HW5

1. Consider the following linear system

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= -12, \\ -x_1 + 4x_2 + 2x_3 &= 20, \\ 2x_1 - 3x_2 + 10x_3 &= 3. \end{aligned}$$

   - Solve this system using Jacobi method and Gauss-Seidel method. (You can write a code to do this.)
   - Do your iterative methods converge in the first part? Justify them by estimating their spectral radii.

2. Consider the following linear system

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1, \\ 2x_1 + 2x_2 + 2x_3 &= 4, \\ -x_1 - x_2 + 2x_3 &= -5. \end{aligned}$$

   - Solve this system using Jacobi method and Gauss-Seidel method. (You can write a code to do this.)
   - Do your iterative methods converge in the first part? Justify them by estimating their spectral radii.

3. Consider the following system

$$\begin{aligned} 4x_1 - x_2 &= 1, \\ -x_1 + 4x_2 - x_3 &= 4, \\ -x_2 + 4x_3 &= -3. \end{aligned}$$

   Solve this system using SOR with different values of ω (by computer code). What is the best choice of ω? Recall that we must need $\omega \in (0, 2)$.

   Let B_J be the matrix in the Jacobi iteration. Compare your optimal ω to

$$\frac{2}{1 + \sqrt{1 - \rho^2(B_J)}}$$

4. Use the Jordan canonical form to show that $\lim_{k\to\infty} \|B^k\| = 0$ if and only if $\rho(B) < 1$ where $\rho(B)$ means the spectral radius of B.

5. Regarding the convergence theorem for iterative method

$$x^{(k+1)} = Bx^{(k)} + f,$$

if we only say "The sequence converges for any initial $x^{(0)}$", we cannot conclude $\rho(B) < 1$.

- Construct B and f such that the iterative method converges for any $x^{(0)}$, while $\rho(B) \geq 1$.
- Show that if the case mentioned in the first part happens, then the matrix has an eigenvalue to be 1.

  The argument is like this: let the limit be x_*. Then, $x_* = Bx_* + f$. Then,

  $$(x^{(k+1)} - x_*) = B(x^{(k)} - x_*) = B^k(x_0 - x_*) \to 0.$$

  If the limit is unique, then $B^k e_0 \to 0$ for any e_0 and we must have $B^k \to 0$ so $\rho(B) < 1$. Hence, the limit cannot be unique. That means there exists two solutions to $x_* = Bx_* + f$. In other words, the equation $v = Bv$ has a nonzero solution, so there is an eigenvalue to be 1.

  Some people say $\rho(B) = 1$ so there is an eigenvalue to be 1. This is wrong. The reason is that there could be complex eigenvalues.

6. Consider $Ax = b$. Suppose A is positive definite. Show that the Gauss-Seidel iterative method for this system converges by showing that the spectral radius is less than 1.

7. Consider the iterative method

$$x^{(k+1)} = Bx^{(k)} + f,$$

where B is of size $n \times n$. If $\rho(B) = 0$, show that $x^{(n)}$ is the exact solution.

The point is to show $B^n = 0$. You can either say its charateristic function is λ^n or use its Jordan canonical form.

8. Consider the following ODE:

$$-(e^{-x}u')' = x^2, \quad u(0) = u(1) = 0.$$

We use the finite difference method to solve this problem. Let u_j be the solution at $x_j := jh$ where $h = \frac{1}{n}$. We use the following finite difference to approximate the equation

$$-\frac{1}{h^2}(e^{-x_{j+1/2}}(u_{j+1} - u_j) - e^{-x_{j-1/2}}(u_j - u_{j-1})) = x_j^2, \quad j = 1, \cdots, n-1. \tag{1}$$

This gives a linear system with positive definite coefficient matrix

$$Au = b.$$

- Write a code for the conjugate gradient descent and apply your code to this system. Find the solution numerically and compare to the exact solution.
- (Bonus: 3 points) Do not construct the matrix A explicitly. Instead, write a function that takes u as the input and outputs Au vector (in your function, there should be no matrix A and doing the matrix-vector multiplication Au). Use your function and do the conjugate gradient descent again.

  The point is to use the finite difference scheme (the left hand side of (1)) to generate the output. For example, if the input is U, you can do $([U(2:n-1), 0] - U).*e^{-(x+h/2)} - e^{-(x-h/2)}.*(U - [0, U(1:n-2)]))/h^2$