Advanced computational methods-Lecture 6

1 A fundamental theorem

Here, we will assume that both b and σ are globally Lipschitz. We consider generally the (strong) numerical scheme given by

$$\bar{X}_{t,x}(t+h) = x + A(t, x, h; W(\theta) - W(t), t \le \theta \le t+h).$$

In other words, the numerical value at $t+h$ is determined by the numerical value x at t, the step size h, and all the noises between t and $t+h$. Hence, it is a Markov chain.

In general, the numerical values are therefore generated by

$$X_{k+1} = \bar{X}_{t_k, X_k}(t_{k+1}).$$

We will use $X_{t,x}(t+h)$ to represent the solution to the SDEs with condition $X(t) = x$.

Theorem 1. *If there exist $K > 0$, $p_2 \ge 1/2$ and $p_1 \ge p_2 + \frac{1}{2}$ such that*

$$|\mathbb{E}(X_{t,x}(t+h) - \bar{X}_{t,x}(t+h))| \le K\sqrt{1+|x|^2}h^{p_1},$$

and

$$\|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)\| = \sqrt{\mathbb{E}|X_{t,x}(t+h) - \bar{X}_{t,x}(t+h)|^2} \le K\sqrt{1+|x|^2}h^{p_2},$$

then one has

$$\|X_k - X(t_k)\| \le K(1+\mathbb{E}|X_0|^2)^{1/2}h^{p_2-1/2}.$$

This theorem basically says that the global order of convergence is reduced by $1/2$ from the local mean square deviation provided that the mean is capured correctly.

Exercise: find an example where $p_2 = 1$ but the scheme diverges (Hint: you must make the first condition fail, hence it is best to construct examples such that the drift term is not captured).

Exercise: Compute p_1 and p_2 for Euler-Maruyama scheme.

Below, we aim to prove this theorem.

1.1 Auxilliary lemmas

The following lemma is some resolved version of a previous result.

Lemma 1. *For the original SDE, one writes*

$$X_{t,x}(t+h) - X_{t,y}(t+h) = x - y + Z.$$

Then,

$$\mathbb{E}|X_{t,x}(t+h) - X_{t,y}(t+h)|^2 \le |x-y|^2(1+Kh)$$

and

$$\mathbb{E}|Z|^2 \le K|x-y|^2 h.$$

In fact, one first applied Itô's formula to $|X_{t,x}(t+h) - X_{t,y}(t+h)|^2$ directly, one has

$$d|X_{t,x}(s)-X_{t,y}(s)|^2 = 2(X_{t,x}(s)-X_{t,y}(s))\cdot\Big([b(s,X_{t,x}(s))-b(s,X_{t,y}(s))]ds$$

$$+[\sigma(s,X_{t,x}(s))-\sigma(s,X_{t,y}(s))]dW\Big)+\mathrm{tr}[\sigma(s,X_{t,x}(s))-\sigma(s,X_{t,y}(s))][\sigma(s,X_{t,x}(s))-\sigma(s,X_{t,y}(s))]^T ds.$$

Integrating and taking expectation, the Grönwall's inequality yields the first result.

For the estimates of Z, you can write out the formula for Z directly. Then, apply Itô's isometry and using Grönwall.

The details are left for your homework.

As a second preparation, we remark that the above estimates and conditions can be made for conditional expectations. In particular, one has the following fact:

Lemma 2. *Suppose $\zeta \in \mathcal{G} \subset \mathcal{F}$ for some σ-algebra $\mathcal{G}$, and the random variable $f(x,\omega)$ is independent of $\mathcal{G}$. Denote*

$$\mathbb{E}f(x,\omega) = \phi(x).$$

Then, it holds that

$$\mathbb{E}(f(\zeta,\omega)|\mathcal{G}) = \phi(\zeta).$$

With this lemma, the above assertions can be made into conditional versions. For example, if $X, Y \in \mathcal{F}_t$, then

$$|\mathbb{E}[(X_{t,X}(t+h) - \bar{X}_{t,X}(t+h))|\mathcal{F}_t]| \le K(1+|X|^2)h^{p_1},$$

and

$$\mathbb{E}\left[|X_{t,X}(t+h)-\bar{X}_{t,X}(t+h)|^2\Big|\mathcal{F}_t\right]\le K(1+|X|^2)h^{2p_2},$$

Moreover, it also holds that

$$\mathbb{E}(|X_{t,X}(t+h)-X_{t,Y}(t+h)|^2|\mathcal{F}_t)\le |X-Y|^2(1+Kh)$$

and similar results hold for the Z variable.

1.2 Controlling the moments

Lemma 3. *Suppose $\mathbb{E}|X_0|^2<\infty$, then there exists a constant $C(T)>0$ such that*

$$\mathbb{E}|X_k|^2\le C(T)(1+\mathbb{E}|X_0|^2).$$

Note that there is no explicit formula for the discrete scheme A, so we have to turn to the assumptions that relates to the solution of the time continuous SDE to prove.

Using the conditional version of the inequality and taking one more expectation,

$$\mathbb{E}\left[|X_{t,X_k}(t+h)-\bar{X}_{t,X_k}(t+h)|^2\right]\le K(1+\mathbb{E}|X_k|^2)h^{2p_2},$$

If X_k has bounded second moment, according to the existence and uniquess theorem for SDEs, we have

$$\mathbb{E}|X_{t,X_k}(t+h)|^2<\infty.$$

This implies that $\mathbb{E}|\bar{X}_{t,X_k}(t+h)|^2<\infty$. Hence, $\mathbb{E}|X_k|^2<\infty\Rightarrow\mathbb{E}|X_{k+1}|^2<\infty$.

Then, we now estimate the moments in detail.

$$X_{k+1}=X_k+[X_{t_k,X_k}(t_{k+1})-X_k]+[\bar{X}_{t_k,X_k}(t_{k+1})-X_{t_k,X_k}(t_{k+1})]$$

Taking the square, we have six terms. We now estimate them each by each.

$$\mathbb{E}|X_{t_k,X_k}(t_{k+1})-X_k|^2\le K(1+\mathbb{E}|X_k|^2)h,$$

and this is due to the property of SDE itself. This will be left as homework. As we have seen

$$\mathbb{E}\left[|X_{t,X_k}(t+h)-\bar{X}_{t,X_k}(t+h)|^2\right]\le K(1+\mathbb{E}|X_k|^2)h^{2p_2}\le K(1+\mathbb{E}|X_k|^2)h.$$

We now move to the cross terms. For

$$\mathbb{E}X_k \cdot [X_{t_k,X_k}(t_{k+1}) - X_k],$$

we have to use the property of conditional expectation as we did above for Euler-Maruyama scheme.

$$\mathbb{E}X_k \cdot \mathbb{E}[X_{t_k,X_k}(t_{k+1}) - X_k|\mathcal{F}_k] \le \|X_k\| \|\mathbb{E}[X_{t_k,X_k}(t_{k+1}) - X_k|\mathcal{F}_k]\| \le K(1+\|X_k\|^2)h,$$

where we used similar estimate

$$\|\mathbb{E}[X_{t_k,X_k}(t_{k+1}) - X_k|\mathcal{F}_k]\| \le \sqrt{K(1+\|X_k\|^2)h^2}.$$

Remark 1. *We have seen this in the E-M scheme. However, here, we are not assuming b, σ to be bounded. This general case will be left as your homework.*

The other cross terms are straightforward using the assumptions. For example,

$$\begin{aligned}\mathbb{E}[X_{t_k,X_k}(t_{k+1}) - X_k] \cdot [\bar{X}_{t_k,X_k}(t_{k+1}) - X_{t_k,X_k}(t_{k+1})] &\le \sqrt{K(1+\mathbb{E}|X_k|^2)h}\sqrt{K(1+\mathbb{E}|X_k|^2)h^{2p_2}} \\ &\le K(1+\mathbb{E}|X_k|^2)h^{p_2+1/2} \le K(1+\mathbb{E}|X_k|^2)h.\end{aligned}$$

Eventually, we have

$$\mathbb{E}|X_{k+1}|^2 \le \mathbb{E}|X_k|^2 + K(1+\mathbb{E}|X_k|^2)h.$$

The discrete Grönwall inequality implies the claim.

1.3 The proof of the theorem

Finally, we move to the proof of the theorem.

Taking $X(0) = X_0$, we aim to control

$$X(t_{k+1}) - X_{k+1} = \Big(X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})\Big) + \Big(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})\Big)$$

The first error is due to error arising from the errors in initial data. To do this, we write

$$X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}) = X(t_k) - \bar{X}_k + Z$$

apply Lemma 1, of course applied using the conditional version.

We take square and esimates.

$$\begin{aligned}\mathbb{E}|X(t_{k+1})-X_{k+1}|^2 =&\mathbb{E}|X_{t_k,X(t_k)}(t_{k+1})-X_{t_k,\bar{X}_k}(t_{k+1})|^2\\&+2\mathbb{E}\Big(X_{t_k,X(t_k)}(t_{k+1})-X_{t_k,\bar{X}_k}(t_{k+1})\Big)\cdot\Big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\Big)\\&+\mathbb{E}|X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})|^2.\end{aligned}$$

By the conditional version,

$$\mathbb{E}\left(|X_{t_k,X(t_k)}(t_{k+1})-X_{t_k,\bar{X}_k}(t_{k+1})|^2\Big|\mathcal{F}_{t_k}\right)\le\mathbb{E}|X(t_k)-X_k|^2(1+Kh).$$

Hence, the first term is controlled by

$$\mathbb{E}|X_{t_k,X(t_k)}(t_{k+1})-X_{t_k,\bar{X}_k}(t_{k+1})|^2\le\mathbb{E}|X(t_k)-X_k|^2(1+Kh)$$

Consider the cross term. Using the decomposition, one has

$$\begin{aligned}&\mathbb{E}\Big(X_{t_k,X(t_k)}(t_{k+1})-X_{t_k,\bar{X}_k}(t_{k+1})\Big)\cdot\Big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\Big)\\&=\mathbb{E}\Big(X(t_k)-\bar{X}_k\Big)\cdot\Big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\Big)+\mathbb{E}Z\cdot\Big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\Big)\end{aligned}$$

For the first, using the conditional expectation first and then the conditional version of the assumption,

$$\le K\|X(t_k)-\bar{X}_k\|\sqrt{\mathbb{E}|X_k|^2}h^{p_1}\le K\sqrt{1+\mathbb{E}|X_0|^2}\|X(t_k)-\bar{X}_k\|h^{p_1}.$$

For the Z term,

$$\mathbb{E}Z\cdot\Big(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\Big)\le\|Z\|\|X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})\|$$

For Z, using the conditional version of the Lemma,

$$\mathbb{E}|Z^2|\le\mathbb{E}(\mathbb{E}(|Z^2||\mathcal{F}_{t_k}))\le K\mathbb{E}|X(t_k)-X_k|^2h.$$

Hence, $\|Z\|\le K\|X(t_k)-X_k\|h^{1/2}$. The second term is similar: it is controlled by $K\sqrt{1+\mathbb{E}|X_0|^2}h^{p_2}$.

Lastly, the last term is controlled directly by the assumption

$$\le K(1+\mathbb{E}|X_k|^2)h^{2p_2}\le K(1+\mathbb{E}|X_0|^2)h^{2p_2}.$$

To summarized, we have (note $p_1\ge p_2+1/2$)

$$E_{k+1}^2\le E_k^2(1+Kh)+CE_kh^{p_2+1/2}+Ch^{2p_2}$$

This implies that

$$E_k^2 \leq K e^{CT} h^{2p_2-1}.$$

The fundamental theorem can be applied directly to Euler-Maruyama schemes to conclude the same claims we have proved. You will do this in homework for how to use this theorem to prove the claims for the Euler-Maruyama schemes.

2 Itô Taylor expansion and Milstein scheme

How do we get high order schemes? For ODE, the high order schemes like RK-r, mid-point method indeed approximate the Taylor expansion at t_n with higher order terms. Hence, a natural approach is to get approximations for higher terms in the Taylor expansion for SDEs, called the Itô-Taylor expansion.

In this section, we will do for 1D case.

2.1 A way for Itô-Taylor expansion

Consider the ODE flow

$$\frac{dX}{dt} = a(X).$$

Now, we want to expand $f(X_t)$ (eventually, we are interested in $f(x) = x$).

The time derivative associated with the ODE flow is given by:

$$\frac{d}{dt} f(X_t) = a(X) \frac{d}{dx} f(X) =: (\mathcal{L}f)(X).$$

Hence, we have

$$\begin{aligned}
f(X) = f(X_{t_0}) + \int_{t_0}^{t} \frac{d}{ds} f(X_s)\, ds &= f(X_{t_0}) + \int_{t_0}^{t} (\mathcal{L}f)(X_s)\, ds \\
&= f(X_{t_0}) + \int_{t_0}^{t} (\mathcal{L}f)(X_{t_0})\, ds + \int_{t_0}^{t} \int_{t_0}^{s_1} \frac{d}{ds_2} \mathcal{L}f(X_{s_2})\, ds_2 ds_1 \\
&= f(X_{t_0}) + (\mathcal{L}f)(X_{t_0})(t - t_0) + \int_{t_0}^{t} (t - s_2) \mathcal{L}^2 f(X_{s_2})\, ds_2.
\end{aligned}$$

Clearly, we can apply this technique repeatedly and get

$$f(X_t) = f(X_{t_0}) + \sum_{l=1}^{r} \frac{(t - t_0)^l}{l!} L^l f(X_{t_0}) + \int_{t_0}^{t} \frac{(t - s)^r}{r!} L^{r+1} f(X_s) ds.$$

For SDE, we can do similar things and obtain the so-called Itô-Taylor expansion.

$$X_t = X_{t_0} + \int_{t_0}^{t} b(X_s)ds + \int_{t_0}^{t} \sigma(X_s)dW_s.$$

Consider a smooth function f. Then, Ito's formula gives

$$\begin{aligned} f(X_t) = f(X_{t_0}) &+ \int_{t_0}^{t} \left(b(X_s)\frac{\partial}{\partial x}f(X_s) + \frac{1}{2}\sigma^2(X_s)\frac{\partial^2}{\partial x^2}f(X_s) \right) ds + \int_{t_0}^{t} \sigma(X_s)\frac{\partial f(X_s)}{\partial x}dW_s \\ &= f(X_{t_0}) + \int_{t_0}^{t} L^0 f(X_s)ds + \int_{t_0}^{t} L^1 f(X_s)dW_s. \end{aligned}$$

We have introduced

$$L^0 = b\frac{d}{dx} + \frac{1}{2}\sigma^2\frac{d^2}{dx^2}, \quad L^1 = \sigma\frac{d}{dx}.$$

Next, we apply this formula for $L^0 f(X_s)$ and $L^1 f(X_s)$ respectively, and obtain

$$\begin{aligned} f(X_t) &= f(X_{t_0}) + L^0 f(X_{t_0})(t - t_0) + L^1 f(X_{t_0})\Delta W \\ &+ \int_{t_0}^{t}\int_{t_0}^{s_1} (L^0)^2 f(X_{s_2})\, ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1} L^1(L^0 f)(X_{s_2})\, dW_{s_2} ds_1 \\ &+ \int_{t_0}^{t}\int_{t_0}^{s_1} L^0(L^1 f)(X_{s_2})\, ds_2 dW_{s_1} + \int_{t_0}^{t}\int_{t_0}^{s_1} (L^1)^2 f(X_{s_2}) dW_{s_2} dW_{s_1}. \quad (1) \end{aligned}$$

Clearly, if we throw away all the double integrals, we get the Euler-Maruyama scheme.

2.2 An estimate for the multiple integrals

In the multi-dimensional case, if we do the substitution for many times, we will have iterated integrals of the form

$$I_{i_1,\cdots,i_j}(h) := \int_t^{t+h} dw_{i_j}(\theta) \int_t^{\theta} dw_{i_{j-1}}(\theta_1) \cdots \int_t^{\theta_{j-2}} dw_{i_1}(\theta_{j-1}).$$

Here, we understand

$$w_0 = t.$$

The expectation is zero when there is one i_p that is nonzero. When they are all there, the expectation is of order h^j.

For the mean square magnitude,

Lemma 4.

$$[\mathbb{E}(I^2_{i_1,\cdots,i_j})]^{1/2} = O(h^{\sum_{k=1}^{j}(2-i'_k)/2}),$$

where

$$i'_k = \begin{cases} 0 & i_k = 0, \\ 1, & i_k = 1. \end{cases}$$

This lemma confirms that one dW contributes 1/2 smallness for the mean square magnitude, while one dt contributes 1. Hence, we can keep the desired orders as we want, by this intuitive understanding.

Proof of the lemma. The proof can be done easily by induction.

In fact, if $i_j = 0$, then by Hölder, one has

$$\mathbb{E}I^2_{i_1,\cdots,i_j} \le h \int_t^{t+h} \mathbb{E}I^2_{i_1,\cdots,i_{j-1}}(s)\, ds$$

The integral contributes 1/2 smallness while the extra h contributes 1/2.

Otherwise, by the Itô's isometry,

$$\mathbb{E}I^2_{i_1,\cdots,i_j} = \int_t^{t+h} \mathbb{E}I^2_{i_1,\cdots,i_{j-1}}(s)\, ds$$

The integral contributes 1/2. Hence, the induction gives the desired result. □

2.3 Several schemes

From large to small, we should have the following $dW > dt \approx (dW)^2 > dtdW \sim (dW)^3 > dt^2$. Let us keep these terms:

We now set $f(x) = x$ in (1).

$$X_t = X_{t_0} + \sigma(X_{t_0})\Delta W + b(X_{t_0})(t - t_0) + \int_{t_0}^{t}\int_{t_0}^{s_1} L^1\sigma(X_{s_2})dW_{s_2}dW_{s_1} + \int_{t_0}^{t}\int_{t_0}^{s_1} L^1 b(X_{s_2})\, dW_{s_2}ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1} L^0\sigma(X_{s_2})\, ds_2 dW_{s_1} + \int_{t_0}^{t}\int_{t_0}^{s_1} L^0 b(X_{s_2})\, ds_2 ds_1 \quad (2)$$

If we only keep dW and dt terms, we get the Euler-Maruyama scheme.

2.3.1 Milstein

If we do approximation for the expansion and keep to $(dW)^2$, we arrive at the following:

$$X_t \approx X_{t_0}+b(X_{t_0})(t-t_0)+\sigma(X_{t_0})\Delta W+L^1\sigma(X(t_0))\int_{t_0}^{t}(W(s_1)-W(t_0))dW(s_1).$$

The Itô integral can be evaluated directly which is $\frac{1}{2}(\Delta W^2-\Delta t)$.

The Milstein scheme is given by

$$X^{n+1}=X^n+b(X^n)\Delta t+\sigma(X^n)\Delta W_n+\frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2-\Delta t).$$

Note that the two ΔW_n's in this equation should be the same instead of two i.i.d random variables.

Note that if $\sigma = const$, it is then reduced to Euler-Maruyama scheme.

2.3.2 Higher order attempts

If we keep to $dtdW$ and dW^3, then one has

$$\begin{aligned}X_t = X_{t_0}+\sigma(X_{t_0})\Delta W+b(X_{t_0})(t-t_0)++\frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2-\Delta t)\\ +\int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}(L^1)^2\sigma(X_{s_3})dW_{s_3}dW_{s_2}dW_{s_1}+\int_{t_0}^{t}\int_{t_0}^{s_1}L^1b(X_{s_2})\,dW_{s_2}ds_1\\ +\int_{t_0}^{t}\int_{t_0}^{s_1}L^0\sigma(X_{s_2})\,ds_2dW_{s_1}+\int_{t_0}^{t}\int_{t_0}^{s_1}L^0b(X_{s_2})\,ds_2ds_1 \quad (3)\end{aligned}$$

The approximation for this is

$$\begin{aligned}X^{n+1}=X^n+b(X^n)\Delta t+\sigma(X^n)\Delta W_n+\frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2-\Delta t)\\ +(L^1)^2\sigma(X^n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}(W(s_1)-W(t_n))dW_{s_1}dW_s\\ +L^1b(X^n)\int_{t_n}^{t_{n+1}}(W(s)-W(t_n))ds+L^0\sigma(X^n)\int_{t_n}^{t_{n+1}}(s-t_n)dW(s).\end{aligned}$$

If we include the dt^2, we get

$$X^{n+1} = X^n + b(X^n)\Delta t + \sigma(X^n)\Delta W_n + \frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2 - \Delta t)$$
$$+(L^1)^2\sigma(X^n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}(W(s_1)-W(t_n))dW_{s_1}dW_s+L^1b(X^n)\int_{t_n}^{t_{n+1}}(W(s)-W(t_n))ds$$
$$+ L^0\sigma(X^n)\int_{t_n}^{t_{n+1}}(s-t_n)dW(s) + \frac{1}{2}L^0b(X^n)\Delta t^2.$$

3 The orders of some schemes

The Milstein scheme is given by

$$X^{n+1} = X^n + b(X^n)\Delta t + \sigma(X^n)\Delta W_n + \frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2 - \Delta t).$$

The approximation for this is

$$X^{n+1} = X^n + b(X^n)\Delta t + \sigma(X^n)\Delta W_n + \frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2 - \Delta t)$$
$$+ (L^1)^2\sigma(X^n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}(W(s_1) - W(t_n))dW_{s_1}dW_s$$
$$+ L^1b(X^n)\int_{t_n}^{t_{n+1}}(W(s) - W(t_n))ds + L^0\sigma(X^n)\int_{t_n}^{t_{n+1}}(s-t_n)dW(s).$$

If we include the dt^2, we get

$$X^{n+1} = X^n + b(X^n)\Delta t + \sigma(X^n)\Delta W_n + \frac{1}{2}\sigma(X^n)\sigma'(X^n)(\Delta W_n^2 - \Delta t)$$
$$+(L^1)^2\sigma(X^n)\int_{t_n}^{t_{n+1}}\int_{t_n}^{s}(W(s_1)-W(t_n))dW_{s_1}dW_s+L^1b(X^n)\int_{t_n}^{t_{n+1}}(W(s)-W(t_n))ds$$
$$+ L^0\sigma(X^n)\int_{t_n}^{t_{n+1}}(s-t_n)dW(s) + \frac{1}{2}L^0b(X^n)\Delta t^2.$$

Let

$$\rho := X_{t,x}(t+h) - \bar{X}_{t,x}(t+h).$$

We recall

$$|\mathbb{E}\rho| \le K\sqrt{1+|x|^2}h^{p_1}, \quad \sqrt{\mathbb{E}|\rho|^2} \le K\sqrt{1+|x|^2}h^{p_2},$$

These coefficients can be read out directly from the remainder terms.

For the Milstein scheme, the mean is given by the L^0 terms.

$$p_1 = 2, \quad p_2 = 3/2.$$

For the second scheme

$$p_1 = 2, \quad p_2 = 2.$$

To satisfy $p_1 \geq p_2 + 1/2$, we can only take $p_2 = 3/2$. Hence, the order is 1.

For the third scheme

$$p_1 = 3, p_2 = 2.$$

The order is 3/2.

Exercise: Compute p_1 and p_2 for $X^{n+1} = X^n + \sigma(X^n)\Delta W_n$. Does this scheme converge?