Homework 1

March 7, 2014, due Mar 11, 2014

Class materials can be found at `http://202.121.182.14:8080/faculty/leizhang`.

Problem 1.

For a positive definite symmetric matrix A, define the matrix 2-norm as

$$\|A\|_2 = \sup_{\|x\|_2} \frac{\|Ax\|_2}{\|x\|_2}$$

Prove that

$$\|A\|_2 = \lambda_{\max}(A)$$

Problem 2. For $V = -2h$, $Q = -h$, $P = 0$, $T = \theta h$, where $0 < \theta \leq 1$. Try to find weights ω_V, ω_Q, ω_T, ω_P, such that for any smooth function z,

$$\frac{1}{h^2}(\omega_V z(V) + \omega_Q z(Q) + \omega_T z(T) - \omega_P z(P)) = \frac{\partial^2 z}{\partial x^2}\Big|_P + O(h^2)$$

Problem 3. To solve the boundary value problem

$$-u_{xx}(x) = f(x), \quad x \in (0,1). \tag{1}$$

with boundary condition $u(0) = u(1) = 0$, $f \in C^0$, we subdivide interval $[0,1]$ into n equal subintervals with $h = 1/n$. Let $x_j = jh$, $j = 0, \ldots, n$, we are looking for u_j, the approximations to the exact solution $u(x_j)$ at x_j.

(a) (Formulation) If we use central differences to approximate u_{xx},

$$u_{xx}(x_i) \simeq \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}. \tag{2}$$

write down the resulting finite difference scheme (including boundary condition), and the associated linear system $Au = F$ for the unknowns, specify A, u and F.

(b) (Existence and Uniqueness) Prove that A is nonsingular, therefore the finite difference scheme has a unique solution. *hint: there are many ways to do this, one way is to show that $v^T Av > 0$ for any $v \neq 0$, namely, A is symmetrically positive definite.*

(c) (Programming) A is a tri-diagonal matrix, $Au = F$ can be efficiently solved by Gaussian elimination method which will be introduced later. In this homework, suppose that $f = (3x + x^2)e^x$, implement the numerical scheme in your familiar programming language (Matlab, Python, C or Fortran). Take $n = 10$, plot the solution you obtain.

(d) (Maximum Principle) For $v \in \mathbb{R}^m$, we say that $v \geq 0$ if $v_i \geq 0$ for $1 \leq i \leq m$. Show that if $Aw = v$ and $v \geq 0$, then $w \geq 0$. Furthermore, this implies that $\alpha_{ij} \geq 0$, where α_{ij} are the entries of A^{-1}. Use this property to show that if $f \geq 0$, then $u_j \geq 0$, for $j = 0, \ldots, n$.

(e) (Discrete Stability) The function $v(x) = \frac{x(1-x)}{2}$ satisfies

$$-\frac{v(x_{j+1}) - 2v(x_j) + v(x_{j-1})}{h^2} = 1. \tag{3}$$

Use this to show that the entries α_{ij} of A^{-1} satisfies

$$0 \leq \sum_{j=1}^{n-1} \alpha_{ij} \leq \frac{1}{8}. \tag{4}$$

Prove that

$$\max_{1 \leq i \leq n-1} |u_i| \leq \frac{1}{8} \max_{1 \leq i \leq n-1} |f(x_i)| \tag{5}$$

(f) (Truncation Error) Like ODE, we can define truncation error,

$$T_j = -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - f(x_j) \tag{6}$$

Calculate the leading order term of T_j.

(g) (Error Equation) Let $e_j = u(x_j) - u_j$ be the discretization error. Show that e_j satisfies the equation $Ae = T$, where $e = (e_1, \ldots, e_{n-1})^T$ and $T = (T_1, \ldots, T_{n-1})^T$. Using (5) to prove the convergence result

$$\max_{1 \leq i \leq n-1} |u(x_i) - u_i| \leq \frac{h^2}{96} \max_{0 \leq x \leq 1} |u^{(4)}(x)|. \tag{7}$$

(h) (Justification) When $f = (3x + x^2)e^x$, the exact solution is $u(x) = x(1-x)e^x$. Take $n = 4, 8, 16, 32, 64, 128, 256$, and compute numerical solutions u^n with Matlab. Calculate $\|u - u^n\|_\infty := \max_{1 \leq i \leq n-1} |u(x_i) - u_i^n|$, plot (log-log) the convergence with resepct to n. Numeically estimate the prefactor in the estimate $\|u - u^n\|_\infty \simeq Ch^\alpha$, compare it with the result in (7)