Advanced computational methods-Lecture 8

1 Stochastic stability and stiff systems

1.1 The notion of stochastic stability

The theory here is an analogy of the stability region for ODE schemes. The model problem for which we apply the scheme is the geometric Brownian motion

$$dX = \lambda X dt + \mu X dW.$$

(Similar to $dX = \lambda X dt$ for ODEs.)

There are several notions of stability. Here, we consider two of them.

Definition 1. *Given $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$, we say the GBM is mean-square stable if $\lim_{t\to\infty} \mathbb{E}|X_t|^2 = 0$. We say it is asymptotically stable if $\mathbb{P}(\lim_{t\to\infty} |X_t| = 0) = 1$.*

The first is satisfies if $Re(\lambda) + \frac{1}{2}|\mu| < 0$ while the second is satisfied if $Re(\lambda - \frac{1}{2}|\mu|^2) < 0$.

We now move to numerical methods. What is the stability condition for Euler-Maruyama scheme? We have the relation

$$X^{n+1} = (1 + \lambda h + \mu \Delta W_n) X^n \Rightarrow \mathbb{E}|X^{n+1}|^2 = (|1 + \lambda k|^2 + |\mu|^2 h)\mathbb{E}|X^n|^2.$$

Denote

$$z = \lambda h, \quad y = |\mu|^2 h.$$

We need

$$|1 + z|^2 + y < 1,$$

for mean-square stable.

For asymptotic stability, one needs

$$\mathbb{E} \log |1 + \lambda k + \mu\sqrt{k} N(0, 1)| < 0.$$

In fact, we need

$$\prod_{i=1}^{n} (1 + \lambda k + \mu\sqrt{k} z_i) \to 0, a.s.$$

We need

$$\limsup_{n\to\infty} Re \sum_{i=1}^{n} \log(1 + \lambda k + \mu\sqrt{k} z_i) = -\infty, a.s.$$

The claim follows by strong LLN.

Hence, for a general numerical method, one can introduce the analogue of stability regions, i.e. the domain for $(z, y) = (\lambda h, |\mu|^2 h)$ such that the numerical solutions are mean-square stable.

2 Runge Kutta methods

Computing the derivative may not be desired in practice: in some applications where we do not know the formula for σ, and we only have an oracle that gives the value of σ for a given x. Then, the following RK type scheme can be used for Milstein scheme:

$$\begin{aligned} X^* &= X^n + \sigma(X^n)\sqrt{h}, \\ X^{n+1} &= X^n + b(X^n)h + \sigma(X^n)\Delta W_n + \frac{1}{2}\frac{\sigma(X^*) - \sigma(X^n)}{\sqrt{h}}(\Delta W_n^2 - h). \end{aligned}$$

The above RK scheme is for the derivatives involving σ. In some higher order schemes, there can be derivatives in b as well. Let us consider the 3/2 order scheme

$$\begin{aligned} X^{n+1} = X^n &+ bh + \sigma\Delta W_n + \frac{1}{2}\sigma\sigma'(\Delta W^2 - h) \\ &+ b'\sigma\Delta Z_n + \frac{1}{2}(bb' + \frac{1}{2}\sigma^2 b'')h^2 + (b\sigma' + \frac{1}{2}\sigma^2\sigma'')(\Delta W_n h - \Delta Z_n) \\ &+ \frac{1}{2}\sigma(\sigma\sigma'' + (\sigma')^2)(\frac{1}{3}\Delta W_n^2 - h)\Delta W_n. \end{aligned}$$

For additive noise, this becomes

$$X^{n+1} = X^n + bh + \sigma\Delta W_n + b'\sigma\Delta Z_n + \frac{1}{2}(bb' + \frac{1}{2}\sigma^2 b'')h^2$$

In higher dimensional case, the last term corresponds to the form

$$\frac{1}{2}(Lb)h^2.$$

This involves second order derivatives of b. The idea is to use the Itô's formula

$$db(X(t)) = L\nabla b(X)\,dt + \sum_r (\Lambda_r b)dW_r,$$

where $\Lambda_r = \sum_i \sigma_{ir}\partial_i$.

This motivates to approximate

$$Lah = b(\bar{X}(t+h)) - b(t,x) - \sum_r (\Lambda_r b)\Delta W_r,$$

where $\bar{X}$ can be obtained by the Euler-Maruyama scheme. This gives a type of Runge-Kutta method. (see Theorem 5.9 in the book of Milstein)

3 General Itô-Taylor expansion*(Not required)

The general Itô-Taylor expansion, unfortunately, is kind of tedious. Here, I only list out the result and attach the proof. We will not go over the proof in class. You can read the book "Numerical solution of stochastic differential equations" by Kloeden and Platen (section 5.1-5.5).

Setup and definitions

To obtain the Itô-Taylor expansions, we need to treat the the multiple Itô integrals carefully.

For a multi-index α, $\mathcal{H}_\alpha$ denotes the set of functions for which one can define the α interated integrals. Suppose α is of length $\ell(\alpha)$. For $1 \le i \le \ell(\alpha)$, $\alpha_i \in \{0, 1, \dots, m\}$ where m is the dimension of the Brownian motion. Let $\mathcal{M}$ be the set of multi-indices. If $\alpha_i = 0$, it means integration on time while $\alpha_i = j$, it means integration on W^j.

The iterated integrals are then defined as integration from the left of the multi-index from left to right. The integral from stopping time ρ to stopping time τ is defined as

$$I_\alpha[f]_{\rho,\tau} = \begin{cases} \int_\rho^\tau I_{\alpha-}[f]_{\rho,s} ds, \ \alpha_\ell = 0, \\ \int_\rho^\tau I_{\alpha-}[f]_{\rho,s} dW^{\alpha_\ell}, \ \alpha_\ell \ge 1, \end{cases}$$

where $\alpha- = (\alpha_1, \dots, \alpha_{\ell-1})$. We use v to mean the empty α ($\ell(\alpha) = 0$) and define $I_v[f]_{\rho,\tau} = f_\tau$, and this will be consistent with the definition above. (This means for $\alpha = v$, the lower bound does not matter).

To combine the integrals, one may define $W^0 = t$ and the integrals can be written uniformly as dW^j.

Remark 1. *A useful fact regarding the multiple integral for $f = 1$, denoted by I_α or $I_{\alpha,t}$, is given by the following proposition.*

Proposition 1.

$$W^j I_{\alpha,t} = \sum_{i=0}^{\ell} I_{(\alpha_1,\ldots,\alpha_i,j,\alpha_{i+1},\ldots,\alpha_\ell),t} + \sum_{i=1}^{\ell} \chi_{\alpha_i=j\neq 0} I_{(\alpha_1,\ldots,\alpha_{i-1},0,\alpha_{i+1},\ldots,\alpha_\ell),t}$$

where χ means the indicator and the left hand side means multiplication.

Proof. Let $X = W^j, Y = I_\alpha$. Then, we have by definition

$$dY = I_{\alpha-} dW^{\alpha_\ell}$$

Hence,

$$d(XY) = XdY + YdX + d[X,Y]$$

We find easily that

$$d[X,Y] = \chi_{\alpha_\ell=j\neq 0} I_{\alpha-} ds$$

Consequently,

$$\begin{aligned} W^j I_\alpha &= \int_0^t I_{\alpha,s} dI_{(j),s} + \int_0^t I_{(j),s} I_{\alpha-,s} W^j_s + \chi_{\alpha_\ell=j\neq 0} \int_0^t I_{\alpha-,s} ds \\ &= I_{(\alpha_1,\ldots,\alpha_\ell,j)} + \int_0^t I_{(j),s} I_{\alpha-,s} dW^j_s + \chi_{\alpha_\ell=j\neq 0} I_{(\alpha_1,\ldots,\alpha_{\ell-1},0)} \end{aligned}$$

If $\ell(\alpha) \leq 1$, the claims follow directly from this formula. For $\ell \geq 2$, one does induction. The key induction step is to replace $I_{(j)} I_{\alpha-}$ with one single I using the induction assumption. □

Now, consider the Ito process,

$$dX = b(t,X)dt + \sum_{j=1}^{m} \sigma^j(t,X) dW^j$$

The generator is

$$L^0 = \partial_t + \sum_{k=1}^{d} b^k \partial_{x_k} + \frac{1}{2} \sum_{k,l=1}^{d} \sum_{j=1}^{m} \sigma^{k,j} \sigma^{l,j} \partial^2_{k,l}$$

We also introduce the following operators for $j \geq 1$:

$$L^j = \sum_{k=1}^{d} \sigma^{k,j} \frac{\partial}{\partial x^k}$$

These are clearly from the Itô formula.

For a smooth function f, one defines the coefficient function

$$f_\alpha = \begin{cases} f, \ \ell = 0, \\ L^{\alpha_1} f_{-\alpha}, \ \ell \geq 1 \end{cases}$$

where $-\alpha$ is the sequence by deleting the first entry: if $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, then $-\alpha = (\alpha_2, \ldots, \alpha_\ell)$.

Definition 2. *Let $\mathcal{A}$ be a nonempty set of multi-indices. It is called a hierarchical set if (1). $\forall \alpha \in \mathcal{A}$, $\ell(\alpha) < \infty$; (2). For $\alpha \in \mathcal{A}$ with $\ell(\alpha) \geq 1$, we have $-\alpha \in \mathcal{A}$.*

The remainder set $\mathcal{B}(\mathcal{A})$ is given by

$$\mathcal{B}(\mathcal{A}) = \{\alpha \notin \mathcal{A}, -\alpha \in \mathcal{A}\}.$$

We have the following Itô-Taylor expansion

Theorem 1. *Let X_t be the Ito process*

$$dX = b(t, X)dt + \sum_{j=1}^{m} \sigma^j(t, X) dW^j$$

that holds for $t \in [t_0, T]$. Let ρ and τ be two stopping times satisfying

$$t_0 \leq \rho(\omega) \leq \tau(\omega) \leq T$$

with probability 1. Let $\mathcal{A} \subset \mathcal{M}$ be a hierarcical set and f be smooth enough. Then, the Itô-Taylor expansion holds

$$f(\tau, X_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(\rho, X_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathcal{B}(\mathcal{A})} I_\alpha[f_\alpha(\cdot, X_\cdot)]_{\rho,\tau}$$

If $\mathcal{A} = \{v\}$, the formula reduces to Itô formula:

$$f(\tau, X_\tau) = f(\rho, X_\rho) + \int_\rho^\tau L^0 f(s, X_s) ds + \sum_{j=1}^{m} \int_\rho^\tau L^j f(s, X_s) dW_s^j,$$

The proof

The proof basically is done by Itô formula and induction.

We first present a lemma that allows induction.

Lemma 1. *Let $t_0 \le \rho \le \tau \le T$. f is smooth enough. Let $\alpha, \beta \in \mathcal{M}$ with $\ell(\beta) \ge 1$. Then,*

$$I_\alpha[f_\beta(\cdot, X_\cdot)]_{\rho,\tau} = I_\alpha[f_\beta(\rho, X_\rho)]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}[f_{(j)*\beta}(\cdot, X_\cdot)]_{\rho,\tau}$$

where $(j) * \alpha = (j, \alpha_1, \ldots, \alpha_\ell)$.

The proof is straightforward using induction on the length of α and the Itô formula.

Proof. If $\ell(\alpha) = 0$, the formula is just Itô's formula (Note that $I_v[f(\cdot, X_\cdot)]_{\rho,\tau} = f(\tau, X_\tau)$ and $I_v[f(\rho, X_\rho)]_{\rho,\tau} = f(\rho, X_\rho)$):

$$f_\beta(\tau, X_\tau) = f_\beta(\rho, X_\rho) + \sum_{j=0}^{m} I_{(j)}[L^j f_\beta(\cdot, X_\cdot)]_{\rho,\tau}$$

Now, assume that the formula holds for all indices α_1 with $\ell(\alpha_1) = k - 1 \ge 0$. Consider the index α with $\ell(\alpha) = k$. Assume $\alpha = (j_1, \ldots, j_k)$.

$$\begin{aligned} I_\alpha[f_\beta(\cdot, X_\cdot)]_{\rho,\tau} = I_{(j_k)}[I_{\alpha-}[f_\beta(\cdot, X_\cdot)]_{\rho,\cdot}]_{\rho,\tau} &= I_{(j_k)}[I_{\alpha-}[f_\beta(\rho, X_\rho)]_{\rho,\cdot}]_{\rho,\tau} \\ &+ \sum_{j=0}^{m} I_{(j_k)}[I_{(j)*\alpha-}[f_{(j)*\beta}(\cdot, X_\cdot)]_{\rho,\cdot}]_{\rho,\tau} \end{aligned}$$

By definition, the right hand side equals

$$I_\alpha[f_\beta(\rho, X_\rho)]_{\rho,\tau} + \sum_{j=0}^{m} I_{(j)*\alpha}[f_{(j)*\beta}(\cdot, X_\cdot)]_{\rho,\tau}$$

□

We now prove the Itô-Taylor expansion by induction on $\ell := \sup_{\alpha \in \mathcal{A}} \ell(\alpha)$.

Proof of the Itô-Taylor expansion. Let

$$\ell := \sup_{\alpha \in \mathcal{A}} \ell(\alpha).$$

If $\ell = 0$, then

$$\mathcal{B}(\mathcal{A}) = \{(0), (1), \ldots, (m)\}.$$

Clearly,

$$\sum_{\alpha\in\mathcal{A}} I_\alpha[f_\alpha(\rho,X_\rho)]_{\rho,\tau} = I_v[f(\rho,X_\rho)]_{\rho,\tau} = f(\rho,X_\rho).$$

It can be seen that the Itô-Taylor expansion is just Itô's formula.

Now, assume the claim holds for $\ell \le k-1$ with $k \ge 1$. We consider $\ell = k$. Note that

$$\mathcal{E} = \{\alpha \in \mathcal{A} :\ \ell(\alpha) \le k-1\}$$

is a hierarchical set. By the induction assumption:

$$f(\tau,X_\tau) = \sum_{\alpha\in\mathcal{E}} I_\alpha[f_\alpha(\rho,X_\rho)]_{\rho,\tau} + \sum_{\alpha\in\mathcal{B}(\mathcal{E})} I_\alpha[f_\alpha(\cdot,X_\cdot)]_{\rho,\tau}.$$

Clearly, $\mathcal{A}\setminus\mathcal{E} \subset \mathcal{B}(\mathcal{E})$. It follows that

$$\begin{aligned} f(\tau,X_\tau) = \sum_{\alpha\in\mathcal{E}} I_\alpha[f_\alpha(\rho,X_\rho)]_{\rho,\tau} &+ \sum_{\alpha\in\mathcal{A}\setminus\mathcal{E}} I_\alpha[f_\alpha(\cdot,X_\cdot)]_{\rho,\tau} \\ &+ \sum_{\alpha\in\mathcal{B}(\mathcal{E})\setminus(\mathcal{A}\setminus\mathcal{E})} I_\alpha[f_\alpha(\cdot,X_\cdot)]_{\rho,\tau}. \end{aligned}$$

For the second term, we can apply the lemma and then get

$$\sum_{\alpha\in\mathcal{A}\setminus\mathcal{E}} I_\alpha[f_\beta(\cdot,X_\cdot)]_{\rho,\tau} = \sum_{\alpha\in\mathcal{A}\setminus\mathcal{E}} I_\alpha[f_\beta(\rho,X_\rho)]_{\rho,\tau} + \sum_{\alpha\in\mathcal{A}\setminus\mathcal{E}} \sum_{j=0}^{m} I_{(j)*\alpha}[f_{(j)*\beta}(\cdot,X_\cdot)]_{\rho,\tau}$$

It is clear that we only need to check whether we have

$$\mathcal{B}(\mathcal{E})\setminus(\mathcal{A}\setminus\mathcal{E}) \cup [\cup_{j=0}^{m}\{(j)*\alpha : \alpha\in\mathcal{A}\setminus\mathcal{E}\}] = \mathcal{B}(\mathcal{A}).$$

The first set is

$$\{\alpha \notin \mathcal{E}\cup\mathcal{A} : -\alpha \in \mathcal{E}\} = \{\alpha \notin \mathcal{A} : -\alpha\in\mathcal{E}\}$$

The second set is

$$\{\alpha : \ell(\alpha) = k+1, -\alpha \in \mathcal{A}\setminus\mathcal{E}\}$$

Hence, the union is exactly, $\mathcal{B}(\mathcal{A})$. □

4 Weak approximations

In this lecture, we study weak schemes for SDEs (Itô equations). Recall that there are roughly two kinds of things we want to do:

1. Approximate the sample paths of the SDE; namely $t \mapsto X(\omega, \cdot)$.
2. Approximate the law of $X(t)$, or the distributions.

We consider a numerical scheme of the following form:

$$\bar{X}_{t,x}(t+h) = x + A(t, x, h; \xi) \tag{1}$$

Unlike in the strong schemes, the random variable ξ can be unrelated to the given Brownian motion $W(t)$. The reason is that we only aim to approximate the distributions.

We then generate the numerical solution by

$$X^{k+1} = X^k + A(t_k, X^k, h; \xi_k), \tag{2}$$

where ξ_0 is independent of X^0, and ξ_k is independent of all the $X^m, m \le k$ and $\xi_\ell, \ell \le k-1$.

The convergence corresponding to approximating distribution is called **weak convergence**.

Definition 3. *Fix time $T > 0$. We say a discrete approximation $\{X^n(k)\}$ of the the solution of the SDE converges in the weak sense with order $p > 0$ if for any $\phi \in C_b^\infty$ (bounded, smooth, every derivative is bounded), there exists $C(\phi, T) > 0$ such that*

$$\sup_{n: nk \le T} |\mathbb{E}\phi(X^n(k)) - \mathbb{E}\phi(X(nk))| \le C(\phi, T) k^p,$$

for all sufficiently small step size k.

A key tool to analyze the weak order is the backward Kolmogorov equation. Recall that

$$u(x,t) = \mathbb{E}_x \phi(X(t))$$

satisfies

$$\partial_t u = \mathcal{L}u = b \cdot \nabla u + \frac{1}{2}\sigma\sigma^T : \nabla^2 u.$$

Then, it follows that

$$u(x,t) = e^{t\mathcal{L}}\phi(x).$$

With some assumptions on b and σ, we have the semigroup expansion

$$u(x,t) = \sum_{j=0}^{n} \frac{t^n}{j!} \mathcal{L}^j \phi + R$$

with the remainder term controlled by $\|R\|_\infty \le Ct^{n+1}\|\mathcal{L}^{n+1}\phi\|_\infty$.

Pick a test function ϕ,

$$\mathbb{E}\phi(X^n) - \mathbb{E}\phi(X(t_n)) = \mathbb{E}u(0, X^n) - \mathbb{E}u(t_n, X^0)$$

The meaning of $\mathbb{E}$ has been changed for the second term–The space for the continuous Brownian motion disappears! With the right hand side, we can work the probability space for the numerical random variables.

Remark 2. *Some people tend to use* $g(t,x) = u(T-t, x)$ *which satisfies*

$$\partial_t g + \mathcal{L}g = 0,$$

with terminal condition $g(T,x) = u(0,x) = \phi(x)$*. Some people also call this the backward Kolmogorov equation. No matter which you use, the ideas are the same.*

4.1 An approach based on semigroup

In the above theorem, the test functions are chosen to have polynomial growth. Nowadays, people prefer to use bounded test functions with bounded derivatives. Bounded test functions are more convenient in theory and proofs. The difference is that bounded test functions induce weaker topology. However, together moment estimates, the convergence with this weaker topology can be generalized to the test functions with polynomial growth.

Below, we focus on bounded test functions and assume that

Assumption 1. *b and σ are bounded and have bounded derivatives.*

This assumption might be strong. However, it is here only for convenience.

The second approach here uses the semigroup expansion and Markov property a lot.

Note that $\{X^n\}$ is a time-homogeneous Markov chain. We define the operator

$$(S^m f)(x) = \mathbb{E}(f(X^m)|X^0 = x).$$

The first key observation is:

Lemma 2. *$\{S^m\}$ is a semigroup. In other words, $S^pS^{m-p} = S^m$.*

To see this, we have

$$(S^m f)(x) = \mathbb{E}(f(X^m)|X^0 = x) = \mathbb{E}(\mathbb{E}(f(X^m)|X^p)|X^0 = x)$$

By Markov property and the time-homogenuity,

$$\mathbb{E}(f(X^m)|X^p = y) = \mathbb{E}(f(X^{m-p})|X^0 = y) = (S^{m-p}f)(y).$$

Hence, the right hand side is $S^p(S^{m-p}f)$.

Exercise: Use this to show that the multiplication for $\{S^\alpha\}$ is associative.

Now, we focus on $X^0 = x$, because the general initial distribution is just a superposition of such initial conditions.

$$\begin{aligned}\mathbb{E}u(0, X^n) - \mathbb{E}u(t_n, X^0) &= (S^n u(0, \cdot))(x) - (S^0 u(t_n, \cdot))(x) \\ &= \sum_{m=0}^{n-1} (S^{m+1}u(t_n - t_{m+1}, \cdot))(x) - (S^m u(t_n - t_m, \cdot))(x).\end{aligned}$$

Clearly, we only need to estimate

$$S^{m+1}u(t_n - t_{m+1}, \cdot) - S^m u(t_n - t_m, \cdot) = S^m(Su(t_n - t_{m+1}, \cdot) - u(t_n - t_m, \cdot))$$

Lemma 3. *S is L^∞-non-expansive. In other words, $\|Sf - Sg\|_\infty \le \|f - g\|_\infty$.*

This is clear and we omit the proof. Then, we have

$$\|S^m(Su(t_n - t_{m+1}, \cdot) - u(t_n - t_m, \cdot))\|_\infty \le \|Su(t_n - t_{m+1}, \cdot) - u(t_n - t_m, \cdot)\|_\infty$$

Theorem 2. *For any formulate the local and global errors into a claim..*

Example: the Euler-Maruyama scheme

By the explicit formula,

$$(Su(t_n - t_{m+1}, \cdot))(x) = \mathbb{E}u(t_n - t_{m+1}, x + b(x)k + \sigma(x)\Delta W).$$

On the other hand, by the semigroup expansion,

$$u(t_n - t_m, x) = (e^{k\mathcal{L}}u(t_n - t_{m+1}, \cdot))(x) = u(t_n - t_{m+1}, x) + k\mathcal{L}u(t_n - t_{m+1}, x) + O(k^2).$$

To compare them, we do Taylor expansion on the first term:

$$\mathbb{E}u(t_n - t_{m+1}, x + b(x)k + \sigma(x)\Delta W) = \int u(t_n - t_{m+1}, x + b(x)k + \sqrt{k}\sigma(x)z)\rho(z)\, dz$$

By Taylor expansion and the fact that the odd moments of z are zero, we find easily that

$$\mathbb{E}u(t_n - t_{m+1}, x + b(x)k + \sigma(x)\Delta W) = u(t_n - t_{m+1}, x) + kb(x) \cdot \nabla u + \frac{k}{2}\sigma(x)\sigma^T(x) : \nabla^2 u + O(k^2)$$

Hence, we find

$$\|Su(t_n - t_{m+1}, \cdot) - u(t_n - t_m, \cdot)\|_\infty \leq Ck^2.$$

Eventually,

$$\|\mathbb{E}_x u(0, X^n) - \mathbb{E}u(t_n, x)\|_\infty \leq C(T)k.$$