HW2

1. Find the linear least square square approximation for $f(x) = x \ln x$ on $[1, 3]$ (You can either use orthogonal polynomials or just the usual basis $\{1, x\}$, whichever you like).

2. Let $\{\phi_j\}_{j=0}^n$ be orthogonal functions on $[a, b]$ with respect to weight function $w(x)$. We have known that the least square approximation for f is given by

$$S(x) = \sum_{j=0}^{n} a_j \phi_j(x),$$

where

$$a_j = \int_a^b w(x) f(x) \phi_j(x)\, dx / \int_a^b w(x) \phi_j^2(x)\, dx.$$

Consider the inner product we defined:

$$\langle f, g \rangle_w = \int_a^b w(x) f(x) g(x)\, dx.$$

(a). Show that

$$\langle f, S \rangle_w = \langle S, S \rangle_w,$$

and thus the Pythagorean theorem holds

$$\langle f - S, f - S \rangle_w + \langle S, S \rangle_w = \langle f, f \rangle_w$$

Try to explain this using the triangles in the plane.

(b). Use Part (a) to show that the Bessel inequality holds, i.e.,

$$\sum_{j=0}^{n} \frac{(\int_a^b w(x) f(x) \phi_j(x)\, dx)^2}{\int_a^b w(x) \phi_j^2(x)\, dx} \leq \int_a^b w(x) f^2(x)\, dx.$$

3. Consider the weight function $w(x) = 1 + x^2$ on interval $[-1, 1]$. Try to use the Gram-Schimdt process to obtain some orthogonal polynomials with degrees $0, 1, 2$.

4. Given the following table of data

W	R	W	R	W	R	W	R	W	R
0.017	0.154	0.025	0.23	0.020	0.181	0.020	0.180	0.025	0.234
0.087	0.296	0.111	0.357	0.085	0.260	0.119	0.299	0.233	0.537
0.174	0.363	0.211	0.366	0.171	0.334	0.210	0.428	0.783	1.47
1.11	0.531	0.999	0.771	1.29	0.87	1.32	1.15	1.35	2.48
1.74	2.23	3.02	2.01	3.04	3.59	3.34	2.83	1.69	1.44
4.09	3.58	4.28	3.28	4.29	3.40	5.48	4.15	2.75	1.84
5.45	3.52	4.58	2.96	5.30	3.88			4.83	4.66
5.96	2.40	4.68	5.10					5.53	6.94

we want to find some function to fit the data. Hence, we desire to write a short code to do this (this means you need to write a code to generate the matrix and the right hand side of the linear system).

It is found that the variable $y = \ln R$ is roughly a polynomial of $x = \ln W$.

(a). Find the linear least square approximation for (x, y).

(b). Find the quadratic least square approximation for (x, y).

5. Find the Padé approximation $R_{2,2}(x)$ of $f(x) = \cos x$ at $a = 0$.

6. (This was not gone over in class. Bonus. 3 pts)

   Using Padé approximation gives some rational function of the form

$$R_{nm}(x) = \frac{P_n(x)}{Q_m(x)},$$

   which has the same derivatives as f up to order $m + n$. If we only use polynomials, we need a polynomial of degree $N = m + n$, denoted by $M_N(x)$.

   (a). Given x, analyze the number of operations (multiplications/divisions and addtions/subtractions), if we compute $R_{nm}(x)$ and $M_N(x)$ directly.

   (b). One can alternatively write $R_{nm}(x)$ as the continued fraction form. Analyze the number of operations for evaluating $R_{nm}(x)$ in the continued fraction form.