HW7

1. Apply forward Euler's method, the midpoint point method (a RK2 method), and a second order Admas-Bashforth method to solve

$$u'(t) = t^2 + t - u, \quad u(0) = 0.$$

Pick $h = 0.1$ and compare your numerical values to the exact solution. (You can code.)

2. Repeat the proof in class and show that the forward Euler's method converges for the following problem

$$u'(t) = \arctan(u), \quad u(0) = 1,$$

if we aim to compute $u(1)$.

3. Consider the following scheme

$$y_{n+1} = \frac{1}{2}(y_n + y_{n-1}) + \frac{h}{4}(4y_{n+1}^2 - y_n^2 + 3y_{n-1}^2).$$

- Try to find the ODE $y' = f(t, y)$ that this scheme is approximating.
- Compute the local truncation error and find the order of convergence.

Comment: This is a linear multistep method.

4. Try to find the stability region of the trapezoidal method (the implicit method, not the explicit RK-2 method).

5. Use forward Euler, backward Euler and Runga-Kutta 4 method to solve the following ODE (you can code)

$$y' = 1 - 100(y - t), \quad y(0) = 0.5.$$

Discuss and compare the methods.

Comment: This is a typical so-called stiff problem

6. We solve the following equation

$$u_t = u_x, \quad u_0(x) = e^{\sin(x)}, u(0, t) = u(2\pi, t),$$

for $x \in [0, 2\pi)$. To solve this, we pick a spatial step $h = \frac{2\pi}{N}$ and set $x_j = jh$. We then get the following ODE system

$$\frac{d}{dt}u_j(t) = \frac{1}{2h}(u_{j+1} - u_{j-1}), \quad j = 0, 1, \cdots, N-1.$$

Note that we regard $u_N = u_0$ and $u_{-1} = u_{N-1}$. This can be formulated into the matrix form as

$$\frac{d}{dt}u = Au,$$

where

$$A = \frac{1}{2h}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & & 0 \\ \cdots & & & & \cdots & \\ 0 & 0 & & -1 & 0 & 1 \\ 1 & 0 & & & -1 & 0 \end{pmatrix}.$$

Our aim is to solve this ODE system. Take

$$h = \frac{2\pi}{2^6}.$$

Apply the forward Euler, RK2 and RK3 to solve this system, with various step sizes. Discuss your numerical results.

Comment: You may find that the methods with stability regions containing some part of imagainary axis may behave better.