i^{(1)} + (1-z)u_i^{(2)})\,dz.$$

This is just $w = \gamma + A\bar{*}(gw)$ where $gw = (g_1w_1, g_2w_2, \cdots)$. Taking pseudo-convolution with $I - R_\lambda$ on the left, Lemma 4.3 then gives

$$w_n = r_n + R_\lambda * (w + gw/\lambda)_n, \tag{5.2}$$

where $r = \gamma - R_\lambda\bar{*}\gamma \ge 0$. Choosing λ large enough, $1 + g_n/\lambda > 0$ for all $n \le N$. Then,

$$(1 - (R_\lambda)_0^n - (R_\lambda)_0^n g_n/\lambda)w_n = r_n + \sum_{j=1}^{n-1}(R_\lambda)_{n-j}^n w_j(1 + g_j/\lambda). \tag{5.3}$$

By Lemma 4.4,

$$1 - (R_\lambda)_0^n + (R_\lambda)_0^n\lambda^{-1}g_n = \lambda^{-1}(a_0^n)^{-1} + (1 - \lambda^{-1}(a_0^n)^{-1})\lambda^{-1}g_n + O(\lambda^{-2}).$$

Since $|g_n| \le M$, it follows that the coefficient of w_n in (5.3) is positive if λ is large enough by the condition $M\sup_{j\le N} a_0^j < 1$. Since $R_\lambda \ge 0$ by the complete positivity, the result then follows by simple induction.

If $h^{(i)}$'s are two constants with $h^{(1)} - h^{(2)} \ge 0$, using Lemma 4.7, one can conclude that $r \ge 0$. The result then follows. □

Below, we discuss the monotonicity of the solutions. Here, we only consider $h(t) \equiv u_0$ and the autonomous cases. The following result is an analogue of Theorem 2.2.

Theorem 5.2 *Assume $h(t) \equiv u_0$ and $f(t,u) \equiv f(u)$. Suppose f is Lipschitz continuous with $|f'(u)| \le M$ for some $M > 0$. If A is local R-CMM with range N for the numerical method, then for $M \sup_{j\le N} a_0^j = M \sup_{j\le N} \tau_j \bar{a}_0^j < 1$, the solution $\{u_n\}$ is monotone for all $0 \le n \le N$.*

Proof If $f(u_0) = 0$, one can see that the solution is always $u_n \equiv u_0$ (note that the numerical solution is uniquely solvable).

Below, we only focus on $f(u_0) > 0$, as the discussion for $f(u_0) < 0$ is similar. Again, we consider the R-CMM sequences without loss of generality (for the local R-CMM ones, one only needs to repeat the argument for $n \le N$).

Consider first $n = 1$. Then,

$$u_1 - a_0^1 f(u_1) = u_1 - \tau_1 \bar{a}_0^1 f(u_1) = u_0.$$

Let $\mu(u) = u - \tau_1 \bar{a}_0^1 f(u)$. Clearly, $\mu(u_0) < u_0 = \mu(u_1)$ and the function is increasing. Hence, one has $u_1 > u_0$. Now, define

$$v_n := u_{n+1} - u_n, \quad n \ge 0.$$

Hence, $v_0 > 0$ and we aim to show $v_n \ge 0$ for $n \ge 1$.

Recall A is R-CMM, and R_λ is its resolvent. Let $u - u_0 := (u_1 - u_0, u_2 - u_0, \cdots)$ and $f(u) = (f(u_1), f(u_2), \cdots)$. Then, $u - u_0 = A \bar{*} f(u)$. Taking pseudo-convolution with $I - R_\lambda$ on the left, one has

$$u_n - u_0 = R_\lambda \bar{*} (u - u_0 + f(u)/\lambda)_n,$$

which gives

$$\begin{aligned} v_n &= (R_\lambda)_n^{n+1}[u_1 - u_0 + f(u_1)/\lambda] + \sum_{j=1}^{n} (R_\lambda)_{n-j}^{n+1} \left[u_{j+1} - u_0 + \frac{f(u_{j+1})}{\lambda} \right] \\ &\quad - \sum_{j=1}^{n} (R_\lambda)_{n-j}^{n} \left[u_j - u_0 + \frac{f(u_j)}{\lambda} \right] =: I_1 + I_2 + I_3. \end{aligned}$$

Mimicking the proof for the uniform mesh case in Theorem 2.3, one would arrange it into the following

$$\begin{aligned} I_2 + I_3 &= \sum_{j=1}^{n} [(R_\lambda)_{n-j}^{n+1} - (R_\lambda)_{n-j}^{n}] \left[u_{j+1} - u_0 + \frac{f(u_{j+1})}{\lambda} \right] \\ &\quad + \sum_{j=1}^{n} (R_\lambda)_{n-j}^{n} (v_j + \lambda^{-1} g_j v_j), \end{aligned} \tag{5.4}$$

where $g_j = \int_0^1 f'(zu_{j+1} + (1-z)u_j)\,dz$. The issue is that the sign of the first term is not easy to determine. Hence, we must rearrange the terms to resolve this. We rewrite

$$u_{j+1} - u_0 = u_1 - u_0 + \sum_{\ell=1}^{j} v_\ell, \quad f(u_{j+1}) = f(u_1) + \sum_{\ell=1}^{j} g_\ell v_\ell.$$

Introduce the notation

$$S_j^n := \sum_{\ell=j}^{n} (R_\lambda)_{n-\ell}^{n+1} - \sum_{\ell=j+1}^{n} (R_\lambda)_{n-\ell}^{n}.$$

By relation (4.10), $S_j^n \geq 0$. Using S_j^n, one then has

$$I_1 + I_2 + I_3 = S_0^n(u_1 - u_0 + f(u_1)/\lambda) + \sum_{j=1}^{n-1} v_j(1 + g_j/\lambda)S_j^n \\ + v_n(1 + g_n/\lambda)(R_\lambda)_0^{n+1},$$

so that

$$v_n(1 - (R_\lambda)_0^{n+1} - (R_\lambda)_0^{n+1}\lambda^{-1}g_n) = \sum_{j=1}^{n-1} v_j(1 + g_j/\lambda)S_j^n + \gamma_n, \tag{5.5}$$

where

$$\gamma_n = S_0^n(u_1 - u_0 + f(u_1)/\lambda).$$

Hence, for λ sufficiently large $u_1 - u_0 + f(u_1)/\lambda > 0$ and thus $\gamma_n \geq 0$.

Now, consider the coefficient of v_n. By Lemma 4.4,

$$\begin{aligned} \epsilon_n &:= 1 - (R_\lambda)_0^{n+1} + (R_\lambda)_0^{n+1}\lambda^{-1}g_n \\ &= \lambda^{-1}(a_0^{n+1})^{-1} + (1 - \lambda^{-1}(a_0^{n+1})^{-1})\lambda^{-1}g_n + O(\lambda^{-2}). \end{aligned}$$

Since $|g_n| \leq M$ and $a_0^{n+1} = \bar{a}_0^{n+1}\tau_{n+1}$, if $Ma_0^{n+1} < 1$, then $\epsilon_n > 0$ for λ large enough.

For each n, one may choose a suitably large λ (depending on n) so that the coefficients are positive to find $v_n \geq 0$, by induction. □

Remark 5.1 For time fractional ODEs, $\bar{a}_0^n \sim \frac{1}{\Gamma(\alpha+1)}\tau_n^{\alpha-1}$. Then, the condition on the stepsize in Theorem 5.2 agrees with the one in [20].

We perform some discussion on the time fractional differential equations here as they are an important class of the Volterra integral equations we considered. The kernels for time fractional ODEs are clearly CM, which is thus log-convex and CMM.

The so-called CM-preserving schemes on uniform meshes were proposed in [20]. These schemes are on the uniform meshes and require the discrete kernel $\{a_j\}$ to be a CM sequence. It has been shown that the CM-preserving methods have many good stability properties and have been used successfully to prove some sharp estimates [3]. The standard $L1$ scheme, the Grünwald-Letnikov method, and the method with averaged kernel are CM-preserving for the time fractional ODEs [20]. The CM-preserving schemes are clearly CMM by Remark 3.2. However, such methods may be restricted in applications, as they are restricted to uniform meshes and high order schemes often break the CM property.

Another option is to consider discretization that are log-convex, which is related to the positive definiteness of the methods and may enjoy some good properties [22]. As indicated in Lemma 3.1, the monotonicity, nonnegativity and the log-convexity $a_{j+1}a_{j-1} \ge a_j^2$ will suffice for the CMM property on uniform meshes. This means that the CMM property is in fact kind of weak for uniform meshes.

As for the nonuniform meshes, the results in Proposition 6.1 and 6.2 later for the time fractional ODE are interesting in the sense that there is no restriction on the ratios τ_{j+1}/τ_j for the stepsizes. This may indicate that the R-CMM property is quite flexible in practice.

6 Application to fractional ODEs

We now consider some schemes for the fractional ODE (1.3). The L1 scheme [24, 34] is the most popular discretization, obtained by linear interpolation of the derivative u' in the differential form:

$$D^\alpha u(t_n) \approx \mathcal{D}_\tau^\alpha u_n := C\bar{*}\nabla_\tau u_n = C\bar{*}L^{(-1)}\bar{*}(u-u_0)_n, \tag{6.1}$$

where

$$c_{n-j}^n = \frac{1}{\tau_j \Gamma(1-\alpha)} \int_{t_{j-1}}^{t_j} (t_n - s)^{-\alpha}\, ds. \tag{6.2}$$

Hence, the implicit discretization using the L1 scheme is then

$$\mathcal{D}_\tau^\alpha u_n = f(t_n, u_n), \quad n \ge 1. \tag{6.3}$$

Clearly, this scheme can be converted into the integral form using the pseudo-convolution, resulting in

$$u_n = u_0 + A\bar{*}f(t_i, u_i) = u_0 + \bar{A}\bar{*}\tau\bar{*}f, \quad A = B^{(-1)}.$$

In this scheme, c_{n-j}^n is the *average* of $(t_n - s)^{-\alpha}/\Gamma(1-\alpha)$ on $[t_{j-1}, t_j]$ while a_{n-j}^n is the *integral* of $(t_n - s)^{\alpha-1}/\Gamma(\alpha)$ on this interval. In fact, the scaling of c_0^n is like $\tau_n^{-\alpha}$ so that the scaling of a_0^n is like τ_n^α. The differential scheme is said to be R-CMM if A is R-CMM.

Proposition 6.1 *The L1 scheme* (6.3) *is always R-CMM.*

We just outline the proof: C is clearly doubly monotone and satisfies $c_{j-1}^{n-1}c_{j+1}^{n} \geq c_j^n c_j^{n-1}$. Then, $C^{(-1)}$ has positive diagonals and nonpositive off-diagonals. Then, A is R-CMM by Theorem 4.1 (b).

Next, we consider the integral Eq. (1.5) for the fractional ODEs. We consider the simplest scheme for the integral on the nonuniform meshes.

$$u_n = u_0 + \sum_{j=1}^{n} a_{n-j}^n f(t_j, u_j), \tag{6.4}$$

with the coefficients given by

$$a_{n-j}^n = \frac{1}{\Gamma(\alpha)} \int_{t_{j-1}}^{t_j} (t_n - s)^{\alpha-1}\, ds. \tag{6.5}$$

That means we approximate $f(s, u(\cdot))$ using the constant interpolation with the right point on each time interval. Such an integral averaged method on uniform meshes has been applied to investigate the time continuous fractional gradient flows and the fractional SDEs in [18].

Proposition 6.2 *The kernel A in the scheme* (6.4), (6.5) *for the fractional ODE is R-CMM.*

Proof We basically apply Proposition 4.1 to show that A is R-CMM. Here, we verify the conditions. Let $A = (a_{n-j}^n)$. It is clear that

$$a_{n-j}^n = \frac{1}{\Gamma(\alpha+1)}((t_n - t_{j-1})^\alpha - (t_n - t_j)^\alpha).$$

Clearly, A is strictly column monotone (strictly decreasing along the columns).

Step 1 We verify the log-convexity condition (4.8). Fixing n, consider the ratio

$$r_j^n = \frac{a_{n-j}^n}{a_{n-1-j}^{n-1}} = \frac{(t_n - t_{j-1})^\alpha - (t_n - t_j)^\alpha}{(t_{n-1} - t_{j-1})^\alpha - (t_{n-1} - t_j)^\alpha}.$$

We now verify that r_j^n is decreasing for j. To do this, we consider the function

$$\theta(x, y) = \frac{(t_n - t_{j-1} - x)^\alpha - (t_n - t_j - y)^\alpha}{(t_{n-1} - t_{j-1} - x)^\alpha - (t_{n-1} - t_j - y)^\alpha}, \quad 0 \leq x \leq \tau_j,\ 0 \leq y \leq \tau_{j+1}.$$

Clearly, $0 \leq \theta \leq 1$. Moreover,

$$\frac{\partial \theta}{\partial x} = [(t_{n-1} - t_{j-1} - x)^\alpha - (t_{n-1} - t_j - y)^\alpha]^{-2} \alpha A_x,$$

where

$$A_x = (t_{n-1} - t_{j-1} - x)^{\alpha-1}[(t_n - t_{j-1} - x)^\alpha - (t_n - t_j - y)^\alpha] \\ - (t_n - t_{j-1} - x)^{\alpha-1}[(t_{n-1} - t_{j-1} - x)^\alpha - (t_{n-1} - t_j - y)^\alpha].$$

To show $A_x \le 0$, consider the mapping $z \mapsto A_x(z)$ for $0 \le z \le \tau_j + y - x$:

$$A_x(z) := (t_{n-1} - t_{j-1} - x)^{\alpha-1}[(t_n - t_{j-1} - x)^\alpha - (t_n - t_{j-1} - x - z)^\alpha] - \\ (t_n - t_{j-1} - x)^{\alpha-1}[(t_{n-1} - t_{j-1} - x)^\alpha - (t_{n-1} - t_{j-1} - x - z)^\alpha].$$

One can find that $A_x(0) = 0$ and

$$A_x'(z) = \alpha(t_{n-1} - t_{j-1} - x)^{\alpha-1}(t_n - t_{j-1} - x - z)^{\alpha-1} \\ -\alpha(t_n - t_{j-1} - x)^{\alpha-1}(t_{n-1} - t_{j-1} - x - z)^{\alpha-1}.$$

Since $(t_n - t_{j-1} - x)(t_{n-1} - t_{j-1} - x - z) > (t_{n-1} - t_{j-1} - x)(t_n - t_{j-1} - x - z)$ for $z > 0$, one has $A_x'(z) < 0$. Hence, $A_x = A_x(\tau_j + y - x) < 0$.

For the derivative on y, the calculation is similar. In fact

$$\frac{\partial\theta}{\partial y} = [(t_{n-1} - t_{j-1} - x)^\alpha - (t_{n-1} - t_j - y)^\alpha]^{-2}\alpha A_y,$$

with

$$A_y = (t_n - t_j - y)^{\alpha-1}[(t_{n-1} - t_{j-1} - x)^\alpha - (t_{n-1} - t_j - y)^\alpha] \\ -(t_{n-1} - t_j - y)^{\alpha-1}[(t_n - t_{j-1} - x)^\alpha - (t_n - t_j - y)^\alpha].$$

Here, we consider $z \mapsto A_y(z)$ for $0 \le z \le y + \tau_j - x$:

$$A_y(z) := (t_n - t_j - y)^{\alpha-1}[(t_{n-1} - t_j - y + z)^\alpha - (t_{n-1} - t_j - y)^\alpha] \\ -(t_{n-1} - t_j - y)^{\alpha-1}[(t_n - t_j - y + z)^\alpha - (t_n - t_j - y)^\alpha].$$

Using similar trick, one can show that $A_y(0) = 0$ and $A_y'(z) < 0$ for $z > 0$. Hence, $A_y = A_y(y + \tau_j - x) < 0$. Hence, θ is decreasing on the region considered. This then verifies that $r_j^n > r_{j+1}^n$, and thus (4.8).

Step 2 We consider $\Gamma(1+\alpha)L^{(-1)}\bar{*}A\bar{*}L = (\beta_{n-j}^n)$. Then, we show the log-convexity condition (4.8) for this kernel. It is not hard to determine that

$$\beta_{n-j}^n = (t_n - t_{j-1})^\alpha - (t_{n-1} - t_{j-1})^\alpha.$$

It is straightforward to see that every element is positive. Note that this is row monotone, and may not be column monotone.

To verify (4.8), the ratio considered would be different

$$\bar{r}_j^n = \frac{\beta_{n-j}^n}{\beta_{n-(j-1)}^n} = \frac{(t_n - t_{j-1})^\alpha - (t_{n-1} - t_{j-1})^\alpha}{(t_n - t_{j-2})^\alpha - (t_{n-1} - t_{j-2})^\alpha}.$$

We will show that this is decreasing in n. The calculation is similar. Define

$$\bar{\theta}(x, y) = \frac{(t_n - t_{j-1} + x)^\alpha - (t_{n-1} - t_{j-1} + y)^\alpha}{(t_n - t_{j-2} + x)^\alpha - (t_{n-1} - t_{j-2} + y)^\alpha}, \quad 0 \le x \le \tau_{n+1}, 0 \le y \le \tau_n.$$

It can be shown similarly that

$$\frac{\partial \bar{\theta}}{\partial x} < 0, \quad \frac{\partial \bar{\theta}}{\partial y} < 0.$$

This then implies that $\bar{\theta}$ is decreasing on the region considered. Hence, we find

$$\bar{r}_j^n > \bar{r}_j^{n+1}.$$

This is equivalent to (4.8).

Hence, A and $L^{(-1)} \bar{*} A \bar{*} L$ verify the conditions in Proposition 4.1, so A is R-CMM. □

7 Numerical experiments

In this section, we perform some numerical experements to support our theory.

7.1 Example 1: An illustrating integral equation

We consider the following integral equation

$$u(t) = h + \int_0^t a(t-s) f(s, u(s))\, ds,$$

where $h \in \mathbb{R}$ is a constant and we take

$$a(t) = \sum_{i=1}^{2} \omega_i \exp(-\lambda_i (t-s)), \quad \omega_1 = 0.6, \omega_2 = 0.4, \lambda_1 = 1, \lambda_2 = 2.$$

We use the approximation

$$u_n = h + \sum_{j=1}^{n} a_{n-j}^n f(t_j, u_j),$$

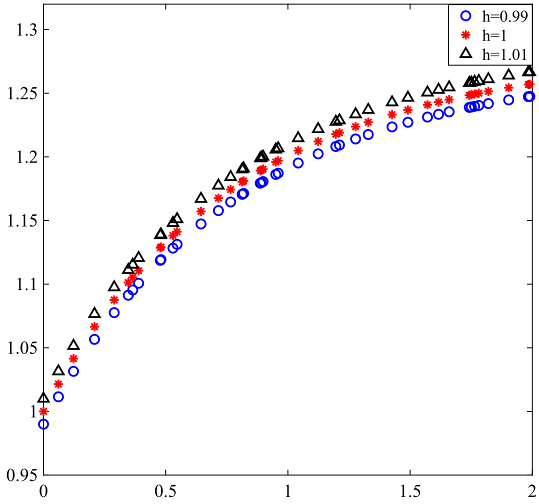

Fig. 1 The numerical solutions with a random mesh and different initial values for the integral equation with kernel $a(t) = \sum_{i=1}^{2} \omega_i \exp(-\lambda_i t)$. The order of the solution curves is preserved, and each solution curve is also monotone

where

$$a_{n-j}^n = \int_{t_{j-1}}^{t_j} a(t_n - s)\, ds = \sum_{i=1}^{2} \frac{\omega_i}{\lambda_i}(e^{-\lambda_i(t_n-t_j)} - e^{-\lambda_i(t_n-t_{j-1})}).$$

With the monotonicity of $a(\cdot)$, the column monotonicity of A is clear (for fixed j, with increasing n, a_{n-j}^n becomes smaller). Similar as in the proof of Proposition 6.2, we may consider

$$r_j^n := \frac{a_{n-j}^n}{a_{n-1-1}^{n-1}} = \frac{a(t_n - s_{jn})}{a(t_{n-1} - s_{jn})}, \quad \text{for some } s_{jn} \in (t_{j-1}, t_j).$$

Here, we have applied the mean value theorem that $\int_{t_{j-1}}^{t_j} f(s)\, ds / \int_{t_{j-1}}^{t_j} g(s)\, ds = f(s')/g(s')$ for some $s' \in (t_{j-1}, t_j)$ if g is positive on (t_{j-1}, t_j). It can be verified directly that $t \mapsto a(\tau + t)/a(t)$ is increasing for any $\tau > 0$ as $t \mapsto a(t)$ is log-convex. Hence, as j is larger, $t_{n-1} - s_{jn}$ is smaller, so that r_j^n is decreasing, which implies that $r_{j+1}^n < r_j^n$. One can similar consider β_{n-j}^n and $\bar{r}_j^n$ as in the proof of Proposition 6.2 and find that $\bar{r}_j^{n+1} < \bar{r}_j^n$. This means that the discretization is R-CMM.

We take

$$f(u) = ue^{-u}$$

and run the numerical experiments with initial values $u_0 = 0.99, 1, 1.01$ for a random mesh $\tau_j \sim 0.1 \cdot U(0, 1)$, where $U(0, 1)$ is the uniform distribution on $(0, 1)$. The results are shown in Fig. 1. Clearly, both types of monotonicity are preserved for the numerical results, agreeing with our theory.

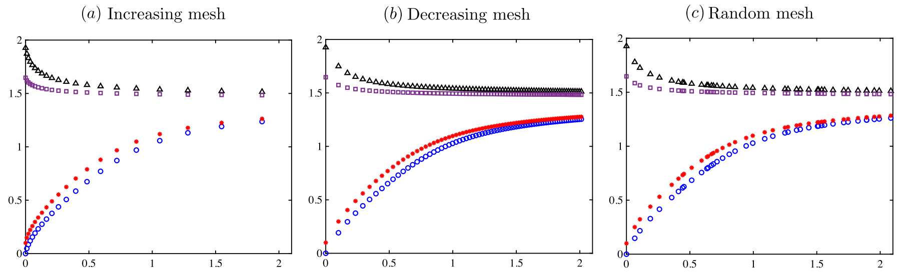

Fig. 2 Numerical solutions with different meshes and different initial values for L1 scheme. **a** increasing mesh; **b** decreasing mesh; **c** random mesh. The orders of the solution curves stay monotone, and each curve is also monotone

7.2 Example 2: A simple fractional ODE

In this example, we perform the numerical simulations for the time fractional ODEs. In particular, we verify numerically that both the L1 discretization (6.3) and the integral averaged scheme (6.4), (6.5) are able to preserve the monotonicity properties of the numerical solutions.

As an example, we consider the following fractional ODE.

$$D_c^{\alpha} u = \sin(1 + u^2), \quad u(0+) = u_0 \Longleftrightarrow u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(u(s))\, ds,$$

where we take $\alpha = 0.6$. We use three types of meshes:

- Increasing mesh with $\tau_1 = 0.01$ and $\tau_{j+1}/\tau_j = 1.2$ for $j \geq 1$;
- Decreasing mesh with $\tau_1 = 0.1$ and $\tau_j = 0.1(1 + 0.5j)^{-1/2}$;
- Random mesh with $\tau_j \sim 0.1 \cdot U(0, 1)$, where $U(0, 1)$ indicates the uniform distribution on $(0, 1)$.

For the L1 scheme, the solution will be determined inductively by solving the implicit equation

$$c_0^n u_n - f(u_n) = c_{n-1}^n u_0 + \sum_{j=1}^{n-1} (c_{n-j-1}^n - c_{n-j}^n) u_j,$$

while the numerical solution to the integral averaged scheme can be determined inductively by the implicit equation

$$u_n - a_0^n f(u_n) = u_0 + \sum_{j=1}^{n-1} a_{n-j}^n f(u_j).$$

The numerical results with initial values $u_0 = 0, 0.1, \sqrt{3\pi/2 - 1}, \sqrt{3\pi/2 - 2}$ are shown in Figs. 2 and 3, for the L1 scheme and the integral averaged scheme

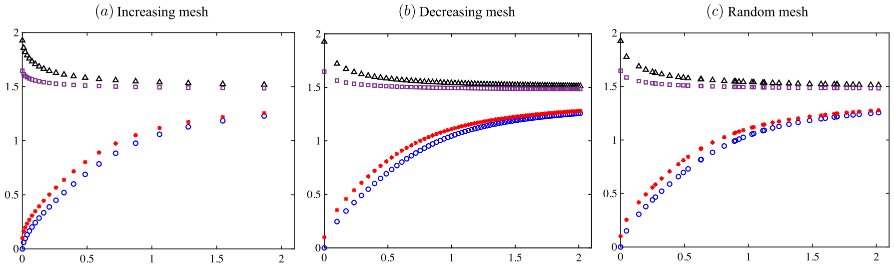

Fig. 3 Numerical solutions with different meshes and different initial values for the integral averaged scheme. **a** increasing mesh; **b** decreasing mesh; **c** random mesh. The orders of the solution curves stay monotone, and each curve is also monotone

respectively. We find that the two monotonicity properties are preserved for these numerical solutions over each nonuniform mesh. (In the figure, the monotonicity for the $u_0 = \sqrt{3\pi/2 - 2}$ curve is not very clear. With the concrete numerical values, we have checked that they are indeed monotone.) This then verifies the theory.

8 Conclusion

We have studied variable-step discretizations of Volterra integral equations with nonincreasing completely positive kernels in this paper. The "complementary monotonicity" has been proposed to describe the nonincreasing completely positive kernels on uniform meshes, and the "right complementary monotone" (R-CMM) kernels has been proposed as the analogue for nonuniform meshes. The properties of these kernels are studied using the resolvent kernels, via the so-called "pseudo-convolution" which is given by the matrix product between the array kernels and is a generalization of the usual convolution on uniform meshes. We then establish the monotonicity properties of the numerical solutions inherited from the continuous equation if the discretization has the R-CMM property.

The tools and results in this paper may be helpful to understand the structure-preserving properties of the numerical solutions on nonuniform meshes. There are many interesting questions for future study. For example, it would be interesting to generalize the results to other types of integral equations. It is also an interesting question whether some high order schemes could have similar properties or not.

A The proofs for the monotonicity properties

We first present the proof for Theorem 2.1.

Proof of Theorem 2.1 Let $v(t) := u_1(t) - u_2(t)$ and $[0, T_b)$ be the common interval of existence. Then,

$$v(t) = \gamma(t) + \int_0^t a(t-s)g(s)v(s)\,ds,$$

$$g(s) := \int_0^1 \partial_u f(s, zu_1(s) + (1-z)u_2(s))\, dz.$$

Convolving both sides with $\delta - r_\lambda$, one has

$$v(t) - r_\lambda * v(t) = (\delta - r_\lambda) * \gamma(t) + \lambda^{-1} r_\lambda * (gv).$$

This implies that

$$v(t) = \beta_\lambda(t) + \int_0^t r_\lambda(t-s)[1 + \lambda^{-1} g(s)]v(s)\, ds. \tag{A.1}$$

For any $t \in [0, T_b)$, u_1, u_2 are bounded on $[0, t]$ and hence one can choose λ large enough to make $1 + g(s)/\lambda > 0$ on $[0, t]$. Recall $\beta_\lambda(t) \geq 0$. Since $\beta_\lambda \equiv 0$ is trivial, we assume that $\beta_\lambda > 0$ somewhere.

If $\beta_\lambda(0) > 0$ (or equivalently $\gamma(0) > 0$), then $v(0) = \beta_\lambda(0) > 0$ so that $v(s) > 0$ for s small enough. If v ever reaches 0 at a first time t_*, then (A.1) tells us that

$$0 = v(t_*) = \beta_\lambda(t_*) + \int_0^{t_*} r_\lambda(t_* - s)[1 + \lambda^{-1} g(s)]v(s)\, ds > 0.$$

This is a contradiction, so $v(t) > 0$ for all $t \in [0, T_b)$.

Now, consider the degenerate case, namely $\beta_\lambda(t)$ is zero on $[0, t_1]$ and $\beta_\lambda(t) > 0$ on $(t_1, t_1 + \delta)$ for some $\delta > 0$. Then, by (2.3), $\gamma = \beta_\lambda + \lambda a * \beta_\lambda$ is also zero on $[0, t_1]$. By the uniqueness of the solution to (1.1), $u_1(t) = u_2(t)$ or $v(t) = 0$ on $[0, t_1]$. Assume for the purpose of contradiction that $v(s) < 0$ on $(t_1, t_1 + \delta_1)$ for some $\delta_1 \leq \delta$. Fix λ such that $1 + h(s)/\lambda > 0$ on $(t_1, t_1 + \delta_1)$. Let $A := \sup_{s \in (t_1, t_1 + \delta_1)} (1 + h(s)/\lambda) > 0$. We take $\epsilon \in (0, \delta_1]$ such that $\int_0^\epsilon r_\lambda(s)\, ds < 1/(2A)$. Let $t_2 = \operatorname{argmin}_{s \in [t_1, t_1 + \epsilon]} v(s) \in (t_1, t_1 + \epsilon]$. Then,

$$\begin{aligned} v(t_2) &= \beta_\lambda(t_2) + \int_{t_1}^{t_2} r_\lambda(t_2 - s)(1 + h(s)/\lambda)v(s)\, ds \\ &\geq \beta_\lambda(t_2) + Av(t_2) \int_{t_1}^{t_2} r_\lambda(t_2 - s)\, ds \geq \beta_\lambda(t_2) + v(t_2)/2. \end{aligned}$$

This is a contradiction. Therefore, $v(t) \geq 0$ for $t \in (t_1, t_1 + \delta_1)$ for some $\delta_1 \leq \delta$, which can be strengthened to $v(t) > 0$ for $t \in (t_1, t_1 + \delta_1)$ by (A.1). Then, (A.1) again implies that $v(t) > 0$ for $t \in (t_1, T_b)$. □

Below we present the proof for Theorem 2.2.

Proof of Theorem 2.2 Convolving both sides of the equation with $\delta - r_\lambda$, one has

$$u(t) = h(t) + (u - h) * r_\lambda + \lambda^{-1} r_\lambda * f(\cdot, u(\cdot)).$$

Taking derivative on both sides which is feasible since u is absolutely continuous, one has

$$u'(t) = h'(t) + r_\lambda(t)(u(0) - h(0)) + r_\lambda * (u' - h') \\ +\lambda^{-1} r_\lambda(t) f(0, u(0)) + \lambda^{-1} r_\lambda * (\partial_t f + \partial_u f u'). \tag{A.2}$$

Sending $t \to 0^+$ in the equation, one has $u(0) = h(0)$. Then, the equation is reduced to

$$u'(t) = \beta(t) + r_\lambda * ([1 + \lambda^{-1}\partial_u f]u'),$$

where

$$\beta(t) := (\delta - r_\lambda) * h' + \lambda^{-1} r_\lambda(t) f(0, h(0)) + \lambda^{-1} r_\lambda * \partial_t f(t, u).$$

The conclusions hold by similar arguments as in the proof of Theorem 2.1. □

B Sketch of the proof for the time continuous CMM property

Proof of Proposition 3.1 Since a is CMM if and only if a^c is CMM, hence if we can show the equivalence between (a) and (b), then the equivalence between (a) and (c) follows automatically.

(a) ⇒ (b): let $a^c = \tilde{\alpha}^c \delta + k$ be the complementary kernel. Then, k is nonnegative and nonincreasing and integrable on $(0, T)$. Consider the kernel a_ϵ defined by

$$a_\epsilon * ((\epsilon + \tilde{\alpha}^c)\delta + k) = 1_{t \ge 0}.$$

Then, a_ϵ is absolutely continuous. Taking $t \to 0^+$, one deduces that

$$\epsilon + \tilde{\alpha}^c = 1/a_\epsilon(0).$$

Let $r_{\epsilon,\lambda}$ be the resolvent for a_ϵ. Using the intuition that $\delta - r_{\epsilon,\lambda} = \lambda^{-1} a_\epsilon^{(-1)} * r_{\epsilon,\lambda}$, one may obtain that

$$1 - 1 * r_{\epsilon,\lambda} = \lambda^{-1} r_{\epsilon,\lambda} * a_\epsilon^c = (\lambda a_\epsilon(0))^{-1} r_{\epsilon,\lambda} + \lambda^{-1} r_{\epsilon,\lambda} * k.$$

This can actually be justified rigorously and see the proof in [4, Theorem 2.2]. With this, one has an equation for $r_{\epsilon,\lambda}$:

$$r_{\epsilon,\lambda} + a_\epsilon(0)(k + \lambda) * r_{\epsilon,\lambda} = \lambda a_\epsilon(0).$$

Since $(k + \lambda)$ is positive and nonincreasing, one can then show that $r_{\epsilon,\lambda} > 0$. Then, $1 - 1 * r_{\epsilon,\lambda} = (\lambda a_\epsilon(0))^{-1} r_{\epsilon,\lambda} + \lambda^{-1} r_{\epsilon,\lambda} * k \ge 0$ follows.

Next, one aims to take the limit $\epsilon \to 0^+$. This limit is not straightforward as the atom may appear. The approach is to convolve with absolutely continuous functions

z with $z(0) = 0$ and $z \geq 0$. Then, show $u_\epsilon := r_{\epsilon,\lambda} * z \to r_\lambda * z =: u$ by the equations they satisfy. In particular, $r_\lambda * (\lambda^{-1} a^c + 1_{t\geq 0}) = 1_{t\geq 0}$, one has

$$(\lambda 1_{t\geq 0} + \tilde{\alpha}^c \delta + k) * u = \lambda 1_{t\geq 0} * z, \quad \epsilon u_\epsilon + (\lambda 1_{t\geq 0} + \tilde{\alpha}^c \delta + k) * u_\epsilon = \lambda 1_{t\geq 0} * z$$

Taking the difference,

$$\epsilon(u_\epsilon - u) + (\lambda 1_{t\geq 0} + \tilde{\alpha}^c \delta + k) * (u_\epsilon - u) = -\epsilon u.$$

One readily shows that $\int_0^t |u_\epsilon - u|\, ds \to 0$. This implies that $u \geq 0$. Then, $r_\lambda \geq 0$. The sign of s is similarly proved using the relation between r_λ and s_λ.

(b) $\Rightarrow$ (a): the main idea is that $\lambda s_\lambda * (\lambda^{-1}\delta + a) = 1_{t\geq 0}$. It is expected that $s_\lambda \to 0$ as $\lambda \to \infty$ since r_λ is close to δ as $\lambda \to \infty$. Hence, the goal is to take certain limit of λs_λ such that the limit would be the complementary kernel, which is then nonnegative and nonincreasing.

By the conditions given, it can be shown that λs_λ is uniformly bounded by $\left(\int_0^t b(\tau)\, d\tau\right)^{-1}$ which is uniformly bounded on $[\epsilon, T]$ for any $\epsilon > 0$. To find a convergent subsequence, one regard λs_λ as a family of measures and then consider the topology tested against the absolutely continuous functions. The narrow limit is then the complementary kernel, which could possibly have an atom at $t = 0$. This intuition can be made rigorous and see the proof of [4, Theorem 2.2]. □

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Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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