Homework 5

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Problem 1. Let $U = [U_0,\ U_1,\ \ldots,\ U_m]^T$ be a vector of function values at equally spaced points on the interval $0 \le x \le 1$, and suppose the underlying function is periodic and smooth. Then we can approximate the first derivative u_x at all of these points by DU, where D is circulant matrix such as

$$D_- = \frac{1}{h}\begin{bmatrix} 1 & & & & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}, \qquad D_+ = \frac{1}{h}\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \\ 1 & & & & -1 \end{bmatrix} \tag{1}$$

for first-order accurate one-sided approximations or

$$D_0 = \frac{1}{2h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & -1 & 0 & 1 & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix} \tag{2}$$

for a second-order accurate centered approximation. (These are illustrated for a grid with $m + 1 = 5$ unknowns and $h = 1/5$.)

The advection equation $u_t + au_x = 0$ on the interval $0 \le x \le 1$ with periodic boundary conditions gives rise to the MOL discretization $U'(t) = -aDU(t)$ where D is one of the matrices above.

(a) Discretizing $U' = -aD_-U$ by forward Euler gives the first order upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_j^n - U_{j-1}^n), \tag{3}$$

where the index i runs from 0 to m with addition of indices performed mod $m + 1$ to incorporate the periodic boundary conditions.

Suppose instead we discretize the MOL equation by the second-order Taylor series method,

$$U^{n+1} = U^n - akD_-U^n + \frac{1}{2}(ak)^2 D_-^2 U^n. \tag{4}$$

Compute D_-^2 and also write out the formula for U_j^n that results from this method.

(b) What are the orders of the methods (3) and (4) derived in part (a)?

(c) Suppose we make the method (3) more symmetric:

$$U^{n+1} = U^n - \frac{ak}{2}(D_+ + D_-)U^n + \frac{1}{2}(ak)^2 D_+ D_- U^n. \tag{5}$$

Write out the formula for U_j^n that results from this method.

(d) Use the numerical methods in (3), (4) and (5) to solve the equation $u_t + u_x = 0$ on $[0,1] \times [0,1]$, with periodic boundary condition and initial value $u(x,0) = \sin(2\pi x)$. Draw the L^2 error with respect to exact solution for $h = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$, and $k = 1/8, 1/16, 1/32, 1/64, 1/128, 1/256$ (You need to run 108 simulations for all the combinations of h and k for 3 methods). Verify the order of convergence for each method.