# Advanced computational methods X071521-SDE
# Lecture 5

## 1 Other weak schemes

Here, I list out some typical weak schemes. If you are interested in them, you can read in details.

### 1.1 Weak Taylor approximations

You can use the Itô-Taylor expansion to generate a lot of schemes. For example, the Milstein scheme has weak order 1. Previously, we have seen a weak 2 scheme. For the reference, you can read Chapter 14 of the book by Kloeden and Platen.

As a side note, for weak approximation, we only care distribution, so you do not have to use i.i.d Gaussian random variables $\Delta W_n$. For example, consider the case $d = 1$ and the SDE

$$dX = b(t, X)dt + \sigma(t, X)dW.$$

Consider

$$X^{n+1} = X^n + b(t_n, X^n)k + \sigma(t_n, X^n)\sqrt{k}\xi_n$$

where $\xi_n$'s are i.i.d Bernoulli random variables, i.e. $P(\xi_n = 1) = P(\xi_n = -1) = \frac{1}{2}$. This scheme is weakly convergent.

*Exercise: Show the above claim by assuming $b$ and $\sigma$ are sufficiently nice. What is the weak order?*

### 1.2 A second order scheme for stationary distribution

In the work "Rational construction of stochastic numerical methods for molecular sampling", Leimkuhler and Matthews obtained a modification of the Euler-Maruyama scheme

$$X^{n+1} = X^n - k\nabla V(x) + \sqrt{2\beta}\frac{1}{2}(\Delta W_n + \Delta W_{n+1})$$

Note that $\{X^n\}$ is not a Markov chain. This scheme solves the SDE still with first order but for sampling the stationary distribution $\pi$ using averages, it is second order. Hence, it is good for MCMC sampling.

### 1.3 Other high order weaks schemes

- In the paper "Weak approximation of stochastic differential equations and application to derivative pricing", Ninomiya and Victoir proposed a splitting type weak second order scheme. The idea behind this is the Baker-Campbell-Hausdorff formula.

- In the paper "A Weak Trapezoidal method for a class of stochastic differential equations", Anderson and Mattingly proposed a Trapezoidal method which is weak second order.

For those interested, you can read.

## 2 An approach to obtain high order weak schemes

Besides, the Itô-Taylor expansion and Runge-Kutta type methods, there are many other methods to obtain high order weak schemes. One way could be the Romberg extrpolation porposed by Talay and Tubaro ("expansion of the global error for numerical schemes solving stochastic differential equations"), which uses the solutions with different time steps. Here, I want to introduce the method proposed by Abdulle et al. in the paper "High weak order methods for stochastic differential equations based on modified equations".

### 2.1 Idea of the method

Suppose that for SDE

$$dX = bdt + \sigma dW$$

we have a numerical method

$$X^{n+1} = \Psi(b, \sigma, X^n, k, \xi_n)$$

that has order $p$.

The idea is to consider a modified equation

$$d\tilde{X} = b_k(\tilde{X})dt + \sigma_k(X)dW$$

where

$$b_k = b + kb_1 + k^2 b_2 + \ldots, \quad \sigma_k = \sigma + k\sigma_1 + k^2\sigma_2 + \ldots$$

We then apply the method to this modified equation and get

$$\tilde{X}^{n+1} = \Psi(b_k, \sigma_k, \tilde{X}^n, k, \xi_n).$$

The hope is that this new sequence better approximates the **original SDE**.

## 2.2 The condition for improving accuracy

As we have seen before, we only need to look at the one step operator $S$:

$$S^{b,\sigma}(\phi) = \mathbb{E}\phi(X^1 | X^0 = x).$$

Suppose that the original scheme has local truncation error of order $p+1$ so that the global order is $p$:

$$S^{b,\sigma}(\phi) - e^{k\mathcal{L}}\phi = O(k^{p+1}).$$

Our goal is to find $b_h$ and $\sigma_h$ such that

$$S^{b_k,\sigma_k}(\phi) - e^{k\mathcal{L}}\phi = O(k^{p+r+1}).$$

**Assumption 1.** *Assume that the method has the expansion about $k$ as*

$$S^{b,\sigma}(\phi) = \phi(x) + kA_0(b,\sigma)\phi(x) + k^2 A_1(b,\sigma)\phi(x) + \dots$$

*where $A_i$'s are differential operators. Moreover, we have*

$$A_i(f + \epsilon f_1, g + \epsilon g_1) = A_i(f,g) + \epsilon \hat{A}_i(f, f_1, g, g_1) + O(\epsilon^2).$$

Since the scheme is consistent, we must have

$$A_0(f,g) = f \cdot \nabla + \frac{1}{2}(gg^T) : \nabla^2.$$

For conveneince, denote

$$b_{k,s} = b(x) + kb_1(x) + \dots + k^s b_s(x), \quad \sigma_{k,s} = \sigma(x) + k\sigma_1(x) + \dots + k^s \sigma_s(x)$$

**Theorem 1.** *Suppose for some $r \geq 1$ that we have found $b_{k,p+r-2}$ and $\sigma_{k,p+r-2}$ such that*

$$\tilde{X}^{n+1} = \Psi(b_{k,p+r-2}, \sigma_{k,p+r-2}, \tilde{X}^n, k, \xi_n).$$

*has weak $(p+r-1)$th order accuracy. If the operator*

$$\mathcal{L}_{p+r-1}\phi = \lim_{k\to 0} \frac{e^{k\mathcal{L}}\phi - S^{b_{k,p+r-1},\sigma_{k,p+r-1}}\phi}{k^{p+r}}$$

*can be written as*

$$\mathcal{L}_{p+r-1} = b_{p+r-1}\cdot\nabla + \frac{1}{2}\sum_{\ell=0}^{p+r-1}(\sigma_\ell\sigma_{p+r-1-\ell}^T) : \nabla^2$$

*for some smooth $b_{p+r-1}$ and $\sigma_{p+r-1}$, then the scheme*

$$\tilde{X}^{n+1} = \Psi(b_{k,p+r-1}, \sigma_{k,p+r-1}, \tilde{X}^n, k, \xi_n)$$

*has $(p+r)$th order weak accuracy.*

**Remark 1.** *If the scheme with $b$ and $\sigma$ already has pth ($p \geq 2$) order accuracy, we want to improve the order by $r = 1$, then we can set set $f_1, \dots, f_{p+r-2}$ to zero. This means the corrected terms appear only at $h^{p+r-1} = h^p$ in $b$ and $\sigma$.*

*Proof of Theorem 1.* Here, I only provide a sketch. You can refer to the paper for the details.

With the assumption about the original scheme, we have

$$\begin{aligned} S^{b_{k,p+r-2},\sigma_{k,p+r-2}}\phi(x) &= \phi(x) + kA_0(b_{k,p+r-2}, \sigma_{k,p+r-2})\phi(x) \\ &\quad + \dots + k^{p+r}A_{p+r-1}(b_{k,p+r-2}, \sigma_{k,p+r-2})\phi(x) + O(k^{p+r+1}) \end{aligned}$$

Inserting the expansions and combining the high order terms into $O(k^{p+r+1})$, this should be

$$\begin{aligned} \phi(x) + k\mathcal{L}(b,\sigma)\phi(x) + \dots + \frac{k^{p+r-1}}{(p+r-1)!}\mathcal{L}^{p+r-1}(b,\sigma)\phi(x) \\ + k^{p+r}B_{p+r}(f,g)\phi(x) + O(k^{p+r+1}) \end{aligned}$$

By the assumption, we know that

$$\mathcal{L}_{p+r-1} = \frac{(\mathcal{L}(b,\sigma))^{p+r}}{(p+r)!} - B_{p+r}(b,\sigma).$$

Now, $b_{k,s}$ and $\sigma_{k,s}$ are modified with $k^{p+r-1}b_{p+r-1}$ and $k^{p+r-1}\sigma_{p+r-1}$, we find that only $A_0$ will contribute $h^{p+r}$ terms to the new scheme. Other $A_i$ terms contribute to higher orders (including $B_{p+r+1}$ and so on).

Clearly, we have

$$A_0(b_{k,p+r-1}, \sigma_{k,p+r-1}) = A_0(b_{k,p+r-2}, \sigma_{k,p+r-2}) + k^{p+r-1}(b_{p+r-1} \cdot \nabla + \frac{1}{2} \sum_{\ell=0}^{p+r-1} (\sigma_\ell \sigma_{p+r-1-\ell}^T) : \nabla^2) + O(k^2).$$

The new terms will cancel exactly $\frac{(\mathcal{L}(b,\sigma))^{p+r}}{(p+r)!} - B_{p+r}(b,\sigma)$ by the assumption, making

$$e^{k\mathcal{L}}\phi(x) - S^{b_{k,p+r-1},\sigma_{k,p+r-1}}\phi(x) = O(k^{p+r+1}).$$

This implies the scheme has $(p+r)$th weak order. □

### 2.3 An example

Consider the $\theta$-Milstein scheme

$$X^{n+1} = X^n + (1-\theta)kb(X^n) + \theta kb(X^{n+1}) + \sigma(X^n)\Delta W_n + \frac{1}{2}\sigma'(X^n)\sigma(X^n)(\Delta W_n^2 - k).$$

Using the fact

$$\mathbb{E}(\Delta W_n)^2 = k, \quad \mathbb{E}(\Delta W_n)^4 = 3k^2,$$

we find

$$S^{b,\sigma}(\phi)(x) = \phi(x) + kA_0(b,\sigma)\phi(x) + k^2 A_1(b,\sigma)\phi(x) + O(h^3),$$

where

$$A_0(b,\sigma) = \mathcal{L}$$

and

$$A_1(b,\sigma)\phi(x) = \theta[b'(x)b(x) + \frac{1}{2}b''(x)\sigma^2(x)]\phi'(x) + \frac{1}{2}[b^2(x) + 2\theta b'(x)\sigma^2(x) + \frac{1}{2}(\sigma'(x)\sigma(x))^2]\phi''(x) + \frac{1}{2}[\sigma'(x)\sigma^3(x) + \sigma^2(x)b(x)]\phi'''(x) + \frac{1}{8}\sigma^4(x)\phi^{(4)}(x).$$

**Remark 2.** *To get this asymptotic expression, we first set*

$$X^1 = x + k(1-\theta)b(x) + \theta kb(X^1) + \sigma(x)\sqrt{k}z + \frac{k}{2}\sigma'(x)\sigma(x)(z^2 - 1).$$

$$S\phi(x) = \mathbb{E}\phi\left(x + k(1-\theta)b(x) + \theta k b(X^1) + \sigma(x)\sqrt{k}z + \frac{k}{2}\sigma'(x)\sigma(x)(z^2-1)\right)$$

*We then do expansion, about $x$. As you imagine, in the $O(h)$ term and $O(h^2)$ terms, we have $f(X^1)$ again. Then, we do expansion again about $x$. Do this repeatedly, we eventual will get $O(h)$ and $O(h^2)$ terms without $X^1$.*

According to the theorem, we perturb

$$b_{k,1} = b + kb_1, \quad \sigma_{k,1} = \sigma + k\sigma_1.$$

Clearly,

$$A_1(b_{k,1}, \sigma_{k,1})\phi(x) = A_1(b, \sigma)\phi(x) + O(k).$$

Hence, the difference for the original $O(k^2)$ terms are unchanged

$$\mathcal{L}_1\phi(x) = \frac{1}{2}\mathcal{L}^2\phi - A_1(b,\sigma)\phi(x)$$

The good thing is that we have new $O(k^2)$ terms from $A_0 = \mathcal{L}$:

$$A_0(b_{k,1}, \sigma_{k,1}) = (b + kb_1)\cdot\nabla + \frac{1}{2}(\sigma + k\sigma_1)(\sigma + k\sigma_1)^T : \nabla^2.$$

The new $O(k^2)$ terms are

$$b_1\partial_x + \sigma\sigma_1\partial_{xx}$$

We require

$$b_1\partial_x + \sigma\sigma_1\partial_{xx} = \frac{1}{2}\mathcal{L}^2\phi - A_1(b,\sigma)\phi(x).$$

This gives

$$b_1 = (\frac{1}{2} - \theta)b'b + \frac{1}{2}(\frac{1}{2} - \theta)b''(x)\sigma^2(x), \quad \sigma_1 = (\frac{1}{2} - \theta)b'\sigma + \frac{1}{2}\sigma' b + \frac{1}{4}\sigma^2\sigma''.$$

The $\theta = 1$ case is suitable for stiff problems.

**Remark 3.** *For Euler-Maruyama scheme, there is no such modified SDE to improve to second order.*

# 3 Stochastic stability

The theory here is an analogy of the stability region for ODE schemes. The model problem for which we apply the scheme is the geometric Brownian motion

$$dX = \lambda X dt + \mu X dW.$$

(Similar to $dX = \lambda X dt$ for ODEs.)

There are several notions of stability. Here, we consider two of them.

**Definition 1.** *Given $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$, we say the GBM is mean-square stable if* $\lim_{t\to\infty} \mathbb{E}|X_t|^2 = 0$. *We say it is asymptotically stable if* $\mathbb{P}(\lim_{t\to\infty} |X_t| = 0) = 1$.

The first is satisfies if $Re(\lambda) + \frac{1}{2}|\mu| < 0$ while the second is satisfied if $Re(\lambda - \frac{1}{2}|\mu|^2) < 0$.

What is the stability condition for Euler-Maruyama? We have the relation

$$X^{n+1} = (1 + \lambda k + \mu \Delta W_n) X^n \Rightarrow \mathbb{E}|X^{n+1}|^2 = (|1 + \lambda k|^2 + |\mu|^2 k)\mathbb{E}|X^n|^2.$$

Denote

$$z = \lambda k, \quad y = \mu^2 k.$$

We need

$$|1 + z|^2 + y < 1,$$

for mean-square stable.

For asymptotic stability, one needs

$$\mathbb{E} \log |1 + \lambda k + \mu \sqrt{k} N(0,1)| < 0.$$

For deterministic cases, the implicit schemes usually have better stability conditions. However, for SDE, this is harder. Often, we only make the deterministic part implicit while the random variable part is still explicit. For example, the stochstic backward Euler reads,

$$X^{n+1} = X^n + b(X^{n+1})k + \sigma(X^n)\Delta W_n.$$