HW4

1. Suppose $\mathcal{L}$ is the generator of the 1D SDE $dX = b(X)dt + \sigma(X)dW$. We know the following semigroup expansion holds

$$e^{k\mathcal{L}}\phi(x) = \sum_{n=0}^{2} \frac{k^n}{n!}\mathcal{L}^n\phi(x) + O(k^3).$$

Compute explicitly $\sum_{n=0}^{2} \frac{k^n}{n!}\mathcal{L}^n\phi(x)$. (From here, you may get a feeling of what we need to improve from first order weak accuracy to second order weak accuracy.)

2. Recall the S^n operator for a time homogeneous Markov chain defined in class:

$$(S^n f)(x) = \mathbb{E}(f(X^n)|X^0 = x).$$

Let $S := S^1$.

- Show that $S^n = S \circ S \ldots \circ S$ (n copies of S).
- Show that S is non-expansive in $L^\infty(\mathbb{R}^d)$.
- Suppose X^n is valued in $\mathbb{Z}$:

$$P(X^{n+1} = p|X^n = q) = \frac{1}{2}(\delta_{q,p+1} + \delta_{q,p-1}).$$

  Find S operator and its conjugate operator S^* (i.e. $\sum_{z\in\mathbb{Z}}(Sf)(z)g(z) = \sum_{w\in\mathbb{Z}} f(w)(S^*g)(w)$). Can you say anything about the significance of S^*?

3. We now consider the price of a European option with exercise price K and expiry date T. It is assumed to be a function of the current stock price S: $V_0 = \varphi(S_0)$. We now want to determine the function φ.

   (a) The stock price is assumed to a geometric Brownian motion:

$$dS = rSdt + \sigma SdW, \quad S(0) = S_0,$$

   where r is the interest rate while σ is the volatility. To determine the function φ, we image that we have a time dependent option price:

$$V = V(S,t), \quad V(S,0) = V_0 = \varphi(S).$$

At $t = T$, we clearly have $V(S,T) = \max(S-K,0)$. According to no-arbitrage assumption, we should have

$$\frac{d}{dt}\mathbb{E}V(S,t) = r\mathbb{E}V(S,t).$$

It follows that we have

$$\varphi(S_0) = e^{-rT}\mathbb{E}\max(S(T)-K,0).$$

Code up and do numerical experiments to compute $\varphi(S_0)$ for some S_0 (pick whatever values of the parameters as you want, but do not make the problem trivial).

(b) Now, use Itô's formula to derive dV. Explain how this naturally yields the Black-Scholes PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

(Of course, rigorous derivation is not in this way, but the key point is still no-artitrage assumption.) This equation is not well-posed for Cauchy problem but it is well-posed for terminal value problem. The terminal condition is given by

$$V(S,T) = \max(S-K,0).$$

Then, it has a solution

$$\begin{aligned} V(S,t) = S\mathcal{N}\Big(&\frac{\ln(S/K) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big) \\ &- Ke^{-r(T-t)}\mathcal{N}\Big(\frac{\ln(S/K) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big) \end{aligned}$$

Here,

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{s^2}{2}}\,ds.$$

Find $\varphi(S)$ using this exact solution and compare it with the numerical result you compute in the first part.

4. Consider the 1D SDE:

$$dX = b(X)dt + \sigma(X)dW,$$

where $X \in \mathbb{R}$, b, σ are smooth functions with bounded derivatives, and W is a standard Brownian motion. Does the following scheme give a weakly consistent scheme for the SDE?

$$X^{n+1} = X^n + b(X^n)k + \sigma(X^n)\sqrt{k}\xi_n,$$

where k is the time step and ξ_n's are i.i.d (independent identically distributed) random variables with $\mathbb{P}(\xi_n = 1) = \mathbb{P}(\xi_n = -1) = \frac{1}{2}$.

If yes, prove rigorously the weak convergence. (Using the S operator in class is a good way.)

5. In this problem, we are going to do numerical tests to check the weak accuracy of two schemes mentioned in class. The first scheme is the strong order 1 scheme:

$$X^* = X^n + \sigma(X^n)\sqrt{k},$$

$$X^{n+1} = X^n + b(X^n)k + \sigma(X^n)\Delta W_n + \frac{1}{2}\frac{\sigma(X^*) - \sigma(X^n)}{\sqrt{k}}(\Delta W_n^2 - k).$$

The second is the following

$$X^{n+1} = X^n + bk + \sigma\Delta W_n + \frac{1}{2}\sigma\sigma'(\Delta W_n^2 - k)$$
$$+ b'\sigma\Delta Z_n + \frac{1}{2}(bb' + \frac{1}{2}\sigma^2 b'')k^2 + (b\sigma' + \frac{1}{2}\sigma^2\sigma'')(\Delta W_n k - \Delta Z_n).$$

where

$$\Delta Z_n = \frac{1}{2}k(\Delta W_n + \Delta V_n/\sqrt{3}),$$

with $\Delta V_n \sim N(0, k)$ independent of ΔW_n.

Let us consider the following SDE

$$dX = -X dt + \sqrt{X^2 + 4}dW.$$

Choose the test function $\phi(x) = x^2$ (though not C_b^∞) and plot the weak error versus the time step k by choosing suitably many sample points. What are the orders of weak accuray you observe for these two schemes?

6. Consider the θ-Milstein scheme with $\theta = 1$:

$$X^{n+1} = X^n + kb(X^{n+1}) + \sigma(X^n)\Delta W_n + \frac{1}{2}\sigma'(X^n)\sigma(X^n)(\Delta W_n^2 - k).$$

- Show that this scheme is of weak first order.
- Discuss the mean-ssquare stability of this scheme.