HW3

1. This is about
$$X(t) = \int_0^t W(s)dW_s.$$

   - Repeat the lecture notes to show that $X_t = \frac{1}{2}(W(t)^2 - t)$. Show that this is a Martingale.
   - Use the fact $dX = WdW$ and Itô's formula to compute the variance of X.
   - Applying the formula, we have
   $$X_t = \frac{1}{2}(W_t^2 - t), \quad X_s = \frac{1}{2}(W_s^2 - s).$$
   By definition,
   $$X_t - X_s = \int_s^t W(\tau)dW_\tau.$$
   It seems we should have $X_t - X_s = \frac{1}{2}((W_t - W_s)^2 - (t-s))$ using the formula by regarding $W_\tau - W_s$ as a new Brownian motion. However, this is not true. Try to explain why.

2. Consider the OU process determined by
$$dX = -Xdt + 2dW_t, \quad X(0) = 1.$$
Compute the mean and variance of $X(t)$. Using the fact that $X(t)$ has a normal distribution, find the density of $X(t)$.

   Hint: Using Itô formula, $d\mathbb{E}f(X) = \mathbb{E}f'(X)dX + \frac{1}{2}(\mathbb{E}f''(X)d[X,X])$. *In* $\mathbb{E}f'(X)dX$*, the martingale part has mean zero, so we have* $-\mathbb{E}f'(X)Xdt$.

3. Consider the geometric Brownian motion determined by
$$dX = rXdt + \sigma XdW.$$
Find $\mathbb{E}X(t)$ and $\mathbb{E}|X|^2$. (Hint: For the latter one, use Itô's formula and establish an ODE for $u(t) = \mathbb{E}|X(t)|^2$.)

   Is the distribution of X a normal distribution?

4. Assume that $X(t)$ solves the SDE
$$dX = b(X)dt + \sigma(X)dW.$$

(a) Fill in the details in lecture notes to show that $u(x,t) = \mathbb{E}_x(X(t))$ satisfies the backward Kolmogorov equation.

(b) Assume $b = \frac{1}{2\pi(x)}\nabla \cdot (\Lambda \pi(x))$ holds. Fill in the details in lecture notes to show that $q(x,t) := p(x,t)/\pi(x)$ satisfies the backward Kolmogorov equation.

5. We now establish the discrete comparison principle.

Given $a_0 > 0$, consider a nonnegative sequence $\{a_n\}$ that satisfies

$$a_n \le a_{n-1} + kf(t_{n-1}, a_{n-1}), \quad n \ge 1.$$

Suppose a sequence $\{b_n\}$ that satisfies $b_0 \ge a_0$

$$b_n = b_{n-1} + kf(t_{n-1}, b_{n-1}), \quad n \ge 1.$$

(a) If $f(t,\cdot)$ is non-decreasing, show that $a_m \le b_m$ for all m.

(b) Assume $f(t,\cdot)$ is non-increasing. Consider the implicit schemes. The sequences satisfy

$$c_n \le c_{n-1} + kf(t_n, c_n), \quad n \ge 1.$$
$$d_n = d_{n-1} + kf(t_n, d_n), \quad n \ge 1.$$

with $c_0 \le d_0$. Show that $c_m \le d_m$ for all m (Hint: The idea is to consider the inequality that $\zeta_n := (c_n - d_n)\wedge 0$ satisfies). Does the explicit scheme in the first part satisfy this comparison principle for non-increasing f?

(c) Use the two results above, and the convergence of Euler scheme, show that if $f(t,\cdot)$ is Lipschitz continuous with uniform Lipschitz constant, there exists $k_0 > 0$ such that for all k small enough,

$$b_n \le C(T).$$

(d) If f is nonnegative and $f(t,\cdot)$ is non-decreasing, show that

$$a_m \le b_m \le u(t_m)$$

where $u(t)$ solves the ODE $\dot{u} = f(t,u)$, $u(0) = b_0$. Then, conclude the discrete Grönwall inequality we used in class.

6. Suppose $X(\cdot)$ solves the following SDE

$$dX = b(X)dt + \sigma(X)dW, \quad X(0) \in L^2(\Omega).$$

Assume that both b and σ are Lipschitz and that the derivatives are bounded. Prove rigorously that

$$\sup_{nk\le T} \|\mathbb{E}(b(X(t_{n+1})) - b(X(t_n))|\mathcal{F}_n)\| \le Ck.$$

7. Consider the geometric Brownian motion determined by

$$dX = -Xdt + XdW.$$

- Plot two sample paths numerically with some chosen time step k.
- Verify the strong order of Euler-Maruyama scheme numerically. (Note that for strong convergence, we must use the same Brownian motion for the accurate solution and your numerical scheme. Here, you can use the formula for the accuracte solution as the reference. However, some people prefer to use the numerical solution with very small time step as the 'exact solution', then for different time steps, the variable ΔW_n must be related by

$$\Delta W_{2n-1}^{(k/2)} + \Delta W_{2n}^{(k/2)} = \Delta W_n^{(k)}.)$$

8. (Bonus 4 pts) Assume that X, Y are two martingales in $\mathcal{M}_2$ that has nontrivial quadratic variations. Justify the formula

$$d(XY) = XdY + YdX + d[X,Y]$$

by assuming that

$$\sum_i X_i(Y_{t_{i+1}} - Y_{t_i}) \to \int_0^t XdY,$$

and

$$\sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \to [X,Y]_t.$$

In other words, you need to justify

$$(XY)_t = (XY)_0 + \int_0^t XdY + \int_0^t YdX + [X,Y]_t.$$