Homework 6

April 10, 2014

Problem 1. An $n \times n$ circulant matrix C takes the form

$$C = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}. \tag{1}$$

Calculate the eigenvectors and eigenvalues of the matrix C.

Problem 2. Study the stability of the upwind method

$$U_j^{n+1} = U_j^n - \frac{ak}{h}(U_{j+1}^n - U_j^n) \tag{2}$$

by writing the method as applying the forward Euler method to the advection-diffusion equation of the form $u_t + au_x = \varepsilon u_{xx}$.

(a) write down the equation for the semi-discretization analysis, $U' = A_\varepsilon U$.

(b) calculate the eigenvalues μ_p of A_ε.

(c) show that all $k\mu_p$ are located on an ellipse in the complex plane, specify the center and semi-axes of the ellipse.

(d) obtain the condition such that the ellipse is within the stability region of forward Euler method, which is the stability condition of the upwind method.

Problem 3. Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0, \quad x \in \mathbb{R}, t \in [0, T] \\ u(x, 0) = u_0(x), \end{cases} \tag{3}$$

where

$$u_0(x) = \begin{cases} 1, & x \le 0, \\ 0, & x > 0. \end{cases} \tag{4}$$

Take $h = 0.01$, $k/h = 0.5$. Use Lax-Fredrichs method to compute the solution up to $t_n = 0.5$. Compare the numerical solution with the exact solution. Notice that in this case, you just need to choose a sufficient large interval, for example, $[0, 1]$ to compute the numerical solution, and the boundary condition can be chosen as $U(0, t) = 1$, and $U(1, t) = 0$.