# Homework 1

March 5, 2014, due Mar 13

**Problem 1.** Given Newton's equation

$$\begin{cases} y' = 1 - 3x + y + x^2 + xy \\ y(0) = 0 \end{cases} \tag{1}$$

(a) Compute its power series solution up to order 6.

(b) Let $P_n$ denote power series solution up to order $n$, plot all the $P_n$ in the interval $[0, 2]$ for $n = 1 \cdots 6$.

(c) Use Euler method to calculate the solution of equation (1). By the convergence Theorem, when we choose $h$ small enough, Euler's method approximates true solution very well. Take $h = 1e - 4$, and let $u_h$ as the 'approximate' true solution. Compare it with $u_{0.1}$, $u_{0.01}$ and the power series solutions $P_n$ for $n = 1 \cdots 6$.

**Problem 2.** Van der Pols equation

$$u'' - \varepsilon(1 - u^2)u' + u = 0 \tag{2}$$

subject to the initial conditions $u(0) = A_1$ and $u'(0) = A_2$, where $A_1$ and $A_2$ are given real numbers, and $\varepsilon > 0$ a parameter, models electrical circuits connected with electronic oscillators. Rewrite the equation as a coupled system of two first-order differential equations with appropriate initial conditions. Formulate Euler's method for this system, when $\varepsilon = 1$, $A_1 = 1/2$ and $A_2 = 1/2$, on the interval $[0, 1]$ using points with uniform spacing $\Delta t$. Compute the Euler approximations to $u(t)$ and $u'(t)$ at point $t = \Delta t$.
(hint: higher order ODEs can be written as a system of first order ODEs).

**Problem 3.** Consider the initial value problem

$$u' = \log\log(4 + u^2), \quad t \in [0, 1], \quad u(0) = 1, \tag{3}$$

and the sequence $(U_n)_{n=0}^N$, $N \geq 1$, generated by the explicit Euler method

$$\frac{U_{n+1} - U_n}{\Delta t} = \log\log(4 + U_n^2), \quad n = 0, \ldots, N-1, \quad U_0 = 1, \tag{4}$$

using the time points $t_n = n\Delta t$, $n = 0, \ldots, N$, with spacing $\Delta t = 1/N$. Here log denotes the logarithm with base $e$.

(a) Let $T_n$ denote the truncation error of Euler's method at $t = t_n$ for this initial value problem. Show that $|T_n| \leq \Delta t/(4e)$.

(b) Verify that

$$|u_{n+1} - U_{n+1}| \leq (1 + \Delta t L)|u_n - U_n| + \Delta t|T_n|, n = 0, \ldots, N - 1,$$

where $L = 1/(2 \log 4)$.

(c) Find a positive integer $N_0$, as small as possible, such that

$$\max_{0 \leq n \leq N} |u_n - U_n| \leq 10^{-4}$$

whenever $N \geq N_0$.

**Problem 4.** Use Lagrange interpolation polynomial to contruct 3-step Adams-Bashforth method, namely, calculate all the coefficients $b_i$ for the multistep scheme

$$y_{n+3} = y_{n+2} + h[b_2 f(t_{n+2}, y_{n+2}) + b_1 f(t_{n+1}, y_{n+1}) + b_0 f(t_n, y_n)] \qquad (5)$$