Advanced computational methods-Lecture 3

In dealing with the solution to SDEs, we also need the integrator to be semimartingales. In other words,

$$X = X_0 + M_t + V_t,$$

with $M_0 = V_0 = 0$, M_t is a (local) martingale, and V is an FV process (with finite variation). The stochastic integration with respect to X can be defined easily by this decomposition.

Proposition 1. *Let X be a cadlag process (right continuous with left limits), and Y is a cadlag semimartinglae. Suppose $0 \le \tau_0^n \le \tau_1^n \le \cdots$ are stopping times such that $\tau_i^n \to \infty$ a.s. as $i \to \infty$, and also $\delta_n = \sup_{0\le i<\infty}(\tau_{i+1}^n - \tau_i^n) \to 0$ a.s. as $n \to \infty$. Then,*

$$S_n(t) = \sum_{i=0}^{\infty} X(\tau_i^n)(Y(\tau_{i+1}^n \wedge t) - Y(\tau_i^n \wedge t))$$

converges to

$$X_- \cdot Y = \int_0^t X(s-)dY,$$

in probability uniformly for t in compact set. In other words,

$$\lim_{n\to\infty} \mathbb{P}(\sup_{0\le t\le T} |S_n(t) - (X_- \cdot Y)_t| \ge \epsilon) = 0.$$

Similarly, we need the following:

Proposition 2. *Let G be a cadlag adapted process, Y and Z be two cadlag semimartingales. Let $\pi = \{0 = t_1 < t_2 < \cdots < t_i \to \infty\}$. Define*

$$R_\pi(t) = \sum_{i=1}^{\infty} G_{t_i}(Y_{t_{i+1}\wedge t} - Y_{t_i\wedge t})(Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}).$$

Then, $R_\pi(t)$ converges uniformly to

$$\int_0^t G_- d[Y, Z]$$

in probability uniformly on compact time interval.

1 Itô formula

As we have seen, the Brownian motion has rough paths and $d[B,B] = dt$. Intuitively, this means $(dB)^2 = dt$. Hence, if we expand $f(B)$, the quadratic variation term will be nontrivial. Then, we have

$$f(B) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, d[B]_s.$$

Note that for nonstandard Brownian motions, B_0 is not necessarily zero.

This is the special case of the Itô's formula. The most general Itô's formula is valid for stochastic integrals with respect to semi-martingales.

1.1 The Itô's formula

The solutions to SDEs are often semimartingales that can be decomposed to be a martingale plus an FV process.

Then, we can have the following Itô's formula regarding a continuous semimartingale:

Theorem 1. *Let D be an open subset of $\mathbb{R}^d$ and $f \in C^2([0,T]\times D)$. Let X be a $\mathbb{R}^d$-valued* **continuous** *semimartingale such that the probability that $\overline{X[0,T]}$ falls out of D is zero. Then,*

$$f(t,X(t)) = f(0,X(0)) + \int_0^t \partial_t f(s,X(s))\, ds + \sum_{j=1}^d \int_{(0,t]} \partial_{x_j} f(s,X(s)) dX_j(s) + \frac{1}{2}\sum_{1\le j,k\le d} \int_{(0,t]} \partial_{x_j,x_k} f(s,X(s)) d[X_j,X_k].$$

Remark 1. *Note that if X is cadlag that can have jumps, then one should change $X(s)$ to $X(s-)$ in the stochastic integrals and add some terms involving jumps.*

Below, we only see briefly how the formula is proved by Taylor expansion in 1D case.

Proof of Theorem 1 with $d=1$. The proof is in fact straightforward. By Taylor's expansion:

$$f(t,y) = f(s,x) + f_t(s,x)(t-s) + f_x(s,x)(y-x) + \frac{1}{2} f_{xx}(s,x)(y-x)^2 + \phi(s,t,x,y).$$

It can be shown by Lagrangian remainder theorem for the first order Taylor polynomial that

$$\frac{\phi(s,t,x,y)}{|s-t|+|x-y|^2} \to 0,\ (s,t,x,y) \to (u,u,z,z).$$

For any partition, one then has

$$\begin{aligned} f(t,X_t) = f(0,X_0) &+ \sum_i f_t(t\wedge t_i, X_{t\wedge t_i})(t\wedge t_{i+1} - t\wedge t_i) \\ &+ \sum_i f_x(t\wedge t_i, X_{t\wedge t_i})(X_{t\wedge t_{i+1}} - X_{t\wedge t_i}) \\ &+ \frac{1}{2}\sum_i f_{xx}(t\wedge t_i, X_{t\wedge t_i})(X_{t\wedge t_{i+1}} - X_{t\wedge t_i})^2 \\ &+ \sum_i \phi(t\wedge t_i, t\wedge t_{i+1}, X_{t\wedge t_i}, X_{t\wedge t_{i+1}}). \end{aligned}$$

Using the propositions above, it is clearly that in probability the first three sums converge uniformly on compact sets to (recall that X is left continuous)

$$\int_0^t f_t(s,Y(s))\,ds + \int_0^t f_x(s,X(s))\,dX + \frac{1}{2}\int_0^t f_{xx}(s,X(s))\,d[X].$$

For the last sum, one can focus on a fixed $\omega \in \Omega$. When the partition is small enough, $|\frac{\phi}{|t\wedge t_{i+1} - t\wedge t_i| + |X_{t\wedge t_{i+1}} - X_{t\wedge t_i}|^2}|$ will be small because the process is continuous. In fact, detailed analysis shows that this term in fact goes to zero. □

In differential form, the Itô's formula can be written formally as

$$df(t,X) = f_t(t,X)dt + f_x(t,X)dX + \frac{1}{2}f_{xx}d[X].$$

For Brownian motions in $\mathbb{R}^d$, one has the following

Theorem 2. *Let $B = (B_1, B_2, \ldots, B_d)$ be a Brownian motion in $\mathbb{R}^d$ with random initial data $B(0)$. Let $f \in C^2(\mathbb{R}^d)$. Then, we have*

$$f(B(t)) = f(B(0)) + \int_0^t \sum_i \partial_i f(B_s)dB_s + \frac{1}{2}\int_0^t \Delta f(B_s)\,ds.$$

To memorize this, you can understand it as

$$dB_i dB_j = \delta_{ij}dt,\quad dB_i dt = 0,\quad (dB_i)^p = 0, p \geq 3.$$

1.2 Some applications

Example Consider $X_t = \mu t + \sigma B$ with $\sigma > 0, \mu \neq 0$. What is the probability that X_t exits (a, b) at b for $a < 0 < b$?

The idea is to find a function h such that

$$Y_t = h(X_t)$$

is a martingale. It is best that $h(a) = 0$. Then, by optional stopping, $h(X_{\tau\wedge t})$ is also a martingale. Hence,

$$\mathbb{E}h(X_{\tau\wedge t}) = h(0).$$

Then, show that $\mathbb{E}h(X_{\tau\wedge t}) \to \mathbb{E}h(X_\tau) = h(b)\mathbb{P}(X_\tau = b)$. To find such h, one uses Itô's formula and find

$$\mu h' + \frac{1}{2}\sigma^2 h'' = 0.$$

Then, h can be solved uniquely. The answer is

$$(e^{-2\mu a/\sigma^2} - 1)/(e^{-2\mu a/\sigma^2} - e^{-2\mu b/\sigma^2}).$$

Example (Lévy's characterization of Brownian motions) Let X be a d-dimensional continuous (local) martingale. Then, $X(t) - X(0)$ is a standard Brownian motion if and only if $[X_i, X_j]_t = \delta_{ij} t$.

Here, we only look at "$\Leftarrow$" direction only as the other direction is trivial. Pick the test function

$$f(t, x) = \exp\left(i\theta \cdot x + \frac{1}{2}|\theta|^2 t\right),\ \theta \in \mathbb{R}^d.$$

Let $Z = f(t, X)$. Applying Itô's formula,

$$dZ = df(t, X) = \frac{1}{2}|\theta|^2 Z dt + i\theta Z \cdot dX - \frac{1}{2}\sum_j \theta_j^2 Z d[X_i, X_i] = i\theta Z \cdot dX.$$

Hence, Z is the stochastic integration with respect to a (local) martingale in Itô's sense. Hence, Z is a (local) martingale. Moreover, Z is bounded, so Z is a martingale. Consequently,

$$\mathbb{E}(Z_t|\mathcal{F}_s) = Z_s \Rightarrow \mathbb{E}(\frac{Z_t}{Z_s}|\mathcal{F}_s) = 1.$$

This implies

$$\mathbb{E}(\exp(i\theta\cdot(X_t-X_s))|\mathcal{F}_s)=\exp(-\frac{1}{2}|\theta|^2(t-s)).$$

This implies that $X_t - X_s$ is a Gaussian variable and should be independent to $\mathcal{F}_s$ by the property of normal distributions.

Example (A special case of Burkholder-Davis-Gundy inequality) Let $p \geq 2$. For all continuous (local) martingales M with $M_0 = 0$, one has

$$\mathbb{E}[|\sup_{0\le s\le t} M_s|^p] \le (p(p-1)e)^{p/2}\mathbb{E}[|[M]_t|^{p/2}].$$

This roughly says the moments of the quadratic variation can be used to control the moments of martingales. In fact, it sufficies to show this for bounded L^2 martingales. For local martingales, introducing stopping times for the localization and taking limit will suffice.

Applying Itô's formula to $|x|^p$ which isC^2. Then, one has

$$|M_t|^p=\int_0^t p|x|^{p-2}X\,dM+\frac{1}{2}p(p-1)\int_0^t |M|^{p-2}d[M].$$

Taking expectation:

$$\mathbb{E}|M_t|^p=\mathbb{E}\frac{1}{2}p(p-1)\int_0^t |M|^{p-2}d[M]\le\frac{1}{2}p(p-1)\mathbb{E}(|\sup_{0\le s\le t}M_s|^{p-2}|[M]_t|).$$

Applying Hölder, one has

$$\mathbb{E}|M_t|^p\le\frac{1}{2}p(p-1)(\mathbb{E}|\sup_{0\le s\le t}M_s|^p)^{1-2/p}(\mathbb{E}|[M]_t|^{p/2})^{2/p}.$$

Lastly, applying Doob's inequality, the claim follows.

2 Stochastic differential equations

The general stochastic differential equations are given by

$$dX=dH+F(t,X)dY$$

where Y is a general cadlag semimargingales. In this course, we only focus on the Itô equations

$$dX=b(t,X)dt+\sigma(t,X)dB_t,\quad X_0=\xi.$$

This equation is defined by the following integral equation

$$X(t) = \xi + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s) dB_s.$$

The integral is in the Itô sense.

b is $\mathbb{R}^d$ valued and is called the drift vector. σ is called the dispersion matrix which has size $d \times m$. The matrix $\sigma\sigma^T$ is called the diffusion matrix. The solution of Itô equations will be called the diffusion processes.

Before we go to the rigorous theory, let us look at two examples.

Example 1: the Ornstein-Uhlenbeck process

$$dX = -\alpha X dt + \sigma dW_t.$$

Assume the initial data X_0 is independent of the Brownian motion.

Mimicking the technique for ODE, we want to try integrating factor. However, the processes we have all nontrivial quadratic variation.

$$d(ZX) = ZdX + Xdz + d[Z, X]$$

Let us try

$$Z = \exp(\alpha t)$$

Then, $[Z, X] = 0$ because $dBdt = 0$. Then,

$$d(ZX) = -\alpha ZXdt + \sigma ZdB + \alpha ZXdt = \sigma ZdB.$$

Hence, we in fact have the usual formula as in ODE.

The OU process is then solved to be

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s.$$

Of course, this is formal guess, you may need to verify that it satisfies the integral equation by Itô formula, which is left for exercise.

Exercise: Compute the mean and the variance of the 1D OU process. Since the dispersion matrix does not depend on the process X, X is a Gaussian process, write out the density of X_t.

Example 2. Geometric Brownian motion

$$dX = \mu X dt + \sigma X dB, \quad X(0) = x_0.$$

For the integrating factor, one may guess to use

$$Z = \exp(-\mu t - \sigma B_t)$$

and get

$$X = X_0 \exp(\mu t + \sigma B_t)$$

This turns out to be wrong. In fact,

$$d(XZ) = XdZ + ZdX + d[Z, X] = \frac{1}{2} XZ\sigma^2 dt + d[Z, X]$$

The quadratic variation part is nonzero:

$$d[X, Y] = -\sigma^2 XZdt.$$

Hence,

$$d(XY) = -\frac{1}{2} XZ\sigma^2 dt.$$

What is the correct integrating factor? Motivated by the above computation, we try into the factor

$$Z = \exp(-\mu t - \sigma B_t + r\sigma^2 t)$$

Then,

$$d(XZ) = XdZ + ZdX + d[X, Z] = (\frac{1}{2} + r)\sigma^2 XZdt - \sigma^2 XZdt.$$

Clearly, we need $r = \frac{1}{2}$.

Hence, the geometric Brownian motion should be solved as

$$X_t = X_0 \exp\left((\mu - \frac{1}{2})\sigma^2 t + \sigma B_t\right).$$

To verify this is a solution, we need to check all the assumptions in the derivation above are valid. Alternatively, one can check directly by inserting this into the integral equation.

Exercise: Use Itô's formula to find an ODE for $u(t) = \mathbb{E}X^2$ for the geometric Brownian motion. Then, find $u(t)$.

3 Existence and uniqueness of strong solutions for Itô equations

Recall that we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$.

Definition 1. *We say a process X on $(\Omega, \mathcal{F}, \mathbb{P})$ is a strong solution to the Itô equation with initial data $\xi \in \mathcal{F}_0$ if*

$$\mathbb{P}\{\forall T > 0, \int_0^T |b(x, X_s)| ds + \int_0^T |\sigma(s, X_s)|^2 \, ds < \infty\} = 1,$$

and the integral equation

$$X_t = \xi + \int_0^t b(s, X_s) ds + \int_0^T \sigma(s, X_s) dB_s$$

holds (in the sense that both sides are distinguishable processes).

There is also definition of weak solutions for Itô equations, and I will skip this here.

There is a classical result regarding the wellposedness of the equation

Theorem 3. *Assume $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ satisfy the Lipschitz condition*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \le L|x - y|$$

and

$$|b(t, x)| + |\sigma(t, x)| \le L(1 + |x|)$$

then there exists a unique continuous process X on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $\{\mathcal{F}_t\}$ that is a strong solution.

Here, we present the proof of a simpler version:

Theorem 4. *Besides the conditions above, if moreover, $\mathbb{E}\xi^2 < \infty$, then there exists a unique continuous process X on $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to the filtration $\{\mathcal{F}_t\}$ that is a strong solution, and*

$$\mathbb{E}(\sup_{t \in [0,T]} |X_t|^2) \le C(1 + \mathbb{E}\xi^2).$$

Moreover, let $\tilde{X}$ be a solution corresponding to initial data $\tilde{\xi} \in L^2(\mathbb{P})$, then

$$\mathbb{E}(\sup_{0 \le s \le t} |X_s - \tilde{X}_s|^2) \le C(t)\mathbb{E}|\xi - \tilde{\xi}|^2.$$

Proof. Consider the Picard iteration $X_0 = \xi$ and

$$X_{n+1}(t) = \xi + \int_0^t b(s, X_n(s))ds + \int_0^t \sigma(s, X_n(s))dB_s.$$

Clearly,

$$\mathbb{E}(\int_0^t |b(s, X_n)|^2 ds + \int_0^t |\sigma(s, X_n(s))|^2 ds) \leq 4L^2 t(1 + \sup_{s\in[0,t]} \mathbb{E}|X_n(s)|^2)$$

Since $\xi \in L^2(\mathbb{P})$, by induction, we have

$$\mathbb{E} \sup_{s\in[0,t]} |X_n(s)|^2 < \infty.$$

This means X_{n+1} is well-defined.

Moreover,

$$\mathbb{E} \sup_{0\leq s\leq t} |X_{n+1}|^2 \leq C\mathbb{E}\xi^2 + C(t)\int_0^t \mathbb{E}|X_n(s)|^2\, ds \leq A + C(t)\int_0^t \mathbb{E} \sup_{0\leq \tau\leq s} |X_n(\tau)|^2\, ds.$$

From here, we find that $y_n(t) := \sup_{0\leq s\leq t} |X_n(s)|^2$ satisfies that

$$y_{n+1}(t) \leq A + C(T)\int_0^t y_n(s)\, ds,$$

for all $t \in [0, T]$.

This gives a uniform bound:

$$y_n(t) \leq u(t)$$

where $u'(t) = C(T)u$ with $u(0) = A$.

Now, we consider

$$z_n(t) := \mathbb{E} \sup_{0\leq s\leq t} |X_{n+1}(t) - X_n(t)|^2$$

Then, we have

$$z_n(t) \leq 2\mathbb{E} \sup_{0\leq s\leq t} (\int_0^s |b(\tau, X_n) - b(\tau, X_{n-1})|\, d\tau)^2 + 2\mathbb{E} \sup_{0\leq s\leq t} (\int_0^s (\sigma(\tau, X_n) - \sigma(\tau, X_{n-1}))dB_\tau)^2$$

The first term is controlled trivially by $t\int_0^t \mathbb{E}(b(\tau, X_n) - b(\tau, X_{n-1}))^2 ds \leq C(T)\int_0^t z_{n-1}\, ds$.

For the second term, we apply Doob's inequality for martingales

$$\mathbb{E}(\sup_{0\le s\le T} M_s^p) \le (\frac{p}{p-1})^p \mathbb{E} M_t^p,$$

to the stochastic integral and have

$$\begin{aligned}
&\mathbb{E}\sup_{0\le s\le t}(\int_0^s (\sigma(\tau, X_n)-\sigma(\tau, X_{n-1}))dB_\tau)^2 \\
&\le 4\mathbb{E}\int_0^t |\sigma(s,X_n)-\sigma(s,X_{n-1})|^2\,ds \\
&\le C\int_0^t z_{n-1}(s)\,ds.
\end{aligned}$$

Direct computation shows $z_0 \le B(T)$. Then,

$$z_n \le B(T)\frac{C^n t^n}{n!}$$

This implies that $\sum_n \|\sup_{0\le s\le t}|X_{n+1}(t)-X_n(t)|\|_2$ converges.

Moreover, by Chebyshev inequality

$$P(\sup_{0\le s\le t}|X_{n+1}(t)-X_n(t)|>2^{-n}) \le \frac{B4^nC^nt^n}{n!}$$

This is summable. The Borel-Cantelli lemma implies that $X_n(t)$ converges in $C([0,T])$ almost surely to some continuous process $X(t)$.

Then, by Fatou's lemma

$$\mathbb{E}\|\sup_{0\le t\le T}|X(s)-X_n(s)|\|_2 \le \liminf_{m\to\infty}\sum_{k=n}^{m-1}\|\sup_{0\le s\le t}|X_{k+1}(t)-X_k(t)|\|_2$$

This is arbitrarily small if n is large enough. Hence, $X_n \to X$ in $L^2(0,T;L^2)$. Moreover, X has the same second moment bounds.

Then, taking $n\to\infty$ in the Picard iteration, we find that X is a solution.

Lastly, the estimate for two solutions is very similar to the estimate of $\mathbb{E}\sup_{0\le s\le t}|X_{n+1}-X_n|^2$ above. We skip the details. The uniqueness then follows from this estimate. □

Remark 2. *The conditions imposed* $b(\cdot)$ *is too strong for many applications. In fact, it is also known that locally Lipschitz and confinement conditions can imply the existence and uniqueness of solutions (For example, in Theorem 2.3.5 of the book 'Stochastic Differential Equations and applications' (Horwood, 97) by X. Mao, it is shown that* $\max(x\cdot b(x), |\sigma|^2)\le C_1 + C_2|x|^2$ *is enough for the well-posedness, which allows* b *like* $-(1+|x|^2)^p x$*).*