Possible Project 2

Consider the following SDEs on sphere

$$dX = b(X)dt + \sigma(X) \circ dW.$$

Here, b and $\sigma_{\cdot,k}$ ($1 \leq k \leq p$) are tangent vectors on $\mathbb{S}^{d-1}$, and W is a p-dimensional Brownian motion in $\mathbb{R}^p$. The "$\circ$" symbol means the integration is in the Stratonovich sense.

Consider the following scheme:

1: Sample $z^n \sim \mathcal{N}(0, I_d)$.
2: Obtain a prediction

$$Y^{n+1} = X^n + b(X^n)k + \sigma \left(\mathbb{Q}\left(X^n + \frac{1}{2}\sigma(X^n)\sqrt{k}z^n\right) \right) \sqrt{k}z^n.$$

3: Project the point back to sphere:

$$X^{n+1} = \mathbb{Q}(Y^{n+1}).$$

Here, we define

$$\mathbb{Q}x := \frac{x}{|x|}.$$

- (Basic part) Show that this scheme is convergent in weak sense. Try to figure out the weak order. Do numerical tests to verify what you found.

- (Advanced part) Show that the error can be controlled uniformly in time.