Advanced computational methods-Lecture 5

1 Motivation for strong schemes

we study strong schemes for SDEs (Itô equations).

When σ does not depend on X (may depend on time) we say the noise is **additive**. This is because the noise is uniformly added to the system in the space.

However, if σ depends on X, we say the noise is **multiplicative**. This is because the noise is amplified by a factor that depends on the state of the system.

For numerical analysis, we assume b and σ are good functions. There are roughly two kinds of things we want to do:

1. Approximate the sample paths of the SDE; namely $t \mapsto X(\omega, \cdot)$.
2. Approximate the law of $X(t)$, or the distributions.

Depending on which we would like to do, there are two types of convergence we can define. The convergence corresponding to the sample paths is called **strong convergence**.

Definition 1. *Fix time $T > 0$. We say a discrete approximation $\{X^n(k)\}$ of the SDE converges in the strong sense with order $p > 0$ if there exists $C(T) > 0$ such that*

$$\sup_{n:nk\le T} \mathbb{E}|X^n(k) - X(nk)| \le C(T)k^p,$$

for all sufficiently small step size k.

We will assume the following:

Assumption 1. *The functions b, σ are Lipschitz continuous.*

2 The basic scheme: Euler-Maruyama scheme

To be convenient, we use $\|\cdot\|$ to mean the $L^2(\mathbb{P})$ norm.

The first scheme we consider is an analogy of the forward Euler scheme, namely an explicit Euler type scheme. This is called Euler-Maruyama scheme given by

$$X^{n+1} = X^n + b(X^n)k + \sigma(X^n)\Delta W_n$$

For the reference, we recall that the solution of the SDE satisfies

$$X(t_{n+1}) = X(t_n) + \int_{t_n}^{t_{n+1}} b(X_\tau)\, d\tau + \int_{t_n}^{t_{n+1}} \sigma(X_\tau) dW_\tau.$$

2.1 Some basic estimates

Here, we start with some basic estimates.

Lemma 1. *The solution $X(t)$ satisfies*

$$(\mathbb{E}|X(t) - X(s)|^2)^{1/2} \le C(T)|t-s|^{1/2}.$$

Proof. By the equation, we have

$$X(t) - X(s) = \int_s^t b(X_\tau)\, d\tau + \int_s^t \sigma(X_\tau) dW_\tau.$$

Then,

$$\mathbb{E}|X(t) - X(s)|^2 \le C(T) \int_s^t \mathbb{E} b(X_\tau)^2\, d\tau + 2\int_s^t \mathbb{E} tr(\sigma(X_\tau)\sigma^T(X_\tau))\, ds \le C_1(T)|t-s|$$

□

2.2 Multiplicative noise

We are easily attempted to do the following

$$\|X^{n+1} - X(t_{n+1})\| \le |X^n - X(t_n)| + \int_{t_n}^{t_{n+1}} \|b(X^n) - b(X_\tau)\|\, d\tau + \|\int_{t_n}^{t_{n+1}} (\sigma(X^n) - \sigma(X_\tau)) dW_\tau\|$$

The first term is easily estimated as

$$\int_{t_n}^{t_{n+1}} \|b(X^n) - b(X_{t_n})\|\, d\tau + \int_{t_n}^{t_{n+1}} \|b(X(t_n)) - b(X_\tau)\|\, d\tau \le Lk\|X^n - X(t_n)\| + Ck^{3/2}$$

For the second term, we get

$$\|\int_{t_n}^{t_{n+1}} (\sigma(X^n) - \sigma(X_\tau)) dW_\tau\| = (\int_{t_n}^{t_{n+1}} \mathbb{E}(\sigma(X^n) - \sigma(X_\tau))^2\, d\tau)^{1/2}$$

We find that this term is like $\sqrt{k}\|X^n - X(t_n)\| + k$. The $O(k)$ local truncation error is killing us, because then the global error is $O(1)$!

Exercise: **if $\sigma = const$, the above estimate gives us $O(k^{1/2})$ error.**

Where is the issue? How can we resolve this? If we look in detail, we see the trouble is the martingale term. Generally speaking, the above estimate is not wrong. The magnitude of the martingale is indeed $O(k)$ on the interval $[t_n, t_{n+1}]$. The point is that the increment of the martingale has conditional expectation to be zero! We have not used this fact.

Let us denote

$$M_n = \int_{t_n}^{t_{n+1}} (\sigma(X^n) - \sigma(X_\tau)) dW_\tau.$$

We have seen $\|M_n\| = O(k)$. Basically, $\sum_m \|M_m\| = O(1)$. This is not what we want. In fact, we can do the following:

$$\|\sum_m M_m\| = \sqrt{\mathbb{E}\sum_{i,j} M_i M_j}$$

Using the property of martingales, the expectation is zero if $i \neq j$. Hence, we in fact have

$$\|\sum_m M_m\| = \sqrt{\sum_m \|M_m\|^2} = C_1(T)k^{1/2}.$$

This is the gain and we can improve the estimate. Of course, M_m contains the error itself, so we cannot do this directly, but this gives us the hint that we should use the sum of the square.

Now, we have

Theorem 1. *The E-M scheme has strong order* 1/2

Proof.

$$\|X^n - X(t_n)\|^2 \le 2\mathbb{E}(\sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} (b(X^m) - b(X_\tau))\, d\tau)^2 + 2\mathbb{E}(\sum_{m=1}^{n} M_m)^2$$

Uisng Holder first for the integral and then Cauchy inequality for the sum, the first term is controlled by

$$T\sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \mathbb{E}(b(X^m) - b(X_\tau))^2\, d\tau \le CT\sum_{m=0}^{n-1} (\|X^m - X(t_m)\|^2 + k)k$$

The second term is controlled by

$$\sum_m \mathbb{E} M_m^2 = \sum_m \int_{t_m}^{t_{m+1}} (\sigma(X^m) - \sigma(X_\tau))\, d\tau \le C \sum_m (\|X^m - X(t_m)\|^2 + k)k$$

Hence,

$$\|X^n - X(t_n)\|^2 \le C(T)k \sum_{m=0}^{n-1} (\|X^m - X(t_m)\|^2 + Ck.$$

This gives that

$$\|X^n - X(t_n)\|^2 \le Ck.$$

The claim is proved. □

2.3 Additive noise

Now, consider $\sigma(X) = \sigma$. The above estimate still gives $\|X^n - X(t_n)\| \le Ck^{1/2}$. Can we improve this result?

In fact, by the same computation, we have

$$\|X^n - X(t_n)\|^2 \le \mathbb{E}(\sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} (b(X^m) - b(X_\tau))\, d\tau)^2.$$

If we control the right hand side as before and get

$$T \sum_{m=0}^{n-1} \int_{t_m}^{t_{m+1}} \mathbb{E}(b(X^m) - b(X_\tau))^2\, d\tau$$

then we have to get the $k^{1/2}$ bound.

Can we use the property of martingale to improve this? The answer is positive.

The key observation is that though

$$\| \int_{t_m}^{t_{m+1}} (b(X(t_m)) - b(X_\tau))\, d\tau\| = O(k^{3/2}),$$

we have

$$\begin{aligned}\|\mathbb{E}(\int_{t_m}^{t_{m+1}} (b(X(t_m)) - b(X_\tau))\, d\tau | \mathcal{F}_{t_m})\| &\le \|\mathbb{E}(\int_{t_m}^{t_{m+1}} b'(X(t_m))(X(\tau) - X(t_m))\, d\tau | \mathcal{F}_{t_m})\| \\ &+ \|\mathbb{E}(\int_{t_m}^{t_{m+1}} Cb''(\xi)(X(\tau) - X(t_m))^2\, d\tau | \mathcal{F}_{t_m})\|\end{aligned}$$

The second term is clearly $O(k^2)$. For the first term, we have

$$\mathbb{E}(\int_{t_m}^{t_{m+1}} b'(X(t_m))(X(\tau)-X(t_m))\,d\tau|\mathcal{F}_{t_m}) = b'(X(t_m))\mathbb{E}(\int_{t_m}^{t_{m+1}} (X(\tau)-X(t_m))\,d\tau|\mathcal{F}_{t_m})$$

The martingale part vanishes in the conditional expectation. Hence, we have the first term to be $O(k^2)$ as well.

Theorem 2. *If $\sigma = const$, the Euler-Maruyama scheme has strong order 1.*

Proof. We first of all split the error part:

$$\begin{aligned}\mathbb{E}(\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}(b(X^m)-b(X_\tau))\,d\tau)^2 &= \mathbb{E}(\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}(b(X^m)-b(X(t_m)))\,d\tau \\ &\quad + \sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}(b(X(t_m))-b(X_\tau))\,d\tau)^2\end{aligned}$$

$$\leq 2\mathbb{E}(\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}(b(X^m)-b(X(t_m)))\,d\tau)^2 + 2\mathbb{E}(\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}(b(X(t_m))-b(X_\tau))\,d\tau)^2$$

The first term is estimated as

$$2T\sum_{m=0}^{n-1}\int_{t_m}^{t_{m+1}}\mathbb{E}((b(X^m)-b(X(t_m))^2\,d\tau \leq C(T)\sum_{m=0}^{n-1}k\|X^m - X(t_m)\|^2.$$

To treat the second term, we define

$$I_m = \int_{t_m}^{t_{m+1}}(b(X(t_m))-b(X_\tau))\,d\tau.$$

and

$$R_n = \sum_{m=0}^{n-1} I_m.$$

Then,

$$\begin{aligned}\mathbb{E}R_n^2 = \mathbb{E}R_{n-1}^2 + 2\mathbb{E}(R_{n-1}I_{n-1}) + \mathbb{E}I_{n-1}^2 &\leq \mathbb{E}R_{n-1}^2 + 2\mathbb{E}(R_{n-1}\mathbb{E}(I_{n-1}|\mathcal{F}_{t_{n-1}})) + Ck^3 \\ &\leq \mathbb{E}R_{n-1}^2 + C\|R_{n-1}\|k^2 + Ck^3\end{aligned}$$

Denote $u_n = \|R_n\|^2 = \mathbb{E}R_n^2$. We then have

$$u_n \leq C\sum_{m=1}^{n-1}\sqrt{u_m}k^2 + Ck^2 \leq C\sum_{m=1}^{n-1}u_m k + (\sum_m Ck^3) + Ck^2.$$

Hence, finally, we get

$$u_n \le C \sum_{m=1}^{n-1} u_m k + Ck^2.$$

By induction, you can prove that $u_n \le u(t_n)$ where $u' = Cu, u(0) = Ck^2$. (This is a version of discrete comparison principle; or discrete Gronwall. This will be left for exercise) This shows that $u_n \le Ck^2$.

Finally, we obtain

$$\|X^n - X(t_n)\|^2 \le C(T) \sum_{m=0}^{n-1} k\|X^m - X(t_m)\|^2 + Ck^2.$$

Using the discrete Gronwall again, we find

$$\|X^n - X(t_n)\|^2 \le C(T)k^2.$$

□

3 A fundamental theorem

In the EM scheme, we have found that the local truncation error is like $O(k)$. Then, we turned to use the properties of martingale to gain the accuracy.In particular, we use the conditional expectation to gain extra smallness when adding these local $O(k)$ errors into global error. The whole order is reduced by $1/2$ only by using the conditional expectations.

This in fact can be generalized into a theorem, which forms the foundation for proving the strong order of schemes. This can be found in section 1.1 of the book "stochastic numerics for mathematical physics".

Here, we will assume that both b and σ are globally Lipschitz. We consider generally the (strong) numerical scheme given by

$$\bar{X}_{t,x}(t+h) = x + A(t, x, h; W(\theta) - W(t), t \le \theta \le t+h).$$

In other words, the numerical value at $t+h$ is determined by the numerical value x at t, the step size h, and all the noises between t and $t+h$. Hence, it is a Markov chain.

In general, the numerical values are therefore generated by

$$X_{k+1} = \bar{X}_{t_k, X_k}(t_{k+1}).$$

We will use $X_{t,x}(t+h)$ to represent the solution to the SDEs with condition $X(t) = x$.

Theorem 3. *If there exist* $K > 0$, $p_2 \geq 1/2$ *and* $p_1 \geq p_2 + \frac{1}{2}$ *such that*

$$|\mathbb{E}(X_{t,x}(t+h) - \bar{X}_{t,x}(t+h))| \leq K\sqrt{1+|x|^2}h^{p_1},$$

and

$$\|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\| = \sqrt{\mathbb{E}|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)|^2} \leq K\sqrt{1+|x|^2}h^{p_2},$$

then one has

$$\|X_k - X(t_k)\| \leq K(1+\mathbb{E}|X_0|^2)^{1/2}h^{p_2-1/2}.$$

This theorem basically says that the global order of convergence is reduced by 1/2 from the local mean square deviation provided that the mean is capured correctly.

Exercise: find an example where $p_2 = 1$ *but the scheme diverges (Hint: you must make the first condition fail, hence it is best to construct examples such that the drift term is not captured).*

Exercise: Compute p_1 *and* p_2 *for Euler-Maruyama scheme.*