HW3

1. Consider the following way for finding the square root of a: let $\bar{x}$ be some approximation and the accurate square root be x_*. Let $\Delta = x_* - \bar{x}$. Then,
$$(\bar{x} + \Delta)^2 = a.$$
Since Δ^2 is very small, we can have
$$\bar{x}^2 + 2\bar{x}\Delta \approx a.$$
Hence, $\Delta = \frac{a-\bar{x}^2}{2\bar{x}}$ and the new approximation is $x_1 = \bar{x} + \Delta$. If x_1 is not good enough, you can repeat this process.

    - Try the above method for finding $\sqrt{17}$ by setting $\bar{x} = 4$. Do the process twice.
    - Is the obtained value close to $\sqrt{17}$? Explain why by identifying the method to one you know and figure out the convergence order.

2. Apply Newton's method to find the negative root of $f(x) = e^x - 3x^2$ and the sequence generated is $\{x_k\}$. Keep your answer such that $|x_k - x_{k-1}|$ is within 10^{-6}. Denote the root by x_*. Try to use x_* to represent the following limits
$$\lim_{k\to\infty} \frac{x_{k+1} - x_*}{(x_k - x_*)^2}$$
and
$$\lim_{k\to\infty} \frac{x_{k+1} - x_k}{(x_k - x_{k-1})^2}.$$
Hint: for the second limit, you may want to use $\lim_{k\to\infty} \frac{x_k - x_*}{(x_{k-1}-x_*)^\beta} = 0$ for any $\beta < 2$.

3. Consider the iteration for some given $a > 0$ and $x_0 \neq 0$, and $|x_0|$ is small so that the limit exists,
$$x_{k+1} = \frac{x_k(x_k^2 + 3a)}{3x_k^2 + a}.$$
Find the limit x_*. What is the value of β such that
$$\lim_{k\to\infty} \frac{x_{k+1} - x_*}{(x_k - x_*)^\beta}$$
is a nonzero real number?

4. Let A be a symmetric real matrix. Show that it is positive definite if and only if all of its eigenvalues are real positive numbers.

5. Suppose the vectors x in $\mathbb{C}^d$ are equipped with some norm $\|\cdot\|$, which satisfies three properties:

   - $\|x\| \geq 0$ and the equality holds if and only if $x = 0$
   - $\|\alpha x\| = |\alpha|\|x\|$, $\alpha \in \mathbb{C}$,
   - $\|x + y\| \leq \|x\| + \|y\|$.

   Such a norm can induce a norm for the matrix

   $$\|A\| := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

   Show that

   $$\max_i |\lambda_i| \leq \|A\|.$$

   In other words, the largest absolute value of the eigenvalues of A is controlled by any norms of the matrices induced by norms of vectors.

6. Use Gauss elimination to solve the following linear system of equations, and find the inverse of the coefficient matrix.

   $$\begin{aligned} 4x_1 - x_2 + x_3 &= 8, \\ 2x_1 + 5x_2 + 2x_3 &= 3, \\ x_1 + 2x_2 + 4x_3 &= 11. \end{aligned}$$

7. (Bonus: 5) To get a sense of how the order of convergence affects the decay rate of errors, let us consider the following three sequences for $a_0 = b_0 = c_0 = 1$:

   $$a_{k+1} = \sin(a_k)$$

   $$b_{k+1} = \frac{1}{2}\sin(b_k)$$

   and

   $$c_{k+1} = c_k - \sin(c_k).$$

   They all converge to zero. However, the first one converges linearly but with the asymptotic ratio being 1, the second one converges linearly but with a smaller asymptotic ratio, while the last one converges superlinearly. Try to find good bounds as functions of n for them. For example, we desire something like $a_n \leq CK^n, Cn^p$ and so on.