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On the mean field limit of the Random Batch Method for interacting particle systems

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Abstract The Random Batch Method proposed in our previous work (Jin et al. J Comput Phys, 2020) is not only a numerical method for interacting particle systems and its mean-field limit, but also can be viewed as a model of the particle system in which particles interact, at discrete time, with randomly selected mini-batch of particles. In this paper, we investigate the mean-field limit of this model as the number of particles $N \to \infty$. Unlike the classical mean field limit for interacting particle systems where the law of large numbers plays the role and the chaos is propagated to later times, the mean field limit now does not rely on the law of large numbers and the chaos is imposed at every discrete time. Despite this, we will not only justify this mean-field limit (discrete in time) but will also show that the limit, as the discrete time interval $\tau \to 0$, approaches to the solution of a nonlinear Fokker-Planck equation arising as the mean-field limit of the original interacting particle system in the Wasserstein distance.

Keywords Random Batch Method, mean field limit, chaos, Wasserstein distance, nonlinear Fokker-Planck equation

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1 Introduction

Many physical, biological and social sciences phenomena, at the microscopic level, are described by interacting particle systems, for example, molecules in fluids [19], plasma [5], swarming [7, 9, 13, 48], chemotaxis [4,26], flocking [1,12,25], synchronization [11,24] and consensus [43]. We consider the following general first order systems:

$$dX^i = b(X^i)dt + \frac{1}{N-1}\sum_{j:j\neq i} K(X^i - X^j)dt + \sqrt{2}\sigma dW^i, \quad i = 1, 2, \dots, N \tag{1.1}$$

with the initial data X_0^i's being independent and identically distributed (i.i.d.), sampled from a common distribution μ_0. W^i's are N independent d-dimensional Wiener processes (standard Brownian motions). Here, we allow $\sigma = 0$ to include systems without noise.

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As is well known, under certain conditions, the mean field limit (i.e., $N \to \infty$) of (1.1) is given by

$$\partial_t \mu = -\nabla \cdot ((b(x) + K * \mu)\mu) + \sigma^2 \Delta \mu. \tag{1.2}$$

The term "mean field limit" means that the empirical measure $\mu_N := N^{-1} \sum_{i=1}^N \delta(x - X^i)$ converges weakly to μ almost surely and the one marginal distribution $\mu_N^{(1)} := \mathscr{L}(X^1)$, the law of X^1, converges to μ. See [10,15,18,37] for some related models and proofs, though the setups in these works do not quite fit our problem as we allow $|b(\cdot)|$ to have polynomial growth. Recall that μ is in general a probability distribution and (1.2) is understood in the distributional sense. We will denote the solution operator to (1.2) by $\mathcal{S}$:

$$\mathcal{S}(\Delta)\mu(t_1) := \mu(t_1 + \Delta), \quad \forall\, t_1 \geqslant 0, \quad \Delta \geqslant 0. \tag{1.3}$$

Clearly, $\{\mathcal{S}(t) : t \geqslant 0\}$ is a nonlinear semigroup.

A direct simulation of (1.1) costs $O(N^2)$ per time step, which is expensive. To reduce the computational cost, in [28], a random algorithm that uses random mini-batches, called the Random Batch Method (RBM), has been proposed to reduce the computation cost per time step from $O(N^2)$ to $O(N)$. The method has been applied to various problems with promising results [28, 32, 35, 36]. However, the understanding of the method is still limited, despite some theoretical proofs [28,29]. The idea of using the "mini-batch" was inspired by the stochastic gradient descent (SGD) method [6, 44] in machine learning. The "mini-batch" was also used for Bayesian inference [50], and similar ideas were used to simulate the mean-field equations for flocking [1]. How to apply the mini-batch depends on the specific problems. The strategy in [28] for interacting particle systems (1.1) is to do random grouping. Intuitively, the method converges due to certain time average in time, and thus the convergence is like the convergence in the law of large number (in time) (see [28] for more details). Compared with the fast multipole method, the accuracy is lower (half order in time step), but the RBM is simpler to implement and is valid for more general potentials (see [29, 35]).

The RBM algorithm corresponding to (1.1) is shown in Algorithm 1. Suppose we aim to do the simulation until time $T > 0$. We first choose a time step $\tau > 0$ and a batch size $p \ll N, p \geqslant 2$ that divides N. Define the discrete time grids $t_k := k\tau$, $k \in \mathbb{N}$. For each time subinterval $[t_{k-1}, t_k)$, there are two steps: (1) at time grid t_{k-1}, we divide the N particles into $n := N/p$ groups (batches) randomly; (2) the particles evolve with interaction inside the batches only. Here, we use the same symbols X^i without causing any confusion. The Wiener process W^i (the Brownian motion) used in (1.4) is the same as in (1.1).

Algorithm 1 The Random Batch Method (RBM)

for k in $1 : [T/\tau]$ **do**
  Divide $\{1, 2, \ldots, N\}$ into $n = N/p$ batches randomly.
  for each batch $\mathcal{C}_q$ **do**
    Update X^i's ($i \in \mathcal{C}_q$) by solving the following stochastic differential equation (SDE) with $t \in [t_{k-1}, t_k)$:

$$dX^i = b(X^i)dt + \frac{1}{p-1} \sum_{j \in \mathcal{C}_q, j \neq i} K(X^i - X^j)dt + \sqrt{2}\sigma dW^i. \tag{1.4}$$

  end for
end for

As pointed out in [28], the RBM is asymptotic-preserving regarding the mean field limit $N \to \infty$ (see [21, 34, 46]); namely, the error bound of the one marginal distribution can be made independent of N so that it can be used for large N as an efficient numerical particle method for (1.2), the mean field nonlinear Fokker-Planck equation of (1.1). While the RBM was introduced as a numerical method, it can also be viewed as a new model for the underlying particle system. A natural question for both numerical and modeling interests is: what is the limiting (mean field) dynamics as $N \to \infty$ for a fixed time step τ?

Intuitively, in a specific realization of the random division of batches, when $N \gg 1$, the probability that two chosen particles are correlated is very small. Hence, in the $N \to \infty$ limit, the two chosen particles will be uncorrelated with probability 1. Since the particles are exchangeable, the marginal distributions of them will be identical. Hence, let us focus on one specific particle, i.e., $i = 1$, to understand the mean field limit. Imagine that there are infinitely many particles as $N \to \infty$. For each time interval, we draw $p - 1$ particles from the infinite set, and they are independent from particle 1 by the intuition just mentioned. They share the same distribution with particle 1. This small group then evolves with interactions between themselves to the next time point so that the distribution of particle 1 has been changed. At this new time point, we draw another $p - 1$ particles to interact with particle 1. In this sense, in the $N \to \infty$ limit, the N-particle system is then reduced to a p-particle system described by the following stochastic differential equation (SDE) system for $t \in [t_k, t_{k+1})$:

$$dY^i = b(Y^i)dt + \frac{1}{p-1}\sum_{j=1, j\neq i}^{p} K(Y^i - Y^j)dt + \sqrt{2}\sigma dW^i, \quad i = 1, \dots, p \tag{1.5}$$

with $\{Y^i(t_k)\}$ being i.i.d., drawn from $\tilde{\mu}(\cdot, t_k)$. We may impose $Y^1(t_k^-) = Y^1(t_k^+)$, and for other particles $i \neq 1$, $Y^i(t)$ in $[t_{k-1}, t_k)$ and $[t_k, t_{k+1})$ are independent so they are not continuous at t_k. In fact, Y^i's ($i \neq 1$) correspond to the batchmates of particle 1 as in Algorithm 1 so they are different particles for different iterations. Then, $\tilde{\mu}(\cdot, t_{k+1}) = \mathscr{L}(Y^1(t_{k+1}^-))$, the law of $Y^1(t_{k+1}^-)$. In terms of the individual particle 1, the rest $N - 1$ particles average out to an infinite pool of independent particles from particle 1 at each time step t_k. This becomes the mean field limit model of the RBM, and one may write out the following mean field limit for the RBM in terms of the probability distribution as shown in Algorithm 2, while (1.5) becomes the microscopic description.

Algorithm 2 Mean field dynamics of the RBM (1.4)

1: $\tilde{\mu}(\cdot, 0) = \mu_0 \in \mathbf{P}(\mathbb{R}^d)$.
2: **for** $k \geqslant 0$ **do**
3: Let $\rho^{(p)}(\dots, t_k) = \tilde{\mu}(\cdot, t_k)^{\otimes p}$ be a probability measure on $(\mathbb{R}^d)^p \cong \mathbb{R}^{pd}$.
4: Evolve the measure $\rho^{(p)}$ by the following Fokker-Planck equation for $t \in [t_k, t_{k+1})$:

$$\partial_t \rho^{(p)} = -\sum_{i=1}^{p} \nabla_{x_i} \cdot \left(\left[b(x_i) + \frac{1}{p-1}\sum_{j=1, j\neq i}^{p} K(x_i - x_j) \right] \rho^{(p)} \right) + \sigma^2 \sum_{i=1}^{p} \Delta_{x_i} \rho^{(p)}. \tag{1.6}$$

5: Set

$$\tilde{\mu}(\cdot, t_{k+1}) := \int_{(\mathbb{R}^d)^{(p-1)}} \rho^{(p)}(\cdot, dy_2, \dots, dy_p, t_{k+1}^-). \tag{1.7}$$

6: **end for**

The dynamics shown in Algorithm 2 naturally defines a nonlinear operator $\mathcal{G}_\infty : \mathbf{P}(\mathbb{R}^d) \to \mathbf{P}(\mathbb{R}^d)$ as

$$\tilde{\mu}(\cdot, t_{k+1}) =: \mathcal{G}_\infty(\tilde{\mu}(\cdot, t_k)). \tag{1.8}$$

As indicated above, the mean field limit here does not rely on the law of large numbers. Instead, it relies on the fact that the particles in one batch are unlikely to be related if $N \gg 1$. In the mean field limit dynamics of the RBM, one starts with a chaotic configuration[1)], the p particles evolve with interaction to each other. Then, at the starting point of the next time interval, one imposes the chaos so that the particles are independent again. This mean field limit is different from the standard mean field limit for the system (1.1), given by (1.2): in the mean field limit of the RBM, the chaos is imposed at every time step; in the classical mean field limit for the interacting particle system, the chaos is propagated to later times. This mechanism may allow the mean-field limit of the RBM to achieve a higher convergence rate than the standard $N^{-1/2}$ convergence rate (at least N^{-1} under Wasserstein-1 as seen in Section 3). In spite of the difference just mentioned, we will show that these two limiting dynamics are in fact close: in Section 4, we will show that as $\tau \to 0$ the dynamics given by $\mathcal{G}_\infty$ can approximate that of the nonlinear

1) By "chaotic configuration", we mean that there exists a one particle distribution f such that for any j, the j-marginal distribution is given by $\mu^{(j)} = f^{\otimes j}$. Such independence in a configuration is then loosely called "chaos". If the j-marginal distribution is more close to $f^{\otimes j}$ for some f, we loosely say "there is more chaos".

Fokker-Planck equation (1.2). We remark that as $\tau \to 0$, the dynamics of the RBM has been shown to converge to the N-particle system (1.1) in [28]. Thus, this result implies that the two limits $\lim_{N\to\infty}$ and $\lim_{\tau\to 0}$ commute (see also Subsection 5.1 and Figure 3).

The argument in this paper for $t \leqslant T$ can be generalized to second order systems, which we omit, but one may see Section 5 for some discussion. Of course, the argument for large time behavior can be different and this is left for future study.

The rest of this paper is organized as follows. We introduce the notations and give a brief review to the Wasserstein distance in Section 2. The mean field limit under the Wasserstein distance is shown in Section 3. Section 4 is devoted to the discussion of the mean field dynamics of the RBM. In particular, we show that it is close to the mean-field nonlinear Fokker-Planck equation. Some discussion is performed in Section 5. We finally conclude the work along with future directions in Section 6.

2 Preliminaries and notations

In this section, we first introduce some assumptions and notations. Then we give a brief introduction to the Wasserstein distance and prove some auxiliary results.

2.1 Mathematical setup of the problem

We first introduce several assumptions that will be used throughout the paper. In these assumptions, "being smooth" means that the functions are infinitely differentiable. Note that the conditions in these assumptions may be stronger than necessary.

Assumption 2.1. *The moments of the initial data are finite*:

$$\int_{\mathbb{R}^d} |x|^q \mu_0(dx) < \infty, \quad \forall\, q \in [1,\infty). \tag{2.1}$$

One of the following two conditions will be used for the external fields and interaction kernels.

Assumption 2.2. *Assume $b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ and $K(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ are smooth. Moreover, $b(\cdot)$ is one-sided Lipschitz*:

$$(z_1 - z_2)\cdot(b(z_1) - b(z_2)) \leqslant \beta|z_1 - z_2|^2 \tag{2.2}$$

for some constant β, and K is Lipschitz continuous

$$|K(z_1) - K(z_2)| \leqslant L|z_1 - z_2|.$$

Assumption 2.3. *The fields $b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ and $K(\cdot): \mathbb{R}^d \to \mathbb{R}^d$ are smooth. Moreover, $b(\cdot)$ is strongly confining*:

$$(z_1 - z_2)\cdot(b(z_1) - b(z_2)) \leqslant -r|z_1 - z_2|^2 \tag{2.3}$$

for some constant $r > 0$, and K is Lipschitz continuous $|K(z_1) - K(z_2)| \leqslant L|z_1 - z_2|$. The parameters r and L satisfy

$$r > 2L. \tag{2.4}$$

Remark 2.4. Compared with our previous works [28, 29], we are not assuming the boundedness of K in this paper to prove the mean-field limit and investigate the limiting dynamics. The boundedness of K in our previous works is a simple condition to guarantee the boundedness of the variance of the random forces (though the boundedness of variance may also be proved without assuming boundedness of K).

Denote $\mathcal{C}_q^{(k)}$ $(1 \leqslant q \leqslant n)$ the batches at t_k so that $\bigcup_q \mathcal{C}_q^{(k)} = \{1, \ldots, N\}$, and

$$\mathcal{C}^{(k)} := \{\mathcal{C}_1^{(k)}, \ldots, \mathcal{C}_n^{(k)}\} \tag{2.5}$$

will denote the random division of batches at t_k. By the Kolmogorov extension theorem [16], there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the random variables $\{X_0^i, W^i, \mathcal{C}^{(k)} : 1 \leqslant i \leqslant N, k \geqslant 0\}$ are defined on this probability space and are all independent. We will use $\mathbb{E}$ to denote the integration on Ω with respect to the probability measure $\mathbb{P}$. For the convenience of the analysis, we introduce the $L^2(\mathbb{P})$ norm as

$$\|v\| := \sqrt{\mathbb{E}|v|^2}. \tag{2.6}$$

Define the filtration $\{\mathcal{F}_k\}_{k\geqslant 0}$ by

$$\mathcal{F}_{k-1} = \sigma(X_0^i, W^i(t), \mathcal{C}^{(j)};\, t \leqslant t_{k-1},\, j \leqslant k-1). \tag{2.7}$$

Clearly, $\mathcal{F}_{k-1}$ is the σ-algebra generated by the initial values X_0^i $(i = 1, \ldots, N)$, $W^i(t)$, $t \leqslant t_{k-1}$, and $\mathcal{C}^{(j)}$, $j \leqslant k-1$. Hence, $\mathcal{F}_{k-1}$ contains the information of how batches are constructed for $t \in [t_{k-1}, t_k)$.

2.2 A review of the Wasserstein distance

Consider a domain $O \subset \mathbb{R}^n$, where n is a positive integer. We denote $\mathbf{P}(O)$ the set of probability measures on O. Let $\mu, \nu \in \mathbf{P}(O)$ be two probability measures and $c : O \times O \to [0, \infty)$ be a cost function. One solves the following optimization problem for the optimal transport:

$$\min_{\gamma} \left\{ \int_{O\times O} c\, d\gamma \,\middle|\, \gamma \in \Pi(\mu,\nu) \right\},$$

where $\Pi(\mu,\nu)$ is the set of "transport plans", i.e., a joint measure on $O \times O$ such that the marginal measures are μ and ν, respectively. If there is a map $T : O \to O$ such that $(I \times T)_{\#}\mu$ minimizes the target function, then T is called an optimal transport map. Here, I is the identity map and

$$(I \times T)_{\#}\mu(E) := \mu((I \times T)^{-1}(E)), \quad \forall\, E \subset O \times O \text{ measurable}. \tag{2.8}$$

Choosing the particular cost function $c(x,y) = |x-y|^{\mathsf{q}}$, $\mathsf{q} \in [1,\infty)$, one can define the Wasserstein-q distance $W_{\mathsf{q}}(\mu,\nu)$ as

$$W_{\mathsf{q}}(\mu,\nu) := \left(\inf_{\gamma\in\Pi(\mu,\nu)} \int_{O\times O} |x-y|^{\mathsf{q}} d\gamma \right)^{1/\mathsf{q}}. \tag{2.9}$$

It has been shown (see [3] and [45, Chapter 5]) that the Wasserstein-q distance between two probability measures μ and ν is also given by

$$W_{\mathsf{q}}^{\mathsf{q}}(\mu,\nu) = \min \left\{ \int_0^1 \|v\|_{L^{\mathsf{q}}(\rho)}^{\mathsf{q}} dt : \partial_t \rho + \nabla\cdot(\rho v) = 0,\, \rho\,|_{t=0} = \mu,\, \rho\,|_{t=1} = \nu \right\}, \tag{2.10}$$

where ρ is a (time-parametrized) nonnegative measure and

$$\|v\|_{L^{\mathsf{q}}(\rho)}^{\mathsf{q}} := \int_O |v|^{\mathsf{q}} \rho(dx). \tag{2.11}$$

Hence, v can be thought as the particle velocity for the optimal transport, as explained in [45, Chapter 5]. With this explanation, one can then understand $\mathbf{P}(O)$ equipped with W_2 distance as a Riemannian manifold so that the Fokker-Planck equations can be formulated as a class of gradient flows on this manifold (see, for example, [30] and [49, Chapter 8]).

Below, we note a useful lemma that relates the total variation distance to the W_{q} distance. This is intrinsically [49, Proposition 7.10] and the version here is more convenient for our purpose in this paper. Recall the Jordan decomposition for a signed measure $\mu = \mu^+ - \mu^-$ defined on a Polish space $\mathcal{E}$. Then, define $|\mu| := \mu^+ + \mu^-$, and the total variation norm of the signed measure by

$$\|\mu\|_{TV} := |\mu|(\mathcal{E}) = \mu^+(\mathcal{E}) + \mu^-(\mathcal{E}). \tag{2.12}$$

Lemma 2.5. *Let $\mu,\nu \in \mathbf{P}(\mathbb{R}^d)$ be two different probability measures on $\mathcal{E}=\mathbb{R}^d$. Let $\delta \geqslant 0$ and $\hat{\mu}$ be a measure such that $|\mu-\nu|(E) \leqslant \delta\hat{\mu}(E)$ for any Borel measurable E. Suppose for $\mathsf{q} \geqslant 1$, $M_{\mathsf{q}} := \inf_{x_0}\int_{\mathbb{R}^d}|x-x_0|^{\mathsf{q}}\hat{\mu}(dx) < \infty$. Then,*

$$W_{\mathsf{q}}(\mu,\nu) \leqslant 2^{1-1/\mathsf{q}}(M_{\mathsf{q}}\delta)^{1/\mathsf{q}}. \tag{2.13}$$

In particular, choosing $\delta = \|\mu-\nu\|_{TV}$ and $\hat{\mu} := \frac{1}{\|\mu-\nu\|_{TV}}|\mu-\nu|$ yields

$$W_{\mathsf{q}}(\mu,\nu) \leqslant 2^{1-1/\mathsf{q}}(M_q\|\mu-\nu\|_{TV})^{1/\mathsf{q}}.$$

Proof. We consider $\mu_m := \mu \wedge \nu$, which is defined by

$$\mu_m(E) = \min(\mu(E),\nu(E)), \quad \forall\, E \text{ measurable}.$$

Define two measures $\mu_1 := \mu - \mu_m$ and $\nu_1 := \nu - \nu_m$. Then,

$$\|\mu-\nu\|_{TV} = \|\mu_1\|_{TV} + \|\nu_1\|_{TV}, \quad \mu_1+\nu_1 \leqslant \delta\,\hat{\mu}. \tag{2.14}$$

Construct the joint distribution (noting $\|\mu_1\|_{TV} = \|\nu_1\|_{TV}$)

$$d\pi := \pi(dx,dy) = \frac{1}{\|\mu_1\|_{TV}}\mu_1(dx)\otimes\nu_1(dy) + Q_{\#}\mu_m(dx,dy)$$

with $Q(x)=(x,x)$ and $Q_{\#}$ is the standard pushforward map as in (2.8). Clearly, the marginal distributions of π are μ and ν, respectively.

Then, fix $x_0 \in \mathbb{R}^d$. We have

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^{\mathsf{q}}d\pi &= \frac{1}{\|\mu_1\|_{TV}}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^{\mathsf{q}}\mu_1(dx)\otimes\nu_1(dy)\\ &\leqslant \frac{2^{\mathsf{q}-1}}{\|\mu_1\|_{TV}}\int_{\mathbb{R}^d\times\mathbb{R}^d}(|x-x_0|^{\mathsf{q}}+|y-x_0|^{\mathsf{q}})\mu_1(dx)\otimes\nu_1(dy)\\ &= 2^{\mathsf{q}-1}\int_{\mathbb{R}^d}|x-x_0|^{\mathsf{q}}[\mu_1+\nu_1](dx).\end{aligned}$$

By noting $\mu_1+\nu_1 \leqslant \delta\,\hat{\mu}$, the claim follows by taking infimum on x_0. □

3 The mean field limit of the RBM with τ fixed

Starting with μ_0, after k steps of the dynamics given in (1.8), one arrives at

$$\mathcal{G}_\infty^k(\mu_0) = \mathcal{G}_\infty \circ \cdots \circ \mathcal{G}_\infty(\mu_0) \quad (k \text{ copies}),$$

which is expected to be the mean field limit of the RBM after k steps. Corresponding to this, one may define the operator $\mathcal{G}_N^{(k)} : \mathbf{P}(\mathbb{R}^d) \to \mathbf{P}(\mathbb{R}^d)$ for the RBM with N particles as follows. Let X_0^i's be i.i.d., drawn from μ_0. Consider (1.4) and define

$$\mathcal{G}_N^{(k)}(\mu_0) := \mathscr{L}(X^1(t_k)), \tag{3.1}$$

where recall that $\mathscr{L}(X^1)$ means the law of X^1, thus the one marginal distribution. By conditioning on a specific sequence of random batches, the particles are not exchangeable. However, when one considers the mixture of all possible sequences of random batches, the laws of the particles $X^i(t_k)$ $(1 \leqslant i \leqslant N)$ are identical. In Figure 1, we illustrate these definitions and various limits.

The semigroup property is closely related to the Markovian property. For the $\mathcal{G}_\infty$ dynamics, knowing the marginal distribution of X^1 can fully determine the probability transition. However, knowing only the marginal distribution is not enough for $\mathcal{G}_N^{(k)}$ dynamics, and the joint distribution must be known. Hence, we remark the following lemma.

Lemma 3.1. *$\{\mathcal{G}_\infty^k : k \geqslant 1\}$ forms a nonlinear semigroup while $\{\mathcal{G}_N^{(k)} : k \geqslant 1\}$ is not a semigroup.*

We first of all introduce some concepts. For each particle i, we define a sequence of lists $\{L_i^{(k)} : k \geqslant 0\}$ associated with i, given as follows:

(a) $L_i^{(0)} = \{i\}$.

(b) For $k \geqslant 1$, let $C_q^{(k-1)}$ be the batch that the particle i stays in for $t \in [t_{k-1}, t_k)$. Then,

$$L_i^{(k)} = \bigcup_{j \in C_q^{(k-1)}} L_j^{(k-1)}. \tag{3.2}$$

Here, $L_i^{(k)}$ can be viewed as the particles that have impacted i for $t < t_k$. Clearly, a particle $i_1 \in L_i^{(k)}$ might not have been a batchmate of i. It could have been a batchmate of i_2, and then i_2 was a batchmate of i at some time. The important observation is that if $L_i^{(k)}$ and $L_j^{(k)}$ do not intersect for a given sequence of random batches, then the particles i and j are independent at t_k^-. Note that we are not claiming all particles in $L_j^{(k)}$ are independent of those in $L_j^{(k)}$ at t_k^-. In fact, it is possible that some $i_1 \in L_i^{(k)}$ and $j_1 \in L_j^{(k)}$ are in the same batch on $[t_{k-1}, t_k)$. However, i_1 and j_1 must be independent at the times when they were added to the batches that eventually impact i and j at t_k^-. This motivates us to define the following definition.

Definition 3.2. We say the particle i is clean on $[t_k, t_{k+1})$ if the batch $\mathcal{C}_q^{(k)}$ that contains i at t_k^+ satisfies the following: (1) any $j \in \mathcal{C}_q^{(k)}$ is clean at t_k^-; (2) any $j, \ell \in \mathcal{C}_q^{(k)}$ with $j \neq \ell$, $L_j^{(k)}$ and $L_\ell^{(k)}$ do not intersect.

Figure 2 gives the illustration for the definitions of $L_i^{(k)}$ and particles being clean. Plainly speaking, a particle i is "clean" at t_k^- if its batchmates at $t < t_k$ were mutually independent and independent to i when they interacted.

$$\begin{array}{ccc}
\text{One marginal of } N \text{ particle system} & & \text{Nonlinear Fokker-Planck} \\
\mu_N^{(1)} & \xrightarrow{N\to\infty} & \mu = \varrho\, dx,\ \mu(\cdot,t) = \mathcal{S}(t)(\mu_0) \\
\uparrow{\scriptstyle \tau\to 0} & & \uparrow{\scriptstyle \tau\to 0} \\
\tilde{\mu}_N^{(1)} & \xrightarrow[N\to\infty]{} & \tilde{\mu} \\
\text{One marginal of the RBM} & & \text{Mean field limit of the RBM} \\
\tilde{\mu}_N^{(1)}(\cdot,t_k) = \mathcal{G}_N^{(k)}(\mu_0) & & \tilde{\mu}(\cdot,t_k) = \mathcal{G}_\infty^k(\mu_0)
\end{array}$$

Figure 1 Illustration of the various operators and the asymptotic limits

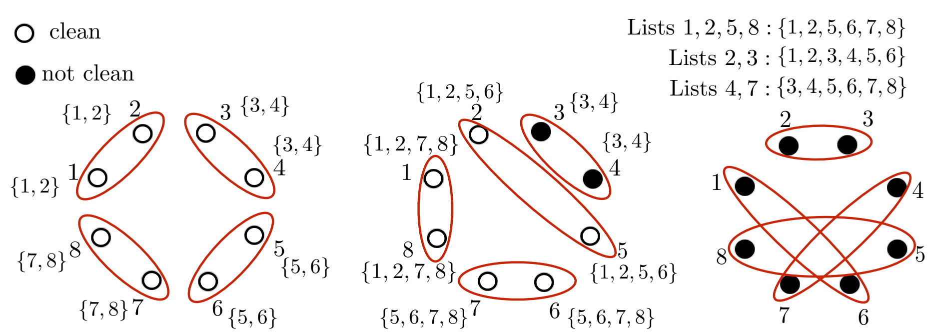

Figure 2 (Color online) Illustration of the definitions of $L_i^{(k)}$ and particles being clean. The three pictures are for t_1^-, t_2^- and t_3^-, respectively with $N = 8$, $p = 2$. The lists (i.e., $\{1, 2\}, \{1, 2, 5, 6\}$ etc.) indicate $L_i^{(k)}$ for the corresponding particles

Let us use the symbol $|A|$ below for a set A to mean the cardinality of A. The following observation is useful for our argument later.

Lemma 3.3. *Consider a fixed sequence of divisions of random batches* $\{\mathcal{C}^{(\ell)}\}_{\ell\leqslant k-1}$.

(i) *It holds that*

$$|L_i^{(k)}| \leqslant p^k,$$

and the particle i *is clean at* t_k^- *if and only if the equality holds.*

(ii) *The distribution of* X^i *for a clean particle* i *at* t_k^- *is* $\mathcal{G}_\infty^k(\mu_0)$.

Proof. The proof is a straightforward induction. Here, let us just mention the proof of the second claim briefly.

For $k=0$, the statement is trivial. Now, suppose the statement is true for all $k\leqslant m-1$. We now consider $k=m$.

For the given sequence of random batches $\{\mathcal{C}^{(\ell)}\}_{\ell\leqslant k-1}$, that a particle i is clean at t_m^- means that on $[t_{m-1},t_m)$, the particles in the batch for i are independent of i at t_{m-1}. By the induction assumption, the distribution of one particle at t_{m-1} is given by $\tilde{\mu}(\cdot,t_{m-1})=\mathcal{G}_\infty^{m-1}(\mu_0)$. By the independence, the joint distribution of them at t_{m-1} is therefore

$$\rho^{(p)}(\dots,t_{m-1})=\tilde{\mu}(\cdot,t_{m-1})^{\otimes p}.$$

From t_{m-1} to t_m, the evolution of the joint distribution obeys the Fokker-Planck equation (1.6). Hence, at t_m^-, the distribution of the particle i is given by $\mathcal{G}_\infty^m(\mu_0)$ by the definition (see (1.8)). □

Let A_k denote the set of particles that are clean at t_k^-. Then,

$$A_1=A_0=\{1,\dots,N\}.$$

For $k\geqslant 2$, one has

$$\begin{aligned} A_k=\{i\in A_{k-1}: i\in\mathcal{C}_q^{(k-1)},\ \forall\, j,\ell\in\mathcal{C}_q^{(k-1)},\ j\neq\ell,\\ j\in A_{k-1},\ \ell\in A_{k-1},\ L_j^{(k-1)}\cap L_\ell^{(k-1)}=\emptyset\}.\end{aligned}\tag{3.3}$$

Denote

$$\epsilon_k:=\mathbb{P}(1\notin A_k).\tag{3.4}$$

Note that by symmetry, ϵ_k is also the probability that the particle i is not clean. We state our main result.

Theorem 3.4. *Let* $\mathsf{q}\in[1,\infty)$. *It holds that*

$$W_{\mathsf{q}}(\mathcal{G}_\infty^k(\mu_0),\mathcal{G}_N^k(\mu_0))\leqslant C\exp(\alpha t_k)\epsilon_k^{1/\mathsf{q}}\tag{3.5}$$

for some $\alpha>0$. *Moreover, in the strong confinement case,* $\alpha=0$.

To prove Theorem 3.4, we need some preparation. We first establish some moments estimates.

Lemma 3.5. *If Assumption* 2.1 *holds, then for* $q\geqslant 1$, *there exists* $\alpha_1(q)\geqslant 0$ *such that*

$$\sup_{k:k\tau\leqslant T}\int_{\mathbb{R}^d}|x|^q\mathcal{G}_\infty^k(\mu_0)(dx)\leqslant C(q)\mathrm{e}^{\alpha_1(q)T}.\tag{3.6}$$

In the strong confinement case (*see Assumption* 2.3), *one can take* $\alpha_1(q)=0$, *i.e., the constant in the upper bound can be uniform in* T.

Proof. We note that $\{\mathcal{G}_\infty^k\}$ is a semigroup, so it suffices to estimate the growth of the moments in one step.

First, consider (1.5) and take $q\geqslant 2$. By Itô's calculus, one has

$$d\mathbb{E}|Y^i|^q=q\mathbb{E}|Y^i|^{q-2}Y^i\cdot\left[b(Y^i)+\frac{1}{p-1}\sum_{j=1,j\neq i}^{p}K(Y^i-Y^j)\right]dt+\mathbb{E}q|Y^i|^{q-2}(d+q-2)\sigma^2dt.\tag{3.7}$$

Using the one-sided Lipschitz condition in Assumption 2.2, one has

$$Y^i \cdot b(Y^i) = (Y^i - 0) \cdot (b(Y^i) - b(0)) + Y^i \cdot b(0) \leqslant \beta |Y^i|^2 + C|Y^i|.$$

Similarly, since K is Lipschitz, one has $|K(Y^i - Y^j)| \leqslant |K(0)| + L(|Y^i| + |Y^j|)$, and thus

$$Y^i \cdot K(Y^i - Y^j) \leqslant |K(0)||Y^i| + L(|Y^i|^2 + |Y^i||Y^j|).$$

It follows that

$$\begin{aligned}\mathbb{E}|Y^i|^{q-2}Y^i \cdot &\left[b(Y^i) + \frac{1}{p-1}\sum_{j=1,j\neq i}^{p} K(Y^i - Y^j)\right] \\ &\leqslant (\beta + L)\mathbb{E}|Y^i|^q + C\mathbb{E}|Y^i|^{q-1} + \frac{1}{p-1}\sum_{j:j\neq i}\mathbb{E}|Y^i|^{q-1}|Y^j|.\end{aligned}$$

By Young's inequality,

$$\mathbb{E}|Y^i|^{q-1}|Y^j| \leqslant \frac{(q-1)\nu}{q}\mathbb{E}|Y^i|^q + \frac{\mathbb{E}|Y^j|^q}{q\nu^{q-1}}$$

for any $\nu > 0$. In particular, one also has

$$\mathbb{E}|Y^i|^{q-1} \leqslant \frac{(q-1)\nu}{q}\mathbb{E}|Y^i|^q + \frac{1}{q\nu^{q-1}}.$$

Similarly, using Young's inequality, $\mathbb{E}|Y^i|^{q-2}$ is also easily controlled by $\delta\mathbb{E}|Y^i|^q + C(\delta)$ for some small δ.

By the exchangeability so that $\mathbb{E}|Y^i|^q = \mathbb{E}|Y^j|^q$, one then has

$$\frac{d}{dt}\mathbb{E}|Y^i|^q \leqslant q(\beta + 2L + \delta)\mathbb{E}|Y^i|^q + C_2.$$

In the strong confinement case as in Assumption 2.3,

$$\begin{aligned}\mathbb{E}|Y^i|^{q-2}X^i \cdot &\left[b(Y^i) + \frac{1}{p-1}\sum_{j=1,j\neq i}^{p} K(Y^i - Y^j)\right] \\ &\leqslant (-r + L)\mathbb{E}|Y^i|^q + C\mathbb{E}|Y^i|^{q-1} + \frac{L}{p-1}\sum_{j:j\neq i}\mathbb{E}|Y^i|^{q-1}|Y^j| \\ &\leqslant \left(-r + L + \frac{(q-1)L}{q}\right)\mathbb{E}|Y^i|^q + \frac{L}{p-1}\sum_{j:j\neq i}\frac{1}{q}\mathbb{E}|Y^j|^q + \delta\mathbb{E}|Y^i|^q + C(\delta) \\ &= (-r + 2L)\mathbb{E}|Y^i|^q + \delta\mathbb{E}|Y^i|^q + C(\delta),\end{aligned}$$

where δ is a sufficiently small but fixed number. The conclusions then follow easily for $q \geqslant 2$.

If $q \in [1, 2)$, one then uses Hölder's inequality $(\mathbb{E}|Y^i|^q)^{1/q} \leqslant (\mathbb{E}|Y^i|^r)^{1/r}$ for $r \geqslant q$ to get the desired result. □

We also need the moment control for the Random Batch Method conditioning on any specific sequence of random batches.

Lemma 3.6. *Consider a fixed sequence of divisions of random batches* $\{\mathcal{C}^{(\ell)}\}$. *Again, consider the solutions* $\{X^i(t)\}_{i=1}^N$ *to* (1.6). *Then,*

$$\sup_{t\leqslant T}\sup_i \mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) \leqslant C(q)\mathrm{e}^{\alpha_1 T}. \tag{3.8}$$

In the strong confinement Assumption 2.2,

$$\sup_{t\geqslant 0}\sup_i \mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) \leqslant C(q), \tag{3.9}$$

where $C(q)$ *and* α_1 *do not depend on the specific sequence of divisions of random batches* $\{\mathcal{C}^{(\ell)}\}$.

Proof. The proof follows the same line as that in Lemma 3.5. The difference is that there is no exchangeability now conditioning on the random batches.

Under Assumption 2.2 and using the similar estimates to those in Lemma 3.5, one has for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) &\leqslant q\left(\beta + L + \frac{(q-1)L}{q} + \delta\right)\mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) \\ &\quad + \frac{qL}{p-1}\sum_{j:j\neq i}\frac{1}{q}\mathbb{E}(|X^j|^q \mid \{\mathcal{C}^{(\ell)}\}) + C(\delta). \end{aligned} \tag{3.10}$$

Under Assumption 2.3, one then has for $t \in [t_k, t_{k+1}]$,

$$\begin{aligned}\frac{d}{dt}\mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) &\leqslant q\left(-r + L + \frac{(q-1)L}{q} + \delta\right)\mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) \\ &\quad + \frac{qL}{p-1}\sum_{j:j\neq i}\frac{1}{q}\mathbb{E}(|X^j|^q \mid \{\mathcal{C}^{(\ell)}\}) + C(\delta). \end{aligned} \tag{3.11}$$

Next, based on (3.10), one easily finds

$$\begin{aligned}\mathbb{E}(|X^i(t)|^q \mid \{\mathcal{C}^{(\ell)}\}) - \mathbb{E}(|X^i|^q(t_k) \mid \{\mathcal{C}^{(\ell)}\}) &\leqslant \left(\beta + L + \frac{(q-1)L}{q} + \delta\right)\int_{t_k}^t \mathbb{E}(|X^i(s)|^q \mid \{\mathcal{C}^{(\ell)}\})ds \\ &\quad + L\int_{t_k}^t \max_{1\leqslant i\leqslant p}\mathbb{E}(|X^i(s)|^q \mid \{\mathcal{C}^{(\ell)}\})ds + \int_{t_k}^t C(\delta)ds.\end{aligned}$$

It follows that

$$a(t) := \max_{1\leqslant i\leqslant p}\mathbb{E}(|X^i|^q \mid \{\mathcal{C}^{(\ell)}\}) \tag{3.12}$$

satisfies

$$a(t) \leqslant a(t_k) + q(\beta + 2L + \delta)\int_{t_k}^t [a(s) + C(\delta)]ds.$$

Grönwall's inequality then yields the first claim with any $\alpha > \beta + 2L$.

For (3.11), defining $r_1 := q(r - L - \frac{q-1}{q}L - \delta) > 0$, one finds that

$$\begin{aligned}\mathbb{E}(|X^i(t)|^q \mid \{\mathcal{C}^{(\ell)}\}) &\leqslant \mathbb{E}(|X^i|^q(t_k) \mid \{\mathcal{C}^{(\ell)}\})\mathrm{e}^{-r_1(t-t_k)} \\ &\quad + \int_{t_k}^t \mathrm{e}^{-r_1(t-s)}\Big[L\max_{1\leqslant i\leqslant p}\mathbb{E}(|X^i(s)|^q \mid \{\mathcal{C}^{(\ell)}\}) + C(\delta)\Big]ds.\end{aligned}$$

Hence, the function a defined in (3.12) satisfies

$$a(t) \leqslant a(t_k)\mathrm{e}^{-r_1(t-t_k)} + \int_{t_k}^t \mathrm{e}^{-r_1(t-s)}[La(s) + C(\delta)]ds.$$

It can be shown easily that $a(t)$ is controlled by $b(t)$ which satisfies the following integral equality:

$$b(t) = a(t_k)\mathrm{e}^{-r_1(t-t_k)} + \int_{t_k}^t \mathrm{e}^{-r_1(t-s)}[Lb(s) + C(\delta)]ds.$$

(One can perturb the initial data $a(t_k) \to a(t_k) + \epsilon$ for $b(\cdot)$ and then take $\epsilon \to 0$.)

Then, one finds

$$b'(t) = (-r_1 + L)b(t) + C(\delta) = q(-r + 2L + \delta)b(t) + C(\delta), \quad b(t_k) = a(t_k).$$

Hence,

$$a(t_{k+1}) \leqslant b(t_{k+1}) \leqslant a(t_k)\mathrm{e}^{-q(r-2L+\delta)\tau} + \frac{C(\delta)}{q(r-2L-\delta)}(1 - \mathrm{e}^{-q(r-2L+\delta)\tau}).$$

The second claim also follows. $\square$

Now, we can prove the main theorem in this section.

Proof of Theorem 3.4. First of all, for the N-particle system, by symmetry, the distribution of any particle is equal to $\mathcal{G}_N^k(\mu_0)$. Now, we focus on a particular particle $i=1$, for example.

By Lemma 3.3,

$$\mathcal{G}_N^k(\mu_0) = \mathbb{P}(1 \in A_k)\mathcal{G}_\infty^k(\mu_0) + \mathbb{P}(1 \notin A_k)\nu_k \tag{3.13}$$

for some probability measure ν_k. To see this, we consider all possible sequences of random batches. Only the first k divisions of batches (i.e., ones at $t_0, \dots, t_{k-1}$) will affect the distribution at t_k. This subsequence (the first k divisions) can take only finitely many values, and let $\{c^\ell\}_{\ell \leqslant k-1}$ be such values. Then, for any $E \subset \mathbb{R}^d$ that is Borel measurable,

$$\mathcal{G}_N^k(\mu_0)[E] = \sum_{\{\mathcal{C}^\ell = c^\ell, \ell \leqslant k-1\}} \mathbb{P}(\mathcal{C}^\ell = c^\ell, \ell \leqslant k-1)\mathbb{P}(X^1 \in E \mid \mathcal{C}^\ell = c^\ell, \ell \leqslant k-1).$$

Lemma 3.3 tells us that if $\{c^\ell\}_{\ell \leqslant k-1}$ is a value such that 1 is clean, then

$$\mathbb{P}(X^1 \in E \mid \mathcal{C}^\ell = c^\ell, \ell \leqslant k-1) = \mathcal{G}_\infty^k(\mu_0)[E].$$

Hence,

$$\begin{aligned}\mathcal{G}_N^k(\mu_0)[E] &= \mathbb{P}(1 \in A_k)\mathcal{G}_\infty^k(\mu_0)[E] \\ &\quad + \sum_{\{\mathcal{C}^\ell = c^\ell, \ell \leqslant k-1\}, 1 \notin A_k} \mathbb{P}(\mathcal{C}^\ell = c^\ell, \ell \leqslant k-1)\mathbb{P}(X^1 \in E \mid \mathcal{C}^\ell = c^\ell, \ell \leqslant k-1) \\ &= \mathbb{P}(1 \in A_k)\mathcal{G}_\infty^k(\mu_0)[E] + \mathbb{P}(1 \notin A_k)\nu_k(E)\end{aligned}$$

with

$$\nu_k(E) = \sum_{\{\mathcal{C}^\ell = c^\ell, \ell \leqslant k-1\}, 1 \notin A_k} \frac{\mathbb{P}(\mathcal{C}^\ell = c^\ell, \ell \leqslant k-1)}{\mathbb{P}(1 \notin A_k)}\mathbb{P}(X^1 \in E \mid \mathcal{C}^\ell = c^\ell, \ell \leqslant k-1).$$

Clearly, ν_k is a convex combination of some conditional marginal distributions of X^1, each being $\mathscr{L}(X^1)$ conditioning on a particular sequence of batches for $\{1 \notin A_k\}$. Hence, ν_k is a probability measure.

By (3.13), it holds that

$$|\mathcal{G}_\infty^k(\mu_0) - \mathcal{G}_N^k(\mu_0)| \leqslant (1 - \mathbb{P}(1 \in A_k))\mathcal{G}_\infty^k(\mu_0) + \mathbb{P}(1 \notin A_k)\nu_k = \epsilon_k(\mathcal{G}_\infty^k(\mu_0) + \nu_k). \tag{3.14}$$

Therefore, the total variation distance between the two measures is controlled by

$$\|\mathcal{G}_\infty^k(\mu_0) - \mathcal{G}_N^k(\mu_0)\|_{TV} \leqslant (1 - \mathbb{P}(1 \in A_k)) + \mathbb{P}(1 \notin A_k) = 2\epsilon_k. \tag{3.15}$$

By Lemma 3.6, for each sequence of batches, one has

$$\sup_i \mathbb{E}(|X^i|^q \mid \mathcal{C}^{(\ell)}) \leqslant C(q)\mathrm{e}^{\alpha_1 t}.$$

Hence, it holds that

$$\int_{\mathbb{R}^d} |x|^q \nu_k(dx) \leqslant C(q)\mathrm{e}^{\alpha_1 t}. \tag{3.16}$$

In the case of strong confinement, $\alpha_1 = 0$. Similarly, by Lemma 3.5, $\mathcal{G}_\infty^k(\mu_0)$ has the same moment control. The application of Lemma 2.5 then yields the desired result. □

Lastly, we close up the estimate.

Theorem 3.7. *For any fixed k, it holds that*

$$\lim_{N \to \infty} \epsilon_k = 0. \tag{3.17}$$

Proof. First of all, clearly, we have

$$\epsilon_0 = \epsilon_1 = 1 - 1 = 0.$$

Now, we do induction on k. Assume

$$\lim_{N\to\infty} \epsilon_k = 0.$$

Consider the batches for $t_k \to t_{k+1}^-$. Assume the batch for particle 1 is $\mathcal{C}_q^{(k)}$. Denote

$$B_k = \{\forall\, j,\ell \in \mathcal{C}_q^{(k)},\, j \neq \ell,\, L_j^{(k)} \cap L_\ell^{(k)} = \emptyset\}.$$

Let $\mathcal{B} = \mathcal{C}_q^{(k)} \setminus \{1\}$ be the set of other particles that share the same batch with particle 1. Then, by the definition of A_{k+1},

$$\begin{aligned}\mathbb{P}(1 \in A_{k+1}) = &\sum_{j_1,\dots,j_{p-1}} \mathbb{P}(\mathcal{B} = \{j_1,\dots,j_{p-1}\})\\ &\times \mathbb{P}\left(B_k \cap \{1 \in A_k\} \bigcap_{\ell=1}^{p-1} \{j_\ell \in A_k\} \,\middle|\, \mathcal{B} = \{j_1,\dots,j_{p-1}\}\right).\end{aligned} \tag{3.18}$$

Denote $E := \{\mathcal{B} = \{j_1,\dots,j_{p-1}\}\}$, where we omit the dependence in $j_\ell, 1 \leqslant \ell \leqslant p-1$ for notational convenience. Conditioning on $B_k \cap E$ (i.e., provided that the event $B_k \cap E$ happens), whether the particles are clean or not are independent. Hence,

$$\begin{aligned}\mathbb{P}\left(B_k \cap \{1 \in A_k\} \bigcap_{\ell=1}^{p-1} \{j_\ell \in A_k\} \,\middle|\, E\right) &= \mathbb{P}(B_k \mid E)\mathbb{P}\left(\{1 \in A_k\} \bigcap_{\ell=1}^{p-1} \{j_\ell \in A_k\} \,\middle|\, E, B_k\right)\\ &= \mathbb{P}(B_k \mid E) \prod_{\ell=1}^{p} \mathbb{P}(j_\ell \in A_k \mid E, B_k),\end{aligned}$$

where we have set $j_p = 1$. Moreover,

$$\mathbb{P}(j_\ell \in A_k \mid E, B_k) = \frac{\mathbb{P}(\{j_\ell \in A_k\} \cap E \cap B_k)}{\mathbb{P}(E \cap B_k)} = \frac{\mathbb{P}(\{1 \in A_k\} \cap E \cap B_k)}{\mathbb{P}(E \cap B_k)} = \frac{\mathbb{P}(\{1 \in A_k\} \cap B_k)}{\mathbb{P}(B_k)}.$$

The second and the last equalities are due to symmetry. For the last equality, $\mathbb{P}(\{\mathcal{B} = \{j_1,\dots,j_{p-1}\}\} \cap B_k)$ should be equal for all possible $j_1,\dots,j_{p-1}$, and the same is true for the numerator. This actually is a kind of independence. Hence, eventually due to the fact

$$\sum_{j_1,\dots,j_{p-1}} \mathbb{P}(\mathcal{B} = \{j_1,\dots,j_{p-1}\})\mathbb{P}(B_k \mid \mathcal{B} = \{j_1,\dots,j_{p-1}\}) = \mathbb{P}(B_k),$$

one has

$$1 - \epsilon_{k+1} = \mathbb{P}(1 \in A_{k+1}) \geqslant \mathbb{P}(B_k)(1 - \epsilon_k/\mathbb{P}(B_k))^p.$$

Hence, it suffices to show

$$\lim_{N\to\infty} \mathbb{P}(B_k) = 1.$$

To get an estimate for this, we consider the following equivalent way to construct $L_i^{(k)}$: one starts with $L_i \leftarrow \{i\}$ and repeat the following for k times:

(1) Set $L_{\mathrm{tmp}} \leftarrow L_i$ and $A = \emptyset$.

(2) Loop the following until L_{tmp} is empty:

(a) Pick a particle $i_1 \in L_{\mathrm{tmp}}$, and then choose $p-1$ particles from $\{1,\dots,N\} \setminus A \cup \{i_1\}$ denoted by $\{i_2,\dots,i_p\}$.

(b) $L_i \leftarrow L_i \cup \{i_2,\dots,i_p\}$.

(c) Set $A \leftarrow A \cup \{i_1, i_2,\dots,i_p\}$.

(d) Set $L_{\mathrm{tmp}} \leftarrow L_{\mathrm{tmp}} \setminus \{i_1, i_2,\dots,i_p\}$.

In the above, we are actually looking back from t_{k-1}. In the j-th iteration, we are constructing batches at t_{k-j}. Hence, this is an equivalent way to construct $L_i^{(k)}$.

Now, we estimate $\mathbb{P}(B_k)$ by constructing the lists $L_{j_\ell}^{(k)} : 1 \leqslant \ell \leqslant p$ for $j_\ell \in C_q^{(k)}$ using the above procedure. Consider that the lists for $j_1, \ldots, j_{\ell-1}$ have been constructed, which have included at most $(\ell-1)p^k$ particles. Now, for $L_{j_\ell}^{(k)}$ not to intersect with the previous lists, one has to choose particles from $\{1, \ldots, N\} \backslash [\bigcup_{z=1}^{\ell-1} L_{j_z}^{(k)} \cup A \cup \{i_1\}]$ in the (2)(a) step. Conditioning on the specific choices of $L_{j_\ell}^{(k)} : 1 \leqslant \ell \leqslant p$ and A with $N_1 := |L_{j_\ell}^{(k)} \cup A \cup \{i_1\}|$, $N_2 := |A|$, this probability is controlled from below by

$$\frac{\binom{N-N_1}{p-1}}{\binom{N-1-N_2}{p-1}} \geqslant \frac{\binom{N-1-\ell p^k}{p-1}}{\binom{N-1}{p-1}}.$$

Hence, as $N \to \infty$,

$$\mathbb{P}(B_k) \geqslant \prod_{\ell=1}^{p} \left[\frac{\binom{N-1-\ell p^k}{p-1}}{\binom{N-1}{p-1}} \right]^k = 1 - O(N^{-1}).$$

Hence, $\lim_{N\to\infty} \epsilon_{k+1} = 0$ and the claim follows. □

As can be seen in the proof, one actually has $\epsilon_k \leqslant C(p,k)N^{-1}$ for some $C(p,k) > 0$. This rate is different from the typical $O(N^{-1/2})$ rate (though under W_2 distance) for the mean field limit of interacting particle systems due to the law of large number results.

Remark 3.8. The current argument of the mean field limit relies on the fact that two particles are unlikely to be related when $N \to \infty$ for finite iterations. This is not enough to get the mean field limit independent of τ. For fixed N, $\epsilon_k \to 1$ as $k \to \infty$. As pointed out in [28], the RBM works due to the averaging effect in time. The regime we consider here (finite iterations and $N \to \infty$) is clearly far before the point when the averaging effect in time comes into play. To study the mean field limit uniform in τ (the averaging mechanism can take effect), one must consider carefully how the correlation decays as k grows when two particles are not totally clean to each other. The study of this creation of chaos will be left for the future.

4 Properties of the limiting dynamics

We consider the limit dynamics given by the operator $\mathcal{G}_\infty$ (defined in (1.8)) and its approximation to the dynamics of the nonlinear Fokker-Planck equation (1.2), the mean-field limit of the interacting particle system (1.1).

As proved in [28], the error between the one marginal distribution of the RBM particle system (1.6) and that of (1.1) are close independent of N under the W_2 distance (the left-hand side in Figure 1). Combining the mean field result in Section 3 and taking $N \to \infty$, one sees that the dynamics of $\mathcal{G}_\infty$ is close to that of (1.2) (the right-hand side in Figure 1). In other words, the two limits $\lim_{N\to\infty}$ and $\lim_{\tau\to 0}$ commute.

A direct application of the strong mean square error in [28] gives an upper bound $O(\sqrt{\tau})$ for the W_2 distance corresponding to the left-hand side in Figure 1, and thus the right-hand side in Figure 1 after taking $N \to \infty$. It is shown in [29] (though for $b(\cdot)$ being bounded) that the weak error is $O(\tau)$. The W_{q} distance is a kind of weak topology as it measures the closeness between distributions instead of the trajectories of particles. Hence, the sharp upper bound for the Wasserstein distance between these two marginal distributions is believed to be $O(\tau)$, even for unbounded $b(\cdot)$. Below, we aim to prove these under W_1 distance.

4.1 Stability of the limiting dynamics

In this subsection, we study the stability and contraction properties of the nonlinear operator $\mathcal{G}_\infty$ for the limiting dynamics.

Proposition 4.1. *Under Assumption 2.2, $\mathcal{G}_\infty$ satisfies for $\mathsf{q} \in [1, \infty)$,*

$$W_{\mathsf{q}}(\mathcal{G}_\infty(\mu_1), \mathcal{G}_\infty(\mu_2)) \leqslant \mathrm{e}^{(\beta+2L)\tau} W_{\mathsf{q}}(\mu_1, \mu_2), \quad \mu_i \in \mathbf{P}(\mathbb{R}^d), \quad i = 1, 2. \tag{4.1}$$

The operator $\mathcal{G}_\infty$ is a contraction in W_{q} under Assumption 2.3:

$$W_{\mathsf{q}}(\mathcal{G}_\infty(\mu_1), \mathcal{G}_\infty(\mu_2)) \leqslant \mathrm{e}^{-(r-2L)\tau} W_{\mathsf{q}}(\mu_1, \mu_2) \tag{4.2}$$

so that $\mathcal{G}_\infty$ has a unique invariant measure π_τ and it holds that for any μ_0,

$$W_{\mathsf{q}}(\mathcal{G}_\infty^n(\mu_0), \pi_\tau) \leqslant \mathrm{e}^{-(r-2L)n\tau} W_{\mathsf{q}}(\mu_0, \pi_\tau). \tag{4.3}$$

Proof. Consider two copies of (1.5): one is

$$dY_1^i = b(Y_1^i)dt + \frac{1}{p-1} \sum_{j=1, j\neq i}^{p} K(Y_1^i - Y_1^j)dt + \sqrt{2}\sigma dW^i, \quad i = 1, \dots, p \tag{4.4}$$

with $(Y_1^1(0), \dots, Y_1^p(0))$ being drawn from $\mu_1^{\otimes p}$; the other one is

$$dY_2^i = b(Y_2^i)dt + \frac{1}{p-1} \sum_{j=1, j\neq i}^{p} K(Y_2^i - Y_2^j)dt + \sqrt{2}\sigma dW^i, \quad i = 1, \dots, p \tag{4.5}$$

with $(Y_2^1(0), \dots, Y_2^p(0))$ being drawn from $\mu_2^{\otimes p}$.

For any $\epsilon > 0$, choose the coupling as follows. First, choose a coupling γ for Y_1^1 and Y_2^1 such that

$$\mathbb{E}|Y_1^1(0) - Y_2^1(0)|^{\mathsf{q}} \leqslant \epsilon + W_{\mathsf{q}}^{\mathsf{q}}(\mu_1, \mu_2).$$

Then, let the samples $(Y_1^i(0), Y_2^i(0))$ be i.i.d., drawn from γ. Let the Brownian motions for the two systems be the same.

Now, to show the claims, it suffices to show that the moments of the SDE system (1.5) are stable. In fact, the joint distribution of $(Y_1^1(\tau), Y_2^1(\tau))$ is a coupling for $\mathcal{G}_\infty(\mu_1)$ and $\mathcal{G}_\infty(\mu_2)$,

$$W_{\mathsf{q}}(\mathcal{G}_\infty(\mu_1), \mathcal{G}_\infty(\mu_2)) \leqslant (\mathbb{E}|Y_1^1(\tau) - Y_2^1(\tau)|^{\mathsf{q}})^{1/\mathsf{q}}.$$

By using the symmetry, it can be computed directly that under Assumption 2.2,

$$\frac{d}{dt}\mathbb{E}|Y_1^1 - Y_2^1|^{\mathsf{q}} \leqslant \mathsf{q}(\beta + 2L)\mathbb{E}|Y_1^1 - Y_2^1|^{\mathsf{q}},$$

and that under Assumption 2.3,

$$\frac{d}{dt}\mathbb{E}|Y_1^1 - Y_2^1|^{\mathsf{q}} \leqslant \mathsf{q}(-r + 2L)\mathbb{E}|Y_1^1 - Y_2^1|^{\mathsf{q}}.$$

For $\mathsf{q} = 1$, one can use $\sqrt{|Y_1^1 - Y_2^1|^2 + \delta}$ to approximate and then take $\delta \to 0^+$. Applying Grönwall's inequality and noticing ϵ is arbitrary, one obtains the first two assertions directly. The last claim follows from the standard contraction mapping theorem [23, Chapter 1]. □

4.2 Basic properties of the nonlinear Fokker-Planck equation

We establish several basic results to (1.2) using a stronger version of Assumption 2.1.

Assumption 4.2. *The measure μ_0 has a density ϱ_0 that is smooth with finite moments*

$$\int_{\mathbb{R}^d} |x|^q \varrho_0 dx < \infty, \quad \forall\, q \geqslant 1,$$

and the entropy is finite, i.e.,

$$H(\mu_0) := \int_{\mathbb{R}^d} \varrho_0 \log \varrho_0 dx < \infty. \tag{4.6}$$

If $\varrho_0(x) = 0$ at some point x, one defines $\varrho_0(x) \log \varrho_0(x) = 0$. We also introduce the following assumption on the growth rate of derivatives of b and K, which will be used below.

Assumption 4.3. *The function b and its derivatives have polynomial growth. The derivatives of K with order at least 2 (i.e., $D^\alpha K$ with $|\alpha| \geqslant 2$) have polynomial growth.*

Based on these conditions, (1.2) can be formulated in terms of the density of μ:

$$\begin{aligned} &\partial_t \varrho = -\nabla \cdot ((b(x) + K * \varrho)\varrho) + \sigma^2 \Delta \varrho, \\ &\varrho(0) = \varrho_0. \end{aligned} \tag{4.7}$$

Then, a weak solution to (4.7) corresponds to a measure solution $\mu = \varrho dx$ to (1.2), where the weak solution is defined as follows.

Definition 4.4. We say $\varrho \in L^\infty([0,T]; L^1(\mathbb{R}^d))$ is a weak solution to (4.7), if $\varrho dx \in C([0,T]; \mathbf{P}(\mathbb{R}^d))$ where $\mathbf{P}(\mathbb{R}^d)$ is equipped with the weak topology, and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, it holds that for any $t \leqslant T$,

$$\begin{aligned} &\int_{\mathbb{R}^d} \varrho(x,t)\varphi(x)dx - \int_{\mathbb{R}^d} \varrho_0(x)\varphi(x)dx \\ &\quad = \int_0^t \int_{\mathbb{R}^d} \nabla\varphi(x) \cdot (b(x) + K * \varrho)\, \rho(x,s)dxds + \sigma^2 \int_0^t \int_{\mathbb{R}^d} \Delta\varphi(x)\varrho(x,s)dxds. \end{aligned} \tag{4.8}$$

Note that the test function used here does not depend on time variable, so we require the integral equation to hold for any $t \leqslant T$. Due to the relation between (4.7) and (1.2), we will not distinguish the measure and its density. For example, we will use $\mathcal{G}_\infty(\varrho_0)$ to mean the nonlinear semigroup acting on the measure μ_0, and will use $W_{\mathsf{q}}(\varrho, \nu)$ to mean the Wasserstein-q distance between $\mu = \varrho dx$ and another measure ν.

We have the following regarding the well-posedness of the nonlinear Fokker-Planck equation (4.7).

Proposition 4.5. *Let Assumption 2.2 or Assumption 2.3 hold, and also $|b| + |\nabla b| \leqslant C(1 + |x|^q)$ for some C and q. Fix any $T > 0$. Assume the initial data ϱ_0 satisfies Assumption 4.2. Then the nonlinear Fokker-Planck equation (4.7) has a unique weak solution satisfying $\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}^d} |x|\varrho dx < \infty$. Moreover, this solution is a strong solution and is smooth together with the moment control*:

$$\sup_{0\leqslant t\leqslant T} \int_{\mathbb{R}^d} |x|^q \varrho dx \leqslant C(q,T) \int_{\mathbb{R}^d} |x|^q \varrho_0 dx. \tag{4.9}$$

Besides, under Assumption 2.3, the moments are uniformly bounded in t, i.e., the constants $C(q,T)$ above can be made independent of T. Moreover, $\mu = \varrho dx$ converges in $W_{\mathsf{q}}, \mathsf{q} \geqslant 1$ to an invariant measure π exponentially as $t \to \infty$.

There are many works on similar models in the literature (see [2, 8, 10, 40] as a few of examples). However, in our case, b and K are not bounded and b can have polynomial growth at infinity, so the proofs in these works do not quite fit our setting here. For example, in the work of [8, 10], $b = -\nabla V$ and they require $\nabla V \cdot x \geqslant C$ for some constant while we allow $b \cdot x \leqslant \beta|x|^2$; also the requirements on the kernel $K(\cdot)$ do not quite match the setup here. In the work [2], a certain class of nonlinear Fokker-Planck equations have been studied via the approach of Crandall and Liggett for m-accretive operators in $L^1(\mathbb{R}^d)$, but the approach cannot be applied directly to our case here. Due to these reasons, we attach a proof of Proposition 4.5 in Appendix A for a reference.

In proving the uniqueness of the solution to (4.7) in Appendix A, we have in fact proved the following mean-field limit.

Proposition 4.6. *With the same assumptions of Proposition 4.5, one has*

$$\sup_{0\leqslant t\leqslant T} W_2(\varrho, \mu_N^{(1)}) \leqslant \frac{C(T)}{\sqrt{N}}, \tag{4.10}$$

where $\mu_N^{(1)}$ is the one marginal distribution of the interacting particle system (1.1). Moreover, if Assumption 2.3 holds, the constant $C(T)$ can be independent of T.

As long as the existence and uniqueness of the solutions to the nonlinear Fokker-Planck equation have been established, one can regard

$$\bar{K}(x,t) := \int_{\mathbb{R}^d} K(x-y)\varrho(y,t)dy, \tag{4.11}$$

as known, and the properties of ϱ can be studied via the *linear* Fokker-Planck equation

$$\partial_t \varrho = -\nabla \cdot [(b(x) + \bar{K}(x,t))\varrho] + \sigma^2 \Delta \varrho. \tag{4.12}$$

By the moment estimates of ϱ, $\bar{K}(0,t)$ is bounded by the first moment of ϱ and it is Lipschitz continuous with uniform Lipschitz constant L. We consider the time continuity of $\bar{K}$.

Lemma 4.7. *Under Assumptions* 2.2, 4.2 *and* 4.3, *we have for any* $\Delta t \in [0,\tau]$,

$$|\bar{K}(x,t+\Delta t) - \bar{K}(x,t)| \leqslant C(M_q(\varrho(t)))(1+|x|^q)\tau \tag{4.13}$$

for some $q>1$, *where* $M_q(\varrho(t))$ *means the q-moment of* ϱ *at* t. *Moreover, if Assumption* 2.2 *is replaced by Assumption* 2.3, $C(M_q(\varrho(t)))$ *has an upper bound independent of time* t.

Proof. It can be computed directly that

$$\begin{aligned}\partial_t \bar{K}(x,t) &= \int_{\mathbb{R}^d} K(x-y)\{-\nabla \cdot [(b(y) + K*\varrho)\varrho] + \sigma^2 \Delta_y \varrho\} dy \\ &= -\int_{\mathbb{R}^d} (b(y) + K*\varrho)\varrho \cdot (\nabla K)(x-y)dy + \int_{\mathbb{R}^d} \sigma^2 (\Delta K)(x-y)\varrho dy.\end{aligned}$$

Since b has polynomial growth and ∇K is bounded, we have

$$\left| -\int_{\mathbb{R}^d} (b(y)+K*\varrho)\varrho \cdot (\nabla K)(x-y)dy \right| \leqslant C\int_{\mathbb{R}^d}(1+|y|^q)\varrho dy + C\iint_{\mathbb{R}^d\times\mathbb{R}^d} |K(x-y)|\varrho(x)\varrho(y)dxdy.$$

This is controlled by the moments of ϱ.

Moreover, since ΔK has polynomial growth,

$$\begin{aligned}\left| \int_{\mathbb{R}^d} \sigma^2 (\Delta K)(x-y)\varrho dy \right| &\leqslant \sigma^2 C \int_{\mathbb{R}^d} (1+|x-y|^q)\varrho dy \\ &\leqslant C\left(1 + \int_{\mathbb{R}^d} (|x|^q + |y|^q)\varrho dy\right) \leqslant C(1+|x|^q),\end{aligned}$$

where C depends on the moments of ϱ.

By using the results in Proposition 4.5, the moments on $[t, t+\Delta t]$ can be controlled by the one at t. Since τ is a fixed small number, we omit the dependence in τ for the amplification constant, the claims then follow. □

Before further discussion, we first establish some auxiliary results regarding the following linear Fokker-Planck equation:

$$\partial_t f = -\nabla \cdot (b_1(x,t)f)dt + \sigma^2 \Delta f =: \mathcal{L}^*_{b_1}(f). \tag{4.14}$$

We will assume $b_1(x,t)$ satisfies

$$(x-y)\cdot(b_1(x,t) - b(y,t)) \leqslant \beta_1 |x-y|^2. \tag{4.15}$$

We say b_1 satisfies the strong confinement condition if $\beta_1 < 0$. We also denote $S_{s,t}$ to be the solution operator from time s to time t, i.e.,

$$S_{s,t} f_s := f_t. \tag{4.16}$$

There are many classical results on the parabolic equation (4.14) with bounded drifts b_1 or drifts with linear growth (see, for example, [33]). However, the results for drifts with polynomial growth seem limited. Below, we will show some results, especially the properties of the fundamental solutions, for drifts with polynomial growth (see Lemma 4.10 and Proposition 4.11) to fulfill our needs.

Lemma 4.8. *Consider* (4.14)*, where* b_1 *satisfies* (4.15)*. Also, assume the derivatives of* $b_1(x,t)$ *have polynomial growth and* $\sup_{t\geqslant 0}|b(0,t)|<\infty$*. Then for* $q\geqslant 1$*, we have the following*:

(i) *For any* $g\in L^1(\mathbb{R}^n)$*, one has*

$$\sup_{\Delta t\leqslant T}\int_{\mathbb{R}^d}(1+|x|^q)|S_{t,t+\Delta t}g|dx\leqslant C(T)\int_{\mathbb{R}^d}(1+|x|^q)|g(x)|dx.$$

(ii) *If* b_1 *satisfies the strong confinement condition* $\beta_1<0$, $C(T)$ *in the item* (i) *can be made independent of* T*. Moreover, when* $\sigma>0$ *and* $\int_{\mathbb{R}^d}gdx=0$, $\beta_1<0$ *implies that*

$$\int_{\mathbb{R}^d}(1+|x|^q)|S_{t,t+\Delta t}g|dx\leqslant P(M_{q_1}(|g|))\mathrm{e}^{-\delta\Delta t},$$

where $\delta>0$ *is independent of* g, $P(\cdot)$ *is some polynomial,* $q_1>q$ *is some suitable number, and* $M_{q_1}(|g|)$ *means the* q_1*-moment of* $|g|$.

(iii) *In the case where* b_1 *does not depend on time so that* $S_{s,t}=\mathrm{e}^{(t-s)\mathcal{L}_{b_1}^*}$*, one also has*

$$\sup_{\Delta t\leqslant T}\int_{\mathbb{R}^d}(1+|x|^q)|(\mathcal{L}_{b_1}^*)^m S_{t,t+\Delta t}g|dx\leqslant C(T)\int_{\mathbb{R}^d}(1+|x|^q)|(\mathcal{L}_{b_1}^*)^m g|dx.$$

Proof. For (i), one decomposes $g:=g^+-g^-$, where $g^+=\max(g,0)$ and $g^-=-\min(g,0)$. Then, $S_{t,t+\Delta t}g=(S_{t,t+\Delta t}g^+)-(S_{t,t+\Delta t}g^-)$ with each of them being nonnegative. The operator $S_{t,t+\Delta t}$ is L^1-contraction, so we focus on the q-moments only. Following similar approaches of Step 1 in Appendix A, one can show that the moments of $S_{t,t+\Delta t}g^\pm$ can be controlled by those of $g^\pm$. Hence, the moments of $S_{t,t+\Delta t}g$ have the desired estimates. We skip the details.

Regarding (ii), we first note that the q moments of $S_{t,t_1}g$ can be uniformly controlled by moments of $|g|$, due to similar reasons. Then, one can consider the measures $\mu^\pm:=\frac{1}{\|g^\pm\|_{L^1}}S_{t,t+\Delta t}g^\pm$. Using standard techniques of Markov chains (see [39, Appendix A] and [41, Chapters 15–16]), one can show that

$$\|\mu^+-\mu^-\|_{TV}\leqslant P_1(M_{q_1}(|\mu|))\mathrm{e}^{-\delta'\Delta t}\Rightarrow\|S_{t,t+\Delta t}g\|_{TV}\leqslant P(M_{q_1}(|g|))\mathrm{e}^{-\delta'\Delta t}$$

for some $q_1>q$ and polynomials $P_1(\cdot)$ and $P(\cdot)$. Then

$$\begin{aligned}\int_{\mathbb{R}^d}|x|^q|S_{t,t+\Delta t}g|(dx)&=\frac{1}{2}\|g\|_{L^1}\int|x|^q|\mu^+-\mu^-|(dx)\\&\leqslant C\|g\|_{L^1}\sqrt{\int|x|^{2q}|\mu^+-\mu^-|(dx)}\sqrt{\|\mu^+-\mu^-\|_{TV}}\\&=C\sqrt{\int_{\mathbb{R}^d}|x|^{2q}|S_{t,t+\Delta t}g|(dx)}\sqrt{\|S_{t,t+\Delta t}g\|_{TV}}.\end{aligned}$$

Furthermore, due to

$$\sqrt{\int_{\mathbb{R}^d}|x|^{2q}|S_{t,t+\Delta t}g|(dx)}\leqslant\frac{1}{2}(1+M_{2q}(|g|)),$$

one can then choose another q_1 large enough such that the claims in (ii) hold.

For (iii), we just note that

$$\partial_t((\mathcal{L}_{b_1}^*)^m f)=\mathcal{L}_{b_1}^*((\mathcal{L}_{b_1}^*)^m f).$$

Then, we apply the property of $\mathrm{e}^{t\mathcal{L}_{b_1}^*}$ proved in the first part (i). □

Remark 4.9. For (ii), if $\sigma=0$, even if the strong confinement condition is satisfied, $\|\mu^+-\mu^-\|_{TV}$ may not decay. However, we believe that when $b_1(x,t)\to b_\infty(x)$, then

$$\int_{\mathbb{R}^d}|x-x_*|^q|S_{t,t+\Delta t}g|dx\leqslant C(M_{q_1}(|g|))\mathrm{e}^{-\delta\Delta t}$$

still holds for the limiting point x_* of the trajectories. We do not explore this in this work.

It is well known that the linear equation (4.14) has a transition density $\Phi(x,t;y,s)$ solving (4.14) for $t>s$ with the initial data $\Phi(x,s;y,s)=\delta(x-y)$. Then,

$$(S_{s,t}g)(x)=\int_{\mathbb{R}^d}\Phi(x,t;y,s)g(y)dy. \tag{4.17}$$

Hence, the property of Φ is important.

Lemma 4.10. *Consider* (4.14) *with* $\sigma>0$*, and* b_1 *satisfying* (4.15)*. Also, assume the derivatives of* $b_1(x,t)$ *have polynomial growth and* $\sup_{t\geqslant 0}|b_1(0,t)|<\infty$*. Then for all* $0\leqslant s<t\leqslant T$*, we have*

$$\int_{\mathbb{R}^d}(1+|x|^q)|\nabla_y\Phi(x,t;y,s)|dx\leqslant C(T)P(|y|)(1+(t-s)^{-1/2}) \tag{4.18}$$

for some polynomial $P(\cdot)$*. If* $\beta_1<0$*,*

$$\int_{\mathbb{R}^d}(1+|x|^q)|\nabla_y\Phi(x,t;y,s)|dx\leqslant CP(|y|)(1+(t-s)^{-1/2})\mathrm{e}^{-\delta(t-s)} \tag{4.19}$$

for some $\delta>0$*.*

The proof of Lemma 4.10 is tedious, and we defer it to Appendix B. Below, we aim to consider the moments of the derivatives of ϱ. Now, we recall the standard multi-index notation used in PDE community:

$$D^{\alpha}:=\prod_{j=1}^{d}\partial_j^{\alpha^j},\quad \alpha=(\alpha^1,\dots,\alpha^d). \tag{4.20}$$

The length of the index is defined as $|\alpha|:=\sum_{j=1}^d\alpha^j$.

The following proposition is helpful for our estimates later.

Proposition 4.11. *Let Assumptions* 2.2, 4.2 *and* 4.3 *hold. Then for any multi-index* α*, it holds that*

$$\sup_{t\leqslant T}\int_{\mathbb{R}^d}(1+|x|^q)|D^{\alpha}\varrho|dx\leqslant C(\alpha,q,T). \tag{4.21}$$

If $\sigma>0$ *and Assumption* 2.3 *holds, then*

$$\sup_{t>0}\int_{\mathbb{R}^d}(1+|x|^q)|D^{\alpha}\varrho|dx<\infty. \tag{4.22}$$

Proof. We set

$$b_1(x,t):=b(x)+\bar{K}(x,t),$$

which is regarded as known (since the existence and uniqueness of ϱ have been established).

In the case $\sigma=0$, consider the characteristics satisfying

$$\dot{Z}=b(Z),\quad Z(0;y)=y.$$

Using the one-sided Lipschitz condition in Assumption 2.2, one has $v\cdot\nabla b_1(x,t)\cdot v\leqslant\beta_1|v|^2$ for any v and x with $\beta_1=\beta+2L$. With this and induction, one can show that $|\partial_{y_i}Z|\leqslant C\mathrm{e}^{\beta_1 t}$ and $D_y^{\alpha}Z(t;y)$ is controlled by polynomials of $|y|$ for higher order α. By using $\varrho=Z_{\#}\varrho_0$, the claim can be proved. We omit the details.

Now, we focus on $\sigma>0$. We do by induction on the derivatives of ϱ. Let $\ell=|\alpha|$. We know already that the claim holds for $\ell=0$.

Suppose the claim is true for $\ell-1$ with $\ell\geqslant 1$. Now, we consider ℓ. One can see that

$$\partial_t D^{\alpha}\varrho=-\nabla\cdot(b_1(x,t)D^{\alpha}\varrho)+\sigma^2\Delta D^{\alpha}\varrho+\sum_{|\beta|\leqslant\ell-1}C_{\beta}\nabla\cdot[f_{\beta}(x)D^{\beta}\varrho].$$

Here, f_β are some functions with polynomial growth. Then, we have

$$D^\alpha \varrho = S_{0,t} D^\alpha \varrho_0 - \int_0^t \sum_{|\beta| \leqslant \ell - 1} C_\beta \int_{\mathbb{R}^d} \nabla_y \Phi(x,t;y,s) \cdot (f_\beta(y) D^\beta \varrho(y,s)) dy ds.$$

The claim follows by a direct application of the induction assumption and Lemma 4.10 with $\beta_1 = \beta + 2L$ or $\beta_1 = -r + 2L$. $\square$

4.3 Approximation of the limiting dynamics to the nonlinear Fokker-Planck equation

To get a feeling about how close the dynamics given by $\mathcal{G}_\infty$ (the mean field limit of the RBM) is to the nonlinear Fokker-Planck equation (1.2), we consider (1.6). Recall that $\rho^{(p)}(\dots, t_k) = \tilde{\mu}(\cdot, t_k)^{\otimes p}$ with order τ error, (1.6) is approximated as

$$\partial_t \rho^{(p)} = -\sum_{i=1}^p \nabla_{x_i} \cdot \left(\left[b(x_i) + \frac{1}{p-1} \sum_{j:j\neq i} K(x_i - x_j) \right] \prod_{j=1}^p \tilde{\mu}(x_j, t_k) \right) + \sigma^2 \sum_{i=1}^p \Delta_{x_i} \rho^{(p)} + O(\tau). \quad (4.23)$$

Since we are curious about how the marginal distribution is evolving, one may take the integrals on $x_2, \dots, x_p$ and have

$$\partial_t \tilde{\rho} = -\nabla_{x_1} \cdot ([b(x_1) + K * \tilde{\mu}(\cdot, t_k)] \tilde{\mu}(x_1, t_k)) + \sigma^2 \Delta_{x_1} \tilde{\rho} + O(\tau).$$

Since $\tilde{\rho} := \int \rho^{(p)} dx_2 \cdots dx_p$ is equal to $\tilde{\mu}(\cdot, t_k)$ initially, one finds that this is close to (1.2) already. Thus, one expects that the overall error between $\mathcal{G}_\infty^k(\varrho_0)$ and $\varrho(k\tau)$ is like $O(\tau)$.

We now state the main results in this section.

Theorem 4.12. *Let ϱ be the solution to the nonlinear Fokker-Planck equation* (4.7)*. Suppose Assumptions* 2.2*,* 4.2 *and* 4.3 *hold. Then,*

$$\sup_{n: n\tau \leqslant T} W_1(\mathcal{G}_\infty^n(\varrho_0), \varrho(n\tau)) \leqslant C(T)\tau. \quad (4.24)$$

If Assumption 2.3 *is assumed in place of Assumption* 2.2 *and also $\sigma > 0$, then*

$$\sup_{n \geqslant 0} W_1(\mathcal{G}_\infty^n(\varrho_0), \varrho(n\tau)) \leqslant C\tau. \quad (4.25)$$

Consequently, the invariant measures (*see Propositions* 4.1 *and* 4.5 *for the related notations*) *satisfy*

$$W_1(\pi_\tau, \pi) \leqslant C\tau. \quad (4.26)$$

Below, we aim to prove Theorem 4.12. We first establish the one-step error and then give the global estimate.

Define

$$M_{q,\ell}^{(k)} := \sum_{|\alpha| \leqslant \ell} \int_{\mathbb{R}^d} (1 + |x|^q) |D^\alpha \varrho(x, t_k)| dx, \quad (4.27)$$

which are the moments of $|D^\alpha \varrho(\cdot, t_k)|$ for $|\alpha| \leqslant \ell$ (see (4.20) for the multi-index notation). In fact, we have the following result provided that ϱ is smooth enough.

Lemma 4.13. *Suppose Assumptions* 2.2 *and* 4.3 *hold. Let $t_k \leqslant T - \tau$. Then,*

$$W_1(\mathcal{G}_\infty(\varrho(\cdot, t_k)), \varrho(\cdot, t_{k+1})) \leqslant g(M_{q,4}^{(k)}) \tau^2$$

for some $q > 1$ and the nondecreasing function $g(\cdot)$, where $M_{q,4}^{(k)}$ is defined in (4.27)*.*

Proof. For the notational convenience in this proof, we denote, only in this proof,

$$\varrho_k(\cdot) \equiv \varrho(\cdot, t_k).$$

Step 1. Consider the SDE corresponding to the nonlinear Fokker-Planck equation (4.7):

$$dX = [b(X) + K(\cdot) * \varrho(\cdot, t)(X)]dt + \sqrt{2}\sigma dW.$$

Denote $\bar{K}(X) := \int_{\mathbb{R}^d} K(X - z)\varrho_k(z)dz$. Then we have

$$dX = [b(X) + \bar{K}(X) + R]dt + \sqrt{2}\sigma dW, \tag{4.28}$$

where by a similar calculation to that in the proof of Lemma 4.7,

$$|R| \leqslant C(M_{q_1,0})(1 + |X(t_k)|^q)\tau$$

for some $q_1 > 1$. In fact, C depends on the moments of $\varrho(\cdot, t)$ for $t \in [t_k, t_{k+1}]$, which can be controlled by the ones at t_k.

We show that the law of X is close in W_1 to the law generated by the following SDE:

$$d\hat{X} = [b(\hat{X}) + \bar{K}(\hat{X})]dt + \sqrt{2}\sigma dW. \tag{4.29}$$

To do this, we estimate $\mathbb{E}|X - \hat{X}|$ under the synchronization coupling (i.e., using the same Brownian motion). In fact,

$$\frac{d}{dt}\mathbb{E}|X - \hat{X}| \leqslant C\mathbb{E}|X - \hat{X}| + C\mathbb{E}|R|.$$

Clearly, $\mathbb{E}|R| \leqslant C(M_{q_2,0})\tau$ for some $q_2 > 1$.

Denote (recall that $\mathscr{L}$ means the law of a random variable)

$$\tilde{\mathcal{S}}(\varrho_k) := \mathscr{L}(\hat{X}(\tau)). \tag{4.30}$$

Then, applying Grönwall's lemma yields

$$W_1(\varrho(\cdot, t_{k+1}), \tilde{\mathcal{S}}(\varrho_k)) \leqslant C(M_{q,0})\tau^2.$$

Step 2. Compare $\tilde{\mathcal{S}}(\varrho_k)$ with $\mathcal{G}_\infty(\varrho_k)$.

We compare the law of $\hat{X}$ in (4.29) (i.e., $\tilde{\mathcal{S}}(\varrho_k)$) with the law of Y^1 (i.e., $\mathcal{G}_\infty(\varrho_k)$), where $(Y^1, \dots, Y^p)$ satisfy

$$dY^i = b(Y^i)dt + \frac{1}{p-1}\sum_{j=1, j\neq i}^{p} K(Y^i - Y^j)dt + \sqrt{2}\sigma dW^i, \quad i = 1, \dots, p \tag{4.31}$$

with the initial data drawn from $\varrho_k^{\otimes p}$. The main strategy is to use Lemma 2.5, so we need to estimate the difference of these two distributions and control the moments of this difference.

It is clear that $\tilde{\mathcal{S}}(\varrho_k) = \mathrm{e}^{\tau\hat{\mathcal{L}}^*}\varrho_k$, where $\hat{\mathcal{L}}^*$ is given by (for ρ in its domain)

$$\begin{aligned}\hat{\mathcal{L}}^*(\rho)(x) &:= -\nabla \cdot \left(\left[b(x) + \int_{\mathbb{R}^d} K(x - x_2)\varrho_k(x_2)dx_2\right]\rho(x)\right) + \sigma^2\Delta_x\rho(x)\\ &= -\int_{\mathbb{R}^d} dx_2\varrho_k(x_2)[\nabla \cdot ([b(x) + K(x - x_2)]\rho(x)) + \sigma^2\Delta_x\rho(x)].\end{aligned} \tag{4.32}$$

Denote the Fokker-Planck operator for the evolution of $(Y^1, \dots, Y^p)$ by

$$\bar{\mathcal{L}}^* := -\sum_{i=1}^{p}\nabla_{x_i} \cdot \left(\left[b(x_i) + \frac{1}{p-1}\sum_{j:j\neq i} K(x_i - x_j)\right] \cdot\right) + \sigma^2\sum_{i=1}^{p}\Delta_{x_i}.$$

Then the law of Y^1 at τ is given by

$$\mathcal{G}_\infty(\varrho_k) = \int_{(\mathbb{R}^d)^{p-1}} \mathrm{e}^{\tau \bar{\mathcal{L}}^*} \prod_{i=1}^p \varrho_k(x_i) dx_2 \cdots dx_p. \tag{4.33}$$

First note

$$\tilde{S}(\varrho_k)(x) = \varrho_k(x) + \tau \hat{\mathcal{L}}^* \varrho_k(x) + \int_0^\tau (\tau - s)(\hat{\mathcal{L}}^*)^2 \mathrm{e}^{s\hat{\mathcal{L}}^*} \varrho_k ds, \tag{4.34}$$

while

$$\begin{aligned}\mathcal{G}_\infty(\varrho_k)(x_1) = &\int_{(\mathbb{R}^d)^{p-1}} \prod_{i=1}^p \varrho_k(x_i) dx_2 \cdots dx_p + \tau \int_{(\mathbb{R}^d)^{p-1}} \bar{\mathcal{L}}^* \prod_{i=1}^p \varrho_k(x_i) dx_2 \cdots dx_p \\ &+ \int_0^\tau (\tau - s) \int_{(\mathbb{R}^d)^{p-1}} (\bar{\mathcal{L}}^*)^2 \mathrm{e}^{s\bar{\mathcal{L}}^*} \prod_{i=1}^p \varrho_k(x_i) dx_2 \cdots dx_p ds. \end{aligned} \tag{4.35}$$

The first line of (4.35) is reduced to

$$\begin{aligned}&\varrho_k(x_1) - \tau \int_{(\mathbb{R}^d)^{p-1}} \nabla_{x_1} \cdot \left(\left[b(x_1) + \frac{1}{p-1} \sum_{j:j\neq 1} K(x_1 - x_j) \right] \prod_{i=1}^p \varrho_k(x_i) \right) dx_2 \cdots dx_p \\ &\quad + \tau \sigma^2 \Delta_{x_1} \varrho_k(x_1) = \varrho_k(x_1) + \tau \hat{\mathcal{L}}^* \varrho_k(x_1), \end{aligned} \tag{4.36}$$

where we used

$$\begin{aligned}&\int_{(\mathbb{R}^d)^{p-1}} \nabla_{x_1} \cdot \left(\left[b(x_1) + \frac{1}{p-1} \sum_{j:j\neq 1} K(x_1 - x_j) \right] \prod_{i=1}^p \varrho_k(x_i) \right) dx_2 \cdots dx_p \\ &\quad = \nabla_{x_1} \cdot \left(\left[b(x_1) + \int_{\mathbb{R}^d} K(x_1 - y) \varrho_k(y) dy \right] \varrho_k(x_1) \right). \end{aligned}$$

Hence, we find

$$\begin{aligned}&|\tilde{S}(\varrho_k)(x) - \mathcal{G}_\infty(\varrho_k)(x)| \\ &\quad \leqslant \int_0^\tau (\tau - s) \left[|(\hat{\mathcal{L}}^*)^2 \mathrm{e}^{s\hat{\mathcal{L}}^*} \varrho_k| + \left| \int_{(\mathbb{R}^d)^{p-1}} (\bar{\mathcal{L}}^*)^2 \mathrm{e}^{s\bar{\mathcal{L}}^*} \left(\prod_{i=1}^p \varrho_k(x_i) \right) dx_2 \cdots dx_p \right| \right] ds. \end{aligned} \tag{4.37}$$

Now, we will apply Lemma 2.5 for $\mathsf{q} = 1$ with $\delta = \tau^2$ and $\hat{\mu} = \hat{\rho} dx$ with

$$\hat{\rho} = \frac{1}{\tau^2} \int_0^\tau (\tau - s) \left[|(\hat{\mathcal{L}}^*)^2 \mathrm{e}^{s\hat{\mathcal{L}}^*} \varrho_k| + \left| \int_{(\mathbb{R}^d)^{p-1}} (\bar{\mathcal{L}}^*)^2 \mathrm{e}^{s\bar{\mathcal{L}}^*} \left(\prod_{i=1}^p \varrho_k(x_i) \right) dx_2 \cdots dx_p \right| \right] ds.$$

The moment M_1 of $\hat{\mu}$ is controlled by $C(M_{q,4})$ for a constant C depending on $M_{q,4}$. To see this, we first remark that for $x \in \mathbb{R}^d$, one has $1 + |x| \leqslant 2 + |x|^2$. Both $\hat{\mathcal{L}}^*$ and $\bar{\mathcal{L}}^*$ are constant operators, and then one has by Lemma 4.8(iii) that for some $q > 1$,

$$\int_{\mathbb{R}^d} (2 + |x|^2) \hat{\rho}\, dx \leqslant C M_{q,4}.$$

To illustrate how this is estimated, we take the second term as an example:

$$\begin{aligned}&\int_{\mathbb{R}^d} (2 + |x|^2) \left| \int_{(\mathbb{R}^d)^{p-1}} (\bar{\mathcal{L}}^*)^2 \mathrm{e}^{s\bar{\mathcal{L}}^*} \prod_{i=1}^p \varrho_k(x_i) dx_2 \cdots dx_p \right| dx \\ &\quad \leqslant \int_{(\mathbb{R}^d)^p} (2 + |x_1|^2) \left| (\bar{\mathcal{L}}^*)^2 \mathrm{e}^{s\bar{\mathcal{L}}^*} \prod_{i=1}^p \varrho_k(x_i) \right| dx_1 \cdots dx_p \\ &\quad = \int_{(\mathbb{R}^d)^p} \left(2 + \frac{1}{p} \sum_i |x_i|^2 \right) \left| (\bar{\mathcal{L}}^*)^2 \mathrm{e}^{s\bar{\mathcal{L}}^*} \prod_{i=1}^p \varrho_k(x_i) \right| dx_1 \cdots dx_p \\ &\quad \leqslant \int_{(\mathbb{R}^d)^p} \left(2 + \frac{1}{p} \sum_i |x_i|^2 \right) \left| (\bar{\mathcal{L}}^*)^2 \prod_{i=1}^p \varrho_k(x_i) \right| dx_1 \cdots dx_p. \end{aligned}$$

This is controlled by $M_{q,4}$. Note that the dependence in τ for the constant $C(\tau)$ in Lemma 4.8 has been omitted since $\tau \lesssim O(1)$.

Lastly, the constants $C(M_{q,0})$ and $C(M_{q,4})$ clearly have an upper bound $g(M_{q,4})$ with g nondecreasing, defined on $[0,\infty)$. □

With the key one-step estimate established in Lemma 4.13 above, we can now finish the proof of Theorem 4.12.

Proof of Theorem 4.12. By the semigroup property of $\{\mathcal{G}_\infty^k\}$, one can find easily that

$$W_1(\mathcal{G}_\infty^n(\varrho_0), \varrho(n\tau)) \leqslant \sum_{m=1}^{n} W_1(\mathcal{G}_\infty^{n-m+1}(\varrho((m-1)\tau)), \mathcal{G}_\infty^{n-m}(\varrho(m\tau))).$$

By Proposition 4.1, under Assumptions 2.2 and 4.3, one has for $n\tau \leqslant T$,

$$\sum_{m=1}^{n} W_1(\mathcal{G}_\infty^{n-m+1}(\varrho((m-1)\tau)), \mathcal{G}_\infty^{n-m}(\varrho(m\tau))) \leqslant \sum_{m=1}^{n} \mathrm{e}^{(\beta+2L)(n-m)\tau} W_1(\mathcal{G}_\infty(\varrho((m-1)\tau)), \varrho(m\tau)).$$

Combining Proposition 4.11 and Lemma 4.13, $W_1(\mathcal{G}_\infty(\varrho((m-1)\tau)), \varrho(m\tau)) \leqslant C(T)\tau^2$ and thus the claim follows.

Under Assumptions 2.3 and 4.3, by using Propositions 4.1 and 4.11, the above estimates can be changed by replacing α with $-(r-2L)$, and $W_1(\mathcal{G}_\infty(\varrho((m-1)\tau)), \varrho(m\tau))$ now is bounded by $C\tau^2$ with C uniform in T. Hence, the conclusions follow easily. □

5 Some helpful discussions

In this section, we perform some helpful discussions to deepen the understanding and extend the results to second order interacting particle systems.

5.1 The mean field limit for $\tau \ll 1$

Formally, as $\tau \to 0$, the equation for Y^1 in (1.5) tends to (i.e., the limit for $\lim_{\tau\to 0}\lim_{N\to\infty}$) the SDE

$$dY = b(Y)dt + \frac{1}{p-1}\sum_{j=1}^{p-1} K(Y - Y_j)dt + \sqrt{2}\sigma dW \tag{5.1}$$

with $Y_j \sim \mathscr{L}(Y)$ being i.i.d., and $\{Y_j(s_i)\}$'s are independent for different time points s_i. Theorem 4.12 essentially tells us that the law of this SDE obeys the same nonlinear Fokker-Planck equation (1.2), which was satisfied by the law of the following seemingly different SDE:

$$dX = b(X)dt + \left(\int_{\mathbb{R}^d} K(X-y)\varrho(y,t)dy\right)dt + \sqrt{2}\sigma dW, \quad \varrho(x,t)dx = \mathscr{L}(X(t)). \tag{5.2}$$

See Figure 3 for illustration (compare with Figure 1).

To understand this, we consider a small but fixed τ, and the following SDEs (with the force field frozen at t_k):

$$\begin{aligned} d\hat{Y} &= b(Y)dt + \frac{1}{p-1}\sum_{j=1}^{p-1} K(\hat{Y} - Y_0^j)dt + \sqrt{2}\sigma dW, \quad Y_0^j \sim \mathscr{L}(Y(t_k)), \\ d\hat{X} &= b(\hat{X})dt + \left(\int_{\mathbb{R}^d} K(\hat{X}-y)\varrho(y,t_k)dy\right)dt + \sqrt{2}\sigma dW. \end{aligned} \tag{5.3}$$

The probability density for the former at $t_k + \tau$ is $\int_{\mathbb{R}^d} dy\varrho(y,t_k)\mathrm{e}^{\tau\mathcal{L}_y^*}\varrho(\cdot,t_k)$, where

$$\mathcal{L}_y^* = -\nabla\cdot([b(x) + K(x-y)]\cdot) + \sigma^2\Delta_x,$$

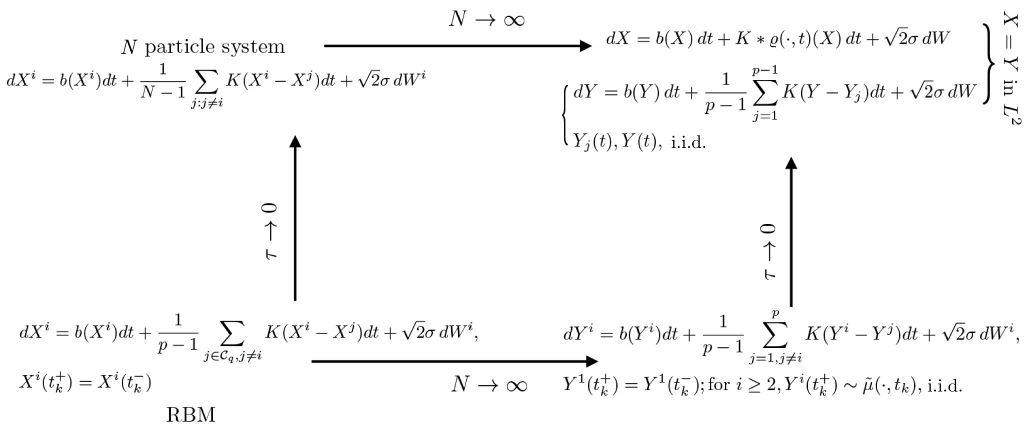

Figure 3 Illustration of the various SDEs in different regime

while the probability density for the latter is $\mathrm{e}^{\tau\hat{\mathcal{L}}^*}\varrho(\cdot,t_k)$ with

$$\hat{\mathcal{L}}^* = -\nabla\cdot\left(\left[b(x) + \int_{\mathbb{R}^d} K(x-y)\varrho(y,t_k)dy\right]\cdot\right) + \sigma^2\Delta_x = \int_{\mathbb{R}^d} dy\varrho(y,t_k)\mathcal{L}_y^*.$$

Clearly, to the leading order, the changing rates of the probability densities are the same.

In Figure 3, we have made a stronger claim that the X and Y processes in the right-upper corner are equal in L^2, instead of "equal in law", if the Brownian motions W used are the same. To see this, one may compute

$$\frac{d}{dt}\mathbb{E}|X-Y|^2 = 2\mathbb{E}(X-Y)\cdot(b(X)-b(Y)) + 2\mathbb{E}(X-Y)\cdot\left(K*\varrho(\cdot,t)(X) - \frac{1}{p-1}\sum_{j=1}^{p-1}K(Y-Y_j)\right).$$

Since $Y_j(t)$ is independent of $Y(t)$ and $X(t)$, one has

$$\mathbb{E}(X-Y)\cdot\left(K*\varrho(\cdot,t)(X) - \frac{1}{p-1}\sum_{j=1}^{p-1}K(Y-Y_j)\right) = \mathbb{E}(X-Y)\cdot(K*\varrho(\cdot,t)(X) - K*\bar{\varrho}(\cdot,t)(Y)),$$

where $\bar{\varrho}$ is the law of Y. By taking $\tau\to 0$ in Theorem 4.12, $\bar{\varrho} = \varrho$. Hence, one actually has

$$\frac{d}{dt}\mathbb{E}|X-Y|^2 \leqslant 2(\beta+L)\mathbb{E}|X-Y|^2.$$

Hence, $X = Y$ in L^2.

5.2 Regarding the approximation in Lemma 4.13

Usually, the Wasserstein distance (especially W_2) was estimated using the SDEs. A natural question is therefore whether one can estimate the Wasserstein distance in Lemma 4.13 via the SDE approach.

Below, we illustrate the issue using the W_2 distance and the approximating problem (5.3) (with the force expressions frozen). Here, we assume the Brownian motions used are the same. The values Y_0^j are i.i.d., drawn from $\varrho(\cdot)$.

We compute that

$$\frac{d}{dt}\mathbb{E}|\hat{X}-\hat{Y}|^2 = \mathbb{E}(\hat{X}-\hat{Y})\cdot(b(\hat{X})-b(\hat{Y})) + D,$$

where

$$D = \mathbb{E}(\hat{X}-\hat{Y})\cdot\left(\bar{K}(\hat{X}) - \frac{1}{p-1}\sum_{j=1}^{p-1}K(\hat{Y}-Y_0^j)\right).$$

Clearly, for fixed x,

$$\mathbb{E}\frac{1}{p-1}\sum_{j=1}^{p-1}K(x-Y_0^j)=\bar{K}(x). \tag{5.4}$$

Hence, if $\hat{Y}$ is independent of Y_0^j's, then this term can be controlled as

$$\mathbb{E}(\hat{X}-\hat{Y})\cdot(\bar{K}(\hat{X})-\bar{K}(\hat{Y}))\leqslant C\mathbb{E}|\hat{X}-\hat{Y}|^2.$$

One is thus tempted to believe that even though that $\hat{Y}$ is not independent of Y_0^j, one can do the Itô-Taylor expansion and the extra term is small enough, which can yield the desired error.

Unfortunately, if one is going to do the Itô-Taylor expansion in $\hat{Y}$, one may find that $D=O(\tau)$. In fact,

$$\begin{aligned}&(\hat{X}-\hat{Y})\cdot\left(\bar{K}(\hat{X})-\frac{1}{p-1}\sum_{j=1}^{p-1}K(\hat{Y}-Y_0^j)\right)\\&=\int_0^t\left(\bar{K}(\hat{X}(s))-\frac{1}{p-1}\sum_{j=1}^{p-1}K(\hat{Y}(s)-Y_0^j)\right)\cdot\left(\bar{K}(\hat{X}(t))-\frac{1}{p-1}\sum_{j=1}^{p-1}K(\hat{Y}(t)-Y_0^j)\right)ds.\end{aligned}$$

If we take expectation, the variance of the random force $\frac{1}{p-1}\sum_{j=1}^{p-1}K(x-X_0^j)$ appears, which gives $D=O(\tau)$. Hence, this estimate is not good and the mean square error is only like

$$\sqrt{\mathbb{E}|\hat{X}-\hat{Y}|^2}=O(\tau).$$

This means that the consistency (5.4) brings no benefit for this mean square error.

Intrinsically, the mean square error above is roughly comparable with

$$\int\varrho(z_1)\cdots\varrho(z_j)W_2^2(\mathrm{e}^{\tau\hat{\mathcal{L}}^*}\varrho,\mathrm{e}^{t\mathcal{L}^*_{z_1,\dots,z_j}}\varrho)dz_1\cdots dz_j.$$

What we care about is the distance between $\mathrm{e}^{\tau\hat{\mathcal{L}}^*}\varrho$ and $\int\varrho(z_1)\cdots\varrho(z_j)\mathrm{e}^{t\mathcal{L}^*_{z_1,\dots,z_j}}\varrho dz_1\cdots dz_j$. The former involves the variance introduced by the random force while the latter does not have this issue and uses the consistency (5.4). This is why we used the total variation norm to obtain the one-step error under the W_1 distance in Lemma 4.13.

5.3 Approximation using weak convergence

The weak convergence is another popular gauge of the convergence of probability $\mathcal{G}_\infty^k(\varrho_0)$ to $\varrho(k\tau)$ [31,42].

Pick a test function φ. By using a consistency condition similar to (4.36), it is not very hard to show

$$\left|\int_{\mathbb{R}^d}\varphi(y)\mathcal{S}(\tau)(\mu)(dy)-\int_{\mathbb{R}^d}\varphi(y)\mathcal{G}_\infty(\mu)(dy)\right|\leqslant C\tau^2 \tag{5.5}$$

for any μ, where we recall $\mathcal{S}(t)$ is the evolution operator for (1.2). Hence, the one-step error is easy to control for weak convergence. However, the difficulty is to get a certain stability property of the nonlinear dynamics under the weak topology. That means, if two measures are close in the weak topology at some time, then let them evolve under $\mathcal{G}_\infty$ for k times, one needs them to be close. Consider

$$U^n(x):=\int_{\mathbb{R}^d}\varphi(y)\mathcal{G}_\infty^n(\delta(y-x))dy.$$

Unlike the linear case (see [17]), it is hard to write U^n as some operator acting on U^{n-1} due to the nonlinearity of $\mathcal{G}_\infty$. Proving the stability of this nonlinear dynamics under weak topology seems challenging, and this is why we chose the Wasserstein metric.

5.4 A remark for second order systems

As shown in [28], the Random Batch Method applied equally well to second order systems on finite time interval. Repeating the proof here, one can show that the similar mean field limit holds for second order systems when $t \in [0, T]$. In particular, let us consider the models for swarming and flocking considered in [1]:

$$\begin{aligned} \dot{x}_i &= v_i, \\ \dot{v}_i &= \frac{1}{N} \sum_j H_\alpha(x_i, x_j, v_i)(v_j - v_i). \end{aligned} \tag{5.6}$$

Here, $H_\alpha(\cdot,\cdot,\cdot)$ is some function modeling interactions between particles. The mean field limit of (5.6) for $t \in [0, T]$ takes the following form (rigorous justification needs some assumptions on H_α [27]):

$$\begin{aligned} &\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (\xi(f) f) = 0, \\ &\xi(f) = \int_{\mathbb{R}^{2d}} H_\alpha(x, y, v)(w - v) f(y, w, t)\, dw dy. \end{aligned} \tag{5.7}$$

Albi and Pareschi [1] developed some stochastic binary interaction algorithms for the dynamics. The symmetric Nanbu algorithm (see [1, Algorithm 4.3]) is like the Random Batch Method when $p = 2$ and the Random Batch Method can be viewed as the generalization of this Nanbu algorithm. When applying the Random Batch Method to the particle system and consider $N \gg 1$, the dynamics is expected to be close to the following limiting dynamics:

Algorithm 3 Mean field dynamics of the RBM for the flocking dynamics (5.6)

1: From t_k to t_{k+1}, the distribution $f_k \in \mathbf{P}(\mathbb{R}^{2d})$ will be transformed into $f_{k+1} = \mathcal{Q}_\infty(f_k)$ as follows.
2: Let $f^{(p)}(\ldots, t_k) = f_k^{\otimes p}$ be a probability measure on $(\mathbb{R}^{2d})^p \cong \mathbb{R}^{2pd}$.
3: Evolve $f^{(p)}$ by time τ according to the following:

$$\begin{aligned} &\partial_t f^{(p)} + \sum_{i=1}^p \nabla_{x_i} \cdot (v_i f^{(p)}) + \sum_{i=1}^p \nabla_{v_i} \cdot (\xi_i f^{(p)}) = 0, \\ &\xi_i = \frac{1}{p-1} \sum_{j: j \neq i} H_\alpha(x_i, x_j, v_i)(v_j - v_i). \end{aligned} \tag{5.8}$$

4: Set

$$f_{k+1} = \mathcal{Q}_\infty(f_k) := \int_{(\mathbb{R}^{2d})^{(p-1)}} f^{(p)}(\cdot, dy_2, \ldots, dy_p; \cdot, dv_2, \ldots, dv_p; t_{k+1}^-). \tag{5.9}$$

We expect that this nonlinear operator will approximate the nonlinear kinetic equation (5.7). In this sense, we believe the $N \to \infty$ limit of [1, Algorithm 4.3] will be an analogue of the dynamics $\mathcal{Q}_\infty$ given in Algorithm 3.

6 Conclusions

We first identified and justified in this work the mean field limit of the RBM for the fixed step size τ. Then we showed that this mean field limit is close to that of the N particle system, though the chaos arises differently in these two dynamics. The current argument of the mean field limit relies on the fact that two particles are unlikely to be related in the RBM when $N \to \infty$ for finite iterations. Hence, this argument cannot give a uniform in τ bound for the speed of the mean field limit. It will be an interesting topic to investigate how mixing and chaos can be created in the RBM after two particles in a batch are separated, so that one may obtain a convergence speed independent of τ.

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Appendix A Proof of Proposition 4.5

Step 1. *A priori* estimates on the moments and the entropy.

We first perform *a priori* estimates on the moments. Fix $q \geqslant 2$. We have

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} |x|^q \varrho dx &= \int_{\mathbb{R}^d} |x|^q \{-\nabla \cdot [(b(x) + K * \varrho)\varrho]\} dx + \int_{\mathbb{R}^d} |x|^q \sigma^2 \Delta \varrho dx \\
&= \int_{\mathbb{R}^d} q|x|^{q-2} x \cdot b(x) \varrho dx + \iint_{\mathbb{R}^d \times \mathbb{R}^d} q|x|^{q-2} x \cdot K(x-y) \varrho(x) \varrho(y) dxdy \\
&\quad + \sigma^2 \int_{\mathbb{R}^d} q(q-2+d)|x|^{q-2} \varrho dx =: I_1 + I_2 + I_3.
\end{aligned}$$

For I_2, one has

$$\begin{aligned}
\iint_{\mathbb{R}^d \times \mathbb{R}^d} q|x|^{q-2} x \cdot K(x-y) \varrho(x) \varrho(y) dxdy &\leqslant q \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{q-2} x \cdot K(0) \varrho(x) \varrho(y) dxdy \\
&\quad + qL \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{q-1} (|x| + |y|) \varrho(x) \varrho(y) dxdy.
\end{aligned}$$

By Young's inequality,

$$q \iint_{\mathbb{R}^d \times \mathbb{R}^d} |x|^{q-2} x \cdot K(0) \varrho(x) \varrho(y) dxdy \leqslant \delta \int_{\mathbb{R}^d} |x|^q \varrho dx + C(\delta).$$

Also, Young's inequality implies that

$$|x|^{q-1}|y| \leqslant \frac{q-1}{q}|x|^q + \frac{1}{q}|y|^q.$$

Hence,

$$I_2 \leqslant q(2L+\delta)\int_{\mathbb{R}^d}|x|^q\varrho dx + C(\delta).$$

If $q=2$, I_3 is a constant. Otherwise if $q>2$, one can use Young's inequality and

$$I_3 \leqslant \delta\int_{\mathbb{R}^d}|x|^q\varrho dx + C(\delta).$$

For I_1, under Assumption 2.2, one has

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^d} q|x|^{q-2}x\cdot(b(x)-b(0))\varrho dx + \int_{\mathbb{R}^d} q|x|^{q-2}x\cdot b(0)\varrho dx \\ &\leqslant \beta q\int_{\mathbb{R}^d}|x|^q\varrho dx + C\int_{\mathbb{R}^d}|x|^{q-1}\varrho dx. \end{aligned}$$

Hence,

$$I_1+I_2+I_3 \leqslant q(\beta+2L+\delta)\int_{\mathbb{R}^d}|x|^q\varrho dx + C(\delta),$$

where the concrete meaning of δ and $C(\delta)$ have changed. By using Grönwall's inequality, the moments can be controlled.

Now, we perform *a priori* estimates on the entropy. Multiply $1+\log\varrho$ on both sides and integrate

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varrho\log\varrho dx = -\int_{\mathbb{R}^d}\varrho(x)\nabla\cdot(b(x)+(K*\varrho)(x))dx - 4\sigma^2\int_{\mathbb{R}^d}|\nabla\sqrt{\varrho}|^2dx.$$

By the moment control, the first term is bounded on $[0,T]$. Hence, the entropy can be controlled.

As a remark, in the case $\sigma=0$, ϱ could be zero at some points. In this case $1+\log\varrho$ is not a good test function. This issue will be explained further in Step 2.

Step 2. The existence in $L^\infty(0,T;L^1(\mathbb{R}^d))\cap C([0,T];\mathbf{P}(\mathbb{R}^d))$.

Take a smooth function $\chi\in C_c[0,\infty)$, i.e., 1 in $[0,1]$ and zero on $[2,\infty)$. Consider the following approximating equation:

$$\begin{aligned} &\partial_t\rho_N = -\nabla\cdot(b(x)\chi(x/N)\rho_N) - \nabla\cdot(\rho_N(K*\rho_N)) + \Delta\rho_N, \\ &\varrho_N\,|_{t=0} = \varrho_0. \end{aligned}$$

Now, $b(x)\chi(x/N)$ and K are Lipschitz functions and $b(x)\chi(x/N)$ is bounded (compactly supported). The existence of a smooth solution is clear (see, for example, [8, Appendix A]). Performing similar estimates to that in Step 1, we have

$$\sup_N\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^d}|x|^2\varrho_N dx \leqslant C(T)$$

and

$$\sup_N\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^d}\varrho_N\log\varrho_N dx \leqslant C(T).$$

Note that for the entropy, the zeros of ϱ_N may make $1+\log(\varrho_N)$ an invalid test function. We instead multiply

$$\frac{\varrho_N}{\varrho_N+\epsilon} + \log(\varrho_N+\epsilon)$$

as the test function for $\epsilon>0$. Then, the left-hand side becomes $\frac{d}{dt}\int\varrho_N\log(\varrho_N+\epsilon)dx$ (note that $\epsilon\to\varrho\log(\varrho+\epsilon)$ is non-decreasing so later one can take $\epsilon\to0$ to get the desired entropy control). For the right-hand side, we note

$$\nabla\left[\frac{\varrho_N}{\varrho_N+\epsilon}+\log(\varrho_N+\epsilon)\right] = \frac{(\varrho_N+2\epsilon)\nabla\varrho_N}{(\varrho_N+\epsilon)^2}.$$

For the transport term,

$$b(x)\chi\left(\frac{x}{N}\right)\frac{\varrho_N(\varrho_N+2\epsilon)\nabla\varrho_N}{(\varrho_N+\epsilon)^2}=(b(x)\chi(x/N))\cdot\nabla\varrho_N+\epsilon^2(b(x)\chi(x/N))\cdot\nabla\left(\frac{1}{\varrho_N+\epsilon}\right).$$

By doing integration by parts and sending $\epsilon\to 0$ first, the second term here will vanish. Through this way, *a priori* estimate on the entropy can be justified for this approximating sequence.

The moment estimates imply that $\{\varrho_N dx\}$ is tight while the entropy estimates imply that $\{\varrho_N\}$ is uniformly integrable. By the Dunford-Pettis theorem, ϱ_N converges weakly to some $\varrho\in L^1_{\rm loc}([0,T]\times\mathbb{R}^d)$ and $\varrho dx\in C([0,T];\mathbf{P}(\mathbb{R}^d))$. Moreover, with the moment control and the uniform integrability

$$\int_{\mathbb{R}^d}K(x-y)\varrho_N(y)dy\to\int_{\mathbb{R}^d}K(x-y)\varrho(y)dy$$

pointwise and actually uniformly on compact sets. With this, then one can easily verify that ϱ is a desired weak solution with the corresponding moment control. This will further imply that $\varrho\in L^\infty([0,T];L^1(\mathbb{R}^d))$.

Step 3. The uniqueness and smoothness of the solution.

We now aim to prove the uniqueness. We divide this step into two sub-steps.

Step 3.1. The weak solution is a strong solution.

Let ϱ be such a weak solution with

$$\sup_{0\leqslant t\leqslant T}\int_{\mathbb{R}^d}|x|\varrho dx<C(T).$$

Then $\bar{K}(x,t):=K*\varrho$ is a smooth function (since K is smooth) and

$$|\bar{K}(0)|\leqslant\left|\int_{\mathbb{R}^d}K(x)\varrho(x)dx\right|\leqslant|K(0)|+LC(T).$$

Moreover, it is easy to see that $\bar{K}(x,t)$ is also Lipschitz with the Lipschitz constant bounded by L.

We claim that for a given ϱ, the solution to

$$\partial_t u=-\nabla\cdot(b(x)u+\bar{K}(x,t)u)+\sigma^2\Delta u,$$
$$u\,|_{t=0}=\varrho_0$$

is unique and thus must be ϱ. In fact, the existence can be justified by the following SDE as its law is a weak solution:

$$dX=(b(X)+\bar{K}(X,t))dt+\sqrt{2}\sigma dW,\quad X_0\sim\varrho_0 dx.$$

For the well-posedness of such SDEs, one can refer to [38, Chapter 2, Theorem 3.5], and also see a recent work with weaker assumptions [47]. Regarding the uniqueness, one considers the difference of two such solutions $u_i, i=1,2$,

$$\partial_t(u_1-u_2)=-\nabla\cdot([b(x)+\bar{K}(x,t)](u_1-u_2))+\sigma^2\Delta(u_1-u_2).$$

We then multiply $h_\epsilon(u_1-u_2):=h((u_1-u_2)/\epsilon)$ on both sides and take integral. Here, $h(\cdot)$ is an odd function that increases monotonely from -1 to 1 on $[-1,1]$. It is 1 on $[1,\infty)$. Hence, $h(\cdot/\epsilon)$ is some approximation for the sign function.

Then,

$$\frac{d}{dt}\int_{\mathbb{R}^d}H_\epsilon(u_1-u_2)dx\leqslant\int_{\mathbb{R}^d}h'\left(\frac{u_1-u_2}{\epsilon}\right)\frac{u_1-u_2}{\epsilon}(b(x)+\bar{K}(x,t))\cdot\nabla(u_1-u_2)dx,$$

where $H_\epsilon(u)=\int_0^u h_\epsilon(s)ds$. The right-hand side goes to zero when $\epsilon\to 0$, because

$$h'\left(\frac{u_1-u_2}{\epsilon}\right)\frac{u_1-u_2}{\epsilon}$$

is bounded and nonzero only on $|u_1 - u_2| \leqslant \epsilon$. Also, $H_\epsilon(u_1 - u_2) \to |u_1 - u_2|$ as $\epsilon \to 0$. Hence, the claim is shown and thus

$$u = \varrho.$$

By the theory of the *linear* PDEs, $u = \varrho$ is in fact a strong solution and smooth. For the general theory of linear parabolic equations, one may refer to [20].

Step 3.2. The uniqueness of the nonlinear Fokker-Planck equation.

For the uniqueness of the nonlinear Fokker-Planck equation, we cannot use the technique in Step 3.1 as we show the uniqueness for the linear PDE, as the term $K * \varrho$ involves the solution ϱ itself. Also, the classical Dobrushin's estimate [14, 22] cannot be used because the flow map is not well defined before we show the uniqueness of ϱ.

Instead, we use the interacting particle system for the mean-field limit and show that any weak solution is close to the one marginal distribution of the N-particle system. This then will result in the uniqueness.

Fix *any* weak solution of the nonlinear Fokker-Planck equation. Consider the following SDEs:

$$dX^i = b(X^i)dt + (K * \varrho)(X^i)dt + \sqrt{2}\sigma dW^i, \quad i = 1, \dots, N. \tag{A.1}$$

According to the argument in Step 3.1, the law of each X^i is exactly the weak solution ϱ used to convolve with K. Moreover, these X^i's are independent.

Now, consider the interacting particle system

$$dY^i = b(Y^i)dt + \frac{1}{N-1}\sum_{j:j\neq i} K(Y^i - Y^j)dt + \sqrt{2}\sigma dW^i, \quad i = 1, \dots, N. \tag{A.2}$$

The next step is to use the technique in the proof of [10, Theorem 3.1]. We compute for fixed i,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\mathbb{E}|X^i - Y^i|^2 &= \mathbb{E}(X^i - Y^i)\cdot(b(X^i) - b(Y^i)) \\ &\quad + \mathbb{E}(X^i - Y^i)\cdot\left(\bar{K}(X^i, t) - \frac{1}{N-1}\sum_{j:j\neq i} K(Y^i - Y^j)\right).\end{aligned} \tag{A.3}$$

The first term is controlled by $\beta\mathbb{E}|X^i - Y^i|^2$. The second term is split as

$$\begin{aligned}&\mathbb{E}(X_i - Y_i)\cdot\left(\bar{K}(X_i, t) - \frac{1}{N-1}\sum_{j:j\neq i} K(Y_i - Y_j)\right) \\ &\quad = \mathbb{E}(X_i - Y_i)\cdot\left(\bar{K}(X_i, t) - \frac{1}{N-1}\sum_{j:j\neq i} K(X_i - X_j)\right) \\ &\qquad + \mathbb{E}(X_i - Y_i)\cdot\left(\frac{1}{N-1}\sum_{j:j\neq i} K(X_i - X_j) - \frac{1}{N-1}\sum_{j:j\neq i} K(Y_i - Y_j)\right) \\ &\quad =: D_1 + D_2.\end{aligned}$$

The term D_2 is easily controlled by $2L\mathbb{E}|X_i - Y_i|^2$ by the exchangeability. For D_1, one can control it as

$$D_1 \leqslant \sqrt{\mathbb{E}|X_i - Y_i|^2}\sqrt{\mathbb{E}\left|\bar{K}(X_i, t) - \frac{1}{N-1}\sum_{j:j\neq i} K(X_i - X_j)\right|^2}.$$

However,

$$\begin{aligned}&\mathbb{E}\left|\bar{K}(X_i, t) - \frac{1}{N-1}\sum_{j:j\neq i} K(X_i - X_j)\right|^2 \\ &\quad = \frac{1}{(N-1)^2}\sum_{j,k:j\neq i,k\neq i}\mathbb{E}(\bar{K}(X_i, t) - K(X_i - X_j))(\bar{K}(X_i, t) - K(X_i - X_k)).\end{aligned}$$

By independence, the terms for $j \neq k$ are zero. Hence, only $N-1$ terms will survive. This means

$$D_1 \leqslant \sqrt{\mathbb{E}|X_i - Y_i|^2}\frac{C_1(T,\varrho)}{\sqrt{N-1}}.$$

Moreover, $C_1(T,\varrho)$ will have an upper bound that is independent of T if Assumption 2.3 holds.

By Grönwall's inequality,

$$\sqrt{\mathbb{E}|X_i - Y_i|^2} \leqslant C(T,\varrho)\frac{1}{\sqrt{N-1}}.$$

Hence for any two weak solutions ϱ_1 and ϱ_2, we have

$$\sup_{0\leqslant t\leqslant T} W_2(\varrho_1,\varrho_2) \leqslant [C(T,\varrho_1) + C(T,\varrho_2)]\frac{1}{\sqrt{N-1}}.$$

Taking $N \to \infty$ yields the uniqueness of the solutions to the nonlinear Fokker-Planck equation.

Step 4. Strong confinement.

Under Assumption 2.3, one in fact has

$$I_1 + I_2 + I_3 \leqslant q(-r + 2L + \delta)\int_{\mathbb{R}^d} |x|^q \varrho dx + C(\delta).$$

The assertions about moments have then been proved with the application of Grönwall's inequality.

Under this condition, the estimate of D_1 term in Step 3 can also be independent of T, because of this uniform moment control. Hence, the mean field limit can be uniform in T.

Lastly, to show the convergence of ϱ as $t \to \infty$, we consider two different initial data $\varrho_{j,0}$ where $j = 1, 2$. Then, one can consider (A.2) with these two initial data. Pick the coupling between $Y_1^i(0)$ and $Y_2^i(0)$ (the data for different i's are independent) such that

$$\mathbb{E}|Y_1^i(0) - Y_2^i(0)|^{\mathsf{q}} \leqslant W_{\mathsf{q}}^{\mathsf{q}}(\varrho_{1,0},\varrho_{2,0}) + \epsilon, \quad \forall\, i = 1,\ldots,N.$$

Then by a similar computation,

$$\frac{d}{dt}\mathbb{E}|Y_1^i(t) - Y_2^i(t)|^{\mathsf{q}} \leqslant \mathsf{q}(-r + 2L)\mathbb{E}|Y_1^i(t) - Y_2^i(t)|^{\mathsf{q}}.$$

By fixing $t > 0$ and taking $N \to \infty$, $\mathscr{L}(Y_j^i(t)) \to \varrho_j(t)$, $j = 1, 2$. Hence, the evolutional nonlinear semigroup for the nonlinear Fokker-Planck equation is a contraction

$$W_{\mathsf{q}}(\varrho_1(t),\varrho_2(t)) \leqslant W_{\mathsf{q}}(\varrho_{1,0},\varrho_{2,0})\mathrm{e}^{-(r-2L)t}.$$

Thus, the last claim follows.

Appendix B Proof of Lemma 4.10

Since $\sigma > 0$, without loss of generality, we will assume

$$\sigma \equiv 1.$$

We first fix $s \geqslant 0$. Consider the trajectory determined by

$$\partial_t Z(t; y, s) = b(Z, t), \quad Z(s; y, s) = y. \tag{B.1}$$

Then, one has

$$\frac{1}{2}\frac{d}{dt}|Z|^2 \leqslant \beta_1|Z|^2 + C|Z|$$

as $b(0,t)$ is bounded. Hence,

$$\frac{d}{dt}|Z| \leqslant \beta_1|Z| + C.$$

This means

$$|Z| \leqslant |y|\mathrm{e}^{\beta_1(t-s)} + C\int_s^t \mathrm{e}^{\beta_1(t-s)} ds. \tag{B.2}$$

Moreover, (4.15) implies that

$$v \cdot \nabla b_1(x,t) \cdot v \leqslant \beta_1 |v|^2, \quad \forall\, v, x \in \mathbb{R}^d, \quad t \geqslant 0.$$

Consequently,

$$|\nabla_y Z| \leqslant \sqrt{d}\mathrm{e}^{\beta_1(t-s)} \tag{B.3}$$

uniformly in y, where

$$|A| := \sqrt{\sum_{ij} A_{ij}^2}$$

is the matrix Frobenius norm.

Assume without loss of generality $|x| \geqslant |y|$. Clearly,

$$|b_1(x,t) - b_1(y,t)| \leqslant |x-y| \left| \int_0^1 \nabla b_1(x\theta + y(1-\theta), t) d\theta \right|.$$

Due to the assumption of polynomial growth of derivatives of b_1,

$$|\nabla b_1(xz + y(1-z), t)| \leqslant C(1 + |x\theta + y(1-\theta)|^q).$$

If $|y| \leqslant \frac{1}{2}|x|$, then

$$|x\theta + y(1-\theta)| \leqslant \frac{3}{2}|x| \leqslant 3|x-y|.$$

Otherwise, we bound this by a polynomial of $|y|$ directly. Hence,

$$|b_1(x,t) - b_1(y,t)| \leqslant \min(P_1(|x|), P_1(|y|))|x-y| + P_2(|x-y|)|x-y| \tag{B.4}$$

for some polynomials P_1 and P_2.

We denote

$$\Phi_0(x,t;y,s) := \frac{1}{(2\pi(t-s))^{d/2}} \exp\left(-\frac{|x - Z(t;y,s)|^2}{2(t-s)}\right). \tag{B.5}$$

Below, we establish an important lemma indicating that Φ_0 is the main term of Φ, and Lemma 4.10 will follow easily.

Lemma B.1. *It holds that*

$$\Phi(x,t;y,s) = \Phi_0(x,t;y,s) + u(x,t;y,s), \tag{B.6}$$

where u satisfies

$$\int_{\mathbb{R}^d} (1 + |x|^q)|\nabla_y u| dx \leqslant h(t-s)P(|y|) \tag{B.7}$$

for some polynomial $P(\cdot)$, some nondecreasing function $h(\cdot)$ defined on $[0,\infty)$.

Moreover, if $\beta_1 < 0$, $h(t-s)$ can be taken as

$$h(t-s) = C\mathrm{e}^{-\delta_1(t-s)} \tag{B.8}$$

for some $\delta_1 > 0$.

Proof. It is not hard to verify

$$\partial_t \Phi_0 + \nabla_x \cdot (b_1(x,t)\Phi_0) - \Delta_x \Phi_0 = \nabla_x \cdot ([b_1(x,t) - b(Z,t)]\Phi_0). \tag{B.9}$$

Hence, letting $u = \Phi - \Phi_0$, one finds

$$\begin{aligned} &\partial_t u + \nabla_x \cdot (b_1(x,t)u) - \Delta_x u = -\nabla_x \cdot ([b_1(x,t) - b(Z,t)]\Phi_0), \\ &u\,|_{t=s} = 0. \end{aligned} \tag{B.10}$$

Letting

$$v := \partial_{y_i} u,$$

one has

$$\begin{aligned} &\partial_t v + \nabla_x \cdot (b_1(x,t)v) - \Delta_x v = R, \\ &u\,|_{t=s} = 0, \end{aligned} \tag{B.11}$$

where

$$R := \nabla_x \cdot b_1(x,t)\nabla\Phi_0 \cdot \partial_{y_i} Z + \partial_{y_i} Z \cdot \nabla b_1(x,t) \cdot \nabla\Phi_0 + (b_1(x,t) - b_1(Z,t)) \cdot \nabla^2\Phi_0 \cdot \partial_{y_i} Z.$$

By writing

$$\nabla_x \cdot b_1(x,t) = [\nabla_x \cdot b_1(x,t) - \nabla \cdot b_1(Z,t)] + \nabla \cdot b_1(Z,t),$$

it is not hard to see (using also (B.3) and (B.4))

$$|R| \leqslant P(|Z|)\frac{1}{(t-s)^{(d+1)/2}} \exp\bigg(-\frac{\gamma|x-Z|^2}{2(t-s)}\bigg)\mathrm{e}^{\beta_1(t-s)}$$

for some polynomial P and $\gamma \in (0,1)$.

We then find

$$v = \int_s^t S_{\lambda,t} R d\lambda.$$

Below, we use $h_i(\cdot)$ to denote some nondecreasing functions defined on $[0,\infty)$. By Lemma 4.8, one has

$$\int_{\mathbb{R}^d} (1+|x|^q)|v|dx \leqslant h_1(t-s)\int_s^t \int_{\mathbb{R}^d} (1+|x|^q)|R(x,\lambda)|dxd\lambda.$$

Clearly,

$$\int_{\mathbb{R}^d} (1+|x|^q)\frac{1}{(t-s)^{(d+1)/2}} \exp\bigg(-\frac{\delta|x-Z|^2}{2(t-s)}\bigg)dx \leqslant C\frac{1+(t-s)^{q/2}}{\sqrt{t-s}}(1+|Z|^q).$$

Moreover, by the stability of trajectory of Z (see (B.2)), $P(|Z|) \leqslant h_2(t-s)P(|y|)$. Hence,

$$\int_{\mathbb{R}^d} (1+|x|^q)|v|dx \leqslant h_3(t-s)P(|y|)\int_s^t \frac{1}{\sqrt{\lambda-s}}d\lambda.$$

If $\beta_1 < 0$, we consider $t \geqslant s+1$ and

$$v = \int_s^t S_{\lambda,t} R d\lambda = S_{(t+s)/2,t}\int_s^{\frac{t+s}{2}} S_{\lambda,(t+s)/2} R d\lambda + \int_{(t+s)/2}^t S_{\lambda,t} R d\lambda. \tag{B.12}$$

The second term is like

$$\begin{aligned} \int_{\mathbb{R}^d} (1+|x|^q)|v|dx &\leqslant C\int_{(s+t)/2}^t \int_{\mathbb{R}^d} (1+|x|^q)|R(x,\lambda)|dxd\lambda \\ &\leqslant C\int_{(s+t)/2}^t \mathrm{e}^{\beta_1(\lambda-s)}P(|Z|)\int_{\mathbb{R}^d} \frac{1+|x|^q}{(t-s)^{(d+1)/2}} \exp\bigg(-\frac{\delta|x-Z|^2}{2(t-s)}\bigg)dxd\lambda. \end{aligned}$$

This is easily controlled by $P(|y|)\mathrm{e}^{-\delta_1(t-s)}$ for some polynomial P and $\delta_1 > 0$ (recall (B.2)).

For the first term in (B.12), we note $\int_s^{\frac{t+s}{2}} S_{\lambda,(t+s)/2} R d\lambda \in L^1$, and

$$\int_{\mathbb{R}^d} \int_s^{\frac{t+s}{2}} S_{\lambda,(t+s)/2} R d\lambda dx = 0$$

since $\int R(x,\lambda)dx = 0$ for all λ. Hence, Lemma 4.8(ii) implies that

$$\int_{\mathbb{R}^d} (1+|x|^q) \left| S_{(t+s)/2,t} \int_s^{\frac{t+s}{2}} S_{\lambda,(t+s)/2} R d\lambda \right| dx \leqslant \mathrm{e}^{-\delta(t-s)/2} P\left(M_{q_1}\left(\left| \int_s^{\frac{t+s}{2}} S_{\lambda,(t+s)/2} R d\lambda \right| \right) \right).$$

For the inside,

$$M_{q_1}\left(\left| \int_s^{\frac{t+s}{2}} S_{\lambda,(t+s)/2} R d\lambda \right| \right) \leqslant C \int_s^{(t+s)/2} \int_{\mathbb{R}^d} (1+|x|^{q_1}) |R| dx d\lambda,$$

where C is independent of time as $\beta_1 < 0$. As has been proved, the integral here is controlled by products of polynomials of $|y|$ and $|t-s|$. Hence, the first term is also controlled similarly. □

As Lemma B.1 is proved, Lemma 4.10 is very straightforward, since $|\nabla_y Z| \leqslant C\mathrm{e}^{\beta_1(t-s)}$.